The third order Benjamin-Ono equation on the torus : well-posedness, traveling waves and stability

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Abstract

We consider the third order Benjamin-Ono equation on the torus

$$\partial_t u = \partial_x \left(-\partial_{xx} u - \frac{3}{2} u H \partial_x u - \frac{3}{2} H(u \partial_x u) + u^3 \right).$$

We prove that for any $t \in \mathbb{R}$, the flow map continuously extends to $H^s_{r,0}(\mathbb{T})$ if $s \geq 0$, but does not admit a continuous extension to $H^{-s}_{r,0}(\mathbb{T})$ if $0 < s < \frac{1}{2}$. Moreover, we show that the extension is not weakly sequentially continuous in $L^2_{r,0}(\mathbb{T})$. We then classify the traveling wave solutions for the third order Benjamin-Ono equation in $L^2_{r,0}(\mathbb{T})$ and study their orbital stability.

Contents

1 Introduction

We are interested in the third equation of the integrable Benjamin-Ono hierarchy on the torus

$$\partial_t u = \partial_x \left(-\partial_{xx} u - \frac{3}{2} u H \partial_x u - \frac{3}{2} H(u \partial_x u) + u^3 \right). \tag{1}$$

The operator H is the Hilbert transform, defined as

$$Hf(x) = \sum_{n \in \mathbb{Z} \setminus 0} -i \operatorname{sgn}(n) \widehat{f}(n) \, \mathrm{e}^{inx}, \quad f = \sum_{n \in \mathbb{Z}} \widehat{f}(n) \, \mathrm{e}^{inx}, \quad \widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) \, \mathrm{e}^{-inx} \, \mathrm{d}x.$$

1.1 Benjamin-Ono equations and integrability

The Benjamin-Ono equation on the torus

$$\partial_t u = H\partial_{xx} u - \partial_x(u^2),$$

was introduced by Benjamin [2] and Ono [12] in order to describe long internal waves in a two-layer fluid of great depth. This equation admits an infinite number of conserved quantities $\mathcal{H}_k, k \geq 1$ (see Nakamura [11] for a proof on the real line). The evolution equations associated to the conservation laws

$$\partial_t u = \partial_x(\nabla \mathcal{H}_k(u)) \tag{2}$$

are the equations for the Benjamin-Ono hierarchy [8].

From Nakamura [10] and Bock, Kruskal [3], we know that the Benjamin-Ono equation admits a Lax pair

$$\frac{\mathrm{d}}{\mathrm{d}t} L_u = [B_u, L_u],$$
$$L_u = Dh - T_u, \quad B_u = iD^2 + 2iT_{D(\Pi u)} - 2iDT_u.$$

Here, $D = -i\partial_x$ and T_u is the Toeplitz operator on the Hardy space

$$L^2_+(\mathbb{T}) = \{h \in L^2(\mathbb{T}) \mid \forall n < 0, \quad \widehat{h}(n) = 0\}$$

defined as

$$T_u : h \in L^2_+(\mathbb{T}) \mapsto \Pi(uh) \in L^2_+(\mathbb{T}),$$

and $\Pi : L^2(\mathbb{T}) \to L^2_+(\mathbb{T})$ is the Szegő projector. The Hamiltonians $\mathcal{H}_k(u)$ are defined from the Lax operator L_u as

$$\mathcal{H}_k(u) = \langle L_u^k \mathbb{1} | \mathbb{1} \rangle. \tag{3}$$

In particular, the Hamiltonian for equation (1) is

$$\mathcal{H}_4(u) + \frac{1}{2}\mathcal{H}_2(u)^2 = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{2}(\partial_x u)^2 - \frac{3}{4}u^2 H\partial_x u + \frac{1}{4}u^4\right) \mathrm{d}x. \tag{4}$$

In [5], Gérard and Kappeler constructed global Birkhoff coordinates for the Benjamin-Ono equation on the torus. In these coordinates, the evolution equations for the Benjamin-Ono hierarchy are easier to understand. Indeed, denote by Φ the Birkhoff map

$$\Phi : u \in L^2_{r,0}(\mathbb{T}) \mapsto (\zeta_n(u))_{n\geq 1} \in h^{\frac{1}{2}}_+,$$

where $L^2_{r,0}(\mathbb{T})$ is the subspace of real valued functions in $L^2(\mathbb{T})$ with zero mean, and

$$h^{\frac{1}{2}}_+ = \Big\{(\zeta_n)_{n\geq 1} \mid \sum_{n\geq 1} n|\zeta_n|^2 < +\infty\Big\}.$$

Then in the Birkhoff coordinates, equation (2) of the hierarchy associated to $\mathcal{H}_k$ becomes

$$\partial_t \zeta_n = i\omega_n^{(k)}\zeta_n, \quad n \geq 1$$

when the frequencies

$$\omega_n^{(k)} = \frac{\partial(\mathcal{H}_k \circ \Phi^{-1})}{\partial|\zeta_n|^2}$$

are well-defined. For instance, this formula is valid if the sequence $\zeta(0) = (\zeta_n(0))_{n\geq 1}$ only has a finite number of nonzero terms, or in other words, if $\Phi^{-1}(\zeta(0))$ is a finite gap potential. In this case, the frequencies $\omega_n^{(k)}$ only depend on the actions $|\zeta_p|^2$, and the evolution simply reads

$$\zeta_n(t) = \zeta_n(0)\, \mathrm{e}^{i\omega_n^{(k)}(\zeta(0))t}, \quad t \in \mathbb{R}, \quad n \geq 1.$$

For the third equation of the hierarchy (1), the frequencies write

$$\omega_n^{(4)}(\zeta) = n^3 + n\sum_{p\geq 1} p|\zeta_p|^2 - 3\sum_{p\geq 1}\min(p,n)^2|\zeta_p|^2 + 3\sum_{p,q\geq 1}\min(p,q,n)|\zeta_p|^2|\zeta_q|^2. \tag{5}$$

More details about the frequencies $\omega_n^{(k)}$ and formula (5) can be found in Appendix A.1.

We refer to Saut [15] for a detailed survey of the Benjamin-Ono equation and of its hierarchy.

1.2 Main results

Our first main result is the determination the well-posedness threshold for the third order Benjamin-Ono equation. For $s \in \mathbb{R}$, we use the notation

$$H^{-s}_{r,0}(\mathbb{T}) = \{u \in H^s(\mathbb{T},\mathbb{R}) \mid \langle u|\mathbb{1}\rangle = 0\}.$$

We prove that the flow map is globally $\mathcal{C}^0$-well-posed (in the sense of Definitions 1 and 2 from [6]) in $H^s_{r,0}(\mathbb{T})$ when $s \geq 0$, but is not globally $\mathcal{C}^0$-well-posed in $H^{-s}_{r,0}(\mathbb{T})$ when $0 < s < \frac{1}{2}$.

Theorem 1.1. *For all $t \in \mathbb{R}$, the flow map for equation (1) $\mathcal{S}^t : u_0 \mapsto u(t)$, defined for finite gap potentials, admits a continuous extension to $H^s_{r,0}(\mathbb{T})$ for all $s \geq 0$, but does not admit a continuous extension to $H^{-s}_{r,0}(\mathbb{T})$ for $0 < s < \frac{1}{2}$.*

Remark 1.2. *Note that if $s \geq \frac{1}{2}$, the maps $t \in \mathbb{R} \mapsto u(t)$ constructed in this way are solutions to equation (1) in the distribution sense.*

We also investigate the question of the sequential weak continuity for the flow map.

Theorem 1.3. *For all $t \in \mathbb{R}$, the extension of flow map for equation (1) $\mathcal{S}^t : u_0 \mapsto u(t)$ is weakly sequentially continuous in $H^s_{r,0}(\mathbb{T})$ for $s > 0$, but is not weakly sequentially continuous in $L^2_{r,0}(\mathbb{T})$.*

In [6], Gérard, Kappeler and Topalov proved that the flow map for the Benjamin-Ono equation is globally $\mathcal{C}^0$-well-posed in $H^s_{r,0}(\mathbb{T})$ for $s > -\frac{1}{2}$, whereas from [13] there is no continuous extension of the flow map to $H^s_{r,0}(\mathbb{T})$ when $s < -\frac{1}{2}$. We expect that the well-posedness threshold on the torus increases by $\frac{1}{2}$ for each new equation in the hierarchy : for the equation corresponding to the k-th Hamiltonian $\mathcal{H}_k$, $k \geq 4$, the threshold should be $H^{\frac{k}{2}-2}_{r,0}(\mathbb{T})$ (see Remarks 2.6 and A.1). Note that all the equations for the Benjamin-Ono hierarchy have critical Sobolev exponent $-\frac{1}{2}$.

Let us mention former approaches to the Cauchy problem for higher order Benjamin-Ono equations. Tanaka [17] considered more general third order type Benjamin-Ono equations on the torus

$$\partial_t u = \partial_x(-\partial_{xx}u - c_1 u H\partial_x u - c_2 H(u\partial_x u) + u^3),$$

and proved local well-posedness in $H^s(\mathbb{T})$ for $s > \frac{5}{2}$. He deduced global well-posedness in $H^s(\mathbb{T})$, $s \geq 3$ for the integrable case $c_1 = c_2 = \frac{3}{2}$.

On the real line, Feng and Han [4] proved local well-posedness in $H^s(\mathbb{R})$, $s \geq 4$ for the third equation of the Benjamin-Ono hierarchy (1). Considering more general third order type Benjamin-Ono equations under the form

$$\partial_t u - bH\partial_{xx}u - a\partial_{xxx}u = cv\partial_x v - d\partial_x(vH\partial_x v + H(v\partial_x v)),$$

Linares, Pilod and Ponce [7] established local well-posedness in $H^s(\mathbb{R})$, $s \geq 2$, then Molinet and Pilod [9] proved global well-posedness in $H^s(\mathbb{R})$, $s \geq 1$.

Concerning Benjamin-Ono equations of fourth order on the torus and on the real line, Tanaka [16] proved local well-posedness in H^s, $s > \frac{7}{2}$ for a more general family of fourth order

type Benjamin-Ono equations, and deduced global well-posedness in H^s, $s \geq 4$ in the integrable case.

Our second main result is the classification of the traveling waves for the third order Benjamin-Ono equation in $L^2_{r,0}(\mathbb{T})$, i.e. the solutions to (1) under the form $u(t,x) = u_0(x+ct)$, $t \in \mathbb{R}$, $x \in \mathbb{T}$, $u_0 \in L^2_{r,0}(\mathbb{T})$.

Definition 1.4. *For $N \geq 1$, we say that $u \in L^2_{r,0}(\mathbb{T})$ is a N gap potential if the set $\{n \geq 1 \mid \zeta_n(u) \neq 0\}$, where $\Phi(u) = (\zeta_n(u))_{n\geq 1}$, is finite and of cardinality N.*

Theorem 1.5. *A potential $u_0 \in L^2_{r,0}(\mathbb{T})$ defines a traveling wave for equation (1) if and only if*

- *either u_0 is a one gap potential ;*
- *either u_0 is a two gap potential, and the two nonzero indexes $p < q$ satisfy, with $\gamma_p = |\zeta_p|^2$ and $\gamma_q = |\zeta_q|^2$,*

$$0 < \gamma_p < \frac{1}{2}\left(p + \sqrt{p^2 + 4q\frac{p+q}{3}}\right)$$

and

$$\gamma_q = \frac{q\frac{p+q}{3} - \gamma_p^2 + p\gamma_p}{2\gamma_p + q}.$$

Note that from [5], the one gap potentials are the only traveling wave solutions to the Benjamin-Ono equation ; they have been characterized by Amick and Toland [1].

Our last main result answers the question of orbital stability for these two types of traveling waves.

Definition 1.6. *Let $u_0 \in L^2_{r,0}(\mathbb{T})$ be a one gap traveling wave. We say the u_0 is orbitally stable if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if v is a solution to (1) with initial condition $v_0 \in L^2_{r,0}(\mathbb{T})$ such that $\|v_0 - u_0\|_{L^2(\mathbb{T})} \leq \delta$, then*

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{T}} \|v(t) - u_0(\cdot + \theta)\|_{L^2(\mathbb{T})} \leq \varepsilon.$$

Theorem 1.7. *The one gap traveling waves are orbitally stable, whereas the two gap traveling waves are orbitally unstable.*

For the Benjamin-Ono equation, Pava and Natali [14] proved the orbital stability of the traveling wave solutions in $H^{\frac{1}{2}}_{r,0}(\mathbb{T})$. In [6], Gérard, Kappeler and Topalov improved the orbital stability of these solutions to $H^{-s}_{r,0}(\mathbb{T})$, $0 \leq s < \frac{1}{2}$.

Plan of the paper The paper is organized as follows. We first prove the well-posedness threshold for the third order Benjamin-Ono equation (1) in Section 2. Finally, in Section 3, we classify the traveling wave solutions and study their orbital stability properties.

In Appendix A.1, we describe how to compute the Hamiltonians $\mathcal{H}_k$ and frequencies $\omega_n^{(k)} = \frac{\partial \mathcal{H}_k \circ \Phi^{-1}}{\partial |\zeta_n|^2}$ in terms of the action variables $|\zeta_p|^2$. In Appendix A.2, we retrieve the Hamiltonian and frequencies of the third order Benjamin-Ono equation (see formulas (4) and (5)) by starting from the definition (3) of the higher order Hamiltonians. In Appendix A.3, we provide an alternative proof of a result from [18] about the structure of the higher order Hamiltonians by using formula (3), which may be of independent interest.

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2 Well-posedness threshold for the fourth Hamiltonian

Let $N \in \mathbb{N}$ and let $\mathcal{U}_N$ be the set

$$\mathcal{U}_N = \{u \in L^2_{r,0}(\mathbb{T}) \mid \zeta_N(u) \neq 0, \quad \zeta_j(u) = 0 \quad \forall j > N\}.$$

We know from [5], Theorem 3, that the restriction of the Birkhoff map Φ to $\mathcal{U}_N$ is a real analytic diffeomorphism onto some Euclidean space. In Birkhoff coordinates, the evolution along the flow of equation (1) for an initial data $u_0 \in \mathcal{U}_N$ writes

$$\begin{cases} \partial_t \zeta_n = i\omega_n^{(4)}(u_0)\zeta_n \\ \zeta_n(0) = \zeta_n(u_0) \end{cases}, \quad n \geq 1,$$

where for all $n \geq 1$, the frequencies $\omega_n^{(4)}(u_0)$ are given by (5)

$$\omega_n^{(4)}(u_0) = n^3 + n\sum_{p\geq 1} p|\zeta_p(u_0)|^2 - 3\sum_{p\geq 1}\min(p,n)^2|\zeta_p(u_0)|^2 + 3\sum_{p,q\geq 1}\min(p,q,n)|\zeta_p(u_0)|^2|\zeta_q(u_0)|^2.$$

This implies that

$$\zeta_n(u(t)) = \zeta_n(u_0)\, \mathrm{e}^{i\omega_n^{(4)}(u_0)t}, \quad t \in \mathbb{R}, \quad n \geq 1.$$

Therefore, for any finite gap inifial data u_0, belonging to some of the sets $\mathcal{U}_N$, the flow map $\mathcal{S}^t : u_0 \in \mathcal{U}_N \mapsto u(t) \in \mathcal{U}_N$ is well-defined.

In part 2.1, we prove that for all $t \in \mathbb{R}$, this flow map extends by continuity to $H^s_{r,0}(\mathbb{T})$ for $s \geq 0$. We also show that the extension is sequentially weakly continuous in $H^s_{r,0}(\mathbb{T})$ for $s > 0$, but not in $L^2_{r,0}(\mathbb{T})$. In part 2.2, we prove that the flow map does not extend by continuity to $H^{-s}_{r,0}(\mathbb{T})$ for $s > 0$. This gives a threshold for the global $\mathcal{C}^0$-well-posedness of the third order Benjamin-Ono equation in the sense of Definitions 1 and 2 from [6].

2.1 Well-posedness in $H^s_{r,0}(\mathbb{T})$, $s \geq 0$

Proposition 2.1. *Let $s \geq 0$. For any $u_0 \in H^s_{r,0}(\mathbb{T})$, there exists a continuous map $t \in \mathbb{R} \mapsto \mathcal{S}^t(u_0) = u(t) \in H^s_{r,0}(\mathbb{T})$ with $u(0) = u_0$ such that the following holds.*

For any finite gap sequence $(u_0^k)_k$ converging to u_0 in $H^s_{r,0}(\mathbb{T})$, for any $t \in \mathbb{R}$, $u_k(t) = \mathcal{S}^t(u_0^k)$ converges to $u(t)$ in $H^s_{r,0}(\mathbb{T})$ as k goes to infinity.

Moreover, the extension of the flow map $\mathcal{S} : u_0 \in H^s_{r,0}(\mathbb{T}) \mapsto (t \mapsto u(t)) \in \mathcal{C}(\mathbb{R}, H^s_{r,0}(\mathbb{T}))$ is continuous.

Recall that from [6], as mentioned in the proofs of Proposition 2 and Theorem 8, we have the following result. For $s \geq 0$, the Birkhoff map Φ defines a homeomorphism between $H^s_{r,0}(\mathbb{T})$ and the space

$$h_+^{\frac{1}{2}+s} = \Big\{(\zeta_n)_{n\geq 1} \mid \sum_{n\geq 1} n^{1+2s}|\zeta_n|^2 < +\infty\Big\}.$$

The proof of Proposition 2.1 therefore relies on the following sequential convergence result obtained after applying the Birkhoff map.

Lemma 2.2. *Fix $s \geq 0$. Let $\zeta^k = (\zeta_n^k)_{n\geq 1}$, $k \in \mathbb{N}$, and ζ be elements of $h_+^{\frac{1}{2}+s}$ such that $\|\zeta^k - \zeta\|_{h_+^{\frac{1}{2}+s}} \underset{k\to+\infty}{\longrightarrow} 0$. Then for all $t \in \mathbb{R}$,*

$$\|(\zeta_n^k\, \mathrm{e}^{i\omega_n^{(4)}(\zeta^k)t})_n - (\zeta_n\, \mathrm{e}^{i\omega_n^{(4)}(\zeta)t})_n\|_{h_+^{\frac{1}{2}+s}} \underset{k\to+\infty}{\longrightarrow} 0,$$

where the convergence is uniform on bounded time intervals.

Proof. Note that since $(\zeta^k)_k$ converges to ζ in $h_+^{\frac{1}{2}+s}$, then for all $n \in \mathbb{N}$, formula (5) for $\omega_n^{(4)}(\zeta^k)$ implies that $\omega_n^{(4)}(\zeta^k)$ converges to $\omega_n^{(4)}(\zeta)$ as k goes to infinity.

Let $\varepsilon > 0$. Fix $K \in \mathbb{N}$ such that for all $k \geq K$,

$$\|\zeta^k - \zeta\|_{h_+^{\frac{1}{2}+s}} \leq \varepsilon.$$

Using that $\zeta \in h_+^{\frac{1}{2}+s}$, fix $N \in \mathbb{N}$ such that

$$\Big(\sum_{n \geq N} n^{1+2s} |\zeta_n|^2 \Big)^{\frac{1}{2}} \leq \varepsilon.$$

Now, if $k \geq K$,

$$\begin{aligned} \|(\zeta_n^k \, \mathrm{e}^{i\omega_n^{(4)}(\zeta^k)t})_n - (\zeta_n \, \mathrm{e}^{i\omega_n^{(4)}(\zeta)t})_n\|_{h_+^{\frac{1}{2}+s}} &\leq \|(\zeta_n^k)_n - (\zeta_n)_n\|_{h_+^{\frac{1}{2}+s}} + \|(\zeta_n(\, \mathrm{e}^{i\omega_n^{(4)}(\zeta^k)t} - \, \mathrm{e}^{i\omega_n^{(4)}(\zeta)t}))_n\|_{h_+^{\frac{1}{2}+s}} \\ &\leq 3\varepsilon + \left(\sum_{n=0}^{N-1} n^{1+2s} |\zeta_n(\, \mathrm{e}^{i\omega_n^{(4)}(\zeta^k)t} - \, \mathrm{e}^{i\omega_n^{(4)}(\zeta)t})|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

which is less than 4ε for k large enough by convergence term by term of the elements in the sum. Moreover, this convergence is uniform on bounded time intervals. □

Proof of Proposition 2.1. Let $s \geq 0$ and $u_0 \in H_{r,0}^s(\mathbb{T})$. Fix $t \in \mathbb{R}$, and a sequence of finite gap initial data $(u_0^k)_k$ converging to u_0 in $H_{r,0}^s(\mathbb{T})$.

We first establish that for all $t \in \mathbb{R}$, $(u_k(t))_k$ has a limit in $H_{r,0}^s(\mathbb{T})$ as k goes to $+\infty$. By assumption, $\Phi(u_0^k)$ converges to $\Phi(u_0)$ in $h_+^{\frac{1}{2}+s}$. Define the sequence $\zeta(t)$ by

$$\zeta_n(t) := \zeta_n(u_0) \, \mathrm{e}^{i\omega_n^{(4)}(u_0)t}, \quad n \in \mathbb{N}.$$

Lemma 2.2 immediately implies that the sequence $(\Phi(u_k(t)))_k$ converges to $\zeta(t)$ in $h_+^{\frac{1}{2}+s}$. Since Φ^{-1} defines a continuous application from $h_+^{\frac{1}{2}+s}$ to $H_{r,0}^s(\mathbb{T})$, we deduce that $u_k(t)$ converges in $H_{r,0}^s(\mathbb{T})$ to $u(t) := \Phi^{-1}(\zeta(t))$. Moreover, the convergence is uniform on bounded time intervals.

We now prove the continuity of the flow map $\mathcal{S}^t$. Let $u_0^k \in H_{r,0}^s(\mathbb{T})$, $k \in \mathbb{N}$, be a sequence of initial data converging to some u_0 in $H_{r,0}^s(\mathbb{T})$. Then $\Phi(u_0^k)$ converges to $\Phi(u_0)$ in $h_+^{\frac{1}{2}+s}$, and the above Lemma 2.2 again implies that $\Phi(u_k(t))$ converges to $\Phi(u(t))$ in $h_+^{\frac{1}{2}+s}$. In other terms, $u_k(t)$ converges to $u(t)$ in $H_{r,0}^s(\mathbb{T})$, where again this convergence is uniform on bounded intervals. □

Corollary 2.3. *For all $s > 0$ and all $t \in \mathbb{R}$, the extension of the flow map restricted to $H_{r,0}^s(\mathbb{T})$: $u_0 \in H_{r,0}^s(\mathbb{T}) \mapsto u(t) \in H_{r,0}^s(\mathbb{T})$ is sequentially weakly continuous.*

Proof. Let $u_0^k \in H_{r,0}^s(\mathbb{T})$, $k \in \mathbb{N}$, be a sequence weakly converging in $H_{r,0}^s(\mathbb{T})$ to $u_0 \in H_{r,0}^s(\mathbb{T})$. Since the embedding $H_{r,0}^s(\mathbb{T}) \hookrightarrow L_{r,0}^2(\mathbb{T})$ is compact, $(u_0^k)_k$ is strongly convergent to u_0 in $L_{r,0}^2(\mathbb{T})$. By continuity of the flow map $\mathcal{S}^t$, one deduces that $(u_k(t))_k$ converges strongly to $u(t)$ in $L_{r,0}^2(\mathbb{T})$. This implies that $(u_k(t))_k$ converges weakly to $u(t)$ in $H_{r,0}^s(\mathbb{T})$. □

Proposition 2.4. *For all $t \in \mathbb{R}^*$, the extension of the flow map restricted to $L_{r,0}^2(\mathbb{T})$: $u_0 \in L_{r,0}^2(\mathbb{T}) \mapsto u(t) \in L_{r,0}^2(\mathbb{T})$ is not sequentially weakly continuous.*

Proof. Fix $t \in \mathbb{R}^*$ and $u_0 \in L^2_{r,0}(\mathbb{T}) \setminus \{0\}$. We construct a sequence $(u_0^k)_k$ in $L^2_{r,0}(\mathbb{T})$ weakly convergent to u_0 in $L^2_{r,0}(\mathbb{T})$ but such that $u_k(t) = \mathcal{S}^t(u_0^k)$ is not weakly convergent to $u(t) = \mathcal{S}^t(u_0)$ in $L^2_{r,0}(\mathbb{T})$.

Let $\alpha > 0$ to be chosen later. For $k \in \mathbb{N}$, we choose $(\zeta_p(u_0^k))_p$ converging weakly to $(\zeta_p(u_0))_p$ in $h^{\frac{1}{2}}_+$ (so that u_0^k converges weakly to u_0 in $L^2_{r,0}(\mathbb{T})$) and such that

$$|\zeta_p(u_0^k)|^2 = |\zeta_p(u_0)|^2 + \frac{\alpha}{p}\delta_{k,p}, \quad p \geq 1.$$

For instance, for $p \neq k$ we choose $\zeta_p(u_0^k) = \zeta_p(u_0)$, and for $p = k$, we choose $\zeta_k(u_0^k) = \sqrt{|\zeta_k(u_0)|^2 + \frac{\alpha}{k}}\frac{\zeta_k(u_0)}{|\zeta_k(u_0)|}$ if $\zeta_k(u_0) \neq 0$ and $\zeta_k(u_0^k) = \frac{\alpha}{k}$ if $\zeta_k(u_0) = 0$.

Fix $t \neq 0$. If $u_k(t)$ was weakly convergent to u_0 in $L^2_{r,0}(\mathbb{T})$, then $(\zeta_p(u_k(t))_p$ would converge weakly to $(\zeta_p(u(t))_p$ in $h^{\frac{1}{2}}_+$, and therefore component by component :

$$\zeta_p(u_0^k)\,\mathrm{e}^{i\omega_p^{(4)}(u_0^k)t} \underset{k\to+\infty}{\longrightarrow} \zeta_p(u_0)\,\mathrm{e}^{i\omega_p^{(4)}(u_0)t}, \quad p \geq 1.$$

In particular, let $p \geq 1$ such that $\zeta_p(u_0) \neq 0$. Then there exists a sequence $(n_k)_k$ of integers such that

$$\omega_p^{(4)}(u_0^k)t + 2\pi n_k \underset{k\to+\infty}{\longrightarrow} \omega_p^{(4)}(u_0)t.$$

From the expression (5) of $\omega_p^{(4)}(u_0^k)$ and the strong convergence of $(\zeta_p(u_0^k))_p$ to $(\zeta_p(u_0))_p$ in $\ell^2_+ = \{(\zeta_p)_{p\geq 1} \mid \sum_{p\geq 1}|\zeta_p|^2 < +\infty\}$ by compactness, we get

$$\sum_{p=1}^{+\infty} p|\zeta_p(u_0)|^2 + \alpha + \frac{2\pi n_k}{t} = \sum_{p=1}^{+\infty} p|\zeta_p(u_0^k)|^2 + \frac{2\pi n_k}{t} \underset{k\to+\infty}{\longrightarrow} \sum_{p=1}^{+\infty} p|\zeta_p(u_0)|^2.$$

We get a contradiction by choosing $\alpha \notin \frac{2\pi}{t}\mathbb{Z}$. □

2.2 Ill-posedness in $H^{-s}_{r,0}(\mathbb{T})$, $s > 0$

Proposition 2.5. *For all $t > 0$, there is no continuous local extension of the flow map $\mathcal{S}^t$ to $H^{-s}_{r,0}(\mathbb{T})$ for $0 < s < \frac{1}{2}$ in the distribution sense.*

Proof. Let us fix $0 < s < \frac{1}{2}$ and an initial data $u_0 \in H^{-s}_{r,0}(\mathbb{T}) \setminus L^2_{r,0}(\mathbb{T})$. From [6], Theorem 5, the Birkhoff map extends by continuity as an homeomorphism

$$\Phi : u \in H^{-s}_{r,0} \mapsto \Phi(u) = (\zeta_n(u))_{n\geq 1} \in h^{\frac{1}{2}-s}_+$$

where

$$h^{\frac{1}{2}-s}_+ = \Big\{(\zeta_n)_{n\geq 1} \mid \sum_{n\geq 1} n^{1-2s}|\zeta_n|^2 < +\infty\Big\}.$$

Therefore, $(\zeta_n(u_0))_{n\geq 1} := \Phi(u_0) \in h^{\frac{1}{2}-s}_+$ is well defined. Let u_0^k, $k \in \mathbb{N}$, be a sequence finite gap initial data, to be chosen later, such that u_0^k converges in $H^{-s}_{r,0}(\mathbb{T})$ to u_0. Write

$$\Phi(u_0^k) = (\zeta_n(u_0^k)\mathbb{1}_{n\leq N_k})_n, \quad k \in \mathbb{N}.$$

Since u_0^k is a finite gap potential, it belongs to $L^2_{r,0}(\mathbb{T})$. Recall that

$$\omega_n^{(4)}(u_0^k) - n\sum_{p=1}^{N_k} p|\zeta_p(u_0^k)|^2 = \widetilde{\omega_n}(u_0^k)$$

where

$$\widetilde{\omega_n}(u_0^k) = n^3 - 3\sum_{p=1}^{N_k} \min(p,n)^2 |\zeta_p(u_0^k)|^2 + 3\sum_{p=1}^{N_k}\sum_{q=1}^{N_k} \min(p,q,n)|\zeta_p(u_0^k)|^2|\zeta_q(u_0^k)|^2.$$

Since u_0^k converges to u_0 in $H^{-s}_{r,0}(\mathbb{T})$, the series $\sum_{p\geq 1}|\zeta_p(u_0)|^2$ is convergent, and

$$\sum_{p\geq 1}|\zeta_p(u_0^k)|^2 \underset{k\to+\infty}{\longrightarrow} \sum_{p\geq 1}|\zeta_p(u_0)|^2.$$

In particular, the term $\widetilde{\omega_n}(u_0^k)$ converges as k goes to infinity to

$$\widetilde{\omega_n}(u_0) = n^3 - 3\sum_{p=1}^{+\infty} \min(p,n)^2 |\zeta_p(u_0)|^2 + 3\sum_{p=1}^{+\infty}\sum_{q=1}^{+\infty} \min(p,q,n)|\zeta_p(u_0)|^2|\zeta_q(u_0)|^2.$$

For $k \in \mathbb{N}$, let

$$\tau_k := \sum_{p=1}^{N_k} p|\zeta_p(u_0^k)|^2 = \frac{1}{2}\|u_0^k\|^2_{L^2(\mathbb{T})}$$

and

$$v_k(t,\cdot) := u_k(t, \cdot - \tau_k t).$$

We use the following identity from the proof of Proposition B.1. in [5] :

$$\zeta_n(u(\cdot+\tau)) = \zeta_n(u)\,\mathrm{e}^{i\tau n}, \quad \tau \in \mathbb{R}, \quad u \in L^2_{r,0}(\mathbb{T}),$$

to deduce that for $n \in \mathbb{N}$,

$$\begin{aligned}\zeta_n(v_k(t)) &= \zeta_n(u_k(t))\,\mathrm{e}^{-in\tau_k t} \\ &= \zeta_n(u_0^k)\,\mathrm{e}^{i(\omega_n^{(4)}(u_0^k) - n\tau_k)t}.\end{aligned}$$

Since

$$\omega_n^{(4)}(u_0^k) - n\tau_k \underset{k\to+\infty}{\longrightarrow} \widetilde{\omega_n}(u_0),$$

the sequence $(\zeta_n(v_k(t)))_k$ is convergent :

$$\zeta_n(v_k(t)) \underset{k\to+\infty}{\longrightarrow} \zeta_n(u_0)\,\mathrm{e}^{i\widetilde{\omega_n}(u_0)t}. \tag{6}$$

Let $t > 0$. If there was a local extension of the flow map $\mathcal{S}^t$ in the distribution sense, then $u_k(t)$ would be weakly convergent to $u(t)$ in $H^{-s}_{r,0}(\mathbb{T})$. Applying the Birkhoff map, which is weakly sequentially continuous (see [6], Theorem 6), $\Phi(u_k(t))$ would converge weakly to $\Phi(u(t))$ in $h_+^{\frac{1}{2}-s}$. In particular, for all n,

$$\zeta_n(v_k(t))\,\mathrm{e}^{i\tau_k nt} = \zeta_n(u_k(t)) \underset{k\to+\infty}{\longrightarrow} \zeta_n(u(t)). \tag{7}$$

We deduce from (6) and (7) that if $\zeta_n(u_0) \neq 0$, then

$$\mathrm{e}^{i\tau_k nt} \underset{k\to+\infty}{\longrightarrow} \frac{\zeta_n(u(t))}{\zeta_n(u_0)}\,\mathrm{e}^{-i\widetilde{\omega_n}(u_0)t}. \tag{8}$$

We construct the sequence $(u_0^k)_k$ in order to contradict this latter point. Let $n \in \mathbb{N}$ such that $\zeta_n(u_0) \neq 0$. Fix $k \in \mathbb{N}$. From the fact that u_0 does not belong to $L^2_{r,0}(\mathbb{T})$,

$$\sum_{p>k} p|\zeta_p(u_0)|^2 = +\infty,$$

therefore one can choose $N_k \geq k+1$ such that

$$\sum_{p=k+1}^{N_k} p|\zeta_p(u_0)|^2 \geq \frac{2\pi}{nt}.$$

Let $0 < \alpha_k < 1$ such that there exists an integer m_k such that

$$\sum_{p\leq k} p|\zeta_p(u_0)|^2 + \alpha_k \sum_{p=k+1}^{N_k} p|\zeta_p(u_0)|^2 = \frac{1}{nt}(k\pi + 2\pi m_k).$$

We define u_0^k by

$$\zeta_p(u_0^k) = \begin{cases} \zeta_p(u_0) \text{ if } p \leq k \\ \sqrt{\alpha_k}\zeta_p(u_0) \text{ if } k < p \leq N_k \\ 0 \text{ if } N_k < p \end{cases}, \quad p \in \mathbb{N}.$$

By construction, u_0^k is finite gap and converges to u_0 in $H^{-s}_{r,0}(\mathbb{T})$. However,

$$\begin{aligned} \tau_k &= \sum_{p\leq k} p|\zeta_p(u_0)|^2 + \alpha_k \sum_{p=k+1}^{N_k} p|\zeta_p(u_0)|^2 \\ &= \frac{1}{nt}(k\pi + 2\pi m_k), \end{aligned}$$

which implies that

$$\mathrm{e}^{i\tau_k nt} = (-1)^k.$$

In particular, the sequence $(\mathrm{e}^{i\tau_k nt})_k$ is not convergent, and we get a contradiction with (8). □

Remark 2.6. *We expect that with a similar argument, one can prove the following fact. For the higher equations of the hierarchy, the well-posedness threshold increases by $\frac{1}{2}$ for each equation (see Remark A.1).*

3 Traveling waves for the fourth Hamiltonian

In this part, we classify all traveling wave solutions to equation (1)

$$\partial_x(-cu - \partial_{xx}u - \frac{3}{2}u^2 H\partial_x u - \frac{3}{2}H(u\partial_x u) + u^3) = 0.$$

A traveling wave of speed $c \in \mathbb{R}$ is a solution to (1) of the form $u(t,x) = u_0(x+ct)$.

The argument of [5], Proposition B.1., applies for the higher equations of the Benjamin-Ono hierarchy, implying that a potential u_0 is a traveling wave solution to (1) of speed $c \in \mathbb{R}$ if and only if for all $t \in \mathbb{R}$ and $n \geq 1$,

$$\mathrm{e}^{icnt}\zeta_n(u_0) = \mathrm{e}^{i\omega_n^{(4)}(u_0)t}\zeta_n(u_0),$$

or in other words :

$$\forall n \geq 1, \text{ if } \zeta_n(u_0) \neq 0, \text{ then } cn = \omega_n^{(4)}(u_0). \tag{9}$$

In particular, all one gap potentials are traveling wave solutions. Note that these potentials are the only traveling wave solutions to the Benjamin-Ono equation and write (see [5], Appendix B)

$$u_0(x) = \frac{pw\,\mathrm{e}^{ipx}}{1 - w\,\mathrm{e}^{ipx}} + \frac{p\overline{w}\,\mathrm{e}^{-ipx}}{1 - \overline{w}\,\mathrm{e}^{-ipx}}$$

with the nonzero gap being at index $p \geq 1$ and $w = \frac{\zeta_p(u_0)}{\sqrt{p+\gamma_p(u_0)}}$. As we will see in this section, the one gap potentials are not the only traveling wave solutions for equation (1).

In part 3.1, we first show that the traveling waves are necessarily one gap and two gap potentials, then provide a classification of the two gap traveling waves in term of their actions. In part 3.2, we prove that the one gap traveling waves are orbitally stable whereas the two gap traveling waves are orbitally unstable.

3.1 Classification of traveling wave solutions

In the following, it will be more convenient to work with the actions $\gamma_p = |\zeta_p(u_0)|^2$. Formula (5) for $\omega_n^{(4)}(u_0)$

$$\omega_n^{(4)}(u_0) = n^3 + n\sum_{p\geq 1} p\gamma_p - 3\sum_{p\geq 1}\min(p,n)^2\gamma_p + 3\sum_{p,q\geq 1}\min(p,q,n)\gamma_p\gamma_q$$

shows that $\frac{\omega_n^{(4)}(u_0)}{n}$ is equivalent to n^2 as n goes to infinity, therefore, from condition (9), the traveling waves for the third equation of the hierarchy (1) are necessarily finite gap solutions.

Proposition 3.1. *Let u_0 be a two gap potential, and $p < q$ be the indices of the two nonzero gaps with gaps $\gamma_p > 0$ and $\gamma_q > 0$. Then u_0 is a traveling wave for equation (1) if and only if*

$$0 < \gamma_p < \frac{1}{2}\left(p + \sqrt{p^2 + 4q\frac{p+q}{3}}\right)$$

and

$$\gamma_q = \frac{q\frac{p+q}{3} + p\gamma_p - \gamma_p^2}{2\gamma_p + q}.$$

Proof. Let u_0 be such a two gap potential. Then u_0 is a traveling wave if and only if $\frac{\omega_p^{(4)}(u_0)}{p} = \frac{\omega_q^{(4)}(u_0)}{q}$, where

$$\frac{\omega_p^{(4)}(u_0)}{p} = p^2 + (p\gamma_p + q\gamma_q) - 3p(\gamma_p + \gamma_q) + 3(\gamma_p^2 + 2\gamma_p\gamma_q + \gamma_q^2)$$

and

$$\frac{\omega_q^{(4)}(u_0)}{q} = q^2 + (p\gamma_p + q\gamma_q) - 3(\frac{p^2}{q}\gamma_p + q\gamma_q) + 3(\frac{p}{q}\gamma_p^2 + 2\frac{p}{q}\gamma_p\gamma_q + \gamma_q^2).$$

Taking the difference of the two terms, $\frac{\omega_p^{(4)}(u_0)}{p} = \frac{\omega_q^{(4)}(u_0)}{q}$, if and only if

$$0 = q^2 - p^2 + 3\left(-p(\frac{p}{q} - 1)\gamma_p - (q-p)\gamma_q + (\frac{p}{q} - 1)\gamma_p^2 + 2(\frac{p}{q} - 1)\gamma_p\gamma_q\right).$$

Dividing by $3(1 - \frac{p}{q})$, this necessary and sufficient condition becomes

$$0 = q\frac{p+q}{3} + p\gamma_p - q\gamma_q - \gamma_p^2 - 2\gamma_p\gamma_q,$$

i.e.

$$(2\gamma_p + q)\gamma_q = q\frac{p+q}{3} + p\gamma_p - \gamma_p^2. \tag{10}$$

Fix $\gamma_p > 0$ and γ_q satisfying this latter equality. We get that $\gamma_q > 0$ if and only if the left-hand side of the equality is positive, i.e.

$$0 < \gamma_p < \frac{1}{2}\left(p + \sqrt{p^2 + 4q\frac{p+q}{3}}\right). \tag{11}$$

Conversely, any two gap solution u_0 satisfying (10) and (11) verifies $\frac{\omega_p^{(4)}(u_0)}{p} = \frac{\omega_q^{(4)}(u_0)}{q}$, therefore is a traveling wave solution. □

Let us give an idea of the form of a two gap potential u_0 with gaps at indices $p < q$. By Theorem 3 in [5], the extension of Πu_0 as an holomorphic function on the unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$ satisfies

$$\Pi u_0(z) = -z\frac{Q'(z)}{Q(z)}$$

where $Q(z) = \det(\mathrm{Id} - zM)$ and $M = (M_{nm})_{0\le n,m\le q-1}$ is a $q \times q$ matrix defined by

$$M_{nm} = \begin{cases} \delta_{m,n+1} & \text{if } \zeta_{n+1} = 0 \\ \sqrt{\mu_{n+1}}\sqrt{\frac{\kappa_m}{\kappa_{n+1}}}\frac{\zeta_m(u_0)\overline{\zeta_{n+1}(u_0)}}{(\lambda_m - \lambda_n - 1)} & \text{if } \zeta_{n+1} \neq 0 \end{cases}.$$

A precise definition of μ_n, κ_n and λ_n can be found in [5]. Therefore, Q and Q' respectively write

$$Q(z) = 1 - z^p M_{p-1,0} - z^q M_{q-1,0}M_{p-1,p}$$

and

$$Q'(z) = -pz^{p-1}M_{p-1,0} - qz^{q-1}M_{q-1,0}M_{p-1,p}$$

This leads to

$$u_0(x) = \frac{p\,\mathrm{e}^{ipx}M_{p-1,0} + q\,\mathrm{e}^{iqx}M_{q-1,0}M_{p-1,p}}{1 - \mathrm{e}^{ipx}M_{p-1,0} - \mathrm{e}^{iqx}M_{q-1,0}M_{p-1,p}} + \frac{p\,\mathrm{e}^{-ipx}\overline{M_{p-1,0}} + q\,\mathrm{e}^{-iqx}\overline{M_{q-1,0}M_{p-1,p}}}{1 - \mathrm{e}^{-ipx}\overline{M_{p-1,0}} - \mathrm{e}^{-iqx}\overline{M_{q-1,0}M_{p-1,p}}}.$$

Proposition 3.2. *There are no three gap traveling waves.*

Proof. Let $p < q < r$ the indices for the nonzero gaps for a three gap potential u_0. The speeds for each mode write

$$\frac{\omega_p^{(4)}(u_0)}{p} = p^2 + (p\gamma_p + q\gamma_q + r\gamma_r) - 3p(\gamma_p + \gamma_q + \gamma_r) + 3(\gamma_p^2 + 2\gamma_p(\gamma_q + \gamma_r) + \gamma_q^2 + 2\gamma_q\gamma_r + \gamma_r^2),$$

$$\frac{\omega_q^{(4)}(u_0)}{q} = q^2 + (p\gamma_p + q\gamma_q + r\gamma_r) - 3(\frac{p^2}{q}\gamma_p + q(\gamma_q + \gamma_r)) + 3(\frac{p}{q}\gamma_p^2 + 2\frac{p}{q}\gamma_p(\gamma_q + \gamma_r) + \gamma_q^2 + 2\gamma_q\gamma_r + \gamma_r^2)$$

and

$$\frac{\omega_r^{(4)}(u_0)}{r} = r^2 + (p\gamma_p + q\gamma_q + r\gamma_r) - 3(\frac{p^2}{r}\gamma_p + \frac{q^2}{r}\gamma_q + r\gamma_r) + 3(\frac{p}{r}\gamma_p^2 + 2\frac{p}{r}\gamma_p(\gamma_q + \gamma_r) + \frac{q}{r}\gamma_q^2 + 2\frac{q}{r}\gamma_q\gamma_r + \gamma_r^2).$$

Let us now subtract the equalities.

$$\frac{\omega_q^{(4)}(u_0)}{q} - \frac{\omega_p^{(4)}(u_0)}{p} = q^2 - p^2 - 3((\frac{p}{q} - 1)p\gamma_p + (q-p)(\gamma_q + \gamma_r)) \\ + 3((\frac{p}{q} - 1)\gamma_p^2 + 2(\frac{p}{q} - 1)\gamma_p(\gamma_q + \gamma_r)).$$

Dividing by $3(1 - \frac{p}{q})$, we get that if $\frac{\omega_q^{(4)}(u_0)}{q} = \frac{\omega_p^{(4)}(u_0)}{p}$, then

$$0 = q\frac{p+q}{3} + p\gamma_p - q(\gamma_q + \gamma_r) - \gamma_p^2 - 2\gamma_p(\gamma_q + \gamma_r),$$

or equivalently

$$(q + 2\gamma_p)(\gamma_q + \gamma_r) = q\frac{p+q}{3} + p\gamma_p - \gamma_p^2. \tag{12}$$

Doing the same for indices p and r,

$$\frac{\omega_r^{(4)}(u_0)}{r} - \frac{\omega_p^{(4)}(u_0)}{p} = r^2 - p^2 - 3((\frac{p}{r} - 1)p\gamma_p + (\frac{q^2}{r} - p)\gamma_q + (r-p)\gamma_r) \\ + 3((\frac{p}{r} - 1)\gamma_p^2 + 2(\frac{p}{r} - 1)\gamma_p(\gamma_q + \gamma_r) + (\frac{q}{r} - 1)\gamma_q^2 + 2(\frac{q}{r} - 1)\gamma_q\gamma_r).$$

Dividing by $3(1 - \frac{p}{r})$, if $\frac{\omega_r^{(4)}(u_0)}{r} = \frac{\omega_p^{(4)}(u_0)}{p}$, then

$$0 = r\frac{r+p}{3} + p\gamma_p + \frac{pr - q^2}{r-p}\gamma_q - r\gamma_r - \gamma_p^2 - 2\gamma_p(\gamma_q + \gamma_r) - \frac{r-q}{r-p}\gamma_q^2 - 2\frac{r-q}{r-p}\gamma_q\gamma_r. \tag{13}$$

Subtracting (13) and (12), if $\frac{\omega_p^{(4)}(u_0)}{p} = \frac{\omega_q^{(4)}(u_0)}{q} = \frac{\omega_r^{(4)}(u_0)}{r}$, then

$$(r - q + 2\frac{r-q}{r-p}\gamma_q)\gamma_r = \frac{r^2 - q^2 + p(r-q)}{3} + (\frac{pr - q^2}{r-p} + q)\gamma_q - \frac{r-q}{r-p}\gamma_q^2.$$

Since

$$\frac{pr - q^2}{r-p} + q = \frac{pr - q^2 + qr - pq}{r-p} = (p+q)\frac{r-q}{r-p},$$

by multiplication by $\frac{r-p}{r-q}$, we get

$$(r - p + 2\gamma_q)\gamma_r = (r-p)\frac{r+q+p}{3} + (p+q)\gamma_q - \gamma_q^2. \tag{14}$$

Now, if u_0 is a traveling wave, the first equality (12) implies

$$\begin{aligned}\gamma_q + \gamma_r &= \frac{q\frac{p+q}{3} + p\gamma_p - \gamma_p^2}{q + 2\gamma_p} \\ &= \frac{p+q}{3} + \frac{-2\gamma_p\frac{p+q}{3} + p\gamma_p - \gamma_p^2}{q + 2\gamma_p} \\ &= \frac{p+q}{3} + \frac{\gamma_p}{3(q + 2\gamma_p)}(p - 2q - 3\gamma_p),\end{aligned}$$

in particular, since $p < q$ and $\gamma_p > 0$, necessarily

$$\gamma_q + \gamma_r < \frac{p+q}{3}. \tag{15}$$

However, the second equality (14) implies that

$$\begin{aligned}\gamma_r &= \frac{(r-p)\frac{r+q+p}{3} + (p+q)\gamma_q - \gamma_q^2}{r-p+2\gamma_q} \\ &= \frac{(r-p)\frac{q+p}{3} + (p+q)\gamma_q - \gamma_q^2}{r-p+2\gamma_q} + \frac{(r-p)\frac{r}{3}}{r-p+2\gamma_q} \\ &= \frac{p+q}{3} + \frac{-2\gamma_q\frac{p+q}{3} + (p+q)\gamma_q - \gamma_q^2}{r-p+2\gamma_q} + \frac{(r-p)\frac{r}{3}}{r-p+2\gamma_q} \\ &= \frac{p+q}{3} + \gamma_q\frac{\frac{p+q}{3} - \gamma_q}{r-p+2\gamma_q} + \frac{(r-p)r}{3(r-p+2\gamma_q)}.\end{aligned}$$

Since from (15),

$$\gamma_q < \frac{p+q}{3},$$

we get from this latter equality for γ_r that

$$\gamma_r > \frac{p+q}{3},$$

but this is a contradiction with (15). □

Corollary 3.3. *There are no N gap traveling waves for $N \geq 3$.*

Proof. The proof is the same as for the three gap traveling waves case, but with some additional terms which might hinder understanding for a first reading. We explain here how to adapt the proof.

Let u_0 be a N gap potential, $N \geq 3$, and $p < q < r < r_4 < \cdots < r_N$ be the indices for the nonzero gaps. Let

$$\Gamma_r := \gamma_r + \sum_{k=4}^{N} \gamma_{r_k}.$$

The speeds for the three smallest modes at indices p, q and r write

$$\begin{aligned}\frac{\omega_p^{(4)}(u_0)}{p} = p^2 + (p\gamma_p + q\gamma_q + r\gamma_r + \sum_{k=4}^{N} r_k\gamma_{r_k}) - 3p(\gamma_p + \gamma_q + \Gamma_r) \\ + 3(\gamma_p^2 + 2\gamma_p(\gamma_q + \Gamma_r) + \gamma_q^2 + 2\gamma_q\Gamma_r + \gamma_r^2),\end{aligned}$$

$$\begin{aligned}\frac{\omega_q^{(4)}(u_0)}{q} = q^2 + (p\gamma_p + q\gamma_q + r\gamma_r + \sum_{k=4}^{N} r_k\gamma_{r_k}) - 3(\frac{p^2}{q}\gamma_p + q(\gamma_q + \Gamma_r)) \\ + 3(\frac{p}{q}\gamma_p^2 + 2\frac{p}{q}\gamma_p(\gamma_q + \Gamma_r) + \gamma_q^2 + 2\gamma_q\Gamma_r + \Gamma_r^2)\end{aligned}$$

and

$$\begin{aligned}\frac{\omega_r^{(4)}(u_0)}{r} = r^2 + (p\gamma_p + q\gamma_q + r\gamma_r + \sum_{k=4}^{N} r_k\gamma_{r_k}) - 3(\frac{p^2}{r}\gamma_p + \frac{q^2}{r}\gamma_q + r\Gamma_r) \\ + 3(\frac{p}{r}\gamma_p^2 + 2\frac{p}{r}\gamma_p(\gamma_q + \Gamma_r) + \frac{q}{r}\gamma_q^2 + 2\frac{q}{r}\gamma_q\Gamma_r + \Gamma_r^2).\end{aligned}$$

The rest of the proof is identical up to replacing γ_r by Γ_r everywhere from this point on. □

3.2 Orbital stability

Proposition 3.4. *The one gap traveling waves in $L^2_{r,0}(\mathbb{T})$ are orbitally stable.*

Proof. Let u_0 be a one gap traveling wave, u_0^k a sequence of initial data converging to u_0 in $L^2_{r,0}(\mathbb{T})$, and t_k a sequence of times. We prove that up to some subsequence,

$$\inf_{\theta\in\mathbb{T}}\|u_k(t_k)-u_0(\cdot+\theta)\|_{L^2_{r,0}(\mathbb{T})}\underset{k\to+\infty}{\longrightarrow}0.$$

It is enough to show that there exists $\theta\in\mathbb{T}$ such that in $L^2_{r,0}(\mathbb{T})$,

$$u_k(t_k)\underset{k\to+\infty}{\longrightarrow}u_0(\cdot+\theta),$$

i.e. such that in $h^{\frac{1}{2}}_+$,

$$\Phi(u_k(t_k))\underset{k\to+\infty}{\longrightarrow}\Phi(u_0(\cdot+\theta)).$$

Recall that

$$\Phi(u_k(t_k))=(\zeta_n(u_0^k)\,\mathrm{e}^{i\omega_n^{(4)}(u_0^k)t_k})_n.$$

Let $p\geq 1$ be the index for which $\zeta_p(u_0)\neq 0$. Up to some subsequence, there exists $\theta\in\mathbb{T}$ such that $\mathrm{e}^{i\omega_p^{(4)}(u_0^k)t_k}\underset{k\to+\infty}{\longrightarrow}\mathrm{e}^{i\theta}$. Moreover, since u_0^k converges to u_0 in $L^2_{r,0}(\mathbb{T})$,

$$p|\zeta_p(u_0^k)-\zeta_p(u_0)|^2+\sum_{n\neq p}n|\zeta_n(u_0^k)|^2\underset{k\to+\infty}{\longrightarrow}0.$$

We deduce that

$$\|\Phi(u_k(t_k))-\Phi(u_0(\cdot+\theta))\|^2_{h^{\frac{1}{2}}_+}=p|\zeta_p(u_0^k)\,\mathrm{e}^{i\omega_n^{(4)}(u_0^k)t_k}-\zeta_p(u_0)\,\mathrm{e}^{i\theta}|^2+\sum_{n\neq p}n|\zeta_n(u_0^k)|^2$$
$$\underset{k\to+\infty}{\longrightarrow}0.$$

□

Proposition 3.5. *The two gap traveling waves in $L^2_{r,0}(\mathbb{T})$ are orbitally unstable.*

Proof. Let u_0 be a two gap traveling wave such that the nonzero terms of the sequence $\Phi(u_0)=(\zeta_n(u_0))_{n\geq 1}$ are $\zeta_p(u_0)$ and $\zeta_q(u_0)$. We define the sequence u_0^k of two gap initial data by their nonzero gaps at indices p and q, denoted $\zeta_p(u_0^k)$ and $\zeta_q(u_0^k)$, as follows. We fix $\zeta_p(u_0^k):=\zeta_p(u_0)$ and choose any sequence of nonzero complex numbers $(\zeta_q(u_0^k))_k$ such that $\zeta_q(u_0^k)\underset{k\to+\infty}{\longrightarrow}\zeta_q(u_0)$ but for all $k\in\mathbb{N}$, $\varepsilon_k:=|\zeta_q(u_0^k)|^2-|\zeta_q|^2\neq 0$. Then we construct $t_k\in\mathbb{R}$ in order to negate the orbital stability of u_0.

Assume by contradiction that

$$\inf_{\theta\in\mathbb{T}}\|u_k(t_k)-u_0(\cdot+\theta)\|_{L^2_{r,0}(\mathbb{T})}\underset{k\to+\infty}{\longrightarrow}0.$$

Then there exists a sequence $\theta_k\in\mathbb{T}$, $k\in\mathbb{N}$, such that in $L^2_{r,0}(\mathbb{T})$,

$$\|u_k(t_k,\cdot-\theta_k)-u_0\|_{L^2_{r,0}(\mathbb{T})}\underset{k\to+\infty}{\longrightarrow}0.$$

Applying the Birkhoff map, which is continuous on $L^2_{r,0}(\mathbb{T})$,

$$\zeta_p(u_0^k)\,\mathrm{e}^{i\omega_p^{(4)}(u_0^k)t_k-ip\theta_k}=\zeta_p(u_k(t_k))\,\mathrm{e}^{-ip\theta_k}\underset{k\to+\infty}{\longrightarrow}\zeta_p(u_0)$$

and

$$\zeta_q(u_0^k)\,\mathrm{e}^{i\omega_q^{(4)}(u_0^k)t_k - iq\theta_k} = \zeta_q(u_k(t_k))\,\mathrm{e}^{-iq\theta_k} \underset{k\to+\infty}{\longrightarrow} \zeta_q(u_0).$$

This implies by taking the arguments that for some integers $n_{p,k}$ and $n_{q,k}$,

$$\omega_p^{(4)}(u_0^k)t_k - p\theta_k + 2\pi n_{p,k} \underset{k\to+\infty}{\longrightarrow} 0$$

and

$$\omega_q^{(4)}(u_0^k)t_k - q\theta_k + 2\pi n_{q,k} \underset{k\to+\infty}{\longrightarrow} 0,$$

therefore

$$pqt_k\left(\frac{\omega_q^{(4)}(u_0^k)}{q} - \frac{\omega_p^{(4)}(u_0^k)}{p}\right) + 2\pi(pn_{q,k} - qn_{p,k}) \underset{k\to+\infty}{\longrightarrow} 0. \tag{16}$$

However, writing $\varepsilon_k = |\zeta_q(u_0^k)|^2 - |\zeta_q(u_0)|^2 = \gamma_q(u_0^k) - \gamma_q(u_0)$, we get that the speeds of the two modes p and q for the initial data u_0^k are given by

$$\frac{\omega_p^{(4)}(u_0^k)}{p} = \frac{\omega_p^{(4)}(u_0)}{p} + q\varepsilon_k - 3p\varepsilon_k + 6\varepsilon_k(\gamma_p(u_0) + \gamma_q(u_0)) + 3\varepsilon_k^2$$

and

$$\frac{\omega_q^{(4)}(u_0^k)}{q} = \frac{\omega_q^{(4)}(u_0)}{q} + q\varepsilon_k - 3q\varepsilon_k + 6\varepsilon_k(\frac{p}{q}\gamma_p(u_0) + \gamma_q(u_0)) + 3\varepsilon_k^2.$$

Since u_0 is a traveling wave, $\frac{\omega_p^{(4)}(u_0)}{p} = \frac{\omega_q^{(4)}(u_0)}{q}$ and therefore

$$\frac{\omega_q^{(4)}(u_0^k)}{q} - \frac{\omega_p^{(4)}(u_0^k)}{p} = -3(q - p + 2(1 - \frac{p}{q})\gamma_p(u_0))\varepsilon_k.$$

Since $\varepsilon_k \neq 0$, then $\frac{\omega_q^{(4)}(u_0^k)}{q} \neq \frac{\omega_p^{(4)}(u_0^k)}{p}$. It is therefore possible to choose a sequence t_k such that the limit (16) does not hold and get a contradiction. For instance, we can choose t_k such that

$$-3pqt_k(q - p + 2(1 - \frac{p}{q})\gamma_p)\varepsilon_k = \pi.$$

□

A Appendices

A.1 About the hierarchy

The aim of this Appendix is to provide a way to compute the Hamiltonians $\mathcal{H}_k$ and frequencies $\omega_n^{(k)}$ for the higher order Benjamin-Ono equations in terms of the actions $\gamma_p = |\zeta_p(u_0)|^2$. In particular, we establish formula (5)

$$\omega_n^{(4)}(u_0) = n^3 + n\sum_{p\geq 1} p\gamma_p - 3\sum_{p\geq 1}\min(p,n)^2\gamma_p + 3\sum_{p,q\geq 1}\min(p,q,n)\gamma_p\gamma_q$$

for finite gap potentials u_0.

We need to recall first some notation. Given $u \in L^2_{r,0}(\mathbb{T})$, we consider its Lax operator $L_u = -i\partial_x - T_u$ acting on $L^2_+(\mathbb{T})$, with domain $H^1_+(\mathbb{T}) = H^1(\mathbb{T}) \cap L^2_+(\mathbb{T})$. The spectrum of L_u is discrete, with eigenvalues

$$\lambda_0(u) < \lambda_1(u) < \cdots < \lambda_n(u) < \cdots.$$

Moreover (see [5]),

$$\gamma_n(u) = \lambda_n(u) - \lambda_{n-1}(u) - 1, \quad n \geq 1$$

is non-negative and satisfies $\gamma_n(u) = |\zeta_n(u)|^2$. We also define $f_n(u) \in H^1_+(\mathbb{T})$ as the L^2-normalized eigenfunction for L_u associated to the eigenvalue $\lambda_n(u)$.

Let u_0 be a finite gap potential and use the above notation. From [5] (3.8), a variant of the generating function, denoted by $\widetilde{\mathcal{H}_\varepsilon}$, is defined as

$$\widetilde{\mathcal{H}_\varepsilon} = \sum_{n=0}^{+\infty} \frac{|\langle \mathbb{1} | f_n \rangle|^2}{1 + \varepsilon \lambda_n}.$$

From the decomposition (2.12) in [5] : $\Pi u = -\sum_{n=1}^{+\infty} \lambda_n \langle \mathbb{1} | f_n \rangle f_n$, we know using formula (3) that

$$\begin{aligned} \mathcal{H}_k(u) &= \sum_{n=0}^{+\infty} |\langle \mathbb{1} | f_n \rangle|^2 \lambda_n^k \\ &= \frac{(-1)^k}{k!} \frac{\mathrm{d}^k}{\mathrm{d}\varepsilon^k}|_{\varepsilon=0} \widetilde{\mathcal{H}_\varepsilon}. \end{aligned}$$

We now make use of the generating function to derive a recurrence formula for the $\mathcal{H}_k$. Set

$$g_\varepsilon := -\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \log \widetilde{\mathcal{H}_\varepsilon} = -\frac{1}{\widetilde{\mathcal{H}_\varepsilon}} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \widetilde{\mathcal{H}_\varepsilon},$$

then from [5] (3.11), g_ε writes

$$g_\varepsilon = \frac{\lambda_0}{1 + \varepsilon \lambda_0} + \sum_{n=1}^{+\infty} \frac{\gamma_n}{(1 + \varepsilon(\lambda_{n-1} + 1))(1 + \varepsilon \lambda_n)}.$$

Using the identity

$$\begin{aligned} -\frac{\mathrm{d}^{k+1}}{\mathrm{d}\varepsilon^{k+1}} \widetilde{\mathcal{H}_\varepsilon} &= \frac{\mathrm{d}^k}{\mathrm{d}\varepsilon^k} \left(g_\varepsilon \widetilde{\mathcal{H}_\varepsilon} \right) \\ &= \sum_{l=0}^{k} \binom{k}{l} \frac{\mathrm{d}^l}{\mathrm{d}\varepsilon^l}(g_\varepsilon) \frac{\mathrm{d}^{k-l}}{\mathrm{d}\varepsilon^{k-l}}(\widetilde{\mathcal{H}_\varepsilon}). \end{aligned}$$

and defining

$$P_l := \frac{(-1)^l}{l!} \frac{\mathrm{d}^l}{\mathrm{d}\varepsilon^l}|_{\varepsilon=0}(g_\varepsilon),$$

we get the recurrence relation

$$\mathcal{H}_{k+1} = \frac{1}{k+1} \sum_{l=0}^{k} P_l \mathcal{H}_{k-l}. \tag{17}$$

Moreover, the frequencies $\omega_n^{(k)} = \frac{\partial \mathcal{H}_k}{\partial \gamma_n}$, satisfy the recurrence formula

$$\omega_n^{(k+1)} = \frac{1}{k+1} \sum_{l=0}^{k} \frac{\partial P_l}{\partial \gamma_n} \mathcal{H}_{k-l} + P_l \omega_n^{(k-l)}. \tag{18}$$

We now simplify P_l and $\frac{\partial P_l}{\partial \gamma_n}$:

$$P_l = \lambda_0^{l+1} + \sum_{n \geq 1} \gamma_n \sum_{m=0}^{l} (\lambda_{n-1} + 1)^m \lambda_n^{l-m},$$

and since $\lambda_{n-1}+1=\lambda_n-\gamma_n$,

$$P_l=\lambda_0^{l+1}+\sum_{n\geq 1}\lambda_n^{l+1}-(\lambda_n-\gamma_n)^{l+1}.$$

From (3.13), $\lambda_n=n-s_{n+1}$ where $s_n=\sum_{k=n}^{\infty}\gamma_n$ for $n\geq 1$, therefore

$$P_l=(-1)^{l+1}s_1^{l+1}+\sum_{n\geq 1}(n-s_{n+1})^{l+1}-(n-s_n)^{l+1}. \tag{19}$$

We deduce

$$\begin{aligned}\frac{1}{l+1}\frac{\partial P_l}{\partial\gamma_n}&=(-1)^{l+1}s_1^l+(n-s_n)^l-\sum_{p<n}(p-s_{p+1})^l-(p-s_p)^l\\&=\sum_{p=1}^{n}(p-s_p)^l-(p-1-s_p)^l\\&=n^l+\sum_{p=1}^{n}\sum_{m=1}^{l-1}\binom{l}{m}(p^m-(p-1)^m)(-1)^{l-m}s_p^{l-m}\\&=n^l+\sum_{m=1}^{l-1}\binom{l}{m}(-1)^{l-m}\sum_{\substack{p,p_1,\dots,p_{l-m}\\1\leq p\leq\min(n,p_1,\dots,p_{l-m})}}(p^m-(p-1)^m)\gamma_{p_1}\dots\gamma_{p_{l-m}},\end{aligned}$$

therefore

$$\frac{1}{l+1}\frac{\partial P_l}{\partial\gamma_n}=n^l+\sum_{m=1}^{l-1}\binom{l}{m}(-1)^{l-m}\sum_{p_1,\dots,p_{l-m}}\min(n,p_1,\dots,p_{l-m})^m\gamma_{p_1}\dots\gamma_{p_{l-m}}.$$

Let us compute the first small terms by using (17) and (18). The Hamiltonian with index 0 is constant $\mathcal{H}_0=1$, leading to $\omega_n^{(0)}=0$, $P_0=0$ and $\frac{\partial P_0}{\partial\gamma_n}=0$.

Concerning index 1, $\mathcal{H}_1=-\langle u|\mathbb{1}\rangle=0$ because we assumed that $u\in L^2_{r,0}(\mathbb{T})$ is of average zero. This leads to $\omega_n^{(1)}=0$, $P_1=2\sum_{p\geq 1}p\gamma_p$ and $\frac{\partial P_1}{\partial\gamma_n}=2n$. Because of cancellations for several of these small terms, the recurrence relations (17) and (18) write, for $k\geq 2$,

$$\mathcal{H}_k=\frac{1}{k}\left(P_1\mathcal{H}_{k-2}+P_2\mathcal{H}_{k-3}+\dots+P_{k-3}\mathcal{H}_2+P_{k-1}\right) \tag{20}$$

and

$$\begin{aligned}\omega_n^{(k)}=\frac{1}{k}\Big(&\frac{\partial P_1}{\partial\gamma_n}\mathcal{H}_{k-2}+\frac{\partial P_2}{\partial\gamma_n}\mathcal{H}_{k-3}+\dots+\frac{\partial P_{k-3}}{\partial\gamma_n}\mathcal{H}_2\\&+P_1\omega_n^{(k-2)}+P_2\omega_n^{(k-3)}+\dots+P_{k-3}\omega_n^{(2)}+\frac{\partial P_{k-1}}{\partial\gamma_n}\Big).\end{aligned} \tag{21}$$

The second index leads to the conservation of the mass

$$\mathcal{H}_2=\frac{\|u\|^2}{2}=\frac{P_1}{2},$$

$$\omega_n^{(2)}=n.$$

Moreover,

$$P_2=-s_1^3+\sum_{p=1}^{\infty}(p-s_{p+1})^3-(p-s_p)^3=3\sum_{p\geq 1}p^2\gamma_p-3\sum_{p\geq 1}s_p^2$$

and

$$\frac{\partial P_2}{\partial \gamma_n} = 3(n^2 - 2\sum_{p=1}^{\infty} \min(p,n)\gamma_p).$$

For the third index, we retrieve identity (8.6) from [5] for the Hamiltonian

$$\mathcal{H}_3 = \frac{P_2}{3} = \sum_{p\geq 1} p^2\gamma_p - \sum_{p\geq 1} s_p^2$$

and formula (8.4) from [5] for the frequencies

$$\omega_n^{(3)} = n^2 - 2\sum_{p=1}^{+\infty} \min(p,n)\gamma_p.$$

We now use that

$$P_3 = s_1^4 + \sum_{p=1}^{+\infty} (p - s_{p+1})^4 - (p - s_p)^4$$

and

$$\frac{\partial P_3}{\partial \gamma_n} = 4(n^3 - 3\sum_{p} \min(p,n)^2\gamma_p + 3\sum_{p,q} \min(p,q,n)\gamma_p\gamma_q)$$

to get the formula for the Hamiltonian of index 4

$$\mathcal{H}_4 = \frac{1}{4}(P_3 + P_1\mathcal{H}_2) = \frac{1}{4}P_3 + \frac{1}{2}\mathcal{H}_2^2$$

and the frequencies

$$\begin{aligned}\omega_n^{(4)} &= \frac{1}{4}\frac{\partial P_3}{\partial \gamma_n} + n\mathcal{H}_2 \\ &= n^3 + n\sum_{p\geq 1} p\gamma_p - 3\sum_{p\geq 1} \min(p,n)^2\gamma_p + 3\sum_{p,q\geq 1} \min(p,q,n)\gamma_p\gamma_q.\end{aligned}$$

In the same way, using that

$$P_4 = -s_1^5 + \sum_{p=1}^{+\infty} (p - s_{p+1})^5 - (p - s_p)^5$$

and

$$\frac{1}{5}\frac{\partial P_4}{\partial \gamma_n} = n^4 - 4\sum_{p\geq 1} \min(p,n)^3\gamma_p + 6\sum_{p,q\geq 1} \min(p,q,n)^2\gamma_p\gamma_q - 4\sum_{p,q,r\geq 1} \min(p,q,r,n)\gamma_p\gamma_q\gamma_r,$$

we can get a formula for the Hamiltonian with index 5

$$\mathcal{H}_5 = \frac{1}{5}(P_4 + P_2\mathcal{H}_2 + P_1\mathcal{H}_3)$$

and the frequencies

$$\begin{aligned}\omega_n^{(5)} &= \frac{1}{5}(\frac{\partial P_1}{\partial \gamma_n}\mathcal{H}_3 + P_1\omega_n^{(3)} + \frac{\partial P_2}{\partial \gamma_n}\mathcal{H}_2 + P_2\omega_n^{(2)} + \frac{\partial P_4}{\partial \gamma_n}) \\ &= \frac{2}{5}n(\sum_{p\geq 1} p^2\gamma_p - \sum_{p\geq 1} s_p^2) + \frac{2}{5}(\sum_{p\geq 1} p\gamma_p)(n^2 - 2\sum_{p\geq 1} \min(p,n)\gamma_p) \\ &+ \frac{3}{5}(n^2 - 2\sum_{p\geq 1} \min(p,n)\gamma_p)(\sum_{p\geq 1} p\gamma_p) + \frac{3}{5}(\sum_{p\geq 1} p^2\gamma_p - \sum_{p\geq 1} s_p^2)n + \frac{1}{5}\frac{\partial P_4}{\partial \gamma_n},\end{aligned}$$

leading to

$$\omega_n^{(5)} = n(\sum_{p\geq 1} p^2\gamma_p - \sum_{p\geq 1} s_p^2) + (\sum_{p\geq 1} p\gamma_p)(n^2 - 2\sum_{p\geq 1}\min(p,n)\gamma_p)$$
$$+ n^4 - 4\sum_{p\geq 1}\min(p,n)^3\gamma_p + 6\sum_{p,q\geq 1}\min(p,q,n)^2\gamma_p\gamma_q - 4\sum_{p,q,r\geq 1}\min(p,q,r,n)\gamma_p\gamma_q\gamma_r. \quad (22)$$

Remark A.1. *Note that formulas (5) and (22) for $\omega_n^{(4)}(u_0)$ and $\omega_n^{(5)}(u_0)$, which have been established for finite gap potentials u_0, still make sense for $\omega_n^{(4)}(u_0)$ if $u_0 \in L^2_{r,0}(\mathbb{T})$ and for $\omega_n^{(5)}(u_0)$ if $u_0 \in H^{\frac{1}{2}}_{r,0}(\mathbb{T})$, but diverge if $u_0 \in H^s_{r,0}(\mathbb{T})$ for s right below these respective exponents ($s < 0$ and $s < \frac{1}{2}$).*

One can actually show by induction the following facts. In the formula (19) for P_k, there is one term $c_1\sum_{p=1}^{+\infty} p^k\gamma_p$, the other terms being convergent if $\sum_{p=1}^{+\infty} p^{k-1}\gamma_p < +\infty$. This implies that in the formula (20) for $\mathcal{H}_k$ appears one term $c_2\sum_{p=1}^{+\infty} p^{k-1}\gamma_p$, the other terms being convergent if $\sum_{p=1}^{+\infty} p^{k-2}\gamma_p < +\infty$. Consequently, in formula (21) for $\omega_n^{(k)}$, $k \geq 4$, appears one term $c_3\sum_{p=1}^{+\infty} p^{k-3}\gamma_p$, the other terms being convergent if $\sum_{p=1}^{+\infty} p^{k-4}\gamma_p < +\infty$.

From these facts, one can see that the formula for $\omega_n^{(k)}$ can be extended by continuity to potentials in $H^{s_k}_{r,0}(\mathbb{T})$ where $s_k = \frac{k}{2} - 2$, however there is no continuous extension to $H^{s_k}_{r,0}(\mathbb{T})$ when $s_k - \frac{1}{2} < s < s_k$. This explains why the well-posedness threshold for the equation associated to the Hamiltonian $\mathcal{H}_k$ in the hierarchy should be $H^{s_k}_{r,0}(\mathbb{T})$.

A.2 Equation for the fourth Hamiltonian

From formula (3) and the decomposition (2.12) in [5] : $\Pi u = -\sum_{n=1}^{+\infty}\lambda_n\langle \mathbb{1}|f_n\rangle f_n$, we see that for $k \geq 2$,

$$\mathcal{H}_k(u) = \langle L_u^{k-2}\Pi u|\Pi u\rangle, \quad (23)$$

where

$$L_u(h) = Dh - \Pi(uh), \quad D = -i\partial_x, \quad h \in H^1_+(\mathbb{T}).$$

For instance,

$$\mathcal{H}_3(u) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}uH\partial_x u - \frac{1}{3}u^3\,\mathrm{d}x$$

leads to the Benjamin-Ono equation

$$\partial_t u = H\partial_x^2 u - \partial_x(u^2).$$

Proposition A.2. *The Hamiltonian for the third order equation of the Benjamin-Ono hierarchy (1) is*

$$\mathcal{H}_4(u) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2}(\partial_x u)^2 - \frac{3}{4}u^2H\partial_x u + \frac{1}{4}u^4\right) - \frac{1}{8}\|u\|^4_{L^2(\mathbb{T})},$$

therefore the third order equation of the Benjamin-Ono hierarchy writes

$$\partial_t u = \partial_x(-\partial_{xx}u - \frac{3}{2}uH\partial_x u - \frac{3}{2}H(u\partial_x u) + u^3)$$

Proof. Let $u_0 \in L^2_{r,0}(\mathbb{T})$. We develop

$$\mathcal{H}_4(u) = \|D\Pi u\|^2_{L^2(\mathbb{T})} - 2\mathrm{Re}\langle D\Pi u|\Pi(u\Pi u))\rangle + \|\Pi(u\Pi u)\|^2_{L^2(\mathbb{T})},$$

and study each term separately.

First, since u is real, $\widehat{u}(-n) = \overline{\widehat{u}(n)}$, therefore

$$\|D\Pi u\|^2_{L^2(\mathbb{T})} = \sum_{n\geq 0} |n|^2 |\widehat{u}(n)|^2 = \frac{1}{2}\sum_{n\in\mathbb{Z}} |n|^2|\widehat{u}(n)|^2 = \frac{1}{2}\|\partial_x u\|^2_{L^2(\mathbb{T})}.$$

Then, u being with average zero, $u = \Pi u + \overline{\Pi u}$, leading to

$$\begin{aligned}\langle D\Pi u | \Pi(u\Pi u)\rangle &= \frac{1}{2\pi}\int_0^{2\pi} D(\Pi u) u\overline{\Pi u}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_0^{2\pi} D(u) u\overline{\Pi u}\,\mathrm{d}x - \frac{1}{2\pi}\int_0^{2\pi} D(\overline{\Pi u}) u\overline{\Pi u}\,\mathrm{d}x\end{aligned}$$

so that

$$\begin{aligned}2\langle D\Pi u | \Pi(u\Pi u)\rangle &= \frac{1}{2\pi}\int_0^{2\pi} D(u^2)\overline{\Pi u}\,\mathrm{d}x - \frac{1}{2\pi}\int_0^{2\pi} D(\overline{\Pi u}^2) u\,\mathrm{d}x \\ &= -\frac{1}{2\pi}\int_0^{2\pi} u^2 D(\overline{\Pi u})\,\mathrm{d}x + \frac{1}{2\pi}\int_0^{2\pi} \overline{\Pi u}^2 Du\,\mathrm{d}x.\end{aligned}$$

Taking the real part,

$$\begin{aligned}4\mathrm{Re}\langle D\Pi u | \Pi(u\Pi u)\rangle &= 2\left(\langle D\Pi u | \Pi(u\Pi u)\rangle + \overline{\langle D\Pi u | \Pi(u\Pi u)\rangle}\right) \\ &= -\frac{1}{2\pi}\int_0^{2\pi} u^2(D(\overline{\Pi u}) + \overline{D\overline{\Pi u}})\,\mathrm{d}x + \frac{1}{2\pi}\int_0^{2\pi} \overline{\Pi u}^2 Du + (\Pi u)^2\overline{Du}\,\mathrm{d}x.\end{aligned}$$

Using that $\overline{Df} = -D\overline{f}$,

$$\begin{aligned}4\mathrm{Re}\langle D\Pi u | \Pi(u\Pi u)\rangle &= -\frac{1}{2\pi}\int_0^{2\pi} u^2(D(\overline{\Pi u}) - D(\Pi u))\,\mathrm{d}x + \frac{1}{2\pi}\int_0^{2\pi} (\overline{\Pi u}^2 - (\Pi u)^2) Du\,\mathrm{d}x \\ &= -\frac{1}{2\pi}\int_0^{2\pi} u^2(D(\overline{\Pi u}) - D(\Pi u))\,\mathrm{d}x + \frac{1}{2\pi}\int_0^{2\pi} (\overline{\Pi u} - \Pi u) u Du\,\mathrm{d}x \\ &= -\frac{1}{2\pi}\int_0^{2\pi} u^2(D(\overline{\Pi u}) - D(\Pi u))\,\mathrm{d}x - \frac{1}{2}\frac{1}{2\pi}\int_0^{2\pi} D(\overline{\Pi u} - \Pi u) u^2\,\mathrm{d}x \\ &= -\frac{3}{2}\frac{1}{2\pi}\int_0^{2\pi} u^2(D(\overline{\Pi u}) - D(\Pi u))\,\mathrm{d}x.\end{aligned}$$

It now remains to remark that $D(\overline{\Pi u}) - D(\Pi u) = -H\partial_x u$ in order to conclude the identity

$$2\mathrm{Re}\langle D\Pi u | \Pi(u\Pi u)\rangle = \frac{3}{4}\langle H\partial_x u | u^2\rangle.$$

Finally, we treat the last term $\|\Pi(u\Pi u)\|^2_{L^2(\mathbb{T})}$. Note that by decomposing $u = \Pi u + \overline{\Pi u}$,

$$\Pi(u\Pi u) = (\Pi u)^2 + \Pi(\overline{\Pi u}\Pi u),$$

therefore

$$|\Pi(u\Pi u)|^2 = |\Pi u|^4 + (\Pi u)^2\overline{\Pi(\overline{\Pi u}\Pi u)} + \overline{\Pi u}^2\Pi(\overline{\Pi u}\Pi u) + |\Pi(\overline{\Pi u}\Pi u)|^2.$$

By removing the useless projections,

$$\begin{aligned}\|\Pi(u\Pi u)\|^2_{L^2(\mathbb{T})} &= \frac{1}{2\pi}\int_0^{2\pi} |\Pi u|^4 + (\Pi u)^2\overline{\Pi u}\Pi u + \overline{\Pi u}^2\overline{\Pi u}\Pi u + |\Pi(\overline{\Pi u}\Pi u)|^2\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_0^{2\pi} (\Pi u)^2\overline{\Pi u}^2 + (\Pi u)^3\overline{\Pi u} + \overline{\Pi u}^3\Pi u + |\Pi(\overline{\Pi u}\Pi u)|^2\,\mathrm{d}x.\end{aligned}$$

But if we take the fourth power of the identity $u = \Pi u + \overline{\Pi u}$

$$u^4 = (\Pi u)^4 + \overline{\Pi u}^4 + 4(\Pi u)^3\overline{\Pi u} + 4\overline{\Pi u}^3\Pi u + 6\overline{\Pi u}^2(\Pi u)^2,$$

and make use of the fact that the mean of u is zero, we get

$$\|u\|^4_{L^4(\mathbb{T})} = \frac{1}{2\pi}\int_0^{2\pi} 4(\Pi u)^3\overline{\Pi u} + 4\overline{\Pi u}^3\Pi u + 6(\Pi u)^2\overline{\Pi u}^2 \,\mathrm{d}x.$$

By subtraction, the following cancellations happen :

$$\|\Pi(u\Pi u)\|^2_{L^2(\mathbb{T})} - \frac{1}{4}\|u\|^4_{L^4(\mathbb{T})} = \frac{1}{2\pi}\int_0^{2\pi} |\Pi(\overline{\Pi u}\Pi u)|^2 \,\mathrm{d}x - \frac{1}{2}\frac{1}{2\pi}\int_0^{2\pi} (\Pi u)^2\overline{\Pi u}^2 \,\mathrm{d}x.$$

To conclude, since $\overline{\Pi u}\Pi u$ is real, we can use the identity $\|f\|^2 + |\langle f|\mathbb{1}\rangle|^2 = 2\|\Pi f\|^2$ for real valued functions $f \in L^2(\mathbb{T})$ to get

$$\frac{1}{2\pi}\int_0^{2\pi} |\Pi(\overline{\Pi u}\Pi u)|^2 \,\mathrm{d}x = \frac{1}{2}\frac{1}{2\pi}\int_0^{2\pi} |\overline{\Pi u}\Pi u|^2 \,\mathrm{d}x - \frac{1}{2}|\langle\overline{\Pi u}\Pi u, \mathbb{1}\rangle|^2,$$

leading to

$$\begin{aligned}\|\Pi(u\Pi u)\|^2_{L^2(\mathbb{T})} - \frac{1}{4}\|u\|^4_{L^4(\mathbb{T})} &= -\frac{1}{2}|\langle\overline{\Pi u}\Pi u, \mathbb{1}\rangle|^2 \\ &= -\frac{1}{2}\|\Pi u\|^4_{L^2(\mathbb{T})} \\ &= -\frac{1}{8}\|u\|^4_{L^2(\mathbb{T})}.\end{aligned}$$

□

A.3 Structure of the higher order Hamiltonians

The aim of this Appendix is to give an alternative proof of Proposition 2.2 in [18].

We first recall the notation introduced in [18] for the sake of completeness. For a smooth function $u \in \mathcal{C}^\infty(\mathbb{T})$, define by induction the sets $\mathcal{P}_n(u)$ as

$$\mathcal{P}_1(u) = \{H^{\varepsilon_1}\partial_x^{\alpha_1}u \mid \varepsilon_1 \in \{0,1\}, \quad \alpha_1 \in \mathbb{N}\},$$

$$\mathcal{P}_2(u) = \{(H^{\varepsilon_1}\partial_x^{\alpha_1}u)(H^{\varepsilon_2}\partial_x^{\alpha_2}u) \mid \varepsilon_1, \varepsilon_2 \in \{0,1\}, \quad \alpha_1, \alpha_2 \in \mathbb{N}\}$$

and for $n \geq 2$,

$$\mathcal{P}_n(u) = \left\{\prod_{l=1}^k H^{\varepsilon_l}p_{j_l}(u) \mid k \in [\![2,n]\!], \quad \varepsilon_1, \dots, \varepsilon_k \in \{0,1\}, \quad \sum_{l=1}^k j_l = n, \quad p_{j_l}(u) \in \mathcal{P}_{j_l}(u)\right\}.$$

Moreover, for $p_n(u) \in \mathcal{P}_n(u)$, the term $\widetilde{p_n}(u)$ is uniquely defined from $p_n(u)$ by removing all the symbols H in the expression of $p_n(u)$ and only keeping the symbols $\partial_x^{\alpha_i}u$. In this case, if

$$\widetilde{p_n}(u) = \prod_{i=1}^n \partial_x^{\alpha_i}u,$$

the maximal order of derivative involved and the sum of these orders are respectively denoted

$$|p_n(u)| = \sup_{i\in[\![1,n]\!]} \alpha_i$$

and

$$\|p_n(u)\| = \sum_{i=1}^n \alpha_i.$$

We now retrieve a proof of the following result (Proposition 2.2 in [18]).

Proposition A.3. *Let $k = 2(m+1)$ be an even integer. Then there exists $c \in \mathbb{R}$ such that the k-th Hamiltonian $\mathcal{H}_k$ writes, for all $u \in \mathcal{C}^\infty(\mathbb{T})$,*

$$\mathcal{H}_{k+2}(u) = \frac{1}{2}\|u\|^2_{\dot{H}^{m+1}(\mathbb{T})} + c\int_0^{2\pi} u(H\partial_x^m u)(\partial_x^{m+1}u)\,\mathrm{d}x + R,$$

where for some real numbers $c(p)$,

$$R = \sum_{j=3}^{2m+4} \sum_{\substack{p(u)\in\mathcal{P}_j(u)\\ \|p(u)\|=2m+4-j\\ |p(u)|\leq m}} c(p)\int_0^{2\pi} p(u)\,\mathrm{d}x.$$

Note that $\mathcal{H}_{k+2}(u) = \frac{1}{2}E_{k/2}(u)$ with the notation from [18].

Proof. Recall formula (23)

$$\begin{aligned}\mathcal{H}_{k+2}(u) &= \langle L_u^k \Pi u | \Pi u\rangle \\ &= \langle L_u^{m+1}\Pi u | L_u^{m+1}\Pi u\rangle \\ &= \langle (D-T_u)^{m+1}\Pi u | (D-T_u)^{m+1}\Pi u\rangle,\end{aligned}$$

where $D = -i\partial_x$ and $T_u : h \in L^2_+(\mathbb{T}) \mapsto \Pi(uh)$.

We expand $\mathcal{H}_{k+2}(u)$ as a sum of terms depending on whether we applied the operator D or the operator T_u when applying L_u.

It is possible to decompose $\mathcal{H}_{k+2}(u)$ as follows :

$$\mathcal{H}_{k+2}(u) = \frac{1}{2}\|u\|^2_{\dot{H}^{m+1}(\mathbb{T})} + A + B,$$

where $\|\Pi u\|^2_{\dot{H}^{m+1}(\mathbb{T})} = \frac{1}{2}\|u\|^2_{\dot{H}^{m+1}(\mathbb{T})}$ is obtained when one only applies operator D, A is obtained when one applies only once the operator T_u and $(2m+1)$ times the operator D

$$\begin{aligned}A &= -2\mathrm{Re}\left(\sum_{j=0}^{m}\langle D^{m-j}\Pi(uD^j\Pi u)|D^{m+1}\Pi u\rangle\right) \\ &= -2\mathrm{Re}\left(\sum_{j=0}^{m}\langle D^{m-j}(uD^j\Pi u)|D^{m+1}\Pi u\rangle\right),\end{aligned}$$

and B is obtained when we apply at least twice in total the operator T_u.

• We first prove that one can decompose A as

$$A = c\int_0^{2\pi} u(H\partial_x^m u)(\partial_x^{m+1}u)\,\mathrm{d}x + \widetilde{A}$$

where for some real numbers $c(p)$,

$$\widetilde{A} = \sum_{\substack{p(u)\in\mathcal{P}_3(u)\\ \|p(u)\|=2m+1\\ |p(u)|\leq m}} c(p)\int_0^{2\pi} p(u)\,\mathrm{d}x. \tag{24}$$

Let $j \in [\![0, m]\!]$. By integration by parts and Leibniz' formula,

$$\begin{aligned}\overline{\langle D^{m-j}(uD^j\Pi u)|D^{m+1}\Pi u\rangle} &= \langle D^{m+1-j}(uD^j\Pi u)|D^m\Pi u\rangle \\ &= \langle uD^{m+1}(\Pi u)|D^m\Pi u\rangle + \langle D^{m+1-j}(u)D^j(\Pi u)|D^m\Pi u\rangle \\ &+ \sum_{k=1}^{m-j}\binom{m+1-j}{k}\langle D^k(u)D^{m+1-k}(\Pi u)|D^m\Pi u\rangle.\end{aligned}$$

We take the real part and sum over the indices j. When distinguishing the cases $j = 0$ and $j \geq 1$, we see that for some suitable $\widetilde{A}$ as in (24), A decomposes as

$$A = -2(m+1)\mathrm{Re}(\langle uD^{m+1}(\Pi u)|D^m\Pi u\rangle) - 2\mathrm{Re}(\langle D^{m+1}(u)\Pi u|D^m\Pi u\rangle) + \widetilde{A}.$$

Write $\Pi u = \frac{u+iHu}{2}$, then there exists some real constants $c(\varepsilon_1, \varepsilon_2)$ such that

$$A = \sum_{\varepsilon_1,\varepsilon_2\in\{0,1\}} c(\varepsilon_1,\varepsilon_2)\int_0^{2\pi} u\partial_x^m(H^{\varepsilon_1}u)\partial_x^{m+1}(H^{\varepsilon_2}u)\,\mathrm{d}x - \mathrm{Re}(\langle D^{m+1}(u)iHu|D^m(\Pi u)\rangle) + \widetilde{A}.$$

On the one hand, the terms in the sum are simplified as follows (see the remark from Tzvetkov and Visciglia [18]). When $\varepsilon_1 = \varepsilon_2$,

$$\begin{aligned}\int_0^{2\pi} u\partial_x^m(H^{\varepsilon_1}u)\partial_x^{m+1}(H^{\varepsilon_1}u)\,\mathrm{d}x &= \frac{1}{2}\int_0^{2\pi} u\partial_x((\partial_x^m(H^{\varepsilon_1}u))^2)\,\mathrm{d}x\\ &= -\frac{1}{2}\int_0^{2\pi}\partial_x(u)(\partial_x^m(H^{\varepsilon_1}u))^2\,\mathrm{d}x\end{aligned}$$

so this term is a remainder term to be added to $\widetilde{A}$. Moreover, by integration by parts,

$$\int_0^{2\pi} u\partial_x^m(u)\partial_x^{m+1}(Hu)\,\mathrm{d}x = -\int_0^{2\pi}\partial_x(u)\partial_x^m(u)\partial_x^m(Hu)\,\mathrm{d}x - \int_0^{2\pi} u\partial_x^{m+1}(u)\partial_x^m(Hu)\,\mathrm{d}x.$$

Therefore, the sum can be written as a linear combination of the term $\int_0^{2\pi} u(H\partial_x^m u)\partial_x^{m+1}u\,\mathrm{d}x$ and other terms that can be added to the remainder $\widetilde{A}$.

On the other end,

$$\begin{aligned}\mathrm{Re}(\langle D^{m+1}(u)iHu|D^m(\Pi u)\rangle) &= \mathrm{Re}(\langle\partial_x^{m+1}(u)Hu|\partial_x^m(\Pi u)\rangle)\\ &= \frac{1}{2}\left(\mathrm{Re}(\langle\partial_x^{m+1}(u)Hu|\partial_x^m(u)\rangle) + \mathrm{Re}(\langle\partial_x^{m+1}(u)Hu|i\partial_x^m(Hu)\rangle)\right)\end{aligned}$$

Since u is real valued, so is Hu, therefore

$$\mathrm{Re}(\langle\partial_x^{m+1}(u)Hu|i\partial_x^m(Hu)\rangle) = 0.$$

By integration by parts, we then write

$$\begin{aligned}\mathrm{Re}(\langle D^{m+1}(u)iHu|D^m(\Pi u)\rangle) &= \frac{1}{4\pi}\int_0^{2\pi}\partial_x^{m+1}(u)Hu\partial_x^m(u)\,\mathrm{d}x\\ &= -\frac{1}{8\pi}\int_0^{2\pi}\partial_x(Hu)(\partial_x^m(u))^2\,\mathrm{d}x\end{aligned}$$

as a remainder term to be added to $\widetilde{A}$.

• We now tackle term B , for which we have applied T_u at least twice. We show that it can be written for some real numbers $c(p)$ as a sum

$$B = \sum_{j=4}^{2m+4}\sum_{\substack{p(u)\in\mathcal{P}_j(u)\\ \|p(u)\|=2m+4-j\\ |p(u)|\leq m}} c(p)\int_0^{2\pi} p(u)\,\mathrm{d}x.$$

Let $\widetilde{B}$ be one of the terms in B obtained by applying T_u $(j-1)$ times on the left side and $(k-1)$ times on the right side.

Assume that we have applied T_u at least once in each side of the brackets, i.e. $j-1 \in [\![1, m+1]\!]$ and $k-1 \in [\![1, m+1]\!]$. Then we can apply Leibniz' rule and decompose the left side as a complex linear combination of terms of the form $p(u)$ where $p(u) \in \mathcal{P}_j(u)$, $\|p(u)\| = m+2-j$ and $|p(u)| \leq m$ (for the right side we just replace j by k). The term $\widetilde{B}$ is therefore a complex linear combination of terms $\int_0^{2\pi} p(u)\,\mathrm{d}x$, where $p(u) \in \mathcal{P}_l(u)$ for some $l = j + k \in [\![4, 2m+4]\!]$, $\|p(u)\| = 2m + 4 - l$ and $|p(u)| \leq m$.

Otherwise, we have applied T_u at least twice in the same side of the brackets, let us say the left, and we only have applied the operator D on the other side : $j-1 \in [\![2, m+1]\!]$ and $k-1 = 0$. Again by Leibniz' rule, $\widetilde{B}$ decomposes as a sum

$$\widetilde{B} = \sum_{j=3}^{m+2} \sum_{\substack{p(u)\in\mathcal{P}_j(u) \\ \|p(u)\|=m+2-j \\ |p(u)|\leq m-1}} c(p)\langle p(u)|D^{m+1}\Pi u\rangle.$$

But then by integration by parts and Leibniz' rule again,

$$\begin{aligned}
\widetilde{B} &= \sum_{j=3}^{m+2} \sum_{\substack{p(u)\in\mathcal{P}_j(u) \\ \|p(u)\|=m+2-j \\ |p(u)|\leq m-1}} c(p)\overline{\langle Dp(u)|D^m\Pi u\rangle} \\
&= \sum_{j=3}^{m+2} \sum_{\substack{p(u)\in\mathcal{P}_j(u) \\ \|p(u)\|=m+3-j \\ |p(u)|\leq m}} c'(p)\overline{\langle p(u)|D^m\Pi u\rangle} \\
&= \sum_{j=4}^{m+3} \sum_{\substack{p(u)\in\mathcal{P}_j(u) \\ \|p(u)\|=2m+4-j \\ |p(u)|\leq m}} c''(p) \int_0^{2\pi} p(u)\,\mathrm{d}x,
\end{aligned}$$

which is of the desired form. □

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