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Abstract: Adaptive observers are recursive algorithms for joint estimation of both state variables and unknown parameters. Usually some persistent excitation (PE) condition is required for the convergence of adaptive observers. However, in practice, it may happen that the PE condition is not satisfied, because the available sensor signals do not contain sufficient information for the considered recursive estimation problem, which is ill-posed. To remedy the lack of PE condition, inspired by typical methods for solving ill-posed inverse problems, this paper proposes a regularized adaptive observer for general linear time varying (LTV) systems. Two regularization terms are introduced in both state and parameter estimation recursions, in order to preserve the state-parameter decoupling transformation involved in the design of the adaptive observer. Like in typical ill-posed inverse problems, regularization implies an estimation bias, which can be reduced by using prior knowledge about the unknown parameters.

Keywords: Adaptive observer, regularization, persistent excitation, linear time varying (LTV) systems, joint state-parameter estimation.

1. INTRODUCTION

Dynamic systems in various engineering fields are often described in state-space form, with only part of the state variables directly accessible through sensor instruments. State estimation is a thus a common task for different engineering purposes. Moreover, some model parameters may be unknown a priori, due to variations in the production of system components, or because of some evolution reflecting aging or degradation of the underlying dynamic system. *Adaptive observers* are recursive algorithms for joint estimation of both state variables and unknown parameters, based on available sensor measurements. The design of adaptive observers is, since a few decades, an active research area. Early contributions about linear time invariant (LTI) systems go back to the seventies (Kreisselmeier, 1977; Ioannou and Kokotovic, 1983), then some classes of nonlinear systems are studied (see e.g. (Bastin and Gevers, 1988; Marino and Tomei, 1995; Cho and Rajamani, 1997; Besançon et al., 2006; Farza et al., 2009)) while the results about LTI and linear time varying (LTV) systems continue to be completed (Zhang, 2018).

In most reported results about adaptive observer design, including those recalled above, in addition to the traditional observability condition for state estimation, a *persistent excitation* (PE) condition is required to ensure the convergence of the algorithms. Roughly speaking, the PE condition implies sufficiently rich information in sensor signals guaranteeing the *well-posedness* of the joint state-parameter recursive estimation problem. It is well known in classical adaptive estimation problems that PE is essential for parameter estimation (Narendra and Annaswamy, 1987; Shimkin and Feuer, 1987; Narendra and Annaswamy, 1989; Astrom and Wittenmark, 1994).

However, in practice, it may happen that the PE condition associated to an adaptive observer is not satisfied, and this situation may persist for different adaptive observers, because intrinsically the available sensor signals do not contain sufficient information for the considered recursive estimation problem. In other words, the considered recursive estimation problem is *ill-posed*. What can be done in this case? The easiest answer would be: it is not possible to apply adaptive observers. However, it is well known that the so-called *inverse problems* in various engineering fields are often ill-posed (Chavent, 2010), yet practical solutions are frequently implemented and applied. Is it possible to do something similar for adaptive observers?

In this paper, like in typical solutions to ill-posed inverse problems, *regularization* will be introduced into the adaptive observer initially presented in (Zhang, 2002) for general multi-input multi-output (MIMO) LTV systems. The regularization technique is also known in robust adaptive control, mainly focused on LTI systems (Ioannou and Sun, 1996). Because the considered LTV system adaptive observer is based on a state-parameter decoupling technique, instead of explicitly minimizing some criterion, the proposed regularization is *not* introduced as a penalty term within a minimized criterion. Moreover, the introduced regularization must preserve the decoupling transformation involved in the design of the adaptive observer.

It is widely acknowledged that the introduction of regularization in estimation problems generally leads to biased estimates. This is the price to pay to solve such ill-posed problems. Nevertheless, *prior knowledge* about model parameters, if available, can be used to reduce the bias. These aspects of regularization will be discussed in this paper after the presentation of the proposed *regularized adaptive observer*.

2. PROBLEM FORMULATION

Throughout this paper, for any vector v, its Euclidean norm is denoted by $\|v\|$. For any matrix M, its matrix norm induced by the Euclidean vector norm is denoted by $\|M\|$. For a symmetric matrix M, the inequality $M > 0$ (or $M \geq 0$) means M is (semi)-positive definite.

This paper will consider continuous-time MIMO LTV systems in the form of

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \Phi(t)\theta + w(t) \tag{1a}$$

$$y(t) = C(t)x(t) + v(t) \tag{1b}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^q$ the input (control), $y(t) \in \mathbb{R}^m$ the output, $\dot{x}(t) = dx(t)/dt$, $\theta \in \mathbb{R}^p$ the unknown constant parameter vector, $A(t), B(t), C(t), \Phi(t)$ are appropriate size matrix-valued functions of the time t, $w(t) \in \mathbb{R}^n$ and $v(t) \in \mathbb{R}^m$ represent uncertainties in the state and output equations. At the initial instant t_0, the initial state $x(t_0)$ is unknown. In order to keep the problem formulation and analysis within the framework of ordinary differential equations (ODE), without resorting to the continuous time stochastic system theory, it is assumed that the uncertainties $w(t)$ and $v(t)$ are unknown arbitrary *bounded* functions of t.

The system matrices $A(t), B(t), C(t)$ are typically constant (case of LTI systems), periodic, or depending on some known scheduling parameter (LPV systems), whereas the matrix $\Phi(t)$ often has a different nature, typically filled with exogenous signals (inputs), sometimes with injected outputs.

In existing adaptive observer design methods, it is usually assumed that the signals contained in $\Phi(t)$ involve sufficiently rich variations so that the parameter vector θ can be estimated by a recursive algorithm. Such an assumption is usually known as a *persistent excitation* (PE) condition, as the one recalled in the next section (see Assumption 3). The lack of excitation may imply that the unknown parameter vector θ is not uniquely determined by the available information assumed above. In other words, the PE condition ensures that the recursive estimation of the parameter vector θ is a well-posed problem.

In practice, it may happen that the PE condition is not satisfied. *The purpose of this paper* is to jointly estimate the state $x(t)$ and the parameter θ from the input $u(t)$, the output $y(t)$ and the time varying matrices $A(t), B(t), E(t), C(t), \Phi(t)$, *despite deficient excitations*. Like in the study of typical inverse problems, regularization will be used to address the lack of PE condition.

Assumption 1. (boundedness and continuity). $A(t), B(t), C(t), \Phi(t)$ are bounded matrix-valued piecewise continuous functions of t, and $u(t)$, $w(t)$, $v(t)$ are bounded vector-valued piecewise continuous functions of t. While $A(t), B(t), C(t), \Phi(t), u(t)$ are known or available from sensor signals, $w(t)$, $v(t)$ are unknown. □

Assumption 2. (observability). The pair $[A(t), C(t)]$ is uniformly completely observable, in the sense of the uniform positive definiteness of the observability Gramian, as defined in (Kalman, 1963). □

3. RECALLING A CONVENTIONAL ADAPTIVE OBSERVER

This section *recalls* a general adaptive observer for MIMO LTV systems, whose convergence has been established under a PE condition. It will serve as the basis for the results in the next section addressing deficient excitations by means of regularization.

The adaptive observer for the LTV system (1) proposed in (Zhang, 2002) is in the form of

$$\begin{aligned}\dot{\hat{x}}(t) = {} & A(t)\hat{x}(t) + B(t)u(t) + \Phi(t)\hat{\theta}(t) \\ & + K(t)[y(t) - C(t)\hat{x}(t)] \\ & + \Upsilon(t)\Gamma\Upsilon^T(t)C^T(t)[y(t) - C(t)\hat{x}(t)]\end{aligned} \tag{2a}$$

$$\dot{\hat{\theta}}(t) = \Gamma\Upsilon^T(t)C^T(t)[y(t) - C(t)\hat{x}(t)] \tag{2b}$$

$$\dot{\Upsilon}(t) = [A(t) - K(t)C(t)]\Upsilon(t) + \Phi(t), \tag{2c}$$

with the initialization

$$\hat{x}(t_0) = \hat{x}_0 \tag{3a}$$

$$\hat{\theta}(t_0) = \hat{\theta}_0 \tag{3b}$$

$$\Upsilon(t_0) = 0_{n\times p}, \tag{3c}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the state estimate, $\hat{\theta}(t) \in \mathbb{R}^p$ the parameter estimate, $\Upsilon(t) \in \mathbb{R}^{n\times p}$ a matrix of auxiliary variables, $K(t) \in \mathbb{R}^{n\times m}$ the state estimation gain matrix, $\Gamma \in \mathbb{R}^{p\times p}$ the parameter gain matrix, $\hat{x}_0 \in \mathbb{R}^n$ and $\hat{\theta}_0 \in \mathbb{R}^p$ are initial values of the state and parameter estimates, and $0_{n\times p}$ is the $n \times p$ zero matrix initializing the auxiliary matrix $\Upsilon(t)$.

Remark 1. The last term in equation (2a) is equal to $\Upsilon(t)\dot{\hat{\theta}}(t)$. It compensates for the error due to the replacement of the true θ by its estimate $\hat{\theta}(t)$ in equation (2a). This extra term plays an important role in the convergence analysis of the adaptive observer. See (Zhang, 2002) for more details. □

In general, the state estimation gain $K(t)$ is chosen such that, if the term $\Phi(t)\theta$ was omitted in the state equation (1a), $K(t)$ would lead to a convergent state observer in the form of

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + K(t)[y(t) - C(t)\hat{x}(t)]. \tag{4}$$

In the time invariant (LTI) case, various methods for designing such observer gains are available. General time varying (LTV) systems are less well studied. In this case, observer gain design can be based on the well-known Kalman filter or on the Kalman-like observer (Besançon et al., 2006). Despite the deterministic framework considered in this paper, the Kalman gain will be used for $K(t)$, as an general LTV observer gain ensuring the convergence of the observer (4) (see Remark 2 below). This gain $K(t)$ is computed as

$$\begin{aligned}\dot{P}(t) = {} & A(t)P(t) + P(t)A^T(t) + Q(t) \\ & - P(t)C^T(t)R^{-1}(t)C(t)P(t)\end{aligned} \tag{5}$$

$$K(t) = P(t)C^T(t)R^{-1}(t) \tag{6}$$

with a positive definite initial matrix $P(t_0)$. In the Kalman filter literature, usually $P(t) \in \mathbb{R}^{n\times n}$ is known as the covariance matrix of the state estimate, $Q(t) \in \mathbb{R}^{n\times n}$ and

$R(t) \in \mathbb{R}^{m\times m}$ are respectively the state and output noise covariance matrices. In this paper, the considered problem is formulated in a deterministic framework, therefore $Q(t), R(t)$ *are treated as tuning parameters*, typically chosen as constant matrices, both symmetric positive definite.

Remark 2. It is known (Kalman, 1963; Jazwinski, 1970) that, under Assumption 2 (uniform complete observability), and under the uniform complete controllability assumption (which is trivially satisfied when $Q(t)$ is chosen to be a positive definite matrix), the Kalman gain $K(t)$ is *bounded* and ensures the *exponential* stability of the error dynamics of the state observer (4). More specifically, the Kalman gain $K(t)$ ensures that the homogenous ODE

$$\dot{\xi}(t) = [A(t) - K(t)C(t)]\xi(t), \tag{7}$$

with $\xi(t) \in \mathbb{R}^n$, is *exponentially* stable. □

Assumption 3. (PE, **assumed in this section only**). The matrix $\Phi(t)$ contains sufficient variations such that the matrix $\Upsilon(t)$, which is driven by $\Phi(t)$ through (2c), satisfies

$$\int_t^{t+T} \Upsilon^T(\tau)C^T(\tau)C(\tau)\Upsilon(\tau)d\tau \geq \alpha I_p, \tag{8}$$

for some positive constants T and α, and for all $t \geq t_0$. □

Remark 3. The auxiliary matrix $\Upsilon(t)$ contains signals obtained by linearly filtering $\Phi(t)$ through the linear filter (2c). Therefore, Assumption 3 is indeed about the properties of $\Phi(t)$, which is usually filled with exogenous signals (inputs), sometimes with injected outputs. For LTI systems (with constant matrices A, C, K), equation (2c) becomes an LTI filter of $\Phi(t)$. In this case the PE assumption can be expressed in terms of the frequency components of $\Phi(t)$.

It is shown in (Zhang, 2002) that, under Assumptions 1, 2 and 3, if $K(t)$ is chosen such that the homogenous system (7) is exponentially stable, then, with any initial state estimate $\hat{x}_0$ and any initial parameter estimate $\hat{\theta}_0$, the state and parameter estimation errors of the adaptive observer (2) both converge exponentially to zero, if the uncertainties $w(t), v(t)$ are omitted ($w(t) = 0$ and $v(t) = 0$). Without omitting $w(t), v(t)$, which are assumed bounded (Assumption 1), the state and parameter estimation errors remain bounded.

4. REGULARIZED ADAPTIVE OBSERVER

In what follows, *no* PE condition will be assumed. In particular, let us forget Assumption 3.

In order to overcome the lack of PE condition, the adaptive observer (2) is modified as

$$\begin{aligned}\dot{\hat{x}}(t) = {} & A(t)\hat{x}(t) + B(t)u(t) + \Phi(t)\hat{\theta}(t) \\ & + K(t)[y(t) - C(t)\hat{x}(t)] \\ & + \Upsilon(t)\Gamma\Upsilon^T(t)C^T(t)[y(t) - C(t)\hat{x}(t)] \\ & -\Upsilon(t)\Gamma\Lambda[\hat{\theta}(t) - \bar{\theta}]\end{aligned} \tag{9a}$$

$$\begin{aligned}\dot{\hat{\theta}}(t) = {} & \Gamma\Upsilon^T(t)C^T(t)[y(t) - C(t)\hat{x}(t)] \\ & -\Gamma\Lambda[\hat{\theta}(t) - \bar{\theta}]\end{aligned} \tag{9b}$$

$$\dot{\Upsilon}(t) = [A(t) - K(t)C(t)]\Upsilon(t) + \Phi(t), \tag{9c}$$

with the same initial conditions as in (3) and the Kalman gain $K(t)$ as in (6). The *only modifications* w.r.t. the adaptive observer (2) are the extra (blue) terms $-\Upsilon(t)\Gamma\Lambda[\hat{\theta}(t) - \bar{\theta}]$ in (9a) and $-\Gamma\Lambda[\hat{\theta}(t) - \bar{\theta}]$ in (9b), with a *symmetric positive definite matrix* $\Lambda \in \mathbb{R}^{p\times p}$ and a vector $\bar{\theta} \in \mathbb{R}^p$.

The positive definite matrix Λ controls the degree of regularization added in the algorithm. The vector $\bar{\theta}$ is a guess of the true value of θ. Naturally, $\bar{\theta}$ can be set to $\hat{\theta}_0$, the initial parameter estimate used in (3b), but this choice is not necessary. It is possible to take $\bar{\theta} = 0$ if no prior knowledge about θ is available.

Usually, to regularize an estimation algorithm designed by minimizing some criterion, a chosen penalty term is added to the minimized criterion. In contrast, in the proposed algorithm (9), *two* regularization terms have been added in (9a) and in (9b), in order to preserve the state-parameter decoupling effect of the transformation defined later in (12), as initially introduced in (Zhang, 2002). This transformation will be used in Proposition 2.

Before analyzing the error dynamics of this regularized adaptive observer, let us first ensure the boundedness of the auxiliary variable $\Upsilon(t) \in \mathbb{R}^{n\times p}$.

Proposition 1. Under Assumptions 1 and 2, the auxiliary variable $\Upsilon(t)$ driven by $\Phi(t)$ through (9c) is bounded. □

Proof. The Kalman gain $K(t)$ is *bounded* and ensures the exponential stability of the homogenous system (7), according to Remark 2. Then, based on Lemma 1 (in the appendix at the end of this paper), the auxiliary variable $\Upsilon(t)$ driven by bounded $\Phi(t)$ through (9c) is bounded. □

The behavior of the state and parameter estimation errors, namely

$$\tilde{x}(t) \triangleq x(t) - \hat{x}(t) \tag{10}$$

$$\tilde{\theta}(t) \triangleq \theta - \hat{\theta}(t), \tag{11}$$

will be analyzed in this paper. As an intermediate step, let us define the *decoupling transformation*, as originally introduced in (Zhang, 2002),

$$\eta(t) \triangleq \tilde{x}(t) - \Upsilon(t)\tilde{\theta}(t), \tag{12}$$

and analyze the behavior of this transformed error variable $\eta(t)$.

Proposition 2. The transformed error variable $\eta(t)$ defined in (12) is governed by the ODE

$$\dot{\eta}(t) = [A(t) - K(t)C(t)]\eta(t) + w(t) - K(t)v(t). \tag{13}$$

Moreover, under Assumptions 1 and 2, the homogeneous part of this ODE is exponentially stable, and there exists a positive constant a such that, for sufficiently large t so that the effect of the initial value $\eta(t_0)$ is negligible,

$$\|\eta(t)\| \leq a\,(\check{w} + \check{K}\check{v}), \tag{14}$$

where $\check{w}, \check{v}$ and $\check{K}$ are respectively upper bounds of $\|w(t)\|, \|v(t)\|$ and $\|K(t)\|$. □

Proof. Rewrite (9a) as

$$\begin{aligned}\dot{\hat{x}}(t) = {} & A(t)\hat{x}(t) + B(t)u(t) + \Phi(t)\hat{\theta}(t) \\ & + K(t)[y(t) - C(t)\hat{x}(t)] \\ & + \Upsilon(t)\dot{\hat{\theta}}(t).\end{aligned} \tag{15}$$

Then it is straightforward to derive from (1) and (15) that

$$\begin{aligned}\dot{\tilde{x}}(t) = {} & [A(t) - K(t)C(t)]\tilde{x}(t) + \Phi(t)\tilde{\theta}(t) - \Upsilon(t)\dot{\hat{\theta}}(t) \\ & + w(t) - K(t)v(t).\end{aligned} \tag{16}$$

Some simple computations with the variable $\eta(t)$ defined in (12) then lead to

$$\begin{aligned}\dot{\eta}(t) &= [A(t) - K(t)C(t)]\eta(t) \\ &\quad + \left\{[A(t) - K(t)C(t)]\Upsilon(t) + \Phi(t) - \dot{\Upsilon}(t)\right\}\tilde{\theta}(t) \\ &\quad - \Upsilon(t)\dot{\hat{\theta}}(t) - \Upsilon(t)\dot{\tilde{\theta}}(t) + w(t) - K(t)v(t) \qquad (17) \\ &= [A(t) - K(t)C(t)]\eta(t) \\ &\quad - \Upsilon(t)\dot{\hat{\theta}}(t) - \Upsilon(t)\dot{\tilde{\theta}}(t) + w(t) - K(t)v(t). \qquad (18)\end{aligned}$$

where the important simplification $\{\cdots\} = 0$ was based on (9c).

The true parameter vector θ is assumed constant, therefore $\dot{\theta} = 0$, and $\dot{\tilde{\theta}}(t) = \dot{\theta} - \dot{\hat{\theta}}(t) = -\dot{\hat{\theta}}(t)$. Then,

$$-\Upsilon(t)\dot{\hat{\theta}}(t) - \Upsilon(t)\dot{\tilde{\theta}}(t) = -\Upsilon(t)[\dot{\hat{\theta}}(t) - \dot{\hat{\theta}}(t)] = 0, \qquad (19)$$

and equation (18) becomes (13), which is now proved.

It then follows from Remark 2 that, under Assumptions 1 and 2, the homogenous part of (13), which is identical to (7), is *exponentially* stable.

Now apply Lemma 1 (in the appendix of this paper) to (13), whose homogenous part is already shown to be *exponentially* stable with some α and β. The terms $w(t) - K(t)v(t)$ correspond to $U(t)$, and therefore $\check{w} + \check{K}\check{v}$ to γ.

For sufficiently large t, the exponentially vanishing last term in (42) is negligible, and the remaining term $\alpha\gamma/\beta$ corresponds to the right hand side of (14), which is then proved with $a = \alpha/\beta$. □

Proposition 3. The parameter estimation error $\tilde{\theta}(t)$ satisfies the ODE

$$\begin{aligned}\dot{\tilde{\theta}}(t) &= -\Gamma[\Upsilon^T(t)C^T(t)C(t)\Upsilon(t) + \Lambda]\tilde{\theta}(t) \\ &\quad - \Gamma\Upsilon^T(t)C^T(t)C(t)\eta(t) \\ &\quad - \Gamma\Upsilon^T(t)C^T(t)v(t) \\ &\quad + \Gamma\Lambda(\theta - \bar{\theta}). \qquad (20)\end{aligned}$$

□

Proof.

$$\begin{aligned}\dot{\tilde{\theta}}(t) &= \dot{\theta} - \dot{\hat{\theta}}(t) = 0 - \dot{\hat{\theta}}(t) \qquad (21) \\ &= -\Gamma\Upsilon^T(t)C^T(t)[y(t) - C(t)\hat{x}(t)] \\ &\quad + \Gamma\Lambda[\hat{\theta}(t) - \bar{\theta}] \qquad (22)\end{aligned}$$

Notice that, notably following (12),

$$\begin{aligned}y(t) - C(t)\hat{x}(t) &= C(t)\tilde{x}(t) + v(t) \qquad (23) \\ &= C(t)[\Upsilon(t)\tilde{\theta}(t) + \eta(t)] + v(t), \qquad (24)\end{aligned}$$

then

$$\begin{aligned}\dot{\tilde{\theta}}(t) &= -\Gamma\Upsilon^T(t)C^T(t)C(t)\Upsilon(t)\tilde{\theta}(t) \\ &\quad - \Gamma\Upsilon^T(t)C^T(t)C(t)\eta(t) \\ &\quad - \Gamma\Upsilon^T(t)C^T(t)v(t) \\ &\quad + \Gamma\Lambda[\hat{\theta}(t) - \bar{\theta}]. \qquad (25)\end{aligned}$$

Rewrite the last term as

$$\begin{aligned}\Gamma\Lambda[\hat{\theta}(t) - \bar{\theta}] &= -\Gamma\Lambda[\theta - \hat{\theta}(t) - \theta + \bar{\theta}] \qquad (26) \\ &= -\Gamma\Lambda\tilde{\theta}(t) + \Gamma\Lambda(\theta - \bar{\theta}), \qquad (27)\end{aligned}$$

and combine $-\Gamma\Lambda\tilde{\theta}(t)$ with the other homogeneous term, then (20) is proved. □

Now let us study the stability of the homogeneous part of the ODE (20).

Proposition 4. Under Assumptions 1 and 2, The homogeneous part of the ODE (20), namely

$$\dot{\vartheta}(t) = -\Gamma[\Upsilon^T(t)C^T(t)C(t)\Upsilon(t) + \Lambda]\vartheta(t), \qquad (28)$$

with $\vartheta(t) \in \mathbb{R}^p$, is *exponentially* stable. More specifically, let $\Xi(t, t_0) \in \mathbb{R}^{p\times p}$ be the associated state transition matrix such that $\vartheta(t_2) = \Xi(t_2, t_1)\vartheta(t_1)$ for all time instants $t_1, t_2 \geq t_0$, then

$$\|\Xi(t, t_0)\| \leq \sqrt{\frac{\gamma_{\max}}{\gamma_{\min}}}e^{-\lambda_{\min}\gamma_{\min}(t-t_0)} \qquad (29)$$

for all $t \geq t_0$, with $\gamma_{\max}$ and $\gamma_{\min}$ denoting respectively the largest and the smallest singular values of Γ, and $\lambda_{\min}$ the smallest singular value of Λ. □

Proof. The matrix $\Upsilon^T(t)C^T(t)C(t)\Upsilon(t)$ is always positive semidefinite, then

$$\Upsilon^T(t)C^T(t)C(t)\Upsilon(t) + \Lambda \geq \Lambda \geq \lambda_{\min}I_p, \qquad (30)$$

where $\lambda_{\min} > 0$ is the smallest singular value of the positive definite matrix Λ. Then, according to Lemma 2 in the appendix of this paper, the homogeneous ODE (28) is exponentially stable, and the associated state transition matrix satisfies (29). □

Proposition 5. Under Assumptions 1 and 2, for sufficiently large t so that the effect of the initial $\tilde{\theta}(t_0)$ is negligible, there exist positive constants c_1, c_2, c_3 such that the parameter estimation error $\|\tilde{\theta}(t)\|$ is upper bounded as

$$\|\tilde{\theta}(t)\| \leq c_1\check{w} + c_2\check{v} + c_3\|\theta - \bar{\theta}\|, \qquad (31)$$

with $\check{w}$ and $\check{v}$ denoting respectively upper bounds of $\|w(t)\|$ and $\|v(t)\|$. □

Proof. The homogeneous part of the ODE (20), namely (28), is exponentially stable, with its state transition matrix satisfying (29).

Apply Lemma 1 to (20) with

$$\alpha = \sqrt{\frac{\gamma_{\max}}{\gamma_{\min}}}, \quad \beta = \lambda_{\min}\gamma_{\min}, \qquad (32)$$

$$\begin{aligned}U(t) &= -\Gamma\Upsilon^T(t)C^T(t)C(t)\eta(t) \\ &\quad - \Gamma\Upsilon^T(t)C^T(t)v(t) + \Gamma\Lambda(\theta - \bar{\theta}), \qquad (33)\end{aligned}$$

then, for sufficiently large t, so that the effect of the initial $\tilde{\theta}(t_0)$ is negligible,

$$\|\tilde{\theta}(t)\| \leq \frac{\sqrt{\frac{\gamma_{\max}}{\gamma_{\min}}}}{\lambda_{\min}\gamma_{\min}}\check{U}, \qquad (34)$$

where $\check{U}$ is an upper bound of $U(t)$ expressed in (33). Basic properties of the Euclidean vector norm and of the induced matrix norm lead to

$$\begin{aligned}\|U(t)\| &\leq \|\Gamma\|\|\Upsilon(t)\|\|C(t)\|^2\|\eta(t)\| \\ &\quad + \|\Gamma\|\|\Upsilon(t)\|\|C(t)\|\|v(t)\| + \|\Gamma\|\|\Lambda\|\|\theta - \bar{\theta}\| \\ &\leq \gamma_{\max}\check{\Upsilon}\check{C}^2\check{\eta} + \gamma_{\max}\check{\Upsilon}\check{C}\check{v} + \gamma_{\max}\lambda_{\max}\|\theta - \bar{\theta}\|\end{aligned}$$

where $\gamma_{\max}$ and $\lambda_{\max}$ are respectively the larges singular values of Γ and Λ, $\check{\Upsilon}, \check{C}, \check{\eta}, \check{v}$ are respectively upper bounds of $\|\Upsilon(t)\|, \|C(t)\|, \|\eta(t)\|, \|v(t)\|$.

Note that $\Upsilon(t)$ is bounded according to Proposition 1. For large t, $\eta(t)$ satisfies (14). Then

$$\begin{aligned}\|U(t)\| &\leq \gamma_{\max}\check{\Upsilon}\check{C}^2\frac{a}{b}(\check{w} + \check{K}\check{v}) + \gamma_{\max}\check{\Upsilon}\check{C}\check{v} \\ &\quad + \gamma_{\max}\lambda_{\max}\|\theta - \bar{\theta}\|. \qquad (35)\end{aligned}$$

The right hand side of this inequality is an upper bound of $\|U(t)\|$, which is a linear combination of $\breve{w}, \breve{v}$ and $\|\theta - \bar{\theta}\|$. This result, combined with (34) then proves (31). □

Remark 4. Proposition 5 indicates that the regularization term implies a bias of the parameter estimate proportional to $\|(\theta - \bar{\theta})\|$. Therefore, whenever possible, prior knowledge should be used for choosing $\bar{\theta}$ as close as possible to the unknown true parameter value θ, in order to reduce the bias. Proposition 4 shows that the convergence rate of the error dynamics depends on Λ and Γ. In general, a faster convergence is related to a higher sensitivity to uncertainties (or noises). The choice of the regularization parameter Λ could be made following the Bayesian approach, if a stochastic model was established. This approach will be studied in future works. □

5. NUMERICAL SIMULATION EXAMPLE

Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Intuitively, the corresponding constant matrix $\Phi(t) = I_3$ may not be sufficiently exciting. This fact is confirmed as follows.

Let $\sigma \in \mathbb{R}$ be *any* constant value. Make a state variable change with $z_2 = x_2 + \sigma$ replacing x_2, whereas x_1 and x_2 remain unchanged. After this variable change, the state and output equations are exactly as before, except that θ_1 and θ_3 are replaced respectively by $(\theta_1 - \sigma)$ and $(\theta_3 + \sigma)$. This result implies that all parameter pairs θ_1, θ_3 corresponding to the same sum value $\theta_1 + \theta_3$ lead to the same input-output relationship! It is thus impossible to uniquely determine θ_1 and θ_3 from input-output data.

For the simulation example presented below, the input $u(t)$

$$u(t) = \sin(t) + \cos(\sqrt{7}t), \tag{36}$$

the parameter vector $\theta = [1; 0.7; 0.5]$, and the initial state $x(0) = [1; 1; 1]$. The uncertainty terms are simulated as

$$w(t) = 0.1[\Delta(t); \Delta(t/2); \Delta(t/3)] \tag{37}$$

$$v(t) = 0.01[\Delta(t); \Delta(t/2)] \tag{38}$$

with the triangular wave function

$$\Delta(t) \triangleq 4\,|t - \lfloor t + 0.5 \rfloor| - 1,$$

and the notation $\lfloor x \rfloor$ denoting the largest integer less than or equal to $x \in \mathbb{R}$.

For the regularized adaptive observer, the state estimation gain $K(t)$ is computed with $P(0) = I_3$, $Q(t) = 0.1 I_3$, $R(t) = 0.01 I_2$, the parameter estimation gain $\Gamma = 20 I_3$, and the regularization parameter matrix $\Lambda = 0.0001 I_3$. The initial state estimate $\hat{x}_0$ and parameter estimate $\hat{\theta}_0$ are set to zero values, as well as the prior parameter guess $\bar{\theta}$.

As shown in Figures 1 and 2, despite the deficient excitation and the simulated uncertainties, the state and parameter estimates are close to their simulated (true) values, after a transient period of about 2 seconds. The two parameter estimates $\hat{\theta}_1(t)$ and $\hat{\theta}_3(t)$ have similar bias

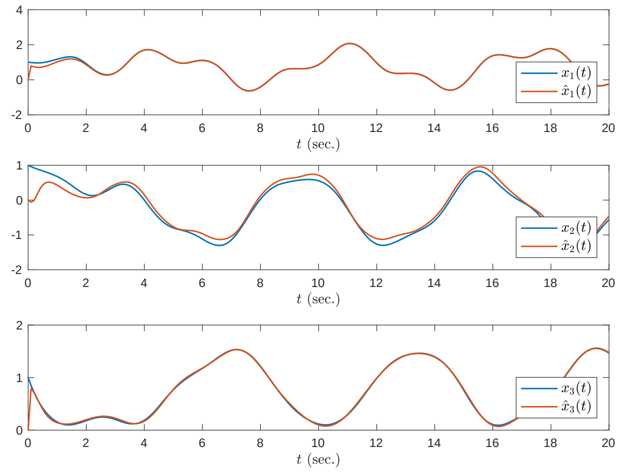

Fig. 1. Simulated states (in blue) and their estimates (in red).

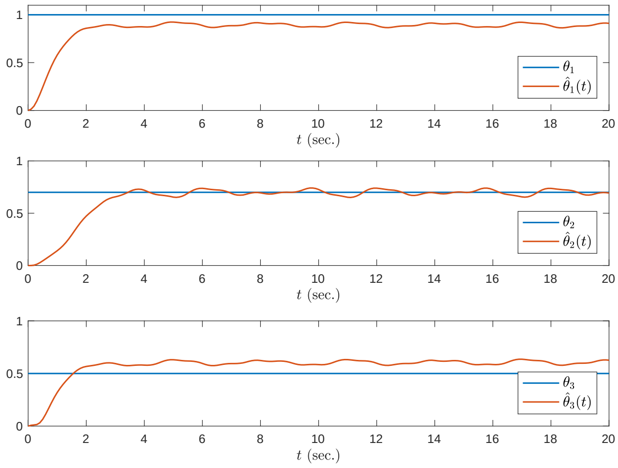

Fig. 2. Simulated parameter values (in blue) and their estimates (in red).

values, but of opposite signs. This fact is due to the indetermination of the two parameters: all parameter pairs θ_1, θ_3 corresponding to the same sum value $\theta_1 + \theta_3$ lead to the same input-output relationship.

6. CONCLUSION

The convergence of adaptive observers is usually ensured by some persistent excitation condition. In order to apply such algorithms when this condition is not satisfied, a regularized adaptive observer has been proposed in this paper, inspired by the usual practice for solving ill-posed inverse problems. In future studies, regularization will be studied in a stochastic framework in order to guide the tuning of the regularization parameter. The regularization method proposed in this paper is not the only possible choice. Other possibilities will also be investigated in future studies.

APPENDIX

Lemma 1. Assume that the homogeneous LTV ODE

$$\dot{\xi}(t) = A(t)\xi(t), \tag{39}$$

with $\xi(t) \in \mathbb{R}^n$ and $A(t) \in \mathbb{R}^{n\times n}$, is exponentially stable, such that its state transition matrix $\Phi(t, t_0) \in \mathbb{R}^n$ satisfies

$$\|\Phi(t, t_0)\| \leq \alpha e^{-\beta(t-t_0)} \tag{40}$$

for some positive constants α, β and for all $t \geq t_0$. Then, the solution of the non homogeneous LTV matrix ODE

$$\dot{X}(t) = A(t)X(t) + U(t), \tag{41}$$

with $X(t) \in \mathbb{R}^{n\times m}$, which is driven by the bounded input $U(t) \in \mathbb{R}^{n\times m}$ satisfying $\|U(t)\| \leq \gamma$ for all $t \geq 0$, satisfies

$$\|X(t)\| \leq \frac{\alpha\gamma}{\beta} + \alpha\left(\|X(t_0)\| - \frac{\gamma}{\beta}\right)e^{-\beta(t-t_0)}. \tag{42}$$

□

Remark that this result holds also for $m = 1$, when $X(t)$ and $U(t)$ are column vectors.

Proof.

$$X(t) = \Phi(t, t_0)X(t_0) + \int_{t_0}^{t} \Phi(t, s)U(s)ds, \tag{43}$$

then

$$\begin{aligned}\|X(t)\| &\leq \|\Phi(t, t_0)\|\|X(t_0)\| + \int_{t_0}^{t} \|\Phi(t, s)\|\|U(s)\|ds \\ &\leq \alpha e^{-\beta(t-t_0)}\|X(t_0)\| + \int_{t_0}^{t} \alpha e^{-\beta(t-s)}\gamma ds \\ &= \alpha\|X(t_0)\|e^{-\beta(t-t_0)} + \frac{\alpha\gamma}{\beta}\left(1 - e^{-\beta(t-t_0)}\right).\end{aligned}$$

Inequality (42) is then proved. □

Lemma 2. Consider the homogeneous LTV state equation

$$\dot{\zeta}(t) = -\Gamma\Omega(t)\zeta(t) \tag{44}$$

where $\zeta(t) \in \mathbb{R}^p$ is the state vector, $\Omega(t) \in \mathbb{R}^{p\times p}$ is a bounded symmetric positive definite matrix-valued piecewise continuous function of t, and Γ is a symmetric positive definite matrix. If $\Omega(t) \geq \beta I_p$ for all $t \geq t_0$, then $\zeta(t)$ converges exponentially to zero with a convergence rate proportional to β. More specifically, let $\Phi(t, t_0) \in \mathbb{R}^{p\times p}$ be the state transition matrix of (44) such that

$$\zeta(t_2) = \Phi(t_2, t_1)\zeta(t_1) \tag{45}$$

for any time instants $t_1, t_2 \geq t_0$, then

$$\|\Phi(t, t_0)\| \leq \sqrt{\frac{\gamma_{\max}}{\gamma_{\min}}}e^{-\beta\gamma_{\min}(t-t_0)} \tag{46}$$

for all $t \geq t_0$, with $\gamma_{\max}$ and $\gamma_{\min}$ denoting respectively the largest and the smallest singular values of Γ. □

Proof. Consider the Lyapunov function candidate

$$V(\zeta(t)) = \zeta^T(t)\Gamma^{-1}\zeta(t), \tag{47}$$

then

$$\begin{aligned}\frac{d}{dt}V(\zeta(t)) &= -2\zeta^T(t)\Omega(t)\zeta(t) &(48)\\ &\leq -2\beta\zeta^T(t)\zeta(t) &(49)\\ &\leq -2\beta\gamma_{\min}\zeta^T(t)\Gamma^{-1}\zeta(t) &(50)\end{aligned}$$

where $\gamma_{\min}$ is the smallest singular value of Γ. Then

$$\frac{d}{dt}V(\zeta(t)) \leq -2\beta\gamma_{\min}V(\zeta(t)). \tag{51}$$

By Grönwall's inequality, for all $t \geq t_0$,

$$V(\zeta(t)) \leq V(\zeta(t_0))e^{-2\beta\gamma_{\min}(t-t_0)}. \tag{52}$$

Let $\gamma_{\max}$ denote the largest singular value of Γ, then

$$\begin{aligned}\zeta^T(t)\gamma_{\max}^{-1}\zeta(t) &\leq V(\zeta(t)) \leq V(\zeta(t_0))e^{-2\beta\gamma_{\min}(t-t_0)} &(53)\\ &\leq \zeta^T(t_0)\gamma_{\min}^{-1}\zeta(t_0)e^{-2\beta\gamma_{\min}(t-t_0)}. &(54)\end{aligned}$$

Therefore,

$$\|\zeta(t)\|^2 \leq \frac{\gamma_{\max}}{\gamma_{\min}}\|\zeta(t_0)\|^2e^{-2\beta\gamma_{\min}(t-t_0)} \tag{55}$$

and

$$\|\zeta(t)\| \leq \sqrt{\frac{\gamma_{\max}}{\gamma_{\min}}}\|\zeta(t_0)\|e^{-\beta\gamma_{\min}(t-t_0)}, \tag{56}$$

which holds *for any* $\zeta(t_0) \in \mathbb{R}^p$, implying (46). □

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