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To cite this version:

Dilshad Surroop, Pascal Combes, Philippe Martin, Pierre Rouchon. Third-order virtual measurements with signal injection. CDC 2019 - 58th IEEE Conference on Decision and Control, Dec 2019, Nice, France. 10.1109/CDC40024.2019.9029751 . hal-02413822

HAL Id: hal-02413822

https://hal.science/hal-02413822v1

Submitted on 16 Dec 2019

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Third-order virtual measurements with signal injection

Dilshad Surroop[1,2], Pascal Combes[2], Philippe Martin[1] and Pierre Rouchon[1]

***Abstract*— Signal injection was conceptualized in [1] as a method to make available an extra "virtual measurement", hence to simplify the design of a control law in particular when the system observability degenerates at a steady-state region of interest. In this paper, we show that the approach of [1] can be extended to produce yet others virtual measurements, thanks to an analysis based on third-order averaging.**

I. Introduction

Signal injection is a control technique that has become widely used for the "sensorless" control of electrical motors at low velocity since its introduction by [2], [3] ("sensorless" meaning only the currents are measured, but neither the rotor position nor its velocity). Its consists in superimposing a fast-varying signal on the control, which creates some small ripple in the measured currents; this ripple contains information about the rotor position that can be used to suitably control the motor. At first sight, the method might seem peculiar to electrical motors, with a usually somewhat heuristic analysis. The essence of the method was then conceptualized in [1] as the creation of new "virtual measurements" which can be extracted from the actual measured output, hence providing means to overcome observability degeneracies; the main ingredient of the analysis is second-order averaging, following the ideas introduced in [4]. Developments following these ideas, with an application to magnetic levitation systems, can be found in [5], [6].

The purpose of this paper is to extend [1] by showing that more virtual measurements can be produced with a finer analysis of the ripple thanks to third-order averaging. To keep the computations as simple as possible and focus on the important ideas, we restrict to Single-Input Single-Output systems with a linear dynamics; nonlinear Multiple-Input Multiple-Output systems could nevertheless be addressed along the same lines. More precisely, consider the system

$$\dot{x} = Ax + Bu \tag{1a}$$
$$y = h(x), \tag{1b}$$

where (x, y, u) belongs to a compact subset of $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$, and A, B are constant matrices; the measured output $y = h(x)$ is assumed smooth enough (e.g. at least $\mathcal{C}^3$). We show that by superimposing to the control u a periodic signal with small period ε, we can make available the so-called virtual measurements

$$Y_1 = H_1(x) := \varepsilon h'(x)B \tag{2a}$$
$$Y_2 = H_2(x) := \frac{\varepsilon^2}{2}\left(h''(x)B\right)B \tag{2b}$$
$$Y_3 = H_3(x) := \varepsilon^2 h'(x)AB, \tag{2c}$$

which can be used in addition to $Y_0 = H_0(x) := h(x)$ to control (1a). The contribution with respect to [1] is threefold:

- an analysis of the output ripple by third-order averaging, with a simpler derivation (section II)
- a much more elaborated procedure to extract the virtual measurements from the output ripple (section III)
- a numerical simulation demonstrating the method is indeed effective, even though the output ripple may be very small and buried into noise (section IV).

II. Signal injection and third-order averaging

Assume we have designed a suitable control law

$$u = \alpha\big(\eta, Y, t\big)$$
$$\dot{\eta} = a\big(\eta, Y, t\big),$$

where $\eta \in \mathbb{R}^q$ and Y is the vector (Y_0, Y_1, Y_2, Y_3). In other words, the closed-loop system

$$\dot{\overline{x}} = A\overline{x} + B\alpha\big(\overline{\eta}, H(\overline{x}), t\big) \tag{3a}$$
$$\dot{\overline{\eta}} = a\big(\overline{\eta}, H(\overline{x}), t\big) \tag{3b}$$

has the desired exponentially stable behaviour, where $H := (H_0, H_1, H_2, H_3)$. We have changed the notation of the state to $(\overline{x}, \overline{\eta})$, so as to distinguish between the solutions of (3) and of (5) below.

Now consider the modified control law

$$u = \alpha\big(\eta, \overline{H}(x), t\big) + s_0\left(\tfrac{t}{\varepsilon}\right) \tag{4a}$$
$$\dot{\eta} = a\big(\eta, \overline{H}(x), t\big) \tag{4b}$$
$$\overline{H}(x) = H\big(x - \varepsilon B s_1\left(\tfrac{t}{\varepsilon}\right) - \varepsilon^2 AB s_2\left(\tfrac{t}{\varepsilon}\right)\big) + \mathcal{O}_\infty(\varepsilon^3), \tag{4c}$$

where s_0 is a 1-periodic function with zero mean, s_1 is the primitive of s_0 with zero mean, and s_2 the primitive of s_1 with zero mean (notice s_1 and s_2 are also 1-periodic); and $\mathcal{O}_\infty$ denotes the uniform "big O" symbol of analysis, namely $f(z, \varepsilon) = \mathcal{O}_\infty(\varepsilon)$ if there exists $K > 0$ independant of z and ε such that $\|f(z, \varepsilon)\| \leq K\varepsilon$. The closed-loop system then reads

$$\dot{x} = Ax + B\alpha\big(\eta, \overline{H}(x), t\big) + Bs_0\left(\tfrac{t}{\varepsilon}\right) \tag{5a}$$
$$\dot{\eta} = a\big(\eta, \overline{H}(x), t\big) \tag{5b}$$
$$\overline{H}(x) = H\big(x - \varepsilon B s_1\left(\tfrac{t}{\varepsilon}\right) - \varepsilon^2 AB s_2\left(\tfrac{t}{\varepsilon}\right)\big) + \mathcal{O}_\infty(\varepsilon^3). \tag{5c}$$

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Theorem 1: Let $\big(x(t),\eta(t)\big)$ and $\big(\overline{x}(t),\overline{\eta}(t)\big)$ be respectively the solutions of (5) and (3), with initial condition $\big(x(0),\eta(0)\big)$ and $\big(\overline{x}(0),\overline{\eta}(0)\big) = \big(x(0) - \varepsilon B s_1(0) - \varepsilon^2 AB s_2(0), \eta(0)\big)$. Then for all $t \geq 0$,

$$x(t) = \overline{x}(t) + \varepsilon B s_1\left(\tfrac{t}{\varepsilon}\right) + \varepsilon^2 AB s_2\left(\tfrac{t}{\varepsilon}\right) + \mathcal{O}_\infty(\varepsilon^3) \quad (6a)$$
$$\eta(t) = \overline{\eta}(t) + \mathcal{O}_\infty(\varepsilon^3) \quad (6b)$$
$$y(t) = H_0\big(\overline{x}(t)\big) + H_1\big(\overline{x}(t)\big) s_1\left(\tfrac{t}{\varepsilon}\right) + H_2\big(\overline{x}(t)\big) s_1^2\left(\tfrac{t}{\varepsilon}\right) + H_3\big(\overline{x}(t)\big) s_2\left(\tfrac{t}{\varepsilon}\right) + \mathcal{O}_\infty(\varepsilon^3). \quad (6c)$$

Proof: The proof is an application of higher-order averaging for differential equations [7, section 2.9], with slow time dependance [7, section 3.3]: consider the two equations

$$\frac{d\overline{X}}{d\sigma}(\sigma) = \varepsilon f^1(\overline{X}, \varepsilon\sigma, \sigma) + \mathcal{O}_\infty(\varepsilon^4)$$
$$\frac{d\widetilde{X}}{d\sigma}(\sigma) = \varepsilon f^1(\widetilde{X}, \varepsilon\sigma, \sigma) + \varepsilon^3 k^3(\widetilde{X}, \varepsilon\sigma, \sigma) + \mathcal{O}_\infty(\varepsilon^4),$$

where f^1, k^3 are T-periodic with respect to their third variable and k^3 has zero-mean with respect to its third variable; according to [7, theorem 2.9.2], the solutions of these two equations starting from the same initial condition are related by

$$\widetilde{X}(\sigma) = \overline{X}(\sigma) + \mathcal{O}_\infty(\varepsilon^3) \quad (7)$$

on the timescale $1/\varepsilon$. It is possible to extend this relation to an infinite timescale, provided that the averaged system has an exponentially stable equilibrium, and that the initial condition is in a compact subset of the region of attraction of this equilibrium, along the lines of [1, lemma 2].

In our case, we first rewrite (5) in the fast timescale $\sigma := t/\varepsilon$

$$\frac{dx}{d\sigma} = \varepsilon\left[Ax + B\alpha\big(\eta, \overline{H}(x), \varepsilon\sigma\big) + B s_0(\sigma)\right] \quad (8a)$$
$$\frac{d\eta}{d\sigma} = \varepsilon a\big(\eta, \overline{H}(x), \varepsilon\sigma\big). \quad (8b)$$

We introduce the new coordinates $(\widetilde{x}, \widetilde{\eta})$ such that

$$\begin{pmatrix}\widetilde{x}\\ \widetilde{\eta}\end{pmatrix} = \begin{pmatrix}x\\ \eta\end{pmatrix} - \varepsilon\begin{pmatrix}B s_1(\sigma)\\ 0\end{pmatrix} - \varepsilon^2\begin{pmatrix}AB s_2(\sigma)\\ 0\end{pmatrix}. \quad (9)$$

Notice (5c) simply reads

$$\overline{H}(x) = H(\widetilde{x}) + \mathcal{O}_\infty(\varepsilon^3).$$

Differentiating with respect to σ and expanding to third order yields the system in the new coordinates

$$\dot{\widetilde{x}} = \varepsilon\left[A\widetilde{x} + B\alpha\big(\widetilde{\eta}, H(\widetilde{x}), \varepsilon\sigma\big)\right] + \varepsilon^3 A^2 B s_2(\sigma) + \mathcal{O}_\infty(\varepsilon^4) \quad (10a)$$
$$\dot{\widetilde{\eta}} = \varepsilon a\big(\widetilde{\eta}, H(\widetilde{x}), \varepsilon\sigma\big) + \mathcal{O}_\infty(\varepsilon^4). \quad (10b)$$

By (7), we then get

$$\widetilde{x} = \overline{x} + \mathcal{O}_\infty(\varepsilon^3)$$
$$\widetilde{\eta} = \overline{\eta} + \mathcal{O}_\infty(\varepsilon^3).$$

Replacing $(\widetilde{x}, \widetilde{\eta})$ by (x, η) using transformation (9) in the two previous equations, we have the desired result

$$x = \overline{x} + \varepsilon B s_1(\sigma) + \varepsilon^2 AB s_2(\sigma) + \mathcal{O}_\infty(\varepsilon^3)$$
$$\eta = \overline{\eta} + \mathcal{O}_\infty(\varepsilon^3).$$

Plugging this expression for x in $y = h(x)$ and Taylor expanding to second order yields (6c). ■

The practical use of the theorem is the following. Assume we can compute a third-order estimate $\widehat{Y}$ of Y from the knowledge of only the actual measurement y. Using the relation between x and $\overline{x}$, we have

$$\widehat{Y} = H(\overline{x}) + \mathcal{O}_\infty(\varepsilon^3) = \overline{H}(x) + \mathcal{O}_\infty(\varepsilon^3)$$

The control law with signal injection

$$u = \alpha\big(\eta, \widehat{Y}, t\big) + s_0\left(\tfrac{t}{\varepsilon}\right) \quad (11a)$$
$$\dot{\eta} = a\big(\eta, \widehat{Y}, t\big) \quad (11b)$$

is then practically implementable and will control (1) as desired in average.

III. Demodulation of the virtual measurements

In this section, we show how to estimate the virtual measurements from the ripple in the actual measurement. Consider the composite signal

$$y(t) = Y_0(t) + Y_1(t) s_1\left(\tfrac{t}{\varepsilon}\right) + Y_2(t) s_1^2\left(\tfrac{t}{\varepsilon}\right) + Y_3(t) s_2\left(\tfrac{t}{\varepsilon}\right) + \mathcal{O}_\infty(\varepsilon^3),$$

where $Y_0, \ldots, Y_3$ are in $\mathcal{C}^3$ and $Y_0^{(3)}, \ldots, Y_3^{(3)}$ are bounded. Theorem 2 below states that $Y_0, \ldots, Y_3$ can be estimated at third order thanks to periodic low-pass filters.

A. Decomposition on an orthogonal basis

We first rewrite y as a decomposition on the periodic orthogonal signals $(1, s_1, s_2, S_1)$,

$$y(t) = \widetilde{Y_0}(t) + \widetilde{Y_1}(t) s_1\left(\tfrac{t}{\varepsilon}\right) + \widetilde{Y_2}(t) S_1\left(\tfrac{t}{\varepsilon}\right) + \widetilde{Y_3}(t) s_2\left(\tfrac{t}{\varepsilon}\right) + \mathcal{O}_\infty(\varepsilon^3). \quad (12)$$

The signals $1, s_1, s_2$ are orthogonal (for the scalar product $\langle f, g\rangle = \int_0^1 f(\tau)g(\tau)d\tau$); indeed, $\langle 1, s_1\rangle = \langle 1, s_2\rangle = 0$ since s_1,s_2 have zero mean, and $\langle s_1, s_2\rangle = \frac{1}{2}\int_0^1 (s_1^2)'(\sigma)d\sigma = 0$. S_1 is then obtained from s_1^2 by Gram-Schmidt orthogonalization,

$$S_1(t) := s_1^2(t) - \overline{s_1^2} - \tfrac{\langle s_1^2, s_1\rangle}{\overline{s_1^2}} s_1(t) - \tfrac{\langle s_1^2, s_2\rangle}{\overline{s_2^2}} s_2(t).$$

As a consequence, the "coordinates" $\widetilde{Y_i}$ are

$$\widetilde{Y_0}(t) = Y_0(t) + \overline{s_1^2} Y_2(t)$$
$$\widetilde{Y_1}(t) = Y_1(t) + \tfrac{\langle s_1^2, s_1\rangle}{\overline{s_1^2}} Y_2(t)$$
$$\widetilde{Y_2}(t) = Y_2(t)$$
$$\widetilde{Y_3}(t) = Y_3(t) + \tfrac{\langle s_1^2, s_2\rangle}{\overline{s_2^2}} Y_2(t).$$

B. Extraction of the Y_i using iterated moving averages

We now turn to the design of the demodulating filters, which are based on iterated moving averages of the form

$$M^1_\varphi(t) := \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} \varphi(\sigma)d\sigma$$
$$M^k_\varphi(t) := \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} M^{k-1}_\varphi(\sigma)d\sigma, \quad k>1.$$

We first recall a basic result on finite differences.

Lemma 1: Let φ be $\mathcal{C}^3$ with $\varphi^{(3)}$ bounded. Then the pth-order backward difference

$$\Delta^p_\varphi(t) := \sum_{i=0}^{p}\binom{p}{i}(-1)^i\varphi(t-i\varepsilon)$$

satisfies

$$\|\Delta^p_{\varphi^{(3-p)}}\|_\infty \lesssim \varepsilon^p\|\varphi^{(3)}\|_\infty, \quad p=0,\dots,3;$$

$\varphi \lesssim \psi$ means there exists $K>0$ such that $\varphi \le K\psi$.

Proof: For simplicity, we just prove as an example the case $p=2$. By the Taylor-Lagrange formula, there exists $t_1\in[t-\varepsilon,t]$ and $t_2\in[t-2\varepsilon,t]$ such that,

$$\varphi'(t-\varepsilon) = \varphi'(t) - \varepsilon\varphi''(t) + \frac{\varepsilon^2}{2}\varphi^{(3)}(t_1)$$
$$\varphi'(t-2\varepsilon) = \varphi'(t) - 2\varepsilon\varphi''(t) + 2\varepsilon^2\varphi^{(3)}(t_2)$$

Therefore,

$$\Delta^p_{\varphi'}(t) = -\varepsilon^2\varphi^{(3)}(t_1) + 2\varepsilon^2\varphi^{(3)}(t_2);$$

hence the desired inequality

$$\|\Delta^p_{\varphi'}\|_\infty \le 3\varepsilon^2\|\varphi^{(3)}\|_\infty.$$

■

This lemma is used to prove the following result, which has an important role on the filter design.

Lemma 2: Let φ be in $\mathcal{C}^3$ with $\varphi^{(3)}$ bounded, and ζ_0 be a 1-periodic function with zero mean. Then,

$$\|M^3_{\varphi\check{\zeta}_0}\|_\infty \lesssim \varepsilon^3\|\varphi^{(3)}\|_\infty\|\zeta_3\|_\infty,$$

with ζ_{j+1} the zero-mean primitive of ζ_j, and $\check{\zeta}(t) := \zeta(\frac{t}{\varepsilon})$.

Proof: Integrating by parts three times and using the periodicity of ζ_j gives

$$M_{\varphi\check{\zeta}_0}(t) = \Delta^1_\varphi(t)\zeta_1\left(\tfrac{t}{\varepsilon}\right) - \varepsilon\Delta^1_{\varphi'}(t)\zeta_2\left(\tfrac{t}{\varepsilon}\right) + \varepsilon^2\Delta^1_{\varphi''}(t)\zeta_3\left(\tfrac{t}{\varepsilon}\right) - \varepsilon^2\int_{t-\varepsilon}^{t}\varphi^{(3)}(\sigma)\zeta_3\left(\tfrac{\sigma}{\varepsilon}\right)d\sigma. \quad (13)$$

We now dominate the last two terms: clearly,

$$\left|\varepsilon^2\int_{t-\varepsilon}^{t}\varphi^{(3)}(\sigma)\zeta_3\left(\tfrac{\sigma}{\varepsilon}\right)\right| \le \varepsilon^3\|\varphi^{(3)}\|_\infty\|\zeta_3\|_\infty;$$

lemma 1 applied to the third term yields

$$\left|\varepsilon^2\Delta^1_{\varphi''}(t)\zeta_3\left(\tfrac{t}{\varepsilon}\right)\right| \lesssim \varepsilon^3\|\varphi^{(3)}\|_\infty\|\zeta_3\|_\infty.$$

Notice that if $\|f\|_\infty \le K$ then $\|M^p_f\|_\infty \le K$; therefore applying M^1 and M^2 to these two terms yields the same bounds.

Consider next the second term of (13); its moving average is

$$-\varepsilon M_{\Delta^1_{\varphi'}\check{\zeta}_2}(t) = -\varepsilon\Delta^2_{\varphi'}\zeta_3\left(\tfrac{t}{\varepsilon}\right) + \varepsilon\int_{t-\varepsilon}^{t}\Delta^1_{\varphi''}(\sigma)\zeta_3\left(\tfrac{\sigma}{\varepsilon}\right)d\sigma.$$

Using again lemma 1, the two terms of the right-hand side are similarly dominated by $\|\varphi^{(3)}\|_\infty\|\zeta_3\|_\infty$, and so are their moving averages.

Finally, consider the first term of (13). Its moving average is

$$M_{\Delta^1_\varphi\check{\zeta}_1}(t) = \Delta^2_\varphi(t)\zeta_2\left(\tfrac{t}{\varepsilon}\right) - \varepsilon\Delta^2_{\varphi'}(t)\zeta_3\left(\tfrac{t}{\varepsilon}\right) + \varepsilon\int_{t-\varepsilon}^{t}\Delta^1_{\varphi''}(\sigma)\zeta_3\left(\tfrac{\sigma}{\varepsilon}\right)d\sigma.$$

Likewise, by lemma 1, the last two terms of the right-hand side are also dominated by $\|\varphi^{(3)}\|_\infty\|\zeta_3\|_\infty$, and so are their moving average; as for the first term, integrating by parts its moving average yields

$$M_{\Delta^2_\varphi\check{\zeta}_2}(t) = \Delta^3_\varphi(t)\zeta_3\left(\tfrac{t}{\varepsilon}\right) - \int_{t-\varepsilon}^{t}\Delta^2_{\varphi'}(\sigma)\zeta_3\left(\tfrac{\sigma}{\varepsilon}\right)d\sigma.$$

Using lemma 1 to the right-hand side ends the proof. ■

We then recover the $\widetilde{Y}_i$ from linear combination of shifted M^3_y thanks to the following lemma.

Lemma 3: Let φ be in $\mathcal{C}^3$ with $\varphi^{(3)}$ bounded. Define

$$P_\varphi(t) := \frac{17}{4}M^3_\varphi(t) - 5M^3_\varphi(t-\varepsilon) + \frac{7}{4}M^3_\varphi(t-2\varepsilon),$$

Then the operator $\varphi \to P_\varphi$ is the identity up to third order,

$$P_\varphi(t) = \varphi(t) + \mathcal{O}_\infty(\varepsilon^3).$$

Proof: Again, consider first a single moving average of φ. Doing a change of variable in the integral and computing the Taylor expansion of φ gives

$$\begin{aligned}\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\varphi(\sigma)d\sigma &= \frac{1}{\varepsilon}\int_0^{\varepsilon}\varphi(t-\sigma)d\sigma\\ &= \frac{1}{\varepsilon}\int_0^{\varepsilon}\left[\varphi(t) - \sigma\varphi'(t) + \frac{\sigma^2}{2}\varphi''(t)\right]d\sigma + \mathcal{O}_\infty(\varepsilon^3)\\ &= \varphi(t) - \frac{\varepsilon}{2}\varphi'(t) + \frac{\varepsilon^2}{6}\varphi''(t) + \mathcal{O}_\infty(\varepsilon^3).\end{aligned}$$

We iterate the previous calculations to get the expressions of M^2_φ and M^3_φ,

$$\begin{aligned}M^2_\varphi(t) &= M_\varphi(t) - \frac{\varepsilon}{2}M_{\varphi'}(t) + \frac{\varepsilon^2}{6}M_{\varphi''}(t) + \mathcal{O}_\infty(\varepsilon^3)\\ &\quad + \varphi(t) - \varepsilon\varphi'(t) + \frac{7}{12}\varphi''(t) + \mathcal{O}_\infty(\varepsilon^3)\\ M^3_\varphi(t) &= \varphi(t) - \frac{3}{2}\varepsilon\varphi'(t) + \frac{15}{12}\varepsilon^2\varphi''(t) + \mathcal{O}_\infty(\varepsilon^3).\end{aligned}$$

Finally, we compute the shifted triple moving average for $k=0,1,2$. This yields

$$\begin{pmatrix} M^3_\varphi(t)\\ M^3_\varphi(t-\varepsilon)\\ M^3_\varphi(t-2\varepsilon)\end{pmatrix} = \underbrace{\begin{pmatrix}1 & -\frac{3}{2} & \frac{15}{12}\\ 1 & -\frac{5}{2} & \frac{39}{12}\\ 1 & -\frac{7}{2} & \frac{75}{12}\end{pmatrix}}_{D}\begin{pmatrix}\varphi(t)\\ \varepsilon\varphi'(t)\\ \varepsilon^2\varphi''(t)\end{pmatrix} + \mathcal{O}_\infty(\varepsilon^3)$$

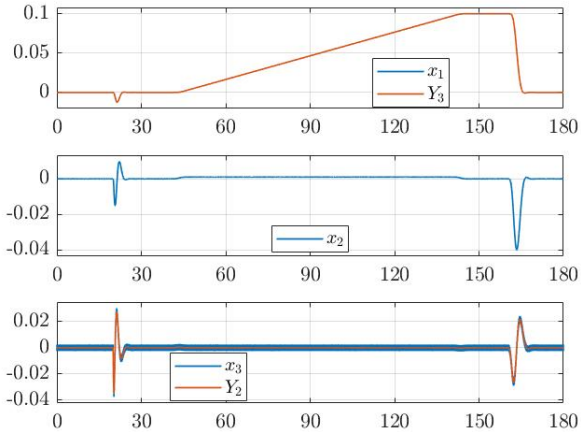

Fig. 1. x_1 and estimate $\hat{Y}_3/\varepsilon^2$ (top); x_2 (middle); x_3 and estimate $\hat{Y}_2/\varepsilon^2$ (bottom).

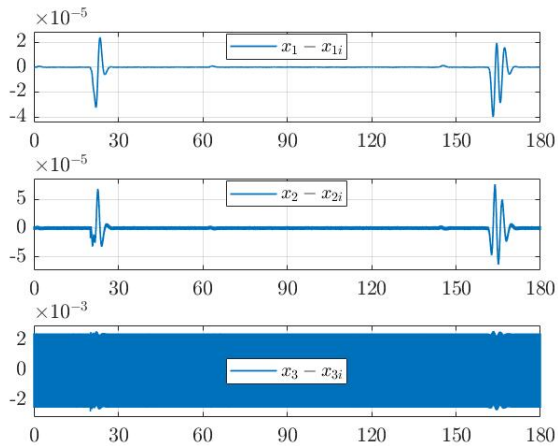

Fig. 2. Difference between x (signal injection) and x_i (ideal control).

Let $\alpha = \begin{pmatrix}1 & 0 & 0\end{pmatrix} D^{-1} = \begin{pmatrix}17/4 & -5 & 7/4\end{pmatrix}$. This yields the desired conclusion

$$P_\varphi(t) = \alpha D \varphi(t) = \varphi(t) + \mathcal{O}_\infty(\varepsilon^3).$$

■

Combining the three lemmas, we estimate the Y_i.

Theorem 2: Considering the operator P defined in lemma 3, we have an estimate for each of the Y_i, namely

$$\begin{aligned}
\widehat{Y_2}(t) &:= \tfrac{1}{\overline{S_1^2}} P_{y\check{S}_1}(t) && \text{(14a)}\\
&= Y_2(t) + \mathcal{O}_\infty(\varepsilon^3)\\
\widehat{Y_3}(t) &:= \tfrac{1}{\overline{s_2^2}} P_{y\check{s}_2}(t) - \tfrac{\langle s_1^2, s_2\rangle}{\overline{s_2^2}} \widehat{Y_2}(t) && \text{(14b)}\\
&= Y_3(t) + \mathcal{O}_\infty(\varepsilon^3)\\
\widehat{Y_1}(t) &:= \tfrac{1}{\overline{s_1^2}} P_{y\check{s}_1}(t) - \tfrac{\langle s_1^2, s_1\rangle}{\overline{s_1^2}} \widehat{Y_2}(t) && \text{(14c)}\\
&= Y_1(t) + \mathcal{O}_\infty(\varepsilon^3)\\
\widehat{Y_0}(t) &:= P_y(t) - \overline{s_1^2}\widehat{Y_2}(t) && \text{(14d)}\\
&= Y_0(t) + \mathcal{O}_\infty(\varepsilon^3).
\end{aligned}$$

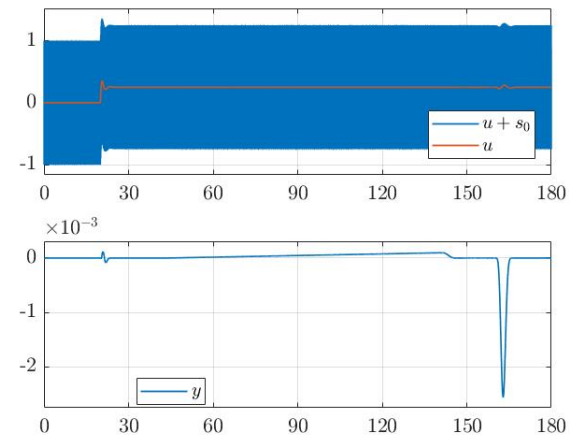

Fig. 3. Control (top) and measured output (bottom)

Proof: Let first determine the estimate for Y_2. Computing $P_{y\check{S}_1}$, we get

$$\begin{aligned}
P_{y\check{S}_1}(t) = {} & P_{\widetilde{Y_0}\check{S}_1}(t) + P_{\widetilde{Y_1}\check{s}_1\check{S}_1}(t) + P_{\widetilde{Y_2}(\check{S}_1^2 - \overline{S_1^2})}(t)\\
& + P_{\widetilde{Y_2}\overline{S_1^2}}(t) + P_{\widetilde{Y_3}\check{s}_2\check{S}_1}(t) + \mathcal{O}_\infty(\varepsilon^3).
\end{aligned}$$

Since P is a linear combination of shifted M^3 and S_1, $s_1 S_1$, $s_2 S_1$ and $S_1^2 - \overline{S_1^2}$ have zero mean, we have by lemma 2

$$P_{y\check{S}_1}(t) = P_{\widetilde{Y_2}\overline{S_1^2}}(t) + \mathcal{O}_\infty(\varepsilon^3).$$

Consequently, using lemma 3, we have the following estimate for $\widetilde{Y_2}$ (which is equal to Y_2)

$$\widehat{Y_2}(t) = \tfrac{1}{\overline{S_1^2}} P_{y\check{S}_1}(t) = Y_2(t) + \mathcal{O}_\infty(\varepsilon^3).$$

Let compute $\widehat{Y_3}$. The same calculations for $P_{y\check{s}_2}$ provide

$$P_{y\check{s}_2}(t) = \overline{s_2^2}\widetilde{Y_3}(t) = \overline{s_2^2}\Big(Y_3(t) + \tfrac{\langle s_1^2, s_2\rangle}{\overline{s_2^2}} Y_2(t)\Big),$$

that is, by definition of $\widehat{Y_3}$ (14b)

$$\widehat{Y_3}(t) = Y_3(t) + \mathcal{O}_\infty(\varepsilon^3).$$

Following the previous lines, we get the estimates $\widehat{Y_0}$, $\widehat{Y_1}$. ■

IV. A WORKED EXAMPLE

We illustrate the interest of the method on the system

$$\begin{aligned}
\dot{x}_1 &= x_2 && \text{(15a)}\\
\dot{x}_2 &= x_3 && \text{(15b)}\\
\dot{x}_3 &= u + d && \text{(15c)}\\
y &= x_1 x_2 + \frac{x_3^3}{3}, && \text{(15d)}
\end{aligned}$$

where d is an unknown disturbance. We would like x_1 to track the reference x_1^{ref} while rejecting the disturbance, with a time response of about a few time units. We want to operate around steady state, i.e. near states of the form $(x_1^{ss}, 0, 0)$ and inputs such that $u^{ss} + d^{ss} = 0$. Notice the system is not first-order observable around steady state because of the

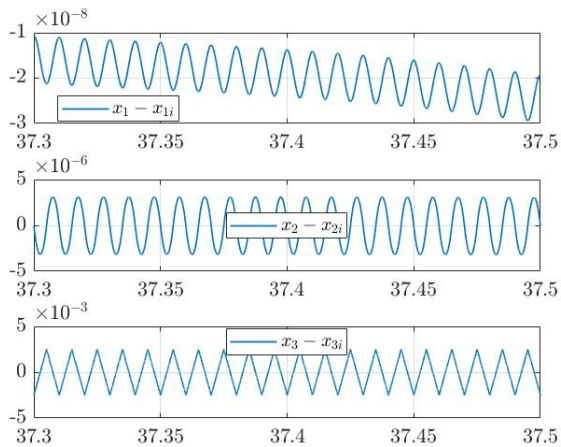

Fig. 4. Zoom on Fig. 2.

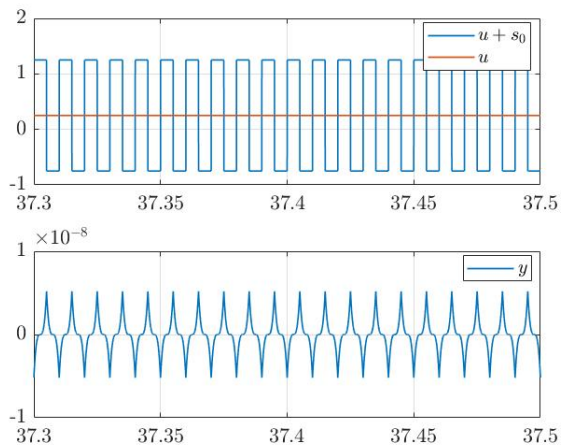

Fig. 5. Zoom on Fig. 3.

very degenerate output y, which makes the control problem far from obvious. Nonetheless, the virtual measurements are

$$
\begin{aligned}
Y_1 &= \varepsilon \begin{pmatrix} x_2 & x_1 & x_3^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} &&= \varepsilon x_3^2 \\
Y_2 &= \frac{\varepsilon^2}{2} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2x_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} &&= \varepsilon^2 x_3 \\
Y_3 &= \varepsilon^2 \begin{pmatrix} x_2 & x_1 & x_3^2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} &&= \varepsilon^2 x_1
\end{aligned}
$$

hence first-order observability is restored thanks to Y_2 and Y_3, without even considering y and Y_1. Notice third-order averaging is paramount, since the virtual measurement Y_1 stemming from second-order averaging is still degenerate.

With $Y_3 = \varepsilon^2 x_1$ and $Y_2 = \varepsilon^2 x_3$, the system is completely linear and can therefore be easily controlled. We design a

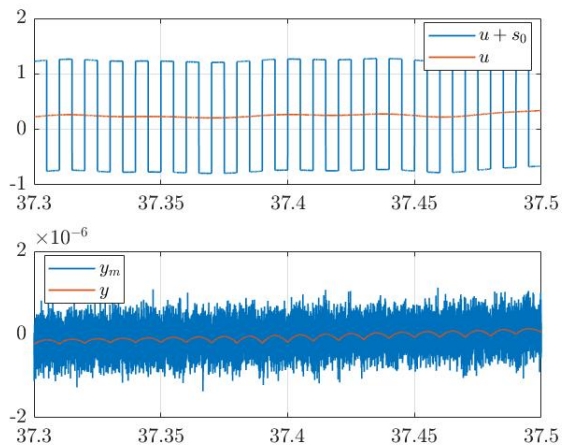

Fig. 6. Control (top) and measured output (bottom) with noise

classical controller-observer design, with observer

$$
\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{pmatrix} = \begin{pmatrix} \hat{x}_2 \\ \hat{x}_3 \\ u \end{pmatrix} + L \begin{pmatrix} Y_3/\varepsilon^2 - \hat{x}_1 \\ Y_2/\varepsilon^2 - \hat{x}_3 \end{pmatrix}
$$

and integral controller

$$
\begin{aligned}
u &= -k_1 \hat{x}_1 - k_2 \hat{x}_2 - k_3 \hat{x}_3 - k_I \eta_I \\
\dot{\eta}_I &= Y_3 - x_1^{ref}.
\end{aligned}
$$

The 3×2 matrix L and (k_1, k_2, k_3, k_I) are chosen such that the eigenvalues of the observer are $(-13.98, -1.57 \pm 1.59i)$ and those of the controller are $(-1.20 \pm 1.77i, -2.00 \pm 0.58i)$, which ensures a time response of a few time units and a reasonable robustness; the observer is faster than the controller, in accordance with Loop Transfer Recovery (at the plant input). Setting $\eta := (\hat{x}_1, \hat{x}_2, \hat{x}_3, \eta_I)^T$, this controller-observer reads

$$
\begin{aligned}
u &= -K\eta \\
\dot{\eta} &= M\eta + N x_1^{ref}(t) + L \begin{pmatrix} Y_3/\varepsilon^2 \\ Y_2/\varepsilon^2 \end{pmatrix}.
\end{aligned}
$$

Following section II, this yields the implementable control law for (15a)–(15c), which is a particular case of (11),

$$
\begin{aligned}
u &= -K\eta + s_0(\tfrac{t}{\varepsilon}) \\
\dot{\eta} &= M\eta + N x_1^{ref}(t) + L \begin{pmatrix} \widehat{Y}_3/\varepsilon^2 \\ \widehat{Y}_2/\varepsilon^2 \end{pmatrix},
\end{aligned}
$$

where $\widehat{Y}_2$ and $\widehat{Y}_3$ are obtained from the actual measurement (15d) by the demodulation procedure of section III. The injected signal s_0 is a square wave of amplitude 1 and frequency 100, which ensures the oscillation is fast with respect to the time constants of the closed-loop system; $n := 2$ is used in the demodulating filter.

The test scenario is the following: at $t = 0$ the system starts at rest at the origin, with the reference x_1^{ref} set to 0; at $t = 20$, a step disturbance d of magnitude -0.25 is applied; for $40 \leq t \leq 140$, x_1^{ref} is a slow (filtered) ramp with

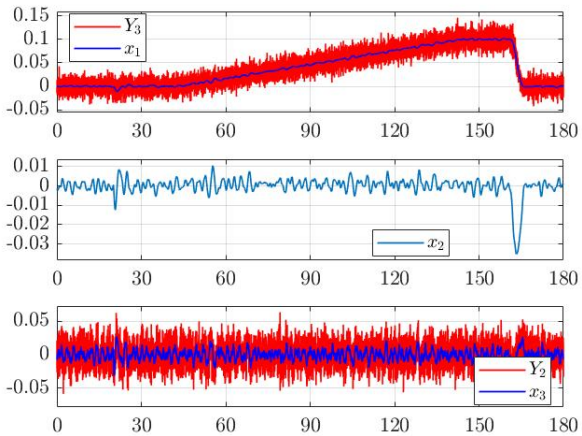

Fig. 7. x_1 and estimate $\hat{Y}_3/\varepsilon^2$ (top); x_2 (middle); x_3 and estimate $\hat{Y}_2/\varepsilon^2$ (bottom).

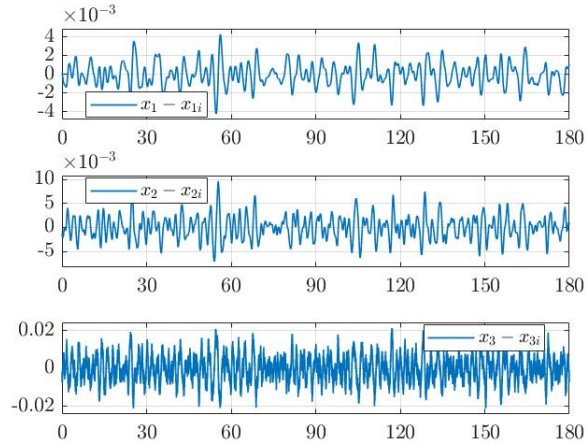

Fig. 8. Difference between x (signal injection) and x_i (ideal control).

slope 10^{-3}, meaning the system must slowly move while nearly first-order unobservable; finally, at $t = 180$, x_1^{ref} is a filtered step so as to quickly return to 0.

Without noise on the actual output y, the performance is excellent: the reference is tracked and the disturbance is rejected (Fig. 1). In fact the system behaves nearly as in the ideal situation where it is controlled directly from x_1, x_3 without signal injection (Fig. 2); the ripple is visible on x_3 because it is directly affected by the input, but is much smaller on x_1, x_2, as anticipated by the averaging analysis. A zoomed view for $37.3 \leq t \leq 37.5$ gives a better insight of the signals: Fig. 4 illustrates (6a); Fig. 5(top) shows the square wave in the control signal; Fig. 5(bottom) shows the ripple in the actual output is really tiny near steady state, with a strange form caused by the nonlinearity of this output.

With a measured output y_m corrupted by noise, one thus might fear the ripple is much too small to be useful; this is not the case since the demodulation process is essentially a low-pass filter. Indeed, the system performs as desired (Fig. 7 and 8), even though the output ripple is buried into noise (Fig. 6). The measurement noise used in the simulation is a band-limited white noise with noise power 1×10^{-18} and sample time 1×10^{-5}.

V. Conclusion

We have shown that it is possible to produce "virtual outputs" thanks to the injection of fast-varying periodic probing signal. These virtual measurements can be used to simplify the design of a feeedback law, in particular when the system observability degenerates. The analysis relies on third-order averaging, and significantly extends the approach of [1]. For simplicity, we have restricted to Single-Input Single-Output systems with linear dynamics; in a future work we will generalize the approach to nonlinear Multi-Input Multi-Output systems.

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