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Weak existence and uniqueness for affine stochastic Volterra equations with L^1 -kernels

Eduardo Abi Jaber*

June 18, 2020

Abstract

We provide existence, uniqueness and stability results for affine stochastic Volterra equations with L^1 -kernels and jumps. Such equations arise as scaling limits of branching processes in population genetics and self-exciting Hawkes processes in mathematical finance. The strategy we adopt for the existence part is based on approximations using stochastic Volterra equations with L^2 -kernels combined with a general stability result. Most importantly, we establish weak uniqueness using a duality argument on the Fourier–Laplace transform via a deterministic Riccati–Volterra integral equation. We illustrate the applicability of our results on Hawkes processes and a class of hyper-rough Volterra Heston models with a Hurst index $H \in (-1/2, 1/2]$.

Keywords: Stochastic Volterra equations, Affine Volterra processes, Riccati–Volterra equations, superprocesses, Hawkes processes, rough volatility.

MSC2010 Classification: 60H20, 60G22, 45D05

1 Introduction

We establish weak existence, uniqueness and stability results for stochastic Volterra equation with locally L^1 -kernels K in the form

$$X_t = G_0(t) + \int_0^t K(t-s)Z_s ds, \quad t \geq 0, \quad (1.1)$$

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for a given function $G_0 : \mathbb{R}_+ \rightarrow \mathbb{R}$ where Z is a real-valued semimartingale, starting from zero, with affine characteristics in X

$$(bX, cX, \nu(d\zeta)X), \tag{1.2}$$

with $b \in \mathbb{R}$, $c \geq 0$, ν a nonnegative measure on \mathbb{R}_+ such that $\int_{\mathbb{R}_+} \zeta^2 \nu(d\zeta) < \infty$, with respect to the ‘truncation function’ $\chi(\zeta) = \zeta$. For L^2 -kernels this formulation was recently introduced in [Abi Jaber et al. \(2019a\)](#), where Z is a semimartingale but whose characteristics are absolutely continuous with respect to the Lebesgue measure. In the L^1 setting, X may fail to be absolutely continuous with respect to the Lebesgue measure, as will be explained in the sequel. For this reason, our study falls beyond the scope of [Abi Jaber et al. \(2019a\)](#).

Our motivation for studying such convolution equations is twofold. Stochastic Volterra equations with kernels that are locally in L^1 but not in L^2 with $c > 0$ $\nu \equiv 0$ arise as scaling limits of branching processes in population genetics and self-exciting Hawkes processes in mathematical finance. As we will highlight in the sequel, the L^1 -framework allows for instance to make sense of fractional dynamics, inspired by the fractional Brownian motion, for negative Hurst indices $H \in (-1/2, 0)$.

- *From branching processes to stochastic Volterra equations.* The link was formulated for the first time in [Mytnik and Salisbury \(2015\)](#) to motivate the study of stochastic Volterra equations with L^2 -kernels. In the sequel we re-formulate the aforementioned introductory exposition linking super-processes with stochastic Volterra equations with L^1 -kernels. Consider a system of n *reactant* particles in one dimension moving independently according to a standard Brownian motion and branching only in the presence of a *catalyst*. The *catalyst* region at a certain time t is defined as the support of some deterministic measure $\rho_t(dx)$. Whenever a particle enters in the *catalyst* region and after spending a random time in the vicinity of the *catalyst*, it will either die or split into two new particles, with equal probabilities. The measure $\rho_t(dx)$ determines the local branching rate in space and time depending on the location and the concentration of the *catalyst*. Two typical examples are $\rho_t(dx) \equiv \bar{\rho}dx$ where the branching occurs in the entire space with constant rate $\bar{\rho}$ and $\rho_t(dx) = \delta_0(dx)$ for a branching occurring with infinite rate only when the particle hits a highly concentrated single point *catalyst* located at 0. In case of branching, the two offspring particles evolve independently with the same spatial movement and branching mechanism as their parent.

One can view the reactant as a rescaled measure-valued process $(\bar{Y}_t^n(dx))_{t \geq 0}$ defined by

$$\bar{Y}_t^n(B) = \frac{\text{number of particles in } B \text{ at time } t}{n}, \quad \text{for every Borel set } B.$$

Sending the number of particles to infinity, one can establish the convergence towards a measure-valued macroscopic *reactant* \bar{Y} , coined *catalytic super-Brownian motion*, which solves an infinite dimensional martingale problem, see [Dawson and Fleischmann \(1991\)](#);

Etheridge (2000); Perkins (2002) and the references therein. Moreover, in the presence of a suitable deterministic *catalyst* $\rho = (\rho_t(dx))_{t \geq 0}$ having no atoms, the measure-valued process \bar{Y} admits a density $\bar{Y}_t(dx) = Y_t(x)dx$ solution to the following stochastic partial differential equation in mild form

$$Y_t(x) = \int_{\mathbb{R}} p_t(x-y)Y_0(y)dy + \int_{[0,t] \times \mathbb{R}} p_{t-s}(x-y)\sqrt{Y_s(y)}W^\rho(ds, dy). \quad (1.3)$$

where Y_0 is an input curve, $p_t(x) = (2\pi t)^{-1/2} \exp(-x^2/(2t))$ is the heat kernel and W^ρ is a space-time noise with covariance structure determined by ρ , we refer to Zähle (2005) for more details. The previous equation is only valid if ρ has no atoms. One could still heuristically set $\rho_t(dx) = \delta_0(dx)$ in (1.3) for the extreme case of a single point *catalyst* at 0, which would formally correspond to the *catalytic super-Brownian motion* of Dawson and Fleischmann (1994). Then, the space-time noise reduces to a standard Brownian motion W so that evaluation at $x = 0$ yields

$$Y_t(0) = g_0(t) + \frac{1}{\sqrt{2\pi}} \int_0^t (t-s)^{-1/2} dZ_s, \quad (1.4)$$

where $dZ_t = \sqrt{Y_t(0)}dW_t$ and $g_0(t) = \int_{\mathbb{R}} p_t(y)Y_0(y)dy$. The link with stochastic Volterra equations of the form (1.1) is established by considering the local occupation time at the catalyst point 0 defined by

$$X_t = \lim_{\varepsilon \rightarrow 0} \int_0^t \int_{\mathbb{R}} p^\varepsilon(y)\bar{Y}_s(dy) ds, \quad t \geq 0, \quad (1.5)$$

where p^ε is a suitable smoothing kernel of the dirac point mass at 0. Integrating both sides of equation (1.4) with respect to time and formally interchanging the integrals lead to

$$X_t = \int_0^t Y_s(0)ds \quad (1.6)$$

$$= \int_0^t g_0(s)ds + \frac{1}{\sqrt{2\pi}} \int_0^t (t-s)^{-1/2} Z_s ds, \quad (1.7)$$

such that Z is a continuous semimartingale with affine characteristics $(0, X, 0)$. Consequently, X solves (1.1) for the kernel

$$K_0(t) = \frac{t^{-1/2}}{\sqrt{2\pi}}, \quad t > 0, \quad (1.8)$$

which is locally in L^1 but not in L^2 . Needless to say, one is not allowed to plug the Dirac measure in (1.3). Indeed, in the presence of a single point *catalyst*, the *catalytic super-Brownian motion* does not admit a density at the *catalyst* position as shown by

Dawson and Fleischmann (1994) and the identities (1.3) and (1.6) break down. The local occupation time X is even singular with respect to the Lebesgue measure, see Dawson et al. (1995); Fleischmann and Le Gall (1995). Still, one can rigorously prove that the local occupation time X defined by (1.5) solves (1.7) by appealing to the martingale problem of the measure-valued process \bar{Y} , we refer to Appendix A for a rigorous derivation.

• *From Hawkes processes to stochastic Volterra equations.* More recently, for particular choices of G_0 and kernels, solutions to (1.1) were obtained in Jusselin and Rosenbaum (2018) as scaling limits of Hawkes processes $(N^n)_{n \geq 1}$ with respective intensities

$$\lambda_t^n = g_0^n(t) + \int_0^t K^n(t-s) dN_s^n, \quad t \geq 0,$$

for some suitable function g_0^n and kernel K^n . The rescaled sequence of integrated accelerated intensities $X^n = \frac{1}{n} \int_0^\cdot \lambda_{n,s}^n ds$ is shown to converge to a continuous process X satisfying (1.1) for the fractional kernel¹

$$K_H(t) = \frac{t^{H-1/2}}{\Gamma(H+1/2)}, \quad t > 0, \tag{1.9}$$

with $H \in (1/2, 1/2]$. We note that for $H = 0$ the fractional kernel reduces to (1.8), up to a normalizing constant. In other words, when $H = 0$, the scaling limit of the integrated intensities of Hawkes processes can be seen as the local occupation time of the *catalytic super-Brownian motion* of Dawson and Fleischmann (1994), provided uniqueness holds. Similarly, when $H \leq 0$, K_H lies locally in L^1 but not in L^2 , and one can also show that in this case X is not absolutely continuous with respect to the Lebesgue measure, see Jusselin and Rosenbaum (2018, Proposition 4.6). For $H > 0$, the kernel (1.9) is nothing else but the kernel that appears in the celebrated Mandelbrot and Van Ness (1968) decomposition of fractional Brownian motion W^H :

$$W_t^H = \int_0^t K_H(t-s) dW_s + \int_{-\infty}^0 (K_H(t-s) - K_H(-s)) dW_s$$

where W is a two-sided standard Brownian motion and $H > 0$ is required to make sense of the stochastic convolution with respect to Brownian motion in the L^2 -theory of Kiyosi Itô. In this sense, the L^1 -framework allows for a generalization of fractional type dynamics with negative Hurst indices.

In both of the motivating cases, one can compute the Laplace transform of X , modulo a deterministic Riccati–Volterra equation of the form

$$\psi(t) = \int_0^t K_H(t-s) \left(\frac{1}{2} \psi^2(s) - 1 \right) ds,$$

¹To be more precise, in Jusselin and Rosenbaum (2018), the limiting kernel is not the fractional kernel but the so-called Mittag-Leffler function and the process Z there has characteristics $(0, X, 0)$. This can be equivalently re-written with the fractional kernel modulo the addition of a suitable drift bX , we refer to Example 2.10 below for more details.

either by using the dual process of the *catalytic super-Brownian motion*, see Dawson and Fleischmann (1994, Equations (4.2.1)-(4.2.2)), or by exploiting the affine structure of the approximating Hawkes processes, see Jusselin and Rosenbaum (2018, Theorem 3.4). Both constructions provide solutions to (1.1), but do not yield uniqueness. Establishing weak uniqueness is one of the main motivation of this work.

In the present paper, we provide a generic treatment of the limiting macroscopic equation (1.1) and we allow for (infinite activity) jumps in Z . For instance, Hawkes processes can be recovered by setting $c = 0$ and $\nu = \delta_1$. The strategy we adopt is based on approximations using stochastic Volterra equations with L^2 kernels, whose existence and uniqueness theory is now well-established, see Abi Jaber et al. (2019a,b) and the references therein. By doing so, we avoid the infinite-dimensional analysis used for super-processes, we also circumvent the need to study scaling limits of Hawkes processes, allowing for more generality in the choice of kernels K and input functions G_0 . Along the way, we derive a general stability result that encompasses the motivating example with Hawkes processes. Most importantly, we establish weak uniqueness using a duality argument on the Fourier–Laplace transform of X via a deterministic Riccati–Volterra integral equation. In particular, this expression extends the one obtained for affine Volterra processes with L^2 -kernels in Abi Jaber et al. (2019b); Cuchiero and Teichmann (2020). We illustrate the applicability of our results on a class of hyper-rough Volterra Heston models with a Hurst index $H \in (-1/2, 1/2]$ and jumps complementing the results of Abi Jaber et al. (2019b); El Euch and Rosenbaum (2019); Jusselin and Rosenbaum (2018). Such models have recently known a growing interest to account for rough volatility, a universal phenomena observed in financial markets, see Gatheral et al. (2018).

Notations Δ_h stands for the shift operator, i.e. $\Delta_h g = g(h + \cdot)$ and dg is the distributional derivative of a right-continuous function g with locally bounded variation, i.e. $dg((s, t]) = g(t) - g(s)$. For a suitable Borel function f the quantity $\int_0^\cdot f(s)dg(s)$ will stand for the Lebesgue–Stieltjes integral, whenever the integral exists. Similarly, for each $t < T$, the convolution $\int_0^t f(t-s)dg(s)$ is defined as the Lebesgue–Stieltjes integral $\int_0^T \mathbf{1}_{[0,t]} f(t-s)dg(s)$ whenever this latter quantity is well-defined.

Outline Section 2 states our main existence, uniqueness and stability results together with the expression for the Fourier–Laplace transform. Section 3 provides a-priori estimates for the solution. In Section 4, we derive a general stability results for stochastic Volterra equations with L^1 -kernels. These results are used to establish weak existence for the stochastic Volterra equation in Section 5. Furthermore, an existence result for Riccati–Volterra equations with L^1 -kernels is derived there. Weak uniqueness is then established by completely characterizing the Fourier–Laplace transform of the solution in terms of the Riccati–Volterra equation of Section 6. In Section 7, we apply our results to obtain existence, uniqueness and the characteristic function of the log-price in hyper-rough Volterra Heston models. Finally, we provide a more rigorous derivation of the stochastic Volterra equation satisfied by the local occupation time of the catalytic super-Brownian motion in

Appendix [A](#).

2 Main results

In this section, we present our main results together with the strategy we adopt. We start by making precise the concept of solution.

We call X a weak solution to [\(1.1\)](#) for the input (G_0, K, b, c, ν) , if there exists a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ supporting a non-decreasing, nonnegative, continuous and adapted process X and a semimartingale Z whose characteristics are given by [\(1.2\)](#) such that [\(1.1\)](#) holds \mathbb{P} -almost surely. We stress that a weak solution is given by the triplet $(X, (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}), Z)$. To ease notations we simply denote the solution by X . In this case, Z admits the following decomposition

$$Z_t = bX_t + M_t^c + M_t^d, \quad t \geq 0, \quad (2.1)$$

where M^c is a continuous local martingale of quadratic variation cX and M^d is a purely discontinuous local martingale given by $\int_{[0,t] \times \mathbb{R}_+} \zeta (\mu^Z(dt, d\zeta) - \nu(d\zeta)dX_t)$ and μ^Z is the jump measure of Z , we refer to [Jacod and Shiryaev \(2003, II.2.5-6\)](#). We say that weak uniqueness holds for the inputs (G_0, K, b, c, ν) if given two weak solutions $(X, (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}), Z)$ and $(X', (\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t \geq 0}, \mathbb{P}'), Z')$, X and X' have the same finite dimensional marginals.

One first notes that the formulation [\(1.1\)](#), when $\nu \equiv 0$, differs from the one given in [Jusselin and Rosenbaum \(2018, Equation \(3\)\)](#), where

$$X_t = G_0(t) + \int_0^t \left(\int_0^{t-s} K(r) dr \right) dZ_s.$$

Although these two formulations are equivalent, thanks to a stochastic Fubini theorem, the advantages of considering the formulation [\(1.1\)](#) as starting point, which is inspired by the ‘martingale problem’ formulation of stochastic Volterra equations recently introduced in [Abi Jaber et al. \(2019a\)](#) will become clear in the sequel.

The following lemma establishes the link with stochastic Volterra equations with L^2 -kernels, as the one studied for instance in [Abi Jaber et al. \(2019a,b\)](#); [Cuchiero and Teichmann \(2020\)](#).

Lemma 2.1. *Fix $K \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$ and $g_0 \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$. Assume that there exists a non-decreasing continuous adapted process X on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ such that*

$$X_t = \int_0^t g_0(s) ds + \int_0^t K(t-s) Z_s ds, \quad (2.2)$$

with Z given by (2.1). Then, $X = \int_0^\cdot Y_s ds$ where Y is a nonnegative weak solution to the following stochastic Volterra equation

$$Y_t = g_0(t) + \int_0^t K(t-s) dZ_s, \quad \mathbb{P} \otimes dt - a.e. \quad (2.3)$$

where the differential characteristics of Z with respect to the Lebesgue measure are given by $(bY, cY, \nu(d\zeta)Y)$.

Conversely, assume there exists a nonnegative weak solution Y to the stochastic Volterra equation (2.3) such that $\int_0^T Y_s ds < \infty$ for all $T > 0$, then $X = \int_0^\cdot Y_s ds$ is a continuous non-decreasing solution to (2.2).

Proof. Fix $t \geq 0$. An application of a stochastic Fubini theorem, see [Abi Jaber et al. \(2019a, Lemma 3.2\)](#), yields

$$\begin{aligned} \int_0^t K(t-s) Z_s ds &= \int_0^t K(s) \left(\int_0^{t-s} dZ_s \right) ds \\ &= \int_0^t \left(\int_0^{t-r} K(s) ds \right) dZ_r \\ &= \int_0^t \left(\int_0^t K(s-r) \mathbf{1}_{\{r \leq s\}} ds \right) dZ_r \\ &= \int_0^t \left(\int_0^s K(s-r) dZ_r \right) ds. \end{aligned}$$

Thus, X admits a density Y with respect to the Lebesgue measure, such that

$$Y_t = g_0(t) + \int_0^t K(t-r) dZ_r,$$

and the characteristics of Z read

$$\left(b \int_0^\cdot Y_s ds, c \int_0^\cdot Y_s ds, \int_{[0, \cdot] \times \mathbb{R}_+} Y_s ds \nu(d\zeta) \right).$$

Since X is non-decreasing almost surely, Y is nonnegative $\mathbb{P} \otimes dt$. The claimed stochastic Volterra equation (2.3) readily follows. The converse direction follows along the same lines by integrating both sides of (2.3) and applying a stochastic Fubini theorem as above to get (2.2). \square

2.1 Uniqueness and Fourier–Laplace transform

We start by stating our first main result concerning the representation of the Fourier–Laplace transform of the joint process (X, M^c, M^d) and the weak uniqueness statement for

(1.1) in terms of a solution to the Riccati–Volterra equation

$$\psi(t) = \int_0^t K(t-s)F(s, \psi(s))ds, \quad (2.4)$$

$$F(s, u) = f_0(s) + \frac{1}{2}cf_1^2(s) + (b + cf_1(s))u + \frac{c}{2}u^2 + \int_{\mathbb{R}_+} \left(e^{(f_2(s)+u)\zeta} - 1 - (f_2(s) + u)\zeta \right) \nu(d\zeta), \quad (2.5)$$

where $f_0, f_1, f_2 : \mathbb{R}_+ \mapsto \mathbb{B}$ are suitable functions. We introduce the following process which enters in the representation of the Fourier–Laplace transform:

$$G_t(s) = G_0(s) + \int_t^s g_t(u)du, \quad t \leq s, \quad (2.6)$$

$$g_t(u) = \int_0^t K(u-r)dZ_r, \quad t < u. \quad (2.7)$$

The stochastic convolution $\int_0^t K(s-r)dZ_r = \int_0^t \Delta_{s-t}K(t-r)dZ_r$ is well-defined as an Itô integral, for all $s > t$, provided that the shifted kernels $\Delta_h K := K(\cdot + h)$ are in $L_{\text{loc}}^2(\mathbb{R}_+, \mathbb{R})$ for any $h > 0$.

Theorem 2.2. Fix $b \in \mathbb{R}, c \geq 0$ and ν a nonnegative measure supported on \mathbb{R}_+ such that $\int_{\mathbb{R}_+} \zeta^2 \nu(d\zeta) < \infty$. Fix a kernel $K \in L_{\text{loc}}^1(\mathbb{R}_+, \mathbb{R})$ such that its shifted kernels $\Delta_h K = K(\cdot + h)$ are in $L_{\text{loc}}^2(\mathbb{R}_+, \mathbb{R})$ for any $h > 0$, and a non-decreasing continuous function $G_0 : \mathbb{R}_+ \mapsto \mathbb{R}$. Fix $T \geq 0$ and three continuous functions $f_0, f_1, f_2 : [0, T] \mapsto \mathbb{C}$. Assume that there exists a solution $\psi \in C([0, T], \mathbb{C})$ to the Riccati–Volterra equation (2.4)–(2.5) such that

$$\sup_{t \leq T} \int_{\mathbb{R}_+} e^{\Re(f_2(t) + \psi(t))\zeta} \zeta^2 \nu(d\zeta) < \infty \quad (2.8)$$

Then, for any weak solution X of (1.1), the joint conditional Fourier–Laplace transform of

$$R_{t,T} = \int_t^T f_0(T-s)dX_s + \int_t^T f_1(T-s)dM_s^c + \int_t^T f_2(T-s)dM_s^d,$$

where M^c and M^d are the local martingales appearing in (2.1), is given by

$$\mathbb{E} \left[\exp(R_{t,T}) \middle| \mathcal{F}_t \right] = \exp \left(\int_t^T F(T-s, \psi(T-s))dG_t(s) \right), \quad t \leq T, \quad (2.9)$$

where G_t is defined as in (2.6). In particular, if the Riccati–Volterra equation (2.4)–(2.5) with $f_2 = f_3 \equiv 0$, admits a continuous solution ψ such that $\sup_{t \leq T} \int_{\mathbb{R}_+} e^{\Re(\psi(t))\zeta} \zeta^2 \nu(d\zeta) < \infty$ for any continuous function $f_0 : [0, T] \mapsto i\mathbb{R}$ and any $T \geq 0$, then weak uniqueness holds for (1.1) for the input (G_0, K, b, c, ν) .

Proof. The proof is detailed in Section 6. □

The following example illustrates the applicability of Theorem 2.2 in the case of pure jump Hawkes processes. The example will be continued in Example 2.10 to illustrate the scaling limits of Hawkes processes mentioned in the introduction. Section 7 provides another example of application of Theorem 2.2 in the context of rough volatility modeling.

Example 2.3. Let N denote a counting process with instantaneous intensity

$$\lambda_t = g_0(s) + \int_0^t K(t-s)dN_s,$$

for some $g_0, K \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R}_+)$. We are interested in the computation of the joint Fourier–Laplace transform of (λ, N) , more precisely of the quantity:

$$\int_0^T h_0(T-s)\lambda_s ds + \int_0^T h_2(T-s)dN_s,$$

for some continuous functions $h_0, h_2 : \mathbb{R}_+ \mapsto \mathbb{C}$. By an application of Lemma 2.1, the integrated intensity $X = \int_0^T \lambda_s ds$ solves (1.1) with $G_0 = \int_0^T g_0(s) ds$ and $Z = N$ with affine characteristics in (1.2) given by

$$(X, 0, \delta_1(d\zeta)X),$$

meaning that $(b, c, \nu) = (1, 0, \delta_1)$ and $M^d = (N - X)$. Under the assumptions of Theorem 2.2, setting $f_0 = h_0 - h_2$, $f_1 \equiv 0$ and $f_2 = h_2$ the joint Fourier-Laplace transform of (Λ, N) is given by

$$\mathbb{E} \left[\exp \left(\int_0^T h_0(T-s)\lambda_s ds + \int_0^T h_2(T-s)dN_s \right) \right] = \exp \left(\int_0^T F(T-s, \psi(T-s))g_0(s) ds \right)$$

where the Riccati–Volterra equations (2.4)–(2.5) read

$$\begin{aligned} \psi(t) &= \int_0^t K(t-s)F(s, \psi(s))ds, \quad t \leq T, \\ F(s, u) &= h_0(s) + e^{(h_2(s)+u)} - 1. \end{aligned}$$

We refer to Theorem 2.5 and Remark 2.6 below for the existence of such ψ . In particular, setting $h_0 \equiv 0$ and $h_2 \equiv ia$ for some $a \in \mathbb{R}$, we recover the formula of Hawkes and Oakes (1974, Theorem 2) derived using cluster representations of Hawkes processes, see also El Euch and Rosenbaum (2019, Theorem 3.1).

Under additional assumptions on K we prove the existence of a solution to the Riccati–Volterra equations (2.4)–(2.5). For this we recall the notion of the resolvent of the first

kind of a kernel: a measure L of locally bounded variation is called a resolvent of the first kind of the kernel $K \in L_{\text{loc}}^1(\mathbb{R}_+, \mathbb{R})$ if

$$\int_{[0,t]} K(t-s)L(ds) = 1, \quad t \geq 0.$$

If such L exists, then it is unique by [Gripenberg et al. \(1990, Theorem 5.2.2\)](#). We consider the following condition on the kernel K :

$$\begin{aligned} & \text{the kernel is nonnegative, non-increasing and continuously} \\ & \text{differentiable on } (0, \infty), \text{ and its resolvent of the first kind } L \text{ is} \\ & \text{nonnegative and non-increasing in the sense that } s \mapsto L([s, s+t]) \text{ is} \\ & \text{non-increasing for all } t \geq 0. \end{aligned} \tag{2.10}$$

We note in [\(2.10\)](#) that any nonnegative and non-increasing kernel that is not identically zero admits a resolvent of the first kind; see [Gripenberg et al. \(1990, Theorem 5.5.5\)](#). The following example provides a large class of kernels for which [\(2.10\)](#) is satisfied.

Example 2.4. If K is completely monotone on $(0, \infty)$, then [\(2.10\)](#) holds due to [Gripenberg et al. \(1990, Theorem 5.5.4\)](#). Recall that a function f is called completely monotone on $(0, \infty)$ if it is infinitely differentiable there with $(-1)^k f^{(k)}(t) \geq 0$ for all $t > 0$ and $k \geq 0$. We also note that, for each $h > 0$, the shifted kernel $\Delta_h K$ are again completely monotone on $[0, \infty)$ so that [\(2.10\)](#) holds also for $\Delta_h K$. In particular, $\Delta_h K \in L_{\text{loc}}^2(\mathbb{R}_+, \mathbb{R})$, for each $h > 0$. This covers, for instance, any constant positive kernel, the fractional kernel $t^{H-1/2}$ with $H \in (-1/2, 1/2]$, and the exponentially decaying kernel $e^{-\eta t}$ with $\eta > 0$. Moreover, sums and products of completely monotone kernels are completely monotone. By combining the above examples we find that the Gamma kernel $K(t) = t^{H-1/2}e^{-\eta t}$ for $H \in (-1/2, 1/2]$ and $\eta \geq 0$ satisfies [\(2.10\)](#).

The following theorem establishes the existence of solutions to the Riccati–Volterra equation [\(2.4\)](#)–[\(2.5\)](#) under structural assumptions on f_0, f_1, f_2 .

Theorem 2.5. Fix $b \in \mathbb{R}, c \geq 0$ and ν a nonnegative measure supported on \mathbb{R}_+ such that $\int_{\mathbb{R}_+} \zeta^2 \nu(d\zeta) < \infty$. Let $f_0, f_1, f_2 : \mathbb{R}_+ \mapsto \mathbb{C}$ be three continuous functions such that

$$\Re(f_0) + \frac{c}{2} \Re(f_1)^2 + \frac{1}{2} \int_{\mathbb{R}_+} \zeta^2 \nu(d\zeta) \Re(f_2)^2 \leq 0 \quad \text{and} \quad \Re(f_2) \leq 0. \tag{2.11}$$

Fix a kernel $K \in L_{\text{loc}}^1(\mathbb{R}_+, \mathbb{R})$ satisfying [\(2.10\)](#). Then, there exists a continuous solution ψ to [\(2.4\)](#)–[\(2.5\)](#) such that $\Re(\psi(t)) \leq 0$, for all $t \geq 0$.

Proof. We refer to [Section 5.2](#). □

Remark 2.6. • Under [\(2.11\)](#), [\(2.8\)](#) follows from the fact that $\Re(\psi) \leq 0$ as stated in [Theorem 2.5](#).

- The condition (2.11) is satisfied for instance if $\Re(f_0) \leq 0$ and $f_1, f_2 : \mathbb{R}_+ \mapsto i\mathbb{R}$. In particular, the second part of Theorem 2.2 provides weak uniqueness for (1.1). Going back to Example 2.3, the existence of a solution to the corresponding Riccati–Volterra equation is ensured provided that $\Re(h_0) \leq 0$ and $h_2 : \mathbb{R}_+ \mapsto i\mathbb{R}$.

Remark 2.7. If K is in $L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$, then Theorems 2.2 and 2.5 agree with Cuchiero and Teichmann (2020, Theorem 5.12) for the jump case; if in addition $\nu \equiv 0$, then one recovers Abi Jaber et al. (2019b, Theorem 7.1) and Abi Jaber and El Euch (2019a, Theorem 2.3) for the continuous case.

2.2 Stability and existence

We now present our existence and stability results for solutions to (1.1). Our strategy for constructing solutions with L^1 -kernels relies on an approximation argument using L^2 -kernels combined with Lemma 2.1. To fix ideas, set $\nu \equiv 0$ and assume $G_0 = \lim_{n \rightarrow \infty} G_0^n$, with $G_0^n = \int_0^\cdot g_0^n(s) ds$, for some sequence of L^1_{loc} -functions $(g_0^n)_{n \geq 1}$. Starting from a L^1_{loc} -kernel K , assume that there exists a sequence of L^2_{loc} -kernels $(K^n)_{n \geq 1}$ such that

$$K^n \rightarrow K, \quad \text{in } L^1_{\text{loc}}, \quad \text{as } n \rightarrow \infty.$$

Then, for each $n \geq 1$, K^n being locally square-integrable, under suitable conditions on (g_0^n, K^n) , the results in Abi Jaber et al. (2019b); Abi Jaber and El Euch (2019a) provide existence of nonnegative solution Y^n for (2.3) with (g_0, K) replaced by (g_0^n, K^n) . Setting $X^n = \int_0^\cdot Y_s^n ds$, Lemma 2.1 provides a solution X^n to (1.1) for the input (G_0^n, K^n) , that is

$$X_t^n = G_0^n(t) + \int_0^t K^n(t-s) Z_s^n ds, \quad (2.12)$$

where the characteristics of Z^n are $(bX^n, cX^n, 0)$. Provided that $(X^n)_{n \geq 1}$ is tight, it will admit a convergent subsequence towards a limiting process X . Finally, sending $n \rightarrow \infty$, one would expect X to solve (1.1) for $\nu \equiv 0$.

We will adapt the same strategy in the case of jumps. Before this, we state our generic stability result.

Theorem 2.8. Assume that there exist sequences of coefficients $(b^n, c^n, \nu^n(d\zeta))_{n \geq 1}$ with $\int_{\mathbb{R}} \zeta^2 \nu^n(d\zeta) < \infty$, non-increasing kernels $(K^n)_{n \geq 1}$ and functions $(G_0^n)_{n \geq 1}$ such that

(i)

$$b^n \rightarrow b, \quad c^n + \int_{\mathbb{R}} \zeta^2 \nu^n(d\zeta) \rightarrow c + \int_{\mathbb{R}} \zeta^2 \nu(d\zeta) \quad \text{and} \quad \int_{\mathbb{R}} h(\zeta) \nu^n(d\zeta) \rightarrow \int_{\mathbb{R}} h(\zeta) \nu(d\zeta),$$

for any continuous and bounded function h vanishing around zero, as $n \rightarrow \infty$, for some $b \in \mathbb{R}$, $c \geq 0$ and $\nu(d\zeta)$ a nonnegative measure such that $\int_{\mathbb{R}} \zeta^2 \nu(d\zeta) < \infty$.

- (ii) $\int_0^t |K^n(s) - K(s)| ds \rightarrow 0$, as $n \rightarrow \infty$, $t \geq 0$, for some non-increasing kernel $K \in L_{\text{loc}}^1(\mathbb{R}_+, \mathbb{R})$.
- (iii) $\sup_{t \leq T} |G_0^n(t) - G_0(t)| \rightarrow 0$, as $n \rightarrow \infty$, $T \geq 0$, for some continuous function G_0 .

Then, any sequence of continuous nonnegative and non-decreasing solutions $(X^n)_{n \geq 1}$ to (1.1) for the respective inputs $(G_0^n, K^n, b^n, c^n, \nu^n)$, is tight on the space of continuous functions $C([0, T], \mathbb{R})$ endowed with the uniform topology, for each $T > 0$. Furthermore, any limit point X is a continuous non-decreasing solution to (1.1) for the input (G_0, K, b, c, ν) .

Proof. We refer to Section 4. □

Remark 2.9. One notes that if K satisfies (2.10), then weak uniqueness holds thanks to Theorems 2.2 and 2.5 so that one gets from Theorem 2.8 that $X^n \Rightarrow X$, where X is the unique solution to (1.1).

The following example illustrates an application of Theorem 2.8 to the scaling limits of Hawkes processes mentioned in the introduction. We stress that the convergence of the second modified characteristic in Theorem 2.8-(i) allows to obtain continuous limiting semimartingales Z from a sequence of jump semimartingales Z^n . We recall that the resolvent of the second kind R of K is the unique L_{loc}^1 function solution to

$$R(t) = K(t) + \int_0^t K(t-s)R(s)ds = K(t) + \int_0^t R(t-s)K(s)ds, \quad t \geq 0. \quad (2.13)$$

The resolvent R exists, for any kernel $K \in L_{\text{loc}}^1$, see Gripenberg et al. (1990, Theorems 2.3.1 and 2.3.5).

Example 2.10. Fix $n \geq 1$ and a sequence of counting processes $(N^n)_{n \geq 1}$ with respective intensities

$$\lambda_t^n = g_0^n(t) + \int_0^t K^n(t-s)dN_s^n, \quad t \geq 0,$$

for some continuous function g_0^n and kernel $K^n \in L_{\text{loc}}^1$. Then, convolving both sides of the equation with the resolvent of the second kind R^n of K^n and using (2.13), leads to

$$\lambda_t^n = g_0^n(t) + \int_0^t R^n(t-s)g_0^n(s)ds + \int_0^t R^n(t-s)dM_s^n, \quad t \geq 0,$$

where $M^n = N^n - \int_0^\cdot \lambda_s^n ds$. Let X^n denote the rescaled sequence of integrated accelerated intensities $X^n = \frac{1}{n} \int_0^\cdot \lambda_{ns}^n ds$. Then, by an application of Lemma 2.1, X^n satisfies

$$X_t^n = G_0^n(t) + \int_0^t \tilde{R}^n(t-s)Z_s^n ds,$$

with $G_0^n(t) = \int_0^t \left(\frac{1}{n} g_0^n(ns) + \int_0^s \tilde{R}^n(s-u) g_0^n(nu) du \right) ds$, $\tilde{R}^n(t) = R^n(nt)$ and $Z_t^n = \frac{1}{n} M_{nt}^n$. Whence Z^n is a pure jump martingale with jump sizes $\frac{1}{n}$ and integrated intensity $\int_0^{nt} \lambda_s^n ds = n^2 X_t^n$ so that its characteristics read $(0, 0, n^2 X^n \delta_{1/n}(d\zeta))$. Set $(b^n, c^n, \nu^n) = (0, 0, n^2 \delta_{1/n})$ and observe that

$$\int_{\mathbb{R}} \zeta^2 \nu^n(d\zeta) = 1 \quad \text{and} \quad \int_{\mathbb{R}} h(z) \nu^n(d\zeta) = h(1/n) n^2 \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

for any continuous and bounded function h vanishing around 0. Whence, Theorem 2.8-(i) is satisfied with $(b, c, \nu) = (0, 1, 0)$. Provided that g_0^n and K^n are chosen such that $(\tilde{R}^n)_{n \geq 1}$ and $(G_0^n)_{n \geq 1}$ satisfy Theorem 2.8-(ii)-(iii) for some \tilde{R} and G_0 , Theorem 2.8 yields that $(X^n)_{n \geq 1}$ is tight on $C([0, T], \mathbb{R})$ such that any limit point satisfies

$$X_t = G_0(t) + \int_0^t \tilde{R}(t-s) Z_s ds$$

with Z a continuous martingale with characteristics $(0, X, 0)$. For instance, setting $g_0^n \equiv 0$, [Jusselin and Rosenbaum \(2018\)](#) construct a sequence of kernels K^n such that \tilde{R}^n converges in $L^1([0, T], \mathbb{R})$ towards the function

$$\tilde{R}_H(t) = \lambda t^{\alpha-1} E_{\alpha, \alpha}(-\lambda t^\alpha),$$

for some $\lambda \in \mathbb{R}$, $\alpha = H + 1/2$ with $H \in (-1/2, 1/2)$ and $E_{\alpha, \alpha}(z) = \sum_{n \geq 0} z^n / \Gamma(\alpha(n+1))$ the so-called Mittag-Leffler function. In particular, \tilde{R}_H is the resolvent of the second kind of λK_H where K_H is the fractional kernel given by (1.9). In this case, using again the resolvent equation (2.13), one obtains

$$X_t = \int_0^t \lambda K_H(t-s) \tilde{Z}_s ds,$$

with $\tilde{Z} = X + Z$.

Remark 2.11. *In the L^2 setting, i.e. when the characteristics of Z are absolutely continuous with respect to the Lebesgue measure, [Abi Jaber et al. \(2019a, Theorem 1.6\)](#) provides a generic stability result for stochastic Volterra equations with jumps using a ‘martingale problem’ formulation, for general coefficients for the differential characteristics of Z going beyond the affine case.*

We now introduce the monotonicity and continuity assumptions needed on the kernel K and the input function G_0 to construct non-decreasing and nonnegative solutions to (1.1). We assume that $K \in L_{\text{loc}}^1(\mathbb{R}_+, \mathbb{R})$ such that (2.10) holds. Concerning the input function G_0 , in the absence of jumps and for $K \in L_{\text{loc}}^2$ [Abi Jaber and El Euch \(2019a\)](#) provide a set \mathcal{G}_K of admissible input curves g_0 defined in terms of the resolvent of the first

kind L to ensure the existence of non-negative solution for (2.3). To guarantee that the approximate solutions (2.12) are non-decreasing, we consider similarly to [Abi Jaber and El Euch \(2019a, Equations \(2.4\)–\(2.5\)\)](#), the following condition²

$$\Delta_h g_0 - (\Delta_h K * L)(0)g_0 - d(\Delta_h K * L) * g_0 \geq 0, \quad h \geq 0, \quad (2.14)$$

where we used the notation $(f * \mu)(t) = \int_0^t f(t-s)\mu(ds)$ for a measure of locally bounded variation μ and a function $f \in L^1_{\text{loc}}$. We then define the space of admissible input curves \mathcal{G}_K to be

$$\mathcal{G}_K = \{g_0 \text{ continuous, satisfying (2.14) such that } g_0(0) \geq 0\}. \quad (2.15)$$

Two notable examples of such admissible input curves are:

- Example 2.12.** (i) g_0 continuous and non-decreasing with $g_0(0) \geq 0$,
(ii) $g_0(t) = x_0 + \int_0^t K(t-s)\theta(s)ds$, for some $x_0 \geq 0$ and $\theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ locally bounded, see e.g. [Abi Jaber and El Euch \(2019a, Example \(2.2\)\)](#).

We are now in place to state the main existence (and uniqueness) result.

Theorem 2.13. *Fix $b \in \mathbb{R}, c \geq 0$ and ν a nonnegative measure supported on \mathbb{R}_+ such that $\int_{\mathbb{R}_+} \zeta^2 \nu(d\zeta) < \infty$. Fix a kernel $K \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$. Assume that K and its shifted kernels $\Delta_{1/n}K$ satisfy (2.10), for any $n \geq 1$. Let $G_0 = \lim_{n \rightarrow \infty} \int_0^\cdot g_0^n(s)ds$ for some functions $g_0^n \in \mathcal{G}_{\Delta_{1/n}K}$, $n \geq 1$, and assume that G_0 is continuous. Then, there exists a unique non-decreasing nonnegative continuous weak solution X to (1.1) for the input (G_0, K, b, c, ν) .*

Proof. The proof for the existence part is given in Section 5.1. The uniqueness statement is obtained from Theorems 2.2 and 2.5, recall Remark 2.6. \square

Remark 2.14. *If K satisfies (2.10) and $g_0 \in \mathcal{G}_K$ as in Example 2.12, the continuous function $G_0(t) = \int_0^t g_0(s)ds$ satisfies the assumption of Theorem 2.13. Indeed:*

- (i) *Take g_0 as in Example 2.12-(i), fix $n \geq 1$ and set $g_0^n = \Delta_{1/n}g_0$. Clearly, g_0^n satisfies again Example 2.12-(i), so that $g_0^n \in \mathcal{G}_{\Delta_{1/n}K}$. Furthermore, we have that $\lim_{n \rightarrow \infty} \int_0^t g_0^n(s)ds = G_0(t)$ by virtue of [Brezis \(2010, Lemma 4.3\)](#).*
- (ii) *Take g_0 as in Example 2.12-(ii) and set $g_0^n(t) = x_0 + \int_0^t \Delta_{1/n}K(t-s)\theta(s)ds$, then, similarly we have that $\lim_{n \rightarrow \infty} \int_0^t g_0^n(s)ds = G_0(t)$ and $g_0^n \in \mathcal{G}_{\Delta_{1/n}K}$.*

²Under (2.10) one can show that $\Delta_h K * L$ is right-continuous and of locally bounded variation (see [Abi Jaber and El Euch \(2019a, Remark B.3\)](#)), thus the associated measure $d(\Delta_h K * L)$ is well defined.

3 A-priori estimates

We first provide a-priori estimates for solutions to (1.1). We make use of the resolvent of the second kind R of K given in (2.13).

Lemma 3.1. *Fix $K \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$ and G_0 locally bounded. Assume that there exists a non-decreasing nonnegative continuous and adapted process \tilde{X} satisfying*

$$\tilde{X}_t = G_0(t) + \int_0^t K(t-s) \tilde{Z}_s ds,$$

where \tilde{Z} is a semimartingale with characteristics $(\tilde{B}_t(\omega), \tilde{C}_t(\omega), \tilde{\nu}_t(\omega, d\zeta))$ such that

$$|\tilde{B}_t(\omega)| + |\tilde{C}_t(\omega)| + \int_{[0,t] \times \mathbb{R}} \zeta^2 \tilde{\nu}_s(\omega, d\zeta) \leq \kappa_L |\tilde{X}_t(\omega)|, \quad t \geq 0, \quad \text{for a.e. } \omega, \quad (3.1)$$

for some constant κ_L . Then, for all $T > 0$,

$$\mathbb{E} \left[\sup_{t \leq T} |\tilde{X}_t|^2 \right] \leq C(T, \kappa_L) \left(1 + \sup_{t \leq T} |G_0(t)|^2 + \|K\|_{L^1([0,T])}^2 \right) (1 + \|R\|_{L^1([0,T])}), \quad (3.2)$$

where $C(T, \kappa_L) > 0$ depends exclusively on (T, κ_L) , and R is the resolvent of the second kind of $C(T, \kappa_L) \|K\|_{L^1([0,T])} |K|$.

Proof. Since \tilde{X} is non-decreasing, we have $\mathbb{E} \left[\sup_{t \leq T} \tilde{X}_t^2 \right] \leq \mathbb{E}[\tilde{X}_T^2]$. It is therefore enough to prove the bound (3.2) for $\mathbb{E}[\tilde{X}_T^2]$. For this, fix $n \geq 1$ and define $\tau_n = \inf\{t \geq 0 : |\tilde{X}_t| \geq n\} \wedge T$. Since \tilde{X} is adapted with continuous sample paths, τ_n is a stopping time such that $\tau_n \rightarrow T$ almost surely as $n \rightarrow \infty$. First observe that

$$|\tilde{X}_t \mathbf{1}_{\{t < \tau_n\}}| \leq |G_0(t)| + \left| \int_0^t K(t-s) \tilde{Z}_s \mathbf{1}_{\{s < \tau_n\}} ds \right|.$$

and set $\tilde{X}_t^n = \tilde{X}_t \mathbf{1}_{\{t \leq \tau_n\}}$. An application of Jensen's inequality on the normalized measure $|K(t)| dt / \|K\|_{L^1([0,T])}$ yields

$$\begin{aligned} |\tilde{X}_t^n|^2 &\leq 2 |G_0(t)|^2 + 2 \left| \int_0^t K(t-s) \tilde{Z}_s \mathbf{1}_{\{s < \tau_n\}} ds \right|^2 \\ &\leq 2 \sup_{r \leq T} |G_0(r)|^2 + 2 \|K\|_{L^1([0,T])} \int_0^t |K(t-s)| \left| \tilde{Z}_s \mathbf{1}_{\{s < \tau_n\}} \right|^2 ds. \end{aligned}$$

\tilde{Z} admits the decomposition $\tilde{Z} = \tilde{B} + \tilde{M}^c + \tilde{M}^d$ such that $\mathbb{E}[|\tilde{M}_t^c|^2] = \mathbb{E}[\tilde{C}_t]$ and $\mathbb{E}[|\tilde{M}_t^d|^2] = \mathbb{E}[\int_{[0,t] \times \mathbb{R}} \zeta^2 \tilde{\nu}_s(\cdot, d\zeta)]$, so that Jensen's inequality combined with the bound (3.1) yield

$$\mathbb{E}[\tilde{Z}_s^2 \mathbf{1}_{\{s < \tau_n\}}] \leq 3(\kappa_L^2 + \kappa_L)(1 + \mathbb{E}[|\tilde{X}_s^n|^2]), \quad s \geq 0.$$

Combining the above, we get for a constant C depending exclusively on (T, κ_L) that may vary from line to line:

$$\begin{aligned} \mathbb{E} \left[|\tilde{X}_t^n|^2 \right] &\leq C \sup_{r \leq T} |G_0(r)|^2 + C \|K\|_{L^1([0, T])} \int_0^t |K(t-s)| \left(1 + \mathbb{E} \left[|\tilde{X}_s^n|^2 \right] \right) ds \\ &\leq C \left(1 + \sup_{r \leq T} |G_0(r)|^2 + \|K\|_{L^1([0, T])}^2 \right) (1 + \|R\|_{L^1([0, T])}). \end{aligned}$$

where the last line follows from the generalised Gronwall inequality for convolution equations with R the resolvent of $C \|K\|_{L^1([0, T])}^{p-1} |K|$, see [Gripenberg et al. \(1990, Theorem 9.8.2\)](#). The claimed estimate (3.2) now follows by sending $n \rightarrow \infty$ and using Fatou's Lemma in the above. \square

Remark 3.2. *If Z is a semimartingale with characteristics (1.2), the growth condition (3.1) is clearly satisfied with $\kappa_L = \left(b + c + \int_{\mathbb{R}_+} \zeta^2 \nu(d\zeta) \right)$, recall the assumption $\int_{\mathbb{R}_+} \zeta^2 \nu(d\zeta) < \infty$. Whence, any solution to (1.1) satisfies the estimate (3.2).*

The following lemma establishes an estimate for the modulus of continuity of the process $\bar{X} = X - G_0$ defined by

$$w_{\bar{X}, T}(\delta) = \sup\{|\bar{X}_s - \bar{X}_t| : s, t \leq T \text{ and } |s - t| \leq \delta\}, \quad 0 < \delta \leq 1. \quad (3.3)$$

Lemma 3.3. *Fix $K \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$ non-increasing and G_0 a locally bounded function. Let X denote a solution to (1.1) for the input (G_0, K, b, c, ν) and set $\bar{X} = X - G_0$. Then, for any $T > 0$ and $\delta \leq 1$,*

$$\mathbb{E} [w_{\bar{X}, T}(\delta)] \leq 3(\kappa_L^2 + \kappa_L) (1 + \mathbb{E}[X_T^2]) \left(\int_0^\delta |K(s)| ds + \int_0^T (K(s) - K(s + \delta)) ds \right)$$

with $\kappa_L = \left(b + c + \int_{\mathbb{R}_+} \zeta^2 \nu(d\zeta) \right)$.

Proof. Fix $s, t \leq T$ such that $|t - s| \leq \delta$. We first write

$$\bar{X}_s - \bar{X}_t = \int_{s \wedge t}^{s \vee t} K(s \vee t - u) Z_u du + \int_0^{s \wedge t} (K(s \vee t - u) - K(s \wedge t - u)) Z_u du.$$

Whence,

$$\begin{aligned} |\bar{X}_s - \bar{X}_t| &\leq \sup_{u \leq T} |Z_u| \left(\int_{s \wedge t}^{s \vee t} |K(s \vee t - u)| du + \int_0^{s \wedge t} |K(s \vee t - u) - K(s \wedge t - u)| du \right) \\ &\leq \sup_{u \leq T} |Z_u| \left(\int_0^\delta |K(u)| du + \int_0^{s \wedge t} |K(s \vee t - s \wedge t + u) - K(u)| du \right) \\ &\leq \sup_{u \leq T} |Z_u| \left(\int_0^\delta |K(u)| du + \int_0^T (K(u) - K(u + \delta)) du \right) \end{aligned}$$

where the last inequality follows from the fact that K is non-increasing and $(s \vee t - s \wedge t) \leq \delta$. The claimed estimate follows upon taking the supremum over s, t and the expectation, using the Burkholder-Davis-Gundy inequality for the local martingale parts of Z and Remark 3.2. \square

4 Tightness and stability

In this section, we prove our general tightness and stability result: Theorem 2.8. One can appreciate the formulation (1.1) and the affine structure of the characteristics (1.2) for the stability argument.

We start with a preliminary lemma.

Lemma 4.1. *Let $(K^n)_{n \geq 1}$ be a sequence of locally integrable kernels K^n such that $\int_0^t |K^n(s) - K(s)| ds \rightarrow 0$, as $n \rightarrow \infty$, $t \geq 0$, for some kernel $K \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$. Then, for all $T > 0$,*

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \left(\int_0^\delta |K^n(s)| ds + \int_0^T |K^n(s + \delta) - K^n(s)| ds \right) = 0. \quad (4.1)$$

Proof. Fix $\varepsilon > 0$ and $T > 0$. Since $K \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$, it is L^1 -continuous, see Brezis (2010, Lemma 4.3), so that we can fix $\delta < 1$ such that

$$\int_0^\delta |K(s)| ds + \int_0^T |K(s + \delta) - K(s)| ds \leq \frac{\varepsilon}{5}.$$

Due to the L^1 -convergence of the kernels $(K^n)_{n \geq 1}$, let n_δ be such that

$$\int_0^{T+1} |K^n(s) - K(s)| ds \leq \frac{\varepsilon}{5}, \quad n \geq n_\delta.$$

Fixing $n \geq n_\delta$ and using the above leads to

$$\int_0^\delta |K^n(s)| ds \leq \int_0^{T+1} |K^n(s) - K(s)| ds + \int_0^\delta |K(s)| ds \leq \frac{2}{5}\varepsilon$$

and

$$\begin{aligned} \int_0^T |K^n(s + \delta) - K^n(s)| ds &\leq 2 \int_0^{T+1} |K^n(s) - K(s)| ds \\ &\quad + \int_0^T |K(s + \delta) - K(s)| ds \\ &\leq \frac{3}{5}\varepsilon. \end{aligned}$$

Whence

$$\int_0^\delta |K^n(s)| ds + \int_0^T |K^n(s + \delta) - K^n(s)| ds \leq \varepsilon,$$

which yields (4.1). \square

Proof of Theorem 2.8. Let $(X^n)_{n \geq 1}$ be a sequence of continuous non-decreasing solutions to (1.1), for the respective inputs $(G_0^n, K^n, b^n, c^n, \nu^n)$, that is, for each $n \geq 1$,

$$X_t^n = G_0^n(t) + \int_0^t K^n(t-s)Z_s^n ds, \quad t \geq 0, \quad (4.2)$$

where Z^n is a semimartingale with characteristics $(b^n X^n, c^n X^n, \nu^n(d\zeta)X^n)$, defined on some filtered probability space $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n)_{t \geq 0}, \mathbb{P}^n)$.

• Fix $T > 0$. We argue *tightness* of (X^n, Z^n) on the space $C([0, T], \mathbb{R}) \times D([0, T], \mathbb{R})$, where $D([0, T], \mathbb{R})$ is the Skorokhod space endowed with the J_1 topology. To prove tightness of $(X^n)_{n \geq 1}$, we start by observing that due to the uniform convergence of $(G_0^n)_{n \geq 1}$ in (iii), it suffices to obtain the tightness of the sequence $\bar{X}^n = X^n - G_0^n$. To this end, we make use of the probabilistic counterpart of the Arzela-Ascoli theorem given in Billingsley (1999, Theorem 7.2). By Chebyshev's inequality, it suffices to prove that

$$\sup_{n \geq 1} \mathbb{E}[\sup_{t \leq T} \bar{X}_t^n] < \infty \quad \text{and} \quad \lim_{\delta \rightarrow 0} \limsup_{n \geq \infty} \mathbb{E}[w_{\bar{X}^n, T}(\delta)] = 0 \quad (4.3)$$

where w is the modulus of continuity defined as in (3.3). By virtue of the continuous dependence of the resolvent on the kernel in L^1 , the L^1 -convergence of K^n in (ii) implies the L^1 -convergence of the respective sequence of resolvents $(R^n)_{n \geq 1}$, see Gripenberg et al. (1990, Theorem 2.3.1). Thus, the sequences $(\|K^n\|_{L^1([0, T])})_{n \geq 1}$ and $(\|R^n\|_{L^1([0, T])})_{n \geq 1}$ are uniformly bounded in n . Furthermore, it follows from (i) that $(b^n, c^n, \int_{\mathbb{R}} \zeta^2 \nu^n(d\zeta))_{n \geq 1}$ are uniformly bounded in n so that the coefficient κ_L appearing in (3.1) for X^n , recall Remark 3.2, does not depend on n . Therefore, recalling (iii), $\sup_{t \leq T} G_0^n(t)$ is uniformly bounded in n and the bound in (3.2) for each X^n does not depend on n , yielding $\sup_{n \geq 1} \mathbb{E}[\sup_{t \leq T} |\bar{X}_t^n|^2] < \infty$. From there, an application of Lemmas 3.3 and 4.1 lead to (4.3) and the tightness of $(X^n)_{n \geq 1}$ on $C([0, T], \mathbb{R})$ follows. We claim that the sequence $(Z^n)_{n \in \mathbb{N}}$ is tight on $D([0, T], \mathbb{R})$. To prove this we verify the conditions in Jacod and Shiryaev (2003, Theorem VI.4.18). We first note that for any $a > 0$, $\varepsilon > 0$, $N \leq T$, we have

$$\mathbb{P} \left(\int_{[0, N] \times \mathbb{R}} \mathbf{1}_{|\zeta| > a} dX_s^n \nu^n(d\zeta) > \varepsilon \right) \leq \frac{1}{a^2 \varepsilon} \sup_{n \geq 1} \int_{\mathbb{R}} |\zeta|^2 \nu^n(d\zeta) \sup_{m \geq 1} \mathbb{E}[X_T^m]$$

Therefore,

$$\lim_{a \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{P} \left(\int_{[0, N] \times \mathbb{R}} \mathbf{1}_{|\zeta| > a} dX_s^n \nu^n(d\zeta) > \varepsilon \right) = 0.$$

Furthermore, since $(X^n)_{n \geq 1}$ is tight on $C([0, T], \mathbb{R})$, the first two ‘modified’ characteristics $(b^n X^n, (c^n + \int_{\mathbb{R}} \zeta^2 \nu^n(d\zeta))X^n)$ of Z^n are C -tight by virtue of (i). In addition, for $p \in \mathbb{N}$ and $h_p(\zeta) = (p|\zeta| - 1)^+ \wedge 1$, $X^n \int_{\mathbb{R}} h_p(\zeta) \nu^n(d\zeta)$ is also C -tight thanks to (i). Whence, we may apply Jacod and Shiryaev (2003, Theorem VI.4.18) to get that $(Z^n)_{n \in \mathbb{N}}$ is tight on

$D([0, T], \mathbb{R})$. Finally, by passing to a further subsequence, we have $(X^n, Z^n) \Rightarrow (X, Z)$ on $C([0, T], \mathbb{R}) \times D([0, T], \mathbb{R})$ for some limiting process (X, Z) defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

• We now move to the *stability* part. We start by proving that Z is a semimartingale with characteristics $(bX, cX, \nu(d\zeta)X)$ with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ generated by (X, Z) . Since X^n and X are continuous, we have $X^n \Rightarrow X$ on $D([0, T], \mathbb{R})$ so that, by virtue of (i), we have, by passing to a further subsequence,

$$\left(X^n, Z^n, b^n X^n, \left(c^n + \int_{\mathbb{R}} \zeta^2 \nu^n(d\zeta) \right) X^n \right) \Rightarrow \left(X, Z, bX, \left(c + \int_{\mathbb{R}} \zeta^2 \nu(d\zeta) \right) X \right)$$

and for any continuous and bounded function h vanishing around zero:

$$\left(X^n, Z^n, \int_{\mathbb{R}_+} h(\zeta) \nu^n(d\zeta) X^n \right) \Rightarrow \left(X, Z, \int_{\mathbb{R}_+} h(\zeta) \nu(d\zeta) X \right).$$

It follows from [Jacod and Shiryaev \(2003, Theorem IX-2.4 and Remark IX-2.21\)](#) that Z is a semimartingale with characteristics $(bX, cX, \nu(d\zeta)X)$ with respect to the filtration \mathbb{F} . An application of Skorokhod's representation theorem provides the existence of a common filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ supporting a sequence of copies $(X^n, Z^n)_{n \geq 1}$ that converges on $C([0, T] \times \mathbb{R}) \times D([0, T] \times \mathbb{R})$, almost surely, along a subsequence, towards a copy of (X, Z) . Keeping the same notations for these copies, we have

$$\|X^n - X\|_{C([0, T], \mathbb{R})} \rightarrow 0, \quad \|Z^n - Z\|_{D([0, T], \mathbb{R})} \rightarrow 0, \quad \mathbb{P} - a.s., \quad \text{as } n \rightarrow \infty.$$

Now fix $t \leq T$ and write

$$\begin{aligned} \int_0^t K^n(t-s) Z_s^n ds - \int_0^t K(t-s) Z_s ds &= \int_0^t (K^n(t-s) - K(t-s)) Z_s^n ds \\ &\quad + \int_0^t K(t-s) (Z_s^n - Z_s) ds \\ &= \mathbf{I}_n + \mathbf{II}_n. \end{aligned}$$

Due to the convergence of $(Z^n)_{n \geq 1}$ on $D([0, T], \mathbb{R})$, $Z_s^n \rightarrow Z_s$ $\mathbb{P} \times dt$ -almost everywhere and $\sup_{n \geq 1} \sup_{s \leq T} |Z_s^n| < \infty$ so that $\mathbf{I}_n \rightarrow 0$ as $n \rightarrow \infty$ by virtue of (ii) and $\mathbf{II}_n \rightarrow 0$ by dominated convergence. This shows that $\int_0^t K^n(t-s) Z_s^n ds \rightarrow \int_0^t K(t-s) Z_s ds$. Combined with (iii), we get, after taking the limit in (4.2), that

$$X_t = \lim_{n \rightarrow \infty} X_t^n = G_0(t) + \int_0^t K(t-s) Z_s ds,$$

for all $t \leq T$. Since X, G_0 and $t \mapsto \int_0^t K(t-s) Z_s ds$ are continuous, one can interchange the quantifiers so that the previous identity holds for all $t \leq T$, \mathbb{P} almost surely. Finally, each X^n being non-decreasing and nonnegative, the limit process X is again non-decreasing and nonnegative, which ends the proof. \square

5 Existence for L^1 -kernels

5.1 Existence for the stochastic Volterra equation

In this section we prove the existence of solutions for the stochastic Volterra equation (1.1), i.e. Theorem 2.13. We proceed in two steps. We first prove the claimed existence for smooth kernels $K \in C^1$ and finite measures ν . Second, we apply a density argument, i.e. Theorem 2.8, to obtain the existence for $K \in L^1_{\text{loc}}$ with possibly infinite activity jumps. We point out that for L^2 -kernels and possibly infinite activity jumps, existence was already obtained by Cuchiero and Teichmann (2020) using infinite dimensional Markovian lifts. However, the set of admissible input curves there is different than \mathcal{G}_K , recall (2.15), and the assumptions on K are different. For this reason, we provide another proof in the L^2 -setting by working directly on the level of the scalar stochastic Volterra equation, in the spirit of Abi Jaber and El Euch (2019a); Abi Jaber et al. (2019b).

To this end, for a stopping time τ we extend the definition of the set \mathcal{G}_K in (5.1) by considering

$$\begin{aligned} \mathcal{G}_K^\tau = \{ & (g(s))_{s \geq 0} \text{ adapted process: satisfying (2.14)} \\ & \text{and continuous on } [\tau(\omega), \infty) \text{ with } g(\tau(\omega)) \geq 0 \text{ a.s.} \}. \end{aligned} \quad (5.1)$$

The following lemma provides some elementary results on \mathcal{G}_K .

Lemma 5.1. *Let K be nonnegative, non-increasing and continuous on $[0, \infty)$ and τ a stopping time.*

- (i) *Let η be a nonnegative random variable, then $s \mapsto \mathbf{1}_{\tau \leq s} K(s - \tau) \eta$ belongs to \mathcal{G}_K^τ ,*
- (ii) *If $g_1, g_2 \in \mathcal{G}_K^\tau$, then $g_1 + g_2 \in \mathcal{G}_K^\tau$. If $\tau \leq \tau'$, then $\mathcal{G}_K^\tau \subset \mathcal{G}_K^{\tau'}$.*

Proof. (ii) is straightforward from the affine structure of (2.14). We prove (i). Clearly $g : s \mapsto \mathbf{1}_{\tau \leq s} K(s - \tau) \eta$ is continuous on $\{\tau(\omega) \leq t\}$ such that $g(\tau) = K(0) \eta \geq 0$ a.s. To argue (2.14), we fix $h \geq 0$. It follows from Abi Jaber and El Euch (2019a, Lemma B.2 and Remark B.3) that

$$\Delta_h K = (\Delta_h K * L)(0)K + d(\Delta_h K * L) * K.$$

Whence, on $\{\tau \leq t\}$:

$$\begin{aligned} \Delta_h g(t) &= \Delta_h K(t - \tau) \eta \\ &= (\Delta_h K * L)(0)K(t - \tau) \eta + (d(\Delta_h K * L) * K)(t - \tau) \eta \\ &= (\Delta_h K * L)(0)g(t) + \int_0^t d(\Delta_h K * L)(ds) \mathbf{1}_{\tau \leq t-s} K(t - s - \tau) \eta \\ &= (\Delta_h K * L)(0)g(t) + (d(\Delta_h K * L) * g)(t) \end{aligned}$$

which yields that the left hand side of (2.14) is zero, leading to (i). \square

We recast the existence results of [Abi Jaber et al. \(2019b\)](#); [Abi Jaber and El Euch \(2019a\)](#) obtained for L^2 -kernels and deterministic input curves $g_0 \in \mathcal{G}_K$ in the absence of jumps in our framework to allow for random input curves.

Lemma 5.2. *Fix $\beta, \sigma \in \mathbb{R}$ and let $K \in C^1[0, \infty)$ satisfying (2.10). Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, \mathbb{P})$ denote a probability space supporting a Brownian motion W . Fix a stopping time τ and a process $\tilde{g}_\tau \in \mathcal{G}_K^\tau$. Then, the equation*

$$Y_t = \tilde{g}_\tau(t) + \int_\tau^t K(t-s)\beta Y_s ds + \int_\tau^t K(t-s)\sigma\sqrt{Y_s}dW_s. \quad (5.2)$$

admits a unique nonnegative continuous and adapted strong solution Y on $[\tau, \infty)$. Furthermore, \mathcal{G}_K^τ is invariant for the process

$$\tilde{g}_t(s) = \tilde{g}_\tau(s) + \int_\tau^t K(s-u)\beta Y_u ds + \int_\tau^t K(s-u)\sigma\sqrt{Y_u}dW_u, \quad \tau \leq t \leq s,$$

meaning that g_t is \mathcal{G}_K^τ -valued on $[[\tau, \infty)) = \{(\omega, t) : \tau(\omega) \leq t\}$.

Proof. We first argue for $\tau \equiv 0$ and deterministic input $g_0 \in \mathcal{G}_K$. The existence of a \mathbb{R}_+ -valued continuous nonnegative weak solution Y to (5.2) follows from [Abi Jaber and El Euch \(2019a, Theorem A.2\)](#)³. The strong uniqueness of Y follows from [Abi Jaber and El Euch \(2019b, Proposition B.3\)](#). This yields the strong existence and uniqueness for (5.2). Finally, an application of the second part of [Abi Jaber and El Euch \(2019a, Theorem 3.1\)](#) yields the invariance of \mathcal{G}_K with respect to $t \mapsto \tilde{g}_t$, after noticing that

$$\tilde{g}_t(s) = \mathbb{E} \left[Y_s - \int_t^s K(s-u)\beta Y_u du \mid \mathcal{F}_t \right], \quad t \leq s.$$

For arbitrary τ and random input $\tilde{g}_\tau \in \mathcal{G}_K^\tau$, the result follows by a straightforward adaptation of the aforementioned results. \square

We now construct a solution to (1.1) when ν is finite and K is continuously differentiable by pasting continuous solutions Y^i to (5.2) on each interval $[\tau_i, \tau_{i+1})$ between two consecutive jumps.

Lemma 5.3. *Let $b \in \mathbb{R}, c \geq 0$ and ν be a nonnegative finite measure supported on \mathbb{R}_+ such that $\int_{\mathbb{R}_+} \zeta^2 \nu(d\zeta) < \infty$. Fix $K \in C^1[0, \infty)$ satisfying (2.10) and let $g_0 \in \mathcal{G}_K$. There exists a non-decreasing nonnegative continuous solution X to (1.1) for the input $G_0(t) = \int_0^t g_0(s) ds$.*

³We note that all the assumptions are met there, except for the local Hölder continuity of g_0 . This assumption is only needed to get Hölder sample paths of X , which we do not require here. Assumption (H_0) there is satisfied with $\gamma = 2$ since K is C^1 .

Proof. Using Lemma 2.1, it is enough to first prove the existence of a càdlàg nonnegative solution Y to the equation

$$Y_t = g_0(t) + \int_0^t K(t-s) dZ_s. \quad (5.3)$$

where Z is a semimartingale with differential characteristics with respect to the Lebesgue measure $(bY, cY, \nu(d\zeta)Y)$, and then set $X = \int_0^\cdot Y_s ds$ to obtain the desired solution to (1.1). Since ν is finite, finding a solution to equation (5.3) is equivalent to solving

$$Y_t = g_0(t) + \sum_{i \geq 0} \mathbf{1}_{\tau_i \leq t} \int_{\tau_i}^{t \wedge \tau_{i+1}} K(t-s) \left(\beta Y_s ds + \sqrt{cY_s} dW_s \right) + \sum_{i \geq 1} \mathbf{1}_{\tau_i \leq t} K(t - \tau_i) J_i, \quad (5.4)$$

where $\beta = b - \int_{\mathbb{R}_+} \zeta \nu(d\zeta)$, J_i are the jump sizes of Z distributed according to $\nu(d\zeta)/\nu(\mathbb{R}_+)$ and arriving at the jump times τ_i with instantaneous intensity $Y_t \nu(\mathbb{R}_+)$ and W is a Brownian motion on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

Our strategy for constructing a solution Y to the above equation consists in pasting continuous solutions Y^i on each interval $[\tau_i, \tau_{i+1})$ between two consecutive jumps. More precisely, fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ supporting a Brownian motion W and a sequence of independent random variables $(E_i, U_i)_{i \geq 1}$ with E_i exponentially distributed with intensity 1 and U_i standard uniform, and let F_ν denote the cumulative distribution function with density $\nu(d\zeta)/\nu(\mathbb{R}_+)$. We set $\tau_0 = 0$ and we assume that for each $i \geq 0$ we have a unique nonnegative continuous solution Y^i on $[\tau_i, \tau_{i+1})$ for the following system of inductive equations

$$Y_t^i = \tilde{g}^i(t) + \int_{\tau_i}^t K(t-s) \left(\beta Y_s^i ds + \sqrt{cY_s^i} dW_s \right) \quad (5.5)$$

$$\tilde{g}^i(t) := g_0(t) + \sum_{j=0}^{i-1} \int_{\tau_j}^{\tau_{j+1}} K(t-s) \left(\beta Y_s^j ds + \sqrt{cY_s^j} dW_s \right) + \sum_{j=1}^i K(t - \tau_j) J_j \quad (5.6)$$

with the convention that $\sum_{i=0}^{-1} = \sum_{j=1}^0 = 0$, $\tau_{i+1} = \tau_i + \delta_i$ with $\delta_i = \inf\{s > 0 : \nu(\mathbb{R}_+) \int_{\tau_i}^{\tau_i+s} Y_u^i du \geq E_i\}$ (with the convention that $\inf \emptyset = \infty$) and $J_i = F_\mu^-(U_i)$. Then, by a localization argument $\tau_i \rightarrow \infty$ and it is straightforward to check that the process Y defined by

$$Y_t = \sum_{i \geq 0} Y_t^i \mathbf{1}_{\tau_i \leq t < \tau_{i+1}}$$

is the unique càdlàg continuous nonnegative solution to (5.4). By the first part of Lemma 5.2, the existence and uniqueness of a solution Y^i is ensured provided that the process \tilde{g}^i is

$\mathcal{G}_K^{\tau_i}$ -valued, for each $i \geq 1$. We now prove this claim by induction using the second part of Lemma 5.2. *Initialization:* for $i = 0$, $\tilde{g}^0 = g_0$ is deterministic and \mathcal{G}_K -valued by assumption. *Heredity:* fix $i \geq 0$ and assume that $\tilde{g}^i \in \mathcal{G}_K^{\tau_i}$ -valued. Fix Y^i the unique nonnegative solution to (5.5) on $[\tau_i, \infty)$ obtained from Lemma 5.2 for the input \tilde{g}^i . The second part of Lemma 5.2 yields that the process

$$\tilde{g}_t^i = \tilde{g}^i + \int_{\tau_i}^t K(\cdot - u) \beta Y_u^i ds + \int_{\tau_i}^t K(\cdot - u) \sqrt{c Y_u^i} dW_u,$$

is $\mathcal{G}_K^{\tau_i}$ -valued on $[\tau_i, \infty)$. In particular, the stopped process $\tilde{g}_{\tau_{i+1}}^i$ belongs to $\mathcal{G}_K^{\tau_i}$. We now observe that (5.6) can be re-written in terms of $\tilde{g}_{\tau_{i+1}}^i$:

$$\begin{aligned} \tilde{g}^{i+1}(t) &= \tilde{g}^i(t) + \int_{\tau_i}^{\tau_{i+1}} K(t-s) \left(\beta Y_s^i ds + \sqrt{c Y_s^i} dW_s \right) + K(t - \tau_{i+1}) J_{i+1} \\ &= \tilde{g}_{\tau_{i+1}}^i(t) + K(t - \tau_{i+1}) J_{i+1}, \end{aligned}$$

on $\{t \geq \tau_{i+1}\}$. Since K is nonnegative and $J_{i+1} \geq 0$, recall that ν is supported on \mathbb{R}_+ , an application of Lemma 5.1 yields that $\tilde{g}^{i+1} \in \mathcal{G}_K^{\tau_{i+1}}$. This proves the induction and ends the proof. \square

For the general case, we use a density argument, i.e. Theorem 2.8, to obtain the existence statement in Theorem 2.13.

Proof of the existence statement in Theorem 2.13. Fix $n \geq 1$. Let $\nu^n(d\zeta) = \mathbf{1}_{\zeta \geq \frac{1}{n}} \nu(d\zeta)$ and $K^n = \Delta_{1/n} K$. Then, ν^n is a nonnegative finite measure supported on \mathbb{R}_+ such that $\int_{\mathbb{R}_+} \zeta^2 \nu^n(d\zeta) < \infty$. An application of Lemma 5.3 yields the existence of a non-decreasing and continuous process X^n solution to (1.1) with the inputs $(G_0^n, K^n, b, c, \nu^n)$, where $G_0^n = \int_0^\cdot g_0^n(s) ds$. Each X^n being non-decreasing, the claimed existence now follows from Theorem 2.8, once we prove that conditions (i)–(iii) are satisfied. (i) is clearly satisfied. (ii) holds by the L^1 -continuity of the kernel K , see Brezis (2010, Lemma 4.3). Finally, to obtain (iii) we first observe that g_0^n is nonnegative, this follows from (2.14) evaluated at $t = 0$. Whence, G_0^n is non-decreasing with a continuous pointwise limit G_0 . An application of Dini's second theorem yields (iii). The proof is complete. \square

5.2 Existence for the Riccati–Volterra equation

In this section we prove Theorem 2.5.

Proof of Theorem 2.5. We first note that the function $z \mapsto \int_{\mathbb{R}_+} (e^{z\zeta} - 1 - z\zeta) \nu(d\zeta)$ is continuous on $\mathcal{U} = \{z \in \mathbb{C} : \Re(z) \leq 0\}$. Whence, since f_2 is continuous and $\Re(f_2) \leq 0$ by (2.11), we obtain that $(s, u) \mapsto \int_{\mathbb{R}_+} (e^{(f_2(s)+u)\zeta} - 1 - (f_2(s)+u)\zeta) \nu(d\zeta)$ is continuous on $[0, T] \times \mathcal{U}$. We define

$$\tilde{F}(s, u) = F(s, \Re(u) \mathbf{1}_{\Re(u) \leq 0} + i\Im(u)),$$

where we recall that F is given by (2.5). Then, \tilde{F} is continuous on $[0, T] \times \mathbb{C}$ so that an application of Gripenberg et al. (1990, Theorem 12.1.1 and the comment below) (on the positive and negative parts) yields the local existence of a continuous solution ψ to the equation

$$\psi(t) = \int_0^t K(t-s)\tilde{F}(s, \psi(s))ds \quad (5.7)$$

on the interval $[0, T_\infty)$ where $T_\infty = \inf\{t : |\psi(t)| = \infty\}$. In order to obtain the claimed existence for (2.4)-(2.5) it suffices to prove that $T_\infty = +\infty$ and that $\Re(\psi(t)) \leq 0$, for all $t \geq 0$.

Step 1. We first prove that $\Re(\psi) \leq 0$ on $[0, T_\infty)$. Fix $T < T_\infty$ and denote by $\tilde{\psi}_r(t) = \Re(\psi(t))\mathbf{1}_{\Re(\psi(t)) \leq 0}$. Taking real parts in (5.7), we get that $\chi = \Re(\psi)$ satisfies

$$\chi(t) = \int_0^t K(t-s)\Re(\tilde{F}(s, \psi(s)))ds, \quad t \leq T, \quad (5.8)$$

with

$$\begin{aligned} \Re(\tilde{F}(s, \psi(s))) &= \Re(f_0(s)) + \frac{1}{2}c\Re(f_1(s))^2 - \frac{c}{2}(\Im(f_1(s)) + \Im(\psi(s)))^2 \\ &\quad + \left(b + c\Re(f_1(s)) + \frac{c}{2}\tilde{\psi}_r(s)\right)\tilde{\psi}_r(s) \\ &\quad + \int_{\mathbb{R}_+} \left(\cos(\Im(f_2(s) + \psi(s))\zeta)e^{(\Re(f_2(s)) + \tilde{\psi}_r(s))\zeta} - 1 - (\Re(f_2(s)) + \tilde{\psi}_r(s))\zeta\right)\nu(d\zeta). \end{aligned}$$

Since $\cos \leq 1$ we obtain, for $v \leq 0$,

$$\cos(a)e^{v\zeta} - 1 - v\zeta \leq e^{v\zeta} - 1 - v\zeta = \zeta^2 \int_v^0 \int_r^0 e^{vl} dl dr \leq \frac{\zeta^2 v^2}{2}.$$

Whence,

$$\Re(\tilde{F}(s, \psi(s))) \leq z(s)\tilde{\psi}_r(s) + w(s), \quad s \leq T, \quad (5.9)$$

with

$$\begin{aligned} z(s) &= b + c\Re(f_1(s)) + \frac{c}{2}\tilde{\psi}_r(s) + \int_{\mathbb{R}_+} \zeta^2 \nu(d\zeta) \left(\Re(f_2(s)) + \frac{1}{2}\tilde{\psi}_r(s)\right) \\ w(s) &= \Re(f_0(s)) + \frac{1}{2}c\Re(f_1(s))^2 + \frac{1}{2} \int_{\mathbb{R}_+} \zeta^2 \nu(d\zeta) (\Re(f_2(s)))^2 - \frac{c}{2}(\Im(f_1(s)) + \Im(\psi(s)))^2. \end{aligned}$$

Observing that $\tilde{\psi}_r(t) = \mathbf{1}_{\Re(\psi(t)) \leq 0}\chi(t)$, recalling that K is nonnegative and combining (5.8)-(5.9) leads to

$$\chi(t) \leq \int_0^t K(t-s)(\tilde{z}(s)\chi(s) + w(s))ds,$$

with $\tilde{z}(s) = \mathbf{1}_{\Re(\psi(s)) \leq 0} z(s)$. Whence,

$$\chi(t) = -h(t) + \int_0^t K(t-s)(\tilde{z}(s)\chi(s) + w(s))ds$$

for some nonnegative function h . Denoting by $\tilde{\chi}$ the solution to the linear equation

$$\tilde{\chi}(t) = \int_0^t K(t-s)(\tilde{z}(s)\tilde{\chi}(s) + w(s))ds \quad (5.10)$$

we obtain that $\delta = (\tilde{\chi} - \chi)$ solves the linear equation

$$\delta(t) = h(t) + \int_0^t K(t-s)\tilde{z}(s)\delta(s)ds.$$

Since, $h \geq 0$, an application of [Abi Jaber and El Euch \(2019b, Theorem C.1\)](#)⁴ leads to $\delta \geq 0$ on $[0, T]$ so that

$$\chi(t) \leq \tilde{\chi}(t), \quad t \leq T. \quad (5.11)$$

We now argue that $\tilde{\chi} \leq 0$. By virtue of [\(2.11\)](#), $w(s) \leq 0$ for all $s \leq T$, so that another application of [Abi Jaber and El Euch \(2019b, Theorem C.1\)](#) on the equation [\(5.10\)](#) leads to $\tilde{\chi}(t) \leq 0$, for all $t \leq T$. Finally, from [\(5.11\)](#), we obtain that $\Re(\psi(t)) = \chi(t) \leq \tilde{\chi}(t) \leq 0$, for all $t \leq T$. The claimed conclusion follows by arbitrariness of $T < T_\infty$.

Step 2. We now argue that $T_\infty = \infty$. Fix $T < T_\infty$. By the above we have $\tilde{\psi}_r = \Re(\psi)$ so that $\tilde{F}(s, \psi(s)) = F(s, \psi(s))$ for all $s < T_\infty$. Using this fact in [\(5.7\)](#), we observe that ψ solves the linear equation

$$h(t) = \int_0^t K(t-s) \left(\frac{c\psi(s)}{2} h(s) + \alpha(s) \right) ds,$$

with α defined by

$$\alpha(s) = F(s, \psi(s)) - \frac{c}{2}\psi(s)^2.$$

Since $\Re(\psi) \leq 0$, an application of [Abi Jaber and El Euch \(2019b, Theorem C.4\)](#)⁵ yields that $\sup_{t \leq T} |\psi(t)| < \infty$. By arbitrariness of T we obtain that $T_\infty = \infty$. The proof is complete. \square

⁴Inspecting the proof of [Abi Jaber and El Euch \(2019b, Theorem C.1\)](#) one can see that the L^2 integrability on the kernel assumed there does not play any role, the result remains clearly valid for $K \in L^1([0, T], \mathbb{R})$. Similarly, the continuity assumption on z there can be weakened to local boundedness.

⁵Again the L^2 integrability on the kernel assumed there does not play any role, the result remains clearly valid for $K \in L^1([0, T], \mathbb{R})$.

6 Weak uniqueness and the Fourier–Laplace transform

In this section, we prove Theorem 2.2.

Throughout this section, we fix $T \geq 0$, $f_0, f_1, f_2 : [0, T] \rightarrow \mathbb{C}$ continuous functions, G_0 a non-decreasing continuous function and $K \in L^1([0, T], \mathbb{R})$. We let $\psi \in C([0, T], \mathbb{C})$ denote a solution to the Riccati equation (2.4)–(2.5) such that (2.8) holds and X be a non-decreasing nonnegative continuous weak solution to (1.1) for the input (G_0, K, b, c, ν) . We recall the decomposition of Z in (2.1) and we define the process V^T :

$$V_t^T = V_0^T + \int_0^t \beta_s^T dX_s + \int_0^t (f_1(T-s) + \psi(T-s)) dM_s^c + \int_0^t (f_2(T-s) + \psi(T-s)) dM_s^d \quad (6.1)$$

$$\beta_s^T = -\frac{c}{2} (f_1(T-s) + \psi(T-s))^2 - \int_{\mathbb{R}_+} \left(e^{(f_2(T-s) + \psi(T-s))\zeta} - 1 - (f_2(T-s) + \psi(T-s))\zeta \right) \nu(d\zeta) \quad (6.2)$$

$$V_0^T = \int_0^T F(T-s, \psi(T-s)) dG_0(s). \quad (6.3)$$

We note that the integral involving ν is well-defined by virtue of (2.8) combined with the local boundedness of (ψ, f_2) and the inequality $|e^{\alpha\zeta} - 1 - \alpha\zeta| \leq e^{\Re(\alpha)\zeta} \alpha\zeta^2/2$. The Lebesgue–Stieltjes integrals are well-defined since (ψ, f_1, f_2) are continuous and (G_0, X) are of locally bounded variation.

$(V^T)_{t \leq T}$ is a semimartingale and a straightforward application of Itô’s Lemma yields that the stochastic exponential $H = \exp(V^T)$ is a complex local martingale with dynamics

$$dH_t = H_{t-} dN_t$$

$$dN_t = (f_1(T-t) + \psi(T-t)) dM_t^c + \int_{\mathbb{R}_+} \left(e^{(f_2(T-t) + \psi(T-t))\zeta} - 1 \right) (\mu^Z(dt, d\zeta) - \nu(d\zeta) dX_t),$$

meaning that $H = \mathcal{E}(N)$, where \mathcal{E} stands for the Doléans–Dade exponential. The following lemma, which extends [Abi Jaber et al. \(2019b\)](#), Lemma 7.3, establishes that H is even a true martingale.

Lemma 6.1. *Let $g_1, g_2 \in L^\infty(\mathbb{R}_+, \mathbb{R})$ such that*

$$\sup_{s \geq 0} \int_{\mathbb{R}_+} e^{g_2(s)\zeta} \zeta^2 \nu(d\zeta) < \infty \quad (6.4)$$

and define

$$U_t = \int_0^t g_1(s) dM_s^c + \int_{[0,t] \times \mathbb{R}_+} \left(e^{g_2(s)\zeta} - 1 \right) (\mu^Z(ds, d\zeta) - \nu(d\zeta) dX_s). \quad (6.5)$$

Then, the Doléans–Dade exponential $\mathcal{E}(U)$ is a martingale. Furthermore, $H = \exp(V^T)$ is a martingale on $[0, T]$.

Proof. Part 1. Martingality of $M := \mathcal{E}(U)$. We first recall that

$$M_t = \mathcal{E} \left(\int_0^\cdot g_1(s) dM_s^c \right) \prod_{0 < s \leq t} (1 + \Delta U_s) e^{-\Delta U_s}.$$

Since $\Delta U_s = (e^{g_2(s)\Delta Z_s} - 1) > -1$, for all $s \geq 0$, M is a nonnegative local martingale. Whence, it is a supermartingale by Fatou’s lemma, and it suffices to show that $\mathbb{E}[M_T] = 1$ for any $T \in \mathbb{R}_+$. To this end, fix $T > 0$ and define the stopping times $\tau_n = \inf\{t \geq 0: X_t > n\} \wedge T$. We first argue that $M^{\tau_n} = M_{\tau_n \wedge \cdot}$ is a uniformly integrable martingale for each n by verifying the condition in [Lépingle and Mémin \(1978, Theorem IV.3\)](#) with the process $y(\omega, t, \zeta) = \mathbf{1}_{t \leq \tau_n(\omega)}(e^{g_2(t)\zeta} - 1)$. Using the bound $|\alpha\zeta e^{\alpha\zeta} + 1 - e^{\alpha\zeta}| \leq \zeta^2 \alpha^2 e^{\alpha\zeta}/2$ and the boundedness of g_1, g_2 , we get that the quantity

$$\int_0^T \mathbf{1}_{s \leq \tau_n} g_1(s) c dX_s + \int_{[0, T] \times \mathbb{R}_+} \mathbf{1}_{s \leq \tau_n} \left(g_2(s) \zeta e^{g_2(s)\zeta} + 1 - e^{g_2(s)\zeta} \right) \nu(d\zeta) dX_s$$

is bounded by $\kappa n \left(1 + \sup_{s \geq 0} \int_{\mathbb{R}_+} e^{g_2(s)\zeta} \zeta^2 \nu(d\zeta) \right)$, for some constant $\kappa > 0$. The upper bound is finite due to condition [\(6.4\)](#). [Lépingle and Mémin \(1978, Theorem IV.3\)](#) can be applied to get that M^{τ_n} is a martingale for each n . Whence,

$$1 = M_0^{\tau_n} = \mathbb{E}_{\mathbb{P}} [M_T^{\tau_n}] = \mathbb{E}_{\mathbb{P}} [M_T \mathbf{1}_{\tau_n \geq T}] + \mathbb{E}_{\mathbb{P}} [M_{\tau_n} \mathbf{1}_{\tau_n < T}],$$

where we made the dependence of the expectation on \mathbb{P} explicit. Since $\mathbb{E}_{\mathbb{P}} [M_T \mathbf{1}_{\tau_n \geq T}] \rightarrow \mathbb{E}_{\mathbb{P}} [M_T]$ as $n \rightarrow \infty$, by dominated convergence, in order to get that $\mathbb{E}_{\mathbb{P}} [M_T] = 1$, it suffices to prove that

$$\mathbb{E}_{\mathbb{P}} [M_{\tau_n} \mathbf{1}_{\tau_n < T}] \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{6.6}$$

To this end, since M^{τ_n} is a martingale, we may define probability measures \mathbb{Q}^n by

$$\frac{d\mathbb{Q}^n}{d\mathbb{P}} = M_{\tau_n}^{\tau_n}.$$

By Girsanov’s theorem, see [Jacod and Shiryaev \(2003, Theorem III.3.24\)](#) (see also the formulation in [Kallsen \(2006, Proposition 2.6\)](#)), the process Z is a semimartingale under \mathbb{Q}^n with characteristics

$$\left(B^n, cX, \int_0^\cdot \mathbf{1}_{\{s \leq \tau_n\}} e^{g_2(s)\zeta} \nu(d\zeta) dX_s \right)$$

where

$$B^n = bX + \int_0^\cdot \mathbf{1}_{\{s \leq \tau_n\}} g_1(s) c dX_s + \int_{[0, \cdot] \times \mathbb{R}_+} \zeta \left(e^{g_2(s)\zeta} - 1 \right) \mathbf{1}_{\{s \leq \tau_n\}} \nu(d\zeta) dX_s.$$

Under \mathbb{Q}^n , we still have

$$X_t = G_0(t) + \int_0^t K(t-s) Z_s ds,$$

and we observe that, due to the boundedness of g_1 , the equality $|e^{\alpha\zeta} - 1| \leq \zeta(1 + e^{\alpha\zeta})$ and (6.4), the characteristics of Z under \mathbb{Q}^n satisfy the growth condition in X as in (3.1) for some constant κ_L independent of n . Therefore, an application of Lemma 3.1 yields the moment bound

$$\mathbb{E}_{\mathbb{Q}^n} [\sup_{t \leq T} |X_t|^2] \leq \eta(\kappa_L, T, K, G_0),$$

where $\eta(\kappa_L, T, K, G_0)$ does not depend on n . We then get by an application of Chebyshev's inequality

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} [M_{\tau_n} \mathbf{1}_{\tau_n \leq T}] &= \mathbb{Q}^n(\tau_n < T) \\ &\leq \mathbb{Q}^n \left(\sup_{t \leq T} X_t > n \right) \\ &\leq \frac{1}{n^2} \mathbb{E}_{\mathbb{Q}^n} \left[\sup_{t \leq T} X_t^2 \right] \\ &\leq \frac{1}{n^2} \eta(\kappa_L, T, K, G_0). \end{aligned}$$

Sending $n \rightarrow \infty$, we obtain (6.6), proving that M is martingale.

Part 2. Martingality of $H = \exp(V^T)$. To show that the local martingale H is a true martingale, it is enough to bound it by a martingale, see Jarrow (2018, Lemma 1.4). We fix $t \leq T$ and define $g_i(s) = \Re(f_i(T-s) + \psi(T-s)) \mathbf{1}_{s \leq T}$, and $m_i(s) = \Im(f_i(T-s) + \psi(T-s)) \mathbf{1}_{s \leq T}$ for $i = 1, 2$. Taking real parts in (6.1) yield

$$\begin{aligned} \Re(V_t^T) &= V_0^T - \frac{c}{2} \int_0^t (g_1^2(s) - m_1^2(s)) dX_s + \int_0^t g_1(s) dM_s^c \\ &\quad + \int_{\mathbb{R}_+} \left(\cos(m_2(s)\zeta) e^{g_2(s)\zeta} - 1 - g_2(s)\zeta \right) \nu(d\zeta) + \int_0^t g_2(s) dM_s^d. \end{aligned}$$

Whence, using that \cos is bounded by 1, we get

$$|H_t| = \exp(\Re(V_t^T)) \leq \exp\left(V_0^T + \frac{c}{2} m_1^2(s)\right) \mathcal{E}(U_t)$$

with U given by (6.5). Since, (ψ, f_1, f_2) are continuous and satisfy (2.8), m_i, g_i are bounded for $i = 1, 2$, and g_2 satisfies (6.4), so that $|H_t| \leq c_T \mathcal{E}(U_t)$ for some constant c_T and $\mathcal{E}(U)$ is a martingale thanks to the first part. This proves that H is a martingale on $[0, T]$. \square

Lemma 6.2. *Assume that the shifted kernels $\Delta_h K$ are in $L^2([0, T], \mathbb{R})$, for all $h > 0$. Set $(G_t)_{t \geq 0}$ and $(g_t)_{t \geq 0}$ as in (2.6)–(2.7). Then, the process $(V_t^T)_{0 \leq t \leq T}$ defined by (6.1)–(6.2)–(6.3) satisfies*

$$\begin{aligned} V_t^T &= \int_0^t f_0(T-s) dX_s + \int_0^t f_1(T-s) dM_s^c + \int_0^t f_2(T-s) dM_s^d \\ &\quad + \int_t^T F(T-s, \psi(T-s)) dG_t(s), \quad t \leq T. \end{aligned} \tag{6.7}$$

Proof. We fix $t \leq T$, $h > 0$ and we define

$$\begin{aligned} X_t^h &= G_0(t) + \int_0^t \Delta_h K(t-s) Z_s ds, \\ g_t^h(s) &= \int_0^t \Delta_h K(s-u) dZ_u, \\ \psi^h(t) &= \int_0^t \Delta_h K(t-s) F(s, \psi(s)) ds, \end{aligned}$$

where we recall that $Z = bX + M^c + M^d$ and F is given by (2.5). We stress that the right-hand sides of all three quantities are defined from X and ψ and do not depend on X^h or ψ^h ; X_t^h and g_t^h are well-defined as Itô integrals since $\Delta_h K \in L^2([0, T], \mathbb{R})$.

Step 1. Convergence of X^h, g^h, ψ^h . It follows from the boundedness of (ψ, f_0, f_1, f_2) and $Z(\omega)$, condition (2.8) and the L^1 -continuity of the kernel K that

$$\sup_{s \leq T} |\psi^h(s) - \psi(s)| \rightarrow 0, \quad |X_t^h - X_t| \rightarrow 0, \quad \mathbb{P} - a.s. \tag{6.8}$$

as $h \rightarrow 0$. Set

$$G_t^h(s) = G_0(s) + \int_t^{s \vee t} g_t^h(u) du.$$

By invoking a stochastic Fubini theorem, justified by the L^2 -integrability of $\Delta_h K$, we get, for all $s > t$,

$$G_t^h(s) - G_t(s) = \int_0^s (\Delta_h K(u) - K(u)) (Z_{t \wedge (s-u)} - Z_{t-u}) du.$$

The boundedness of $Z(\omega)$ and the L^1 -continuity of the kernel K , lead to

$$G_t^h(s) \rightarrow G_t(s), \quad \mathbb{P} - a.s. \tag{6.9}$$

as $h \rightarrow 0$, for all $s \in (t, T]$.

Step 2. Proving (6.7). An application of a stochastic Fubini theorem, see Veraar (2012,

Theorem 2.2) – justified by the L^2 -integrability of $\Delta_h K$, the boundedness of ψ , f and $X(\omega)$ – yields

$$\begin{aligned}
\int_0^t \psi^h(T-s) dZ_s &= \int_0^t \left(\int_0^{T-s} F(u, \psi(u)) \Delta_h K(T-s-u) du \right) dZ_s \\
&= \int_0^T F(u, \psi(u)) \left(\int_0^{t \wedge (T-u)} \Delta_h K(T-u-s) dZ_s \right) du \\
&= \int_0^{T-t} F(u, \psi(u)) \left(\int_0^t \Delta_h K(T-u-s) dZ_s \right) du \\
&\quad + \int_{T-t}^T F(u, \psi(u)) \left(\int_0^{T-u} \Delta_h K(T-u-s) dZ_s \right) du \\
&= \int_t^T F(T-s, \psi(T-s)) g_t^h(s) ds \\
&\quad + \int_0^t F(T-s, \psi(T-s)) d \left(X_s^h - G_0(s) \right) \\
&= \int_t^T F(T-s, \psi(T-s)) d \left(G_t^h(s) - G_0(s) \right) \\
&\quad + \int_0^t F(T-s, \psi(T-s)) d \left(X_s^h - G_0(s) \right),
\end{aligned}$$

where we used in the fourth identity that $(X_s^h - G_0(s)) = \int_0^s (\int_0^r \Delta_h K(r-u) dZ_u) dr$, due to Lemma 2.1 since $\Delta_h K \in L_{\text{loc}}^2$, for $h > 0$. Recalling (6.8)-(6.9) and sending $h \rightarrow 0$ in the previous identity yields, by invoking the dominated convergence for the left-hand side and Helly's second theorem on (X^h, G_t^h) for the right-hand side (see Natanson (2016, Theorem 7.3)), we obtain that

$$\begin{aligned}
\int_0^t \psi(T-s) dZ_s &= \int_t^T F(T-s, \psi(T-s)) dG_t(s) + \int_0^t F(T-s, \psi(T-s)) dX_s \\
&\quad - \int_0^T F(T-s, \psi(T-s)) dG_0(s).
\end{aligned} \tag{6.10}$$

Using (2.5), we can rewrite β^T given in (6.2) as

$$\beta_s^T = f_0(T-s) + b\psi(T-s) - F(T-s, \psi(T-s)).$$

This shows that V^T given by (6.1) can be re-expressed in the form

$$\begin{aligned}
V_t^T &= V_0^T + \int_0^t f_0(T-s) dX_s + \int_0^t \psi(T-s) dZ_s \\
&\quad - \int_0^t F(T-s, \psi(T-s)) dX_s + \int_0^t f_1(T-s) dM_s^c + \int_0^t f_2(T-s) dM_s^d.
\end{aligned}$$

Plugging (6.10) in the previous expression and recalling (6.3) yields (6.7). \square

Combining the two previous Lemma, we prove Theorem 2.2.

Proof of Theorem 2.2. Throughout the proof we fix $\psi \in C([0, T], \mathbb{C})$ a solution to the Riccati equation (2.4)-(2.5) satisfying (2.8). We first prove (2.9), and then deduce the weak uniqueness statement. Let X be a non-decreasing nonnegative continuous weak solution to (1.1) for the input (G_0, K, b, c, ν) . By Lemma 6.1 $\exp(V^T)$ is a true martingale on $[0, T]$. Its terminal value can be computed using (6.7):

$$V_T^T = \int_0^T f_0(T-s) dX_s + \int_0^T f_1(T-s) dM_s^c + \int_0^T f_2(T-s) dM_s^d.$$

Whence, by the martingality property, we have that

$$\mathbb{E} \left[\exp(V_T^T) \middle| \mathcal{F}_t \right] = \exp(V_t^T), \quad (6.11)$$

for all $t \leq T$. This proves (2.9). To argue uniqueness, we first observe that V_0^T given in (6.3) does not depend on the process X , but only depends on G_0, K and ψ . We let Y denote another non-decreasing nonnegative continuous weak solution to (1.1) for the same inputs (G_0, K, b, c, ν) and we set $f_1 = f_2 \equiv 0$. Then, (6.11) holds for Y with the same function ψ , so that evaluating the expression at $t = 0$ gives

$$\mathbb{E} \left[\exp \left(\int_0^T f_0(T-s) dY_s \right) \right] = \exp(V_0^T) = \mathbb{E} \left[\exp \left(\int_0^T f_0(T-s) dX_s \right) \right],$$

for any continuous function $f_0 : [0, T] \mapsto i\mathbb{R}$. This yields that the finite-dimensional marginals $(X_{t_1}, \dots, X_{t_m})$ and $(Y_{t_1}, \dots, Y_{t_m})$ are equal for any m , which proves weak uniqueness. \square

7 Application: Hyper-rough Volterra Heston models with jumps

In this section, we apply our main results to a class of hyper-rough Volterra Heston models with jumps. We fix (G_0, K, b, c, ν) as in Theorem 2.13 and we let X denote the unique non-decreasing nonnegative continuous weak solution to (1.1) with Z the semimartingale with characteristics (1.2) given by Theorem 2.13. We recall the martingales M^c and M^d that appear in the decomposition (2.1) of Z . After a possible extension of the filtered probability space, we let $M^{c,\perp}$ denote a continuous martingale independent of M^c , such

that $\langle M^{c,\perp} \rangle = X$ and we set $M^S = \frac{\rho}{\sqrt{c}}M^c + \sqrt{1-\rho^2}M^{c,\perp}$ for some $\rho \in [-1, 1]$. We consider the following model for the log-price S

$$\log S_t = \log S_0 - \frac{1}{2}X_t + M_t^S, \quad S_0 > 0, \quad (7.1)$$

$$X_t = G_0(t) + \int_0^t K(t-s)Z_s ds, \quad (7.2)$$

where we recall the characteristics of Z are $(bX, cX, \nu(d\zeta)X)$. For instance, if $c = 0$, then X can be interpreted as the ‘integrated intensity’ of a self-exciting jump process, e.g. a Hawkes process, recall Example 2.3; if $\nu = 0$, then Z is continuous and X can be seen as a hyper-rough process, see Remark 7.2 below.

The chief example we have in mind for (K, G_0) for applications is the following:

Example 7.1. • K is proportional to the fractional kernel:

$$K(t) = K_1(t)K_H(t)$$

where

$$K_H(t) = \frac{t^{H-1/2}}{\Gamma(H+1/2)}, \quad t > 0,$$

for some $H \in (-1/2, 1/2]$ and K_1 is a completely monotone kernel on $[0, \infty)$, e.g. $K_1 \equiv 1$ or $K_1(t) = e^{-\eta t}$, for some $\eta > 0$. Under such specification, the assumptions of Theorem 2.13 needed on the kernel are satisfied due to Example 2.4.

• G_0 is absolutely continuous:

$$G_0(t) = \int_0^t g_0(s) ds, \quad t \geq 0, \quad \text{for some } g_0 \in \mathcal{G}_K,$$

as in Example 2.12, recall Remark 2.14.

The following remark shows that X can be thought of as the ‘integrated variance’ process in the absence of jumps.

Remark 7.2. Assume that $\nu = 0$ and $K \in L_{loc}^2$ (e.g. $H > 0$ in the specification of Example (7.1)). It follows from Lemma 2.1 that $X_t = \int_0^t V_s ds$ where (S, V) is a rough Volterra Heston model in the terminology of [Abi Jaber et al. \(2019b, Section 7\)](#); [El Euch and Rosenbaum \(2019\)](#) satisfying

$$\begin{aligned} d \log S_t &= -\frac{1}{2}V_t dt + \sqrt{V_t} d\tilde{B}_t, \quad S_0 > 0, \\ V_t &= g_0(t) + \int_0^t K(t-s)bV_s ds + \int_0^t K(t-s)\sqrt{cV_s} d\tilde{W}_s, \end{aligned}$$

for some Brownian motions \tilde{B} and \tilde{W} obtained from standard martingale representation theorems on a possible extension of the probability space, see for instance [Revuz and Yor \(2013, Proposition V.3.8\)](#). For the fractional kernel with $H \in (0, 1/2)$, the sample paths of V are Hölder continuous of any order strictly less than H and the process V is said to be ‘rough’.

If K_H is no longer in L^2_{loc} (e.g. $H < 0$ in the specification of [Example \(7.1\)](#)), not only Fubini’s interchange breaks down, but it can also be shown that X is nowhere differentiable almost surely, see [Jusselin and Rosenbaum \(2018, Proposition 4.6\)](#). In this case, one cannot really make sense of the spot variance V and is stuck with the ‘integrated variance’ formulation [\(7.2\)](#), justifying the appellation hyper-rough for such equations.

We are now in place to provide the joint Fourier–Laplace transform of $(\log S, X)$ in [\(7.1\)](#)–[\(7.2\)](#).

Theorem 7.3. *Let (K, G_0) be as in [Theorem 2.13](#) and $h_0, h_1 : \mathbb{R}_+ \mapsto \mathbb{C}$ be continuous functions such that*

$$\Re(h_0) \leq 0 \quad \text{and} \quad 0 \leq \Re(h_1) \leq 1.$$

The joint Fourier–Laplace transform of $(X, \log S)$ in [\(7.1\)](#)–[\(7.2\)](#) is given by

$$\mathbb{E} \left[\exp \left(\int_t^T h_0(T-u) dX_u + \int_t^T h_1(T-u) d \log S_u \right) \middle| \mathcal{F}_t \right] = \exp \left(\int_t^T F(s, \psi(T-s)) dG_t(s) \right)$$

for all $t \leq T$, where G_t is given by [\(2.6\)](#) and ψ solves the Riccati–Volterra equation

$$\psi(t) = \int_0^t K(t-s) F(s, \psi(s)) ds, \quad t \geq 0, \tag{7.3}$$

$$\begin{aligned} F(s, u) = & h_0(s) + \frac{1}{2}(h_1^2(s) - h_1(s)) + (b + \rho\sqrt{c}h_1(s))u + \frac{c}{2}u^2 \\ & + \int_{\mathbb{R}_+} (e^{u\zeta} - 1 - u\zeta) \nu(d\zeta) \end{aligned} \tag{7.4}$$

Proof. It suffices to prove that the Fourier-Laplace transform

$$L_t = \mathbb{E} \left[\exp \left(\int_t^T h_0(T-u) dX_u + \int_t^T h_1(T-u) d \log S_u \right) \middle| \mathcal{F}_t \right] \tag{7.5}$$

can be written as

$$L_t = \mathbb{E} \left[\exp \left(\int_t^T f_0(T-s) dX_s + \int_t^T f_1(t-s) dM_s^c \right) \middle| \mathcal{F}_t \right] \tag{7.6}$$

where

$$f_0(t) = h_0(t) + \frac{1}{2}(h_1^2(t) - h_1(t)) - \frac{1}{2}\rho^2 h_1^2(t) \quad \text{and} \quad f_1(t) = \frac{\rho}{\sqrt{c}} h_1(t).$$

Indeed, if this the case, the Riccati-Volterra equations (2.4)–(2.5) reduce to (7.3)–(7.4) and the claimed expression for the Fourier-Laplace transform together with the existence of the corresponding solution ψ follow from Theorem 2.13 (with $f_2 \equiv 0$), since

$$\Re(f_0) + \frac{c}{2}\Re(f_1)^2 = \Re(h_0) + \frac{1}{2}(\Re(h_1)^2 - \Re(h_1) - \Im(h_1)^2) \leq 0,$$

since $\Re(f_0) \leq 0$ and $\Re(h_1) \in [0, 1]$. It remains to prove (7.6) by means of a projection argument. For this, we fix $t \leq T$ and we write the variation of (7.1) between t and T , recall that $M^S = \frac{\rho}{\sqrt{c}}M^c + \sqrt{1 - \rho^2}M^{c,\perp}$, to get

$$d \log S_u = -\frac{1}{2}dX_u + \rho dM_u^c + \sqrt{1 - \rho^2}dM_u^{c,\perp}. \quad (7.7)$$

We then observe that

$$\begin{aligned} M_t &:= \mathbb{E} \left[\exp \left(\sqrt{1 - \rho^2} \int_t^T h_1(T - s) dM_s^{c,\perp} \right) \middle| \mathcal{F}_t \vee \mathcal{F}^X \right] \\ &= \exp \left(\frac{(1 - \rho^2)}{2} \int_t^T h_1(T - s)^2 dX_s \right) \end{aligned} \quad (7.8)$$

so that, using successively (7.7), the tower property of the conditional expectation and the fact that X and Z are \mathcal{F}^X -measurable, L_t given by (7.5) satisfies

$$\begin{aligned} L_t &= \mathbb{E} \left[\exp \left(\int_t^T h_0(T - u) dX_u + \int_t^T h_1(T - u) d \log S_u \right) \middle| \mathcal{F}_t \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\exp \left(\int_t^T h_0(T - u) dX_u + \int_t^T h_1(T - u) d \log S_u \right) \middle| \mathcal{F}_t \vee \mathcal{F}^X \right] \middle| \mathcal{F}_t \right] \\ &= \mathbb{E} \left[\exp \left(\int_t^T (h_0 - \frac{1}{2}h_1)(T - s) dX_s + \int_t^T \frac{\rho}{\sqrt{c}} h_1(T - s) dM_u^c \right) M_t \middle| \mathcal{F}_t \right] \end{aligned}$$

leading to (7.6) due to (7.8). This ends the proof. \square

In particular, we consider the specification of G_0 as in Example 7.1, and set

$$h_0(t) \equiv u_0 \quad \text{and} \quad h_1(t) \equiv u_1, \quad \text{with} \quad \Re(u_0) \leq 0, \quad \Re(u_1) \in [0, 1].$$

For $t = 0$ and $S_0 = 1$, we have $X_0 = 0$, $\log S_0 = 0$ and $dG_0(s) = g_0(s)ds$ so that the unconditional Fourier-Laplace transform reads

$$\mathbb{E} [\exp (u_0 X_T + u_1 \log S_T)] = \exp \left(\int_0^T F(u_1, u_2, \psi(T - s)) g_0(s) ds \right),$$

with

$$\begin{aligned}\psi(t) &= \int_0^t K(t-s)F(u_0, u_1, \psi(s))ds \\ F(u_0, u_1, u_2) &= u_0 + \frac{1}{2}(u_1^2 - u_1) + (b + \rho\sqrt{c}u_1u_2) + \frac{c}{2}u_2^2 + \int_{\mathbb{R}_+} \left(e^{u_2\zeta} - 1 - u_2\zeta \right) \nu(d\zeta).\end{aligned}$$

If in addition $g_0(t) = x_0 + \theta \int_0^t K(s)ds$, for some $x_0, \theta \geq 0$ (recall Example 2.12), then, Fubini's theorem leads to

$$\int_0^T F(u_0, u_1, \psi(T-s))g_0(s)ds = x_0 \int_0^T F(u_0, u_1, \psi(s))ds + \theta \int_0^T \psi(s)ds$$

so that

$$\mathbb{E}[\exp(u_0X_T + u_1 \log S_T)] = \exp\left(x_0 \int_0^T F(u_0, u_1, \psi(s))ds + \theta \int_0^T \psi(s)ds\right).$$

Remark 7.4. *Using Theorem 2.8, one can prove the convergence of the multifactor Markovian approximations designed in [Abi Jaber \(2019\)](#); [Abi Jaber and El Euch \(2019b\)](#) towards the hyper-rough Heston model, where the kernel K_H is approximated by a suitable weighted sum of exponentials $K^n(t) = \sum_{i=1}^n c_i^n e^{-\gamma_i^n t}$. These approximations are therefore still valid for non-positive values of the Hurst index $H \in (-1/2, 0]$, which would allow the simulation of the process X and the numerical approximation of the Riccati–Volterra equations, we refer to the aforementioned articles for more details.*

A Catalytic super–Brownian motion and its local occupation time

In this section, we sketch a rigorous derivation of equation (1.7) satisfied by the local occupation time X given by (1.5) formally derived in the introduction. We will make use of the notation $\langle \mu, \phi \rangle$ to denote the quantity $\int_{\mathbb{R}} \mu(dx)\phi(x)$.

We recall that the super–Brownian motion with a single point catalyst \bar{Y} solves the following martingale problem

$$\langle \bar{Y}_t, \phi \rangle = \langle \bar{Y}_0, \phi \rangle + \frac{1}{2} \int_0^t \langle \bar{Y}_s, \Delta\phi \rangle ds + \phi(0)Z_t,$$

where $\Delta = \partial^2/\partial x^2$, ϕ is a suitable test function and Z is a continuous martingale with quadratic variation

$$\langle Z \rangle_t = X_t,$$

where X is the local occupation time defined by (1.5), see Dawson and Fleischmann (1994, Theorem 1.2.7).

In order to make the link with stochastic Volterra equations, we first reformulate the martingale problem in its ‘mild form’.

Lemma A.1. *Assume that $\psi \in C^2(\mathbb{R}, \mathbb{R})$ has a Gaussian decay, that is $\sup_{x \in \mathbb{R}} |\psi(x)| e^{cx^2} < \infty$, for some constant c . Then,*

$$\langle \bar{Y}_t, \psi \rangle = \langle S_t \bar{Y}_0, \psi \rangle + \int_0^t (S_{t-s} \psi)(0) dZ_s,$$

where

$$(S_t \mu)(x) = \int_{\mathbb{R}} p_t(x-y) \mu(dy) \quad \text{and} \quad p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right), \quad x \geq 0.$$

Sketch of proof. Let $\xi : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable function and set $\phi_t(x) = \xi(t) \phi^0(x)$ for some C^2 function ϕ^0 having a Gaussian decay. An application of Itô’s Lemma gives

$$\begin{aligned} d\langle \bar{Y}_t, \phi^0 \rangle \xi(t) &= \xi(t) d\langle \bar{Y}_t, \psi \rangle + \langle \bar{Y}_t, \phi^0 \rangle \xi'(t) dt \\ &= \langle \bar{Y}_t, \frac{1}{2} \Delta \phi_t + \partial_t \phi_t \rangle dt + \phi_t(0) dZ_t. \end{aligned}$$

Thus,

$$\langle \bar{Y}_t, \phi_t \rangle = \langle \bar{Y}_0, \phi_0 \rangle + \int_0^t \langle \bar{Y}_s, \frac{1}{2} \Delta \phi_s + \partial_t \phi_s \rangle ds + \int_0^t \phi_s(0) dZ_s. \quad (\text{A.1})$$

Fix $t \geq 0$ and consider $\phi_s = S_{t-s} \psi$ for all $s \in [0, t]$. Noticing that $\phi_t = \psi$ and $\partial_t \phi_s = -\frac{1}{2} \Delta \phi_s$, the claimed identity follows from (A.1) with this specific test function combined with a density argument. \square

For each $\varepsilon > 0$, let $p_\varepsilon : x \rightarrow (2\pi\varepsilon)^{-1/2} \exp(-x^2/(2\varepsilon))$ be Gaussian density approximations of the dirac mass at 0. It follows from Lemma A.1 that

$$\langle \bar{Y}_t, p^\varepsilon \rangle = \langle S_t \bar{Y}_0, p^\varepsilon \rangle + \int_0^t (S_{t-s} p^\varepsilon)(0) dZ_s.$$

Integrating both sides with respect to time and invoking stochastic Fubini’s theorem, see Lemma 2.1, leads to

$$\int_0^t \langle \bar{Y}_s, p^\varepsilon \rangle ds = \int_0^t \langle S_s \bar{Y}_0, p^\varepsilon \rangle ds + \int_0^t (S_{t-s} p^\varepsilon)(0) Z_s ds.$$

Sending $\varepsilon \rightarrow 0$ yields

$$X_t = \lim_{\varepsilon \rightarrow 0} \int_0^t \langle \bar{Y}_s, p^\varepsilon \rangle ds = \int_0^t (S_s \bar{Y}_0)(0) ds + \int_0^t p_{t-s}(0) Z_s ds,$$

showing that X solves (1.7) with the function $g_0(t) = (S_t \bar{Y}_0)(0)$.

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