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Weak existence and uniqueness for affine stochastic Volterra equations with L^1 -kernels

Eduardo Abi Jaber*

December 15, 2019

Abstract

We provide existence, uniqueness and stability results for affine stochastic Volterra equations with L^1 -kernels. Such equations arise as scaling limits of branching processes in population genetics and self-exciting Hawkes processes in mathematical finance. The strategy we adopt for the existence part is based on approximations using stochastic Volterra equations with L^2 -kernels. Most importantly, we establish weak uniqueness using a duality argument on the Fourier–Laplace transform via a deterministic Riccati–Volterra integral equation. We illustrate the applicability of our results on a class of hyper-rough Volterra Heston models with a Hurst index $H \in (-1/2, 1/2]$.

Keywords: Stochastic Volterra equations, Affine Volterra processes, Riccati-Volterra equations, superprocesses, rough volatility.

MSC2010 Classification: 60H20, 60G22, 45D05

1 Introduction

We establish weak existence, uniqueness and stability results for stochastic Volterra equation with locally L^1 -kernels K in the form

$$X_t = G_0(t) + \int_0^t K(t-s)Z_s ds, \quad t \geq 0, \quad (1.1)$$

where Z is a real-valued continuous semimartingale with affine characteristics

$$(bX, cX, 0), \quad (1.2)$$

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with $b \in \mathbb{R}$, $c \geq 0$ and a given function $G_0 : \mathbb{R}_+ \rightarrow \mathbb{R}$. For L^2 -kernels this formulation was recently introduced in [Abi Jaber et al. \(2019a\)](#), where Z is a semimartingale with jumps but whose characteristics are absolutely continuous with respect to the Lebesgue measure. In the L^1 setting, X may fail to be absolutely continuous with respect to the Lebesgue measure, as will be explained in the sequel. For this reason, our study falls beyond the scope of [Abi Jaber et al. \(2019a\)](#).

Our motivation for studying such convolution equations is twofold. Stochastic Volterra equations with kernels that are locally in L^1 but not in L^2 arise as scaling limits of branching processes in population genetics and self-exciting Hawkes processes in mathematical finance.

- *From branching processes to stochastic Volterra equations.* The link was formulated for the first time in [Mytnik and Salisbury \(2015\)](#) to motivate the study of stochastic Volterra equations with L^2 -kernels. In the sequel we re-formulate the aforementioned introductory exposition linking super-processes with stochastic Volterra equations with L^1 -kernels. Consider a system of n *reactant* particles in one dimension moving independently according to a standard Brownian motion and branching only in the presence of a *catalyst*. The *catalyst* region at a certain time t is defined as the support of some deterministic measure $\rho_t(dx)$. Whenever a particle enters in the *catalyst* region and after spending a random time in the vicinity of the *catalyst*, it will either die or split into two new particles, with equal probabilities. The measure $\rho_t(dx)$ determines the local branching rate in space and time depending on the location and the concentration of the *catalyst*. Two typical examples are $\rho_t(dx) \equiv \bar{\rho}dx$ where the branching occurs in the entire space with constant rate $\bar{\rho}$ and $\rho_t(dx) = \delta_0(dx)$ for a branching occurring with infinite rate only when the particle hits a highly concentrated single point *catalyst* located at 0. In case of branching, the two offspring particles evolve independently with the same spatial movement and branching mechanism as their parent.

One can view the reactant as a rescaled measure-valued process $(\bar{Y}_t^n(dx))_{t \geq 0}$ defined by

$$\bar{Y}_t^n(B) = \frac{\text{number of particles in } B \text{ at time } t}{n}, \quad \text{for every Borel set } B.$$

Sending the number of particles to infinity, one can establish the convergence towards a measure-valued macroscopic *reactant* \bar{Y} , coined *catalytic super-Brownian motion*, which solves an infinite dimensional martingale problem, see [Dawson and Fleischmann \(1991\)](#); [Etheridge \(2000\)](#); [Perkins \(2002\)](#) and the references therein. Moreover, in the presence of a suitable deterministic *catalyst* $\rho = (\rho_t(dx))_{t \geq 0}$ having no atoms, the measure-valued process \bar{Y} admits a density $\bar{Y}_t(dx) = Y_t(x)dx$ solution to the following stochastic partial differential equation in mild form

$$Y_t(x) = \int_{\mathbb{R}} p_t(x-y)Y_0(y)dy + \int_{[0,t] \times \mathbb{R}} p_{t-s}(x-y)\sqrt{Y_s(y)}W^\rho(ds, dy). \quad (1.3)$$

where Y_0 is an input curve, $p_t(x) = (2\pi t)^{-1/2} \exp(-x^2/(2t))$ is the heat kernel and W^ρ is a

space-time noise with covariance structure determined by ρ , refer to [Zähle \(2005\)](#) for more details. The previous equation is only valid if ρ has no atoms. One could still heuristically set $\rho_t(dx) = \delta_0(dx)$ in [\(1.3\)](#) for the extreme case of a single point *catalyst* at 0, which would formally correspond to the *catalytic super-Brownian motion* of [Dawson and Fleischmann \(1994\)](#). Then, the space-time noise reduces to a standard Brownian motion W so that evaluation at $x = 0$ yields

$$Y_t(0) = g_0(t) + \frac{1}{\sqrt{2\pi}} \int_0^t (t-s)^{-1/2} dZ_s, \quad (1.4)$$

where $dZ_t = \sqrt{\bar{Y}_t(0)} dW_t$ and $g_0(t) = \int_{\mathbb{R}} p_t(y) Y_0(y) dy$. The link with stochastic Volterra equations of the form [\(1.1\)](#) is established by considering the local occupation time at the catalyst point 0 defined by

$$X_t = \lim_{\varepsilon \rightarrow 0} \int_0^t \int_{\mathbb{R}} p^\varepsilon(y) \bar{Y}_s(dy) ds, \quad t \geq 0, \quad (1.5)$$

where p^ε is a suitable smoothing kernel of the dirac point mass at 0. Integrating both sides of equation [\(1.4\)](#) with respect to time and formally interchanging the integrals lead to

$$X_t = \int_0^t Y_s(0) ds \quad (1.6)$$

$$= \int_0^t g_0(s) ds + \frac{1}{\sqrt{2\pi}} \int_0^t (t-s)^{-1/2} Z_s ds. \quad (1.7)$$

where $\langle Z \rangle = X$. Consequently, X solves [\(1.1\)](#) for the kernel

$$K_0(t) = \frac{t^{-1/2}}{\sqrt{2\pi}}, \quad t > 0, \quad (1.8)$$

which is locally in L^1 but not in L^2 . Needless to say, one is not allowed to plug the Dirac measure in [\(1.3\)](#). Indeed, in the presence of a single point *catalyst*, the *catalytic super-Brownian motion* does not admit a density at the *catalyst* position as shown by [Dawson and Fleischmann \(1994\)](#) and the identities [\(1.3\)](#) and [\(1.6\)](#) break down. The local occupation time X is even singular with respect to the Lebesgue measure, see [Dawson et al. \(1995\)](#); [Fleischmann and Le Gall \(1995\)](#). Still, one can rigorously prove that the local occupation time X defined by [\(1.5\)](#) solves [\(1.7\)](#) by appealing to the martingale problem of the measure-valued process \bar{Y} , we refer to [Appendix A](#) for a rigorous derivation.

• *From Hawkes processes to stochastic Volterra equations.* More recently, for particular choices of G_0 and kernels, solutions to [\(1.1\)](#) were obtained in [Jusselin and Rosenbaum \(2018\)](#) as scaling limits of Hawkes processes $(N^n)_{n \geq 1}$ with respective intensities

$$\lambda_t^n = g_0^n(t) + \int_0^t K^n(t-s) dN_s^n, \quad t \geq 0,$$

for some suitable function g_0^n and kernel K^n . The rescaled sequence of integrated accelerated intensities $X^n = \int_0^\cdot \lambda_{n_s}^n ds$ is shown to converge to a continuous process X satisfying (1.1) for the fractional kernel

$$K_H(t) = \frac{t^{H-1/2}}{\Gamma(H+1/2)}, \quad t > 0,$$

with $H \in (1/2, 1/2]$. We note that for $H = 0$ the fractional kernel reduces to (1.8), up to a normalizing constant. In other words, when $H = 0$, the scaling limit of the integrated intensities of Hawkes processes can be seen as the local occupation time of the *catalytic super-Brownian motion* of Dawson and Fleischmann (1994), provided uniqueness holds. Similarly, when $H \leq 0$, K_H lies locally in L^1 but not in L^2 , and one can also show that in this case X is not absolutely continuous with respect to the Lebesgue measure, see Jusselin and Rosenbaum (2018, Proposition 4.6).

In both cases, one can compute the Laplace transform of X , modulo a deterministic Riccati–Volterra equation of the form

$$\psi(t) = \int_0^t K_H(t-s) \left(\frac{1}{2} \psi^2(s) - 1 \right) ds,$$

either by using the dual process of the *catalytic super-Brownian motion*, see Dawson and Fleischmann (1994, Equations (4.2.1)-(4.2.2)), or by exploiting the affine structure of the approximating Hawkes processes, see Jusselin and Rosenbaum (2018, Theorem 3.2). Both constructions provide solutions to (1.1), but do not yield uniqueness. Establishing weak uniqueness is one of the main motivation of this work.

In the present paper, we provide a generic treatment of the limiting macroscopic equation (1.1). The strategy we adopt is based on approximations using stochastic Volterra equations with L^2 kernels, whose existence and uniqueness theory is now well-established, see Abi Jaber et al. (2019a,b) and the references therein. By doing so, we avoid the infinite-dimensional analysis used for super-processes, we also circumvent the need to study scaling limits of Hawkes processes, allowing for more generality in the choice of kernels K and input functions G_0 . Most importantly, we establish weak uniqueness using a duality argument on the Fourier–Laplace transform of X via a deterministic Riccati–Volterra integral equation. In particular, this expression extends the one obtained for affine Volterra processes with L^2 -kernels in Abi Jaber et al. (2019b). We illustrate the applicability of our results on a class of hyper-rough Volterra Heston models with a Hurst index $H \in (-1/2, 1/2]$ extending the results of Abi Jaber et al. (2019b); El Euch and Rosenbaum (2019); Jusselin and Rosenbaum (2018). Such models have recently known a growing interest to account for rough volatility, a universal phenomena observed in financial markets, see Gatheral et al. (2018).

Notations Δ_h stands for the shift operator, i.e. $\Delta_h g = g(h + \cdot)$ and dg is the distributional derivative of a right-continuous function g with locally bounded variation,

i.e. $dg((s, t]) = g(t) - g(s)$. For a suitable borel function f the quantity $\int_0^t f(s)dg(s)$ will stand for the Lebesgue–Stieltjes integral, whenever the integral exists. Similarly, for each $t < T$, the convolution $\int_0^t f(t - s)dg(s)$ is defined as the Lebesgue–Stieltjes integral $\int_0^T \mathbf{1}_{[0, t]} f(t - s)dg(s)$ whenever this latter quantity is well–defined.

Outline Section 2 states our main existence and uniqueness result together with the expression for the Fourier–Laplace transform. Section 3 provides a-priori estimates for the solution. In Section 4, we derive a general stability results for stochastic Volterra equations with L^1 –kernels. These results are used to establish weak existence for the stochastic Volterra equation in Section 5. Furthermore, an existence result for Riccati–Volterra equations with L^1 –kernels is derived there. Weak uniqueness is then established by completely characterizing the Fourier–Laplace transform of the solution in terms of the Riccati–Volterra equation of Section 6. In Section 7, we apply our results to obtain existence, uniqueness and the characteristic function of the log-price in hyper-rough Volterra Heston models. Finally, we provide a more rigorous derivation of the stochastic Volterra equation satisfied by the local occupation time of the catalytic super-Brownian motion in Appendix A.

2 Main result

In this section, we present our main result together with the strategy we adopt. We start by making precise the concept of solution.

We call X a weak solution to (1.1) for the input (G_0, K) , if there exists a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ supporting a non-decreasing, continuous and adapted process X and a continuous semimartingale Z whose characteristics are given by (1.2) such that (1.1) holds \mathbb{P} –almost surely. Note that, by virtue of standard martingale representation theorems, there exists an enlargement of the probability space supporting a Brownian motion W such that Z admits the following decomposition

$$Z_t = bX_t + W_{cX_t}, \quad t \geq 0. \tag{2.1}$$

One first notes that the formulation (1.1) differs from the one given in Jusselin and Rosenbaum (2018, Equation (3)), where

$$X_t = G_0(t) + \int_0^t \left(\int_0^{t-s} K(r)dr \right) dZ_s.$$

Although these two formulation are equivalent, thanks to stochastic Fubini’s theorem, the advantages of considering the formulation (1.1) as starting point, which is inspired by the ‘martingale problem’ formulation of stochastic Volterra equations recently introduced in Abi Jaber et al. (2019a) will become clear in the sequel.

The following lemma establishes the link with stochastic Volterra equations with L^2 –kernels, as the one studied for instance in Abi Jaber et al. (2019b).

Lemma 2.1. Fix $K \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$ and $g_0 \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$. Assume that there exists a non-decreasing continuous adapted process X and a Brownian motion W on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ such that

$$X_t = \int_0^t g_0(s) ds + \int_0^t K(t-s) Z_s ds, \quad (2.2)$$

with Z given by (2.1) and such that $\sup_{t \leq T} \mathbb{E}[|X_t|] < \infty$, for all $T > 0$. Then, $X = \int_0^\cdot Y_s ds$ where Y is a nonnegative weak solution to the following stochastic Volterra equation

$$Y_t = g_0(t) + \int_0^t K(t-s) b Y_s ds + \int_0^t K(t-s) \sqrt{c Y_s} d\tilde{W}_s, \quad \mathbb{P} \otimes dt - a.e. \quad (2.3)$$

Conversely, assume there exists a nonnegative weak solution Y to the stochastic Volterra equation (2.3) such that $\sup_{t \leq T} \mathbb{E}[|Y_t|] < \infty$ for all $T > 0$, then $X = \int_0^\cdot Y_s ds$ is a continuous non-decreasing solution to (2.2).

Proof. Fix $t \geq 0$. An application of stochastic Fubini's theorem, see [Abi Jaber et al. \(2019b, Lemma 2.1\)](#), yields

$$\begin{aligned} \int_0^t K(t-s) Z_s ds &= \int_0^t K(s) \left(\int_0^{t-s} dZ_s \right) ds \\ &= \int_0^t \left(\int_0^{t-r} K(s) ds \right) dZ_r \\ &= \int_0^t \left(\int_0^t K(s-r) \mathbf{1}_{\{r \leq s\}} ds \right) dZ_r \\ &= \int_0^t \left(\int_0^s K(s-r) dZ_r \right) ds. \end{aligned}$$

Thus, X admits a density Y with respect to the Lebesgue measure, such that

$$Y_t = g_0(t) + \int_0^t K(t-r) dZ_r,$$

and the characteristics of Z read

$$\left(b \int_0^\cdot Y_s ds, c \int_0^\cdot Y_s ds, 0 \right).$$

Since X is non-decreasing almost surely, Y is nonnegative $\mathbb{P} \otimes dt$. Since $\langle W_{cX} \rangle_t = c \int_0^t Y_s ds$, standard martingale representation theorems, e.g. [Revuz and Yor \(2013, Proposition V.3.8\)](#), yield the existence of a Brownian motion \tilde{W} such that $dW_{cX_t} = \sqrt{c Y_t} d\tilde{W}_t$. The claimed stochastic Volterra equation (2.3) readily follows. The converse direction follows along the same lines by integrating both sides of (2.3) and applying stochastic Fubini's theorem as above to get (2.2). \square

Our strategy for constructing solutions to (1.1) with L^1 -kernels relies on an approximation argument using L^2 -kernels combined with Lemma 2.1. To fix ideas, assume $G_0 = \lim_{n \rightarrow \infty} G_0^n$, with $G_0^n = \int_0^\cdot g_0^n(s) ds$, for some sequence of L^1_{loc} -functions $(g_0^n)_{n \geq 1}$. Starting from a L^1_{loc} -kernel K , assume that there exists a sequence of L^2_{loc} -kernels $(K^n)_{n \geq 1}$ such that

$$K^n \rightarrow K, \quad \text{in } L^1_{\text{loc}}, \quad \text{as } n \rightarrow \infty.$$

Then, for each $n \geq 1$, K^n being locally square-integrable, under suitable conditions on (g_0^n, K^n) , the results in [Abi Jaber et al. \(2019b\)](#); [Abi Jaber and El Euch \(2019a\)](#) provide existence of nonnegative solution Y^n for (2.3) with (g_0, K) replaced by (g_0^n, K^n) . Setting $X^n = \int_0^\cdot Y_s^n ds$, Lemma 2.1 provides a solution X^n to (1.1) for the input (G_0^n, K^n) , that is

$$X_t^n = G_0^n(t) + \int_0^t K^n(t-s) Z_s^n ds, \quad (2.4)$$

where the characteristics of Z^n are $(bX^n, cX^n, 0)$. Provided that $(X^n)_{n \geq 1}$ is tight, it will admit a convergent subsequence towards a limiting process X . Finally, sending $n \rightarrow \infty$, one would expect X to solve (1.1). The kernel approximation philosophy plays also a key role in deriving the Fourier–Laplace transform of X and establishing weak uniqueness.

We now introduce the assumptions needed on the kernel K and the input function G_0 . First, we assume that $K \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$ such that

$$\int_0^h |K(s)| ds + \int_0^T |K(s+h) - K(s)| ds \leq Ch^\gamma, \quad h, T > 0, \quad (2.5)$$

for some $\gamma, C > 0$. We will also need the following L^2 -condition on the right-shifted kernels $\Delta_\varepsilon K = K(\cdot + \varepsilon)$: for each $\varepsilon > 0$ there exists $\gamma_\varepsilon, C_\varepsilon > 0$ such that

$$\int_0^h |\Delta_\varepsilon K(s)|^2 ds + \int_0^T |\Delta_\varepsilon K(s+h) - \Delta_\varepsilon K(s)|^2 ds \leq C_\varepsilon h^{\gamma_\varepsilon}, \quad h, T > 0. \quad (2.6)$$

Before going further, let us list some examples of kernels that satisfy (2.5)-(2.6).

Example 2.2. (i) Locally Lipschitz kernels K clearly satisfy (2.5)-(2.6) with $\gamma = \gamma_\varepsilon = 1$, for all $\varepsilon > 0$.

(ii) The fractional kernel $K(t) = t^{H-1/2}$ with $H \in (-1/2, 1/2]$ satisfies (2.5)-(2.6). (2.6) follows from (i) since $\Delta_\varepsilon K$ is locally Lipschitz for each $\varepsilon > 0$, and (2.5) is satisfied with $\gamma = H + 1/2$. Indeed, we have $\int_0^h K(t) dt = h^{H+1/2}/(H + 1/2)$ and

$$\int_0^T |K(t+h) - K(t)| dt = h^{H+1/2} \int_0^\infty \left(t^{H-1/2} - (t+1)^{H-1/2} \right) dt,$$

where the integral appearing on the right-hand side is finite. This example shows that (2.6) does not necessarily imply that $K \in L^2_{\text{loc}}$, indeed when $H \leq 0$, $t \mapsto t^{H-1/2}$ is not locally square-integrable.

(iii) If K_1 and K_2 satisfy (2.5)-(2.6), then so does $K_1 + K_2$. If in addition K_2 is locally Lipschitz then one can show that $K_1 K_2$ also satisfies (2.5)-(2.6).

Second, monotonicity and continuity assumptions on (K, G_0) are needed to construct approximating sub-sequences of the form (2.4) with L^2 -kernels, that are non-decreasing. For this we recall the notion of the resolvent of the first kind of a kernel: a measure L of locally bounded variation is called a resolvent of the first kind of the kernel K if

$$\int_{[0,t]} K(t-s)L(ds) = 1, \quad t \geq 0.$$

In addition to (2.5)-(2.6), we consider the following condition on the shifted-kernels

$$\begin{aligned} &\text{the kernel is nonnegative, non-increasing and continuous on } (0, \infty), \text{ and} \\ &\text{its resolvent of the first kind } L \text{ is nonnegative and non-increasing in the} \\ &\text{sense that } s \mapsto L([s, s+t]) \text{ is non-increasing for all } t \geq 0. \end{aligned} \quad (2.7)$$

We note in (2.7) that any nonnegative and non-increasing kernel that is not identically zero admits a resolvent of the first kind; see [Gripenberg et al. \(1990, Theorem 5.5.5\)](#).

Example 2.3. If K is completely monotone on $(0, \infty)$, then (2.7) holds due to [Gripenberg et al. \(1990, Theorem 5.5.4\)](#). Recall that a function f is called completely monotone on $(0, \infty)$ if it is infinitely differentiable there with $(-1)^k f^{(k)}(t) \geq 0$ for all $t > 0$ and $k \geq 0$. For each $\varepsilon > 0$, the shifted kernel $\Delta_\varepsilon K$ is again completely monotone on $[0, \infty)$ so that (2.7) holds also for $\Delta_\varepsilon K$. In particular, $\Delta_\varepsilon K$ is locally Lipschitz and (2.6) is satisfied by Example 2.2-(i). This covers, for instance, any constant positive kernel, the fractional kernel $t^{H-1/2}$ with $H \in (-1/2, 1/2]$, and the exponentially decaying kernel $e^{-\eta t}$ with $\eta > 0$. Moreover, sums and products of completely monotone kernels are completely monotone. By combining the above examples we find that the Gamma kernel $K(t) = t^{H-1/2}e^{-\eta t}$ for $H \in (-1/2, 1/2]$ and $\eta \geq 0$ satisfies (2.5), (2.6) and (2.7).

Concerning the input function G_0 , [Abi Jaber and El Euch \(2019a\)](#) provide a set \mathcal{G}_K of admissible input curves g_0 defined in terms of the resolvent of the first kind L to ensure the existence of non-negative solution for (2.3). To guarantee that the approximate solutions (2.4) are non-decreasing, we consider similarly to [Abi Jaber and El Euch \(2019a, Equations \(2.4\)–\(2.5\)\)](#), the following condition¹

$$\Delta_h g_0 - (\Delta_h K * L)(0)g_0 - d(\Delta_h K * L) * g_0 \geq 0, \quad h \geq 0, \quad (2.8)$$

¹Under (2.7) one can show that $\Delta_h K * L$ is right-continuous and of locally bounded variation (see [Abi Jaber and El Euch \(2019a, Remark B.3\)](#)), thus the associated measure $d(\Delta_h K * L)$ is well defined.

where we used the notation $(f * \mu)(t) = \int_0^t f(t-s)\mu(ds)$ for a measure of locally bounded variation μ and a function $f \in L^1_{\text{loc}}$. We then define the space of admissible input curves \mathcal{G}_K to be

$$\mathcal{G}_K = \{g_0 \text{ continuous, satisfying (2.8) such that } g_0(0) \geq 0\}. \quad (2.9)$$

Two notable examples of such admissible input curves are:

Example 2.4. (i) g_0 continuous and non-decreasing with $g_0(0) \geq 0$,

(ii) $g_0(t) = x_0 + \int_0^t K(t-s)\theta(s)ds$, for some $x_0 \geq 0$ and $\theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ locally bounded, see e.g. [Abi Jaber and El Euch \(2019a, Example \(2.2\)\)](#).

Finally, we introduce the following process which enters in the representation of the Fourier–Laplace transform:

$$G_t(s) = G_0(s) + \int_t^{s \vee t} g_t(u)du, \quad s, t \geq 0, \quad (2.10)$$

$$g_t(s) = \int_0^t K(s-r)dZ_r, \quad s > t. \quad (2.11)$$

We note that since the shifted kernels are in L^2 , the stochastic convolution $\int_0^t K(s-r)dZ_r = \int_0^t \Delta_{s-t}K(t-r)dZ_r$ is well-defined as an Itô integral, for all $s > t$.

We are now in place to state the main result of the paper.

Theorem 2.5. *Fix a nonnegative and non-increasing kernel $K \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$ satisfying (2.5). Assume that its shifted kernels $\Delta_\varepsilon K$ satisfy (2.6) and (2.7), for any $\varepsilon > 0$. Let $G_0 = \lim_{n \rightarrow \infty} \int_0^\cdot g_0^n(s)ds$ for some functions $g_0^n \in \mathcal{G}_{\Delta_{1/n}K}$, $n \geq 1$, and assume that G_0 is continuous. Then, there exists a unique non-decreasing continuous weak solution X to (1.1) for the input (G_0, K) . Furthermore, for $a \in \mathbb{C}$ and $f : [0, T] \rightarrow \mathbb{C}$ continuous such that*

$$a \in i\mathbb{R} \quad \text{and} \quad \Re(f) \leq 0, \quad (2.12)$$

the joint conditional Fourier–Laplace transform of (X, W_X) , where W is the Brownian motion appearing in (2.1), is given by

$$\mathbb{E} \left[\exp \left(\int_t^T f(T-s)dX_s + a(W_{X_T} - W_{X_t}) \right) \middle| \mathcal{F}_t \right] = \exp \left(\int_t^T F(T-s, \psi(T-s))dG_t(s) \right),$$

for all $t \leq T$, where G_t is defined as in (2.10) and $\psi \in C([0, T], \mathbb{C})$ solves the Riccati–Volterra equation

$$\psi(t) = \int_0^t K(t-s)F(s, \psi(s))ds, \quad t \leq T, \quad (2.13)$$

$$F(s, u) = f(s) + \frac{1}{2}a^2 + (b + \sqrt{c}a)u + \frac{c}{2}u^2, \quad (2.14)$$

such that

$$\Re(\psi(t)) \leq 0, \quad t \leq T.$$

Proof. Theorems 5.2 and 5.4 give respectively the existence statement for the stochastic Volterra equation and for the Riccati–Volterra equation. Theorem 6.3 provides the expression for the Fourier–Laplace transform together with the weak uniqueness of X . \square

Remark 2.6. If K satisfies (2.7) and $g_0 \in \mathcal{G}_K$, the continuous function $G_0(t) = \int_0^t g_0(s)ds$ satisfies the assumption of Theorem 2.5. Indeed, fix $n \geq 1$ and set $g_0^n = \Delta_{1/n}g_0$. We start by showing that $g_0^n \in \mathcal{G}_{\Delta_{1/n}K}$. First, since g_0 satisfies (2.8), $g_0^n(0) = g_0(1/n) \geq (\Delta_{1/n}K * L)(0)g_0(0) \geq 0$. Second, we fix $t, h' > 0$ and evaluate the condition (2.8) for g_0 with $h = h' + 1/n$ and t to get that g_0^n satisfies (2.8) with K replaced by $\Delta_{1/n}K$. Furthermore, we have that $\lim_{n \rightarrow \infty} \int_0^t g_0^n(s)ds = G_0(t)$ by virtue of Brezis (2010, Lemma 4.3).

Remark 2.7. If K is in L_{loc}^2 , then one recovers *Abi Jaber et al. (2019b, Theorem 7.1)* and *Abi Jaber and El Euch (2019a, Theorem 2.3)*.

3 A-priori estimates and Hölder regularity

We first provide a-priori estimates for solutions to (1.1). For this, we recall that the resolvent of the second kind R of K is the unique L_{loc}^1 function solution to

$$R(t) = K(t) + \int_0^t K(t-s)R(s)ds = K(t) + \int_0^t R(t-s)K(s)ds, \quad t \geq 0.$$

The resolvent R exists, for any kernel $K \in L_{\text{loc}}^1$, see *Gripenberg et al. (1990, Theorems 2.3.1 and 2.3.5)*.

Lemma 3.1. Fix $K \in L_{\text{loc}}^1$ and assume that there exists a continuous and adapted solution X to (1.1). Then, for all $T > 0$,

$$\sup_{t \leq T} \mathbb{E} [|X_t|^p] \leq C \left(1 + \sup_{t \leq T} |G_0(t)|^p + \|K\|_{L^1([0,T])}^p \right) (1 + \|R\|_{L^1([0,T])}), \quad p \geq 1, \quad (3.1)$$

where C is a constant only depending on (p, T, b, c) , and R is the resolvent of the second kind of $C\|K\|_{L^1([0,T])}^{p-1}|K|$. Furthermore, denoting by $\tilde{X} = (X - G_0)$, we have for all $s \leq t \leq T$,

$$\begin{aligned} \mathbb{E} [|\tilde{X}_t - \tilde{X}_s|^p] &\leq C \left(\left(\int_0^T |K(t-r) - K(s-r)|dr \right)^p + \left(\int_0^{t-s} |K(r)|dr \right)^p \right) \\ &\quad \times \left(1 + \sup_{r \leq T} \mathbb{E} [|X_r|^p] \right) \end{aligned} \quad (3.2)$$

where C is a constant only depending on (p, T, b, c) .

Proof. Fix $p, n \geq 1$ and define $\tau_n = \inf\{t \geq 0 : |X_t| \geq n\} \wedge T$. Since X is adapted with continuous sample paths, τ_n is a stopping time such that $\tau_n \rightarrow T$ almost surely as $n \rightarrow \infty$. First observe that

$$|X_t \mathbf{1}_{\{t < \tau_n\}}| \leq |G_0(t)| + \left| \int_0^t K(t-s) Z_s \mathbf{1}_{\{s < \tau_n\}} ds \right|.$$

and set $X_t^n = X_t \mathbf{1}_{\{t \leq \tau_n\}}$. An applications of Jensen's inequality on the normalized measure $|K(t)|dt / \|K\|_{L^1([0,T])}$ yields

$$\begin{aligned} |X_t^n|^p &\leq 2^{p-1} |G_0(t)|^p + 2^{p-1} \left| \int_0^t K(t-s) Z_s \mathbf{1}_{\{s < \tau_n\}} ds \right|^p \\ &\leq 2^{p-1} \sup_{r \leq T} |G_0(r)|^p + 2^{p-1} \|K\|_{L^1([0,T])}^{p-1} \int_0^t |K(t-s)| |Z_s \mathbf{1}_{\{s < \tau_n\}}|^p ds. \end{aligned}$$

Taking expectation together with the Burkholder-Davis-Gundy's inequality, we get for a constant C depending exclusively on (p, T, b, c) that may vary from line to line:

$$\begin{aligned} \mathbb{E} [|X_t^n|^p] &\leq C \sup_{r \leq T} |G_0(r)|^p + C \|K\|_{L^1([0,T])}^{p-1} \int_0^t |K(t-s)| (1 + \mathbb{E} [|X_s^n|^p]) ds \\ &\leq C \left(1 + \sup_{r \leq T} |G_0(r)|^p + \|K\|_{L^1([0,T])}^p \right) (1 + \|R\|_{L^1([0,T])}) \end{aligned}$$

where the last line follows from the generalised Gronwall inequality for convolution equations with R the resolvent of $C \|K\|_{L^1([0,T])}^{p-1} |K|$, see [Gripenberg et al. \(1990, Theorem 9.8.2\)](#). The claimed estimate [\(3.1\)](#) now follows by sending $n \rightarrow \infty$ and using Fatou's Lemma. To prove [\(3.2\)](#), we fix $s \leq t \leq T$ and we write thanks to Jensen's inequality

$$\begin{aligned} |\tilde{X}_t - \tilde{X}_s|^p &\leq 2^{p-1} \left| \int_0^s |K(t-r) - K(s-r)| |Z_r| dr \right|^p + 2^{p-1} \left| \int_s^t |K(t-r)| |Z_r| dr \right|^p \\ &\leq 2^{p-1} \left(\int_0^T |K(t-r) - K(s-r)| dr \right)^{p-1} \int_0^s |K(t-r) - K(s-r)| |Z_r|^p dr \\ &\quad + 2^{p-1} \left(\int_0^{t-s} |K(r)| dr \right)^{p-1} \int_s^t |K(t-r)| |Z_r|^p dr. \end{aligned}$$

Taking expectation and invoking once again Burkholder-Davis-Gundy's inequality, we get

$$\begin{aligned}
& \mathbb{E} \left[|\tilde{X}_t - \tilde{X}_s|^p \right] \\
& \leq C \left(\int_0^T |K(t-r) - K(s-r)| dr \right)^{p-1} \int_0^s |K(t-r) - K(s-r)| (1 + \mathbb{E} [|X_r|^p]) dr \\
& \quad + C \left(\int_0^{t-s} |K(r)| dr \right)^{p-1} \int_s^t |K(t-r)| (1 + \mathbb{E} [|X_r|^p]) dr \\
& \leq C \left(\left(\int_0^T |K(t-r) - K(s-r)| dr \right)^p + \left(\int_0^{t-s} |K(r)| dr \right)^p \right) \\
& \quad \times \left(1 + \sup_{r \leq T} \mathbb{E} [|X_r|^p] \right).
\end{aligned}$$

□

It follows that (3.1) is finite whenever G_0 is locally bounded. Furthermore, (3.2) combined with Kolmogorov–Chenstov continuity theorem, see Revuz and Yor (2013, Theorem I.2.1), provides the existence of a version of $(X - G_0)$ with locally Hölder sample paths, provided the kernel satisfies (2.5). This is the object of the next lemma.

Lemma 3.2. *Assume that K satisfies (2.5) for some $\gamma > 0$ and that G_0 is locally bounded, then any continuous and adapted solution X to (1.1) satisfies*

$$\sup_{t \leq T} \mathbb{E} [|X_t|^p] < \infty, \quad p \geq 1, \quad T > 0. \tag{3.3}$$

Furthermore, if G_0 is locally Hölder continuous with order γ , then X has locally Hölder sample paths of any order strictly less than γ .

Proof. Fix $p \geq 1$, $s \leq t \leq T$. By local boundedness of G_0 , (3.3) readily follows from (3.1). Under (2.5) and the γ -Hölder continuity of G_0 on $[0, T]$, (3.2) reads

$$\mathbb{E} [|X_t - X_s|^p] \leq C \left(1 + \sup_{r \leq T} \mathbb{E} [|X_r|^p] \right) (t - s)^{\gamma p}$$

so that Revuz and Yor (2013, Theorem I.2.1) yields the existence of a unique version with Hölder sample paths on $[0, T]$ of any order strictly less than $(\gamma - 1/p)$. By continuity of X , X corresponds to this version. The claimed result follows by arbitrariness of $p \geq 1$. □

Remark 3.3. *It is clear from the proof that the moment bound (3.3) in Lemma 3.2 holds also for state and time-dependent predictable characteristics $b(x, t, \omega)$ and $c(x, t, \omega)$, provided they satisfy a linear growth condition uniformly in (t, ω) , that is,*

$$|b(x, t, \omega)| \vee |c(x, t, \omega)| \leq c_{\text{LG}}(1 + |x|), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_+, \quad \omega \in \Omega,$$

for some constant c_{LG} .

The following remark shows that the constant γ in Lemma 3.2 is not in general the optimal Hölder constant.

Remark 3.4. For the fractional kernel $K(t) = \frac{t^{H-1/2}}{\Gamma(H+1/2)}$, with $H \in (-1/2, 1/2]$, $\gamma = H + 1/2$ due to Example 2.2-(ii). The Riemman-Liouville fractional operator I^γ is defined by

$$I^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s) ds.$$

The operator is well defined for any $f \in L_{\text{loc}}^p$, for some $p \geq 1$. Further, $I^\gamma f$ is $(\gamma + 1/p)$ -Hölder continuous, see Samko et al. (1993, Theorem 3.6). This is consistent with Lemma 3.2, where we relied in the proof on the L^∞ -property of the process Z . The result can be strengthened by exploiting that the process Z admits a version \tilde{Z} with locally Hölder continuous sample paths of any order strictly less than $1/2$. Then, again standard mapping theorems for I^α yield that $I^\gamma \tilde{Z}$ is locally Hölder continuous of any order strictly less than $\gamma + 1/2$, see Samko et al. (1993, Theorem 3.3).

4 Tightness and stability

In this section, we state and prove our general tightness and stability result. One can appreciate the formulation (1.1) for the stability argument.

Theorem 4.1. Let K be a kernel satisfying (2.5) for some $\gamma > 0$. Assume that there exist sequences of kernels $(K^n)_{n \geq 1}$ and functions $(G_0^n)_{n \geq 1}$ such that

- (i) K^n satisfies (2.5) for γ , uniformly in $n \geq 1$ (i.e. the constant C appearing in (2.5) does not depend on n);
- (ii) $\int_0^t |K^n(s) - K(s)| ds \rightarrow 0$, as $n \rightarrow \infty$, $t \geq 0$.
- (iii) $\sup_{t \leq T} |G_0^n(t)| < C'_0(T)$ for some $C'_0(T) > 0$, for all $T > 0$, uniformly in $n \geq 1$,
- (iv) $G_0^n(t) \rightarrow G_0(t)$, as $n \rightarrow \infty$, $t \geq 0$, for some continuous function G_0 .

Then, any sequence of continuous non-decreasing solutions $(X^n)_{n \geq 1}$ to (1.1), for the respective inputs (G_0^n, K^n) , is tight on the space of continuous functions $C([0, T], \mathbb{R})$ endowed with the uniform topology, for each $T > 0$. Furthermore, any limit point X is a continuous non-decreasing solution to (1.1) for the input (G_0, K) .

Proof. Let $(X^n)_{n \geq 1}$ be a sequence of continuous non-decreasing solutions to (1.1), for the respective inputs (G_0^n, K^n) , that is

$$X_t^n = G_0^n(t) + \int_0^t K^n(t-s) Z_s^n ds, \quad t \geq 0, \quad (4.1)$$

where $Z^n = bX^n + W_{cX^n}^n$ and (X^n, W^n) are defined on some filtered probability space $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n)_{t \geq 0}, \mathbb{P}^n)$, for $n \geq 1$.

• To argue *tightness*, it is sufficient to prove that the bound (3.2) for $\tilde{X}^n = (X^n - G_0^n)$ is uniform in n , in the sense that there exists $C(p, T) > 0$ such that

$$\mathbb{E} \left[|\tilde{X}_t^n - \tilde{X}_s^n|^p \right] \leq C(p, T)(t - s)^{\gamma p}, \quad n \geq 1, \quad p \geq 1, \quad T > 0. \quad (4.2)$$

Indeed, if this is the case, then Kolmogorov's tightness criterion implies that the sequence $(\tilde{X}^n)_{n \geq 1}$ is tight on $C([0, T], \mathbb{R})$, leading to the tightness of $(X^n)_{n \geq 1}$ by using (iv) and invoking Prokhorov's theorem. To get (4.2), we first observe that thanks to (i), for all $n \geq 1$, the bound (3.2) reads

$$\mathbb{E} \left[|\tilde{X}_t^n - \tilde{X}_s^n|^p \right] \leq C(t - s)^{\gamma p} \left(1 + \sup_{r \leq T} \mathbb{E} [|X_r^n|^p] \right),$$

with C independent of n . Whence, it suffices to prove that there exists a constant $C(p, T) > 0$ such that

$$\sup_{n \geq 1} \sup_{t \leq T} \mathbb{E} [|X_t^n|^p] \leq C(p, T), \quad p \geq 1, \quad T > 0. \quad (4.3)$$

By virtue of the continuous dependence of the resolvent on the kernel in L^1 , the L^1 -convergence of K in (ii) implies the L^1 -convergence of the respective sequence of resolvents $(R^n)_{n \geq 1}$, see Gripenberg et al. (1990, Theorem 2.3.1). Thus, the sequences $(\|K^n\|_{L^1([0, T])})_{n \geq 1}$ and $(\|R^n\|_{L^1([0, T])})_{n \geq 1}$ are uniformly bounded in n . Therefore, recalling (iii), the bound in (3.1) for each X^n does not depend on n , yielding (4.3).

• We now move to the *stability* part. Fix $T > 0$ and let X be such that $X^n \Rightarrow X$ on $C([0, T], \mathbb{R})$, possibly along a subsequence. It follows from Jacod and Shiryaev (2003, Theorem VI-4.13) that the sequence of local martingales $(W_{cX^n}^n)_{n \geq 1}$ is tight. Furthermore, any limit process M is a local martingale with quadratic variation cX , which can be written as $M = W_{cX}$ for some Brownian motion W , see Jacod and Shiryaev (2003, Theorem VI-6.26) and Jacod and Shiryaev (2003, Corollary IX-1.19). An application of Skorokhod's representation theorem provides the existence of a common filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ supporting a sequence of copies $(X^n, W_{cX^n}^n)_{n \geq 1}$ that converges uniformly on $[0, T]$, almost surely, along a subsequence, towards a copy of (X, W_{cX}) . Keeping the same notations for these copies, we have

$$\sup_{t \leq T} |X_t^n - X_t| \rightarrow 0, \quad \sup_{t \leq T} |W_{cX_t^n}^n - W_{cX_t}| \rightarrow 0, \quad \mathbb{P} - a.s., \quad \text{as } n \rightarrow \infty. \quad (4.4)$$

Now set $Z = bX + W_{cX}$, fix $t \leq T$ and write

$$\begin{aligned} \int_0^t K^n(t-s)Z_s^n ds - \int_0^t K(t-s)Z_s ds &= \int_0^t K^n(t-s)(Z_s^n - Z_s) ds \\ &\quad - \int_0^t (K^n(t-s) - K(t-s))Z_s ds \\ &= \mathbf{I}_n + \mathbf{II}_n. \end{aligned}$$

Z being continuous, it is bounded almost surely so that $\mathbf{II}_n \rightarrow 0$ as $n \rightarrow \infty$ by virtue of (ii). Moreover,

$$|\mathbf{I}_n| \leq \|K^n\|_{L^1([0,T])} \left(b \sup_{t \leq T} |X_t^n - X_t| + \sup_{t \leq T} |W_{cX_t}^n - W_{cX_t}| \right)$$

which goes to 0 as $n \rightarrow \infty$ thanks to (4.4) and the uniform boundedness of $(\|K^n\|_{L^1([0,T])})_{n \geq 1}$ due to the convergence (i). This shows that $\int_0^t K^n(t-s)Z_s^n ds \rightarrow \int_0^t K(t-s)Z_s ds$. Combined with (iii), we get, after taking the limit in (4.1), that

$$X_t = \lim_{n \rightarrow \infty} X_t^n = G_0(t) + \int_0^t K(t-s)Z_s ds,$$

for all $t \leq T$. Since X is continuous, one can interchange the quantifiers so that the previous identity holds for all $t \leq T$, \mathbb{P} almost surely. Finally, each X^n being non-decreasing, the limit process X is again non-decreasing, which ends the proof. \square

5 Existence for L^1 -kernels

Throughout this section, we let $(K^n)_{n \geq 1}$ denote the sequence of right-shifted kernels

$$K^n = \Delta_{1/n} K \quad n \geq 1.$$

We will prove the existence of solutions for the stochastic Volterra equation (1.1) and the Riccati-Volterra equation (2.13)-(2.14) with L^1 -kernels using a density argument.

5.1 Existence for the stochastic Volterra equation

In preparation for the approximation argument, we first recast the existence results of [Abi Jaber et al. \(2019b,a\)](#) for L^2 -kernels in our framework. We recall the definition of the set \mathcal{G}_K in (2.9).

Lemma 5.1. *Fix $n \geq 1$. Assume that the shifted kernel K^n satisfies (2.6) and (2.7). Let $g_0^n \in \mathcal{G}_{K^n}$ and set $G_0^n = \int_0^\cdot g_0^n(s) ds$. Then, there exists a non-decreasing continuous weak*

solution X^n to (1.1) for the input (G_0^n, K^n) . This solution is given by $X^n = \int_0^\cdot Y_s^n ds$, where Y^n is a nonnegative weak continuous solution to

$$Y_t^n = g_0^n(t) + \int_0^t K^n(t-s)bY_s^n ds + \int_0^t K^n(t-s)\sqrt{cY_s^n}dW_s^n. \quad (5.1)$$

Proof. The existence of a \mathbb{R}_+ -valued continuous weak solution to (5.1) follows from [Abi Jaber and El Euch \(2019a, Theorem A.2\)](#)². Furthermore, this solution satisfies

$$\sup_{t \leq T} \mathbb{E}[|Y_t^n|^p] < \infty, \quad p \geq 1, \quad T > 0.$$

An application of the converse part in [Lemma 2.1](#) ends the proof. \square

Theorem 5.2. *Fix a nonnegative and non-increasing kernel $K \in L_{\text{loc}}^1$ satisfying (2.5) for some C_0, γ_0 . Assume its shifted kernels $K^n = \Delta_{1/n}K$ satisfy (2.6) and (2.7), for each $n \geq 1$. Let $G_0 = \lim_{n \rightarrow \infty} \int_0^\cdot g_0^n(s)ds$ for some functions $g_0^n \in \mathcal{G}_{K^n}$ and assume that G_0 is continuous. Then, there exists a non-decreasing continuous weak solution X to (1.1).*

Proof. Fix $n \geq 1$. An application of [Lemma 5.1](#) yields the existence of a non-decreasing and continuous process X^n solution to (1.1) with the inputs (G_0^n, K^n) , where $G_0^n = \int_0^\cdot g_0^n(s)ds$. Each X^n being non-decreasing, the claimed existence now follows from [Theorem 4.1](#), once we prove that conditions (i)-(iv) are satisfied. To prove (i), we observe that since K is nonnegative, non-increasing and satisfies (2.5), we have

$$\begin{aligned} \int_0^h |K^n(s)|ds + \int_0^T |K^n(s+h) - K^n(s)|ds &\leq \int_0^h |K(s)|ds \\ &\quad + \int_0^T |K(s+1/n+h) - K(s+1/n)|ds \\ &\leq C_0 h^{\gamma_0}. \end{aligned} \quad (5.2)$$

(ii) holds by the L^1 -continuity of the kernel K , see [Brezis \(2010, Lemma 4.3\)](#). (iv) follows from the assumption on G_0 . Finally, to obtain (iii) we first observe that g_0^n is nonnegative, this follows from (2.8) evaluated at $t = 0$ (see [Remark 2.6](#)). Whence, G_0^n is non-decreasing so that $\sup_{t \leq T} G_0^n(t) \leq G_0^n(T)$. The right-hand side is uniformly bounded in n by virtue of the convergence of the sequence $(G_0^n(T))_{n \geq 1}$. The proof is complete. \square

5.2 Existence for the Riccati–Volterra equation

Similarly, we obtain existence of solutions to Riccati–Volterra equations with L^1 -kernels by a density argument. We start by recalling the existence of solutions for the shifted Riccati–Volterra equations (2.13)-(2.14).

²We note that all the assumptions are met there, except for the local Hölder continuity of g_0 . This assumption is only needed to get Hölder sample paths of X , which we do not require here.

Lemma 5.3. *Let $a \in i\mathbb{R}$ and $f : \mathbb{R}_+ \rightarrow \mathbb{C}$ measurable and locally bounded such that $\Re(f) \leq 0$. Assume that the shifted kernels K^n satisfy (2.6) and (2.7), for all $n \geq 1$. Then, for each $n \geq 1$, there exists a continuous solution ψ^n to (2.13)-(2.14) for the kernel K^n such that $\Re(\psi^n(t)) \leq 0$ and*

$$\sup_{t \leq T} |\psi^n(t)| \leq \left(\sup_{t \leq T} |f(t)| + \frac{1}{2}|a|^2 \right) \int_0^T E_b^n(s) ds, \quad (5.3)$$

for any $T > 0$, where $E_b^n = R_b^n/b$ when $b \neq 0$ and $E_0^n = K^n$ with R_b^n the resolvent of the second kind of bK^n .

Proof. Fix $T > 0$ and $n \geq 0$. Since all shifted kernels $(K^n)_{n \geq 1}$ are continuous on $(0, T]$, one readily obtains that K^n is continuous on $[0, T]$ for each $n \geq 1$. Fix $n \geq 1$. An application of [Abi Jaber et al. \(2019b, Lemma 6.3\)](#) yields the existence of a solution $\psi^n \in L^2([0, T], \mathbb{C})$ such that $\Re(\psi^n(t)) \leq 0$, for all $t \leq T$. To argue continuity of ψ^n , we fix $t \leq T$ and $h > 0$ and write

$$\begin{aligned} \psi^n(t+h) - \psi^n(t) &= \int_0^t (K^n(t+h-s) - K^n(t-s)) F(s, \psi^n(s)) ds \\ &\quad + \int_t^{t+h} K^n(t+h-s) F(s, \psi^n(s)) ds. \end{aligned}$$

Using the continuity of K^n on $[0, T]$ and the L^2 integrability of ψ^n an application of the dominated convergence theorem yields that the right hand side goes to 0 as $h \rightarrow 0$. This shows that $\psi^n \in C([0, T], \mathbb{C})$ for all $n \geq 1$. Finally, the bound (5.3) follows from [Abi Jaber and El Euch \(2019b, Corollary C.4\)](#) by noticing that ψ^n solves the linear equation

$$\chi = \int_0^t K^n(t-s) (z(s)\chi(s) + w(s)),$$

with $w(s) = f(s) + \frac{1}{2}a^2$ and $z(s) = (b + a\sqrt{c} + \frac{c}{2}\psi^n(s))$ so that $\Re(z) \leq b$. \square

Theorem 5.4. *Let $a \in i\mathbb{R}$ and $f : \mathbb{R}_+ \rightarrow \mathbb{C}$ measurable and locally bounded such that $\Re(f) \leq 0$. Fix a nonnegative and non-increasing kernel $K \in L^1_{\text{loc}}$ satisfying (2.5) for some C_0, γ_0 . Assume that its shifted kernels $K^n = \Delta_{1/n}K$ satisfy (2.6) and (2.7), for each $n \geq 1$. Then, there exists a continuous solution ψ to (2.13)-(2.14) such that $\Re(\psi(t)) \leq 0$, for all $t \geq 0$.*

Proof. Let $(\psi^n)_{n \geq 1}$ denote the sequence of continuous solutions produced by [Lemma 5.3](#) for the respective shifted kernels $K^n = \Delta_{1/n}K$ and fix $T > 0$.

- First, we prove that the sequence $(\psi^n)_{n \geq 1}$ is relatively compact on $C([0, T], \mathbb{C})$ using the Arzelà–Ascoli theorem. Since $(K^n)_{n \geq 1}$ converges to K in L^1 , the respective sequence of resolvents $(R_b^n)_{n \geq 1}$ converge in L^1 , by virtue of the continuous dependence of the resolvent

on the kernel in L^1 , see [Gripenberg et al. \(1990, Theorem 2.3.1\)](#). Thus, the sequences $(\|K^n\|_{L^1([0,T])})_{n \geq 1}$ and $(\|R_b^n\|_{L^1([0,T])})_{n \geq 1}$ are uniformly bounded in n so that [\(5.3\)](#) implies

$$\sup_{n \geq 1} \sup_{t \leq T} |\psi^n(t)| < \infty. \quad (5.4)$$

Fix $n \geq 1$, $s \leq t \leq T$ and write

$$\begin{aligned} \psi^n(t) - \psi^n(s) &= \int_0^s (K^n(t-u) - K^n(s-u)) F(u, \psi^n(u)) du \\ &\quad + \int_s^t K^n(t-u) F(u, \psi^n(u)) du \\ &= \mathbf{I} + \mathbf{II}. \end{aligned}$$

Recall that the shifted kernels K^n satisfy the condition [\(5.2\)](#) uniformly in n with C_0 and γ_0 . Whence, using the uniform bound [\(5.4\)](#) combined with the boundedness of f we readily get, by the triangle inequality, that

$$|\psi^n(t) - \psi^n(s)| \leq |\mathbf{I}| + |\mathbf{II}| \leq C(t-s)^{\gamma_0},$$

for some constant C independent of n . An application of the Arzelà–Ascoli theorem yields the existence of $\psi \in C([0, T], \mathbb{R})$ such that

$$\sup_{t \leq T} |\psi^n(t) - \psi(t)| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (5.5)$$

In particular, since $\Re(\psi^n) \leq 0$, we have $\Re(\psi(t)) \leq 0$, for all $t \leq T$.

• Second, we show that ψ solves the Riccati–Volterra equation with the kernel K . For this we fix $t \leq T$ and we write

$$\begin{aligned} \psi^n(t) - \int_0^t K(t-s) F(s, \psi(s)) ds &= \int_0^t K^n(t-s) (F(s, \psi^n(s)) - F(s, \psi(s))) ds \\ &\quad + \int_0^t (K^n(t-s) - K(t-s)) F(s, \psi(s)) ds \\ &= \mathbf{1}_n + \mathbf{2}_n. \end{aligned}$$

Clearly $\mathbf{2}_n \rightarrow 0$ by virtue of the L^1 -convergence of the kernels and the boundedness of f and ψ . To argue that $\mathbf{1}_n \rightarrow 0$, it suffices to observe that, for all $s \leq T$,

$$|F(s, \psi^n(s)) - F(s, \psi(s))| \leq C(1 + \sup_{t \leq T} |\psi(t)| + \sup_{t \leq T} |\psi^n(t)|) (\sup_{t \leq T} |\psi^n(t) - \psi(t)|),$$

for a constant C independent of n . It follows from [\(5.4\)](#) and [\(5.5\)](#) that the right hand side goes to 0. Combined with the uniform boundedness of $(\|K^n\|_{L([0,T])})_{n \geq 1}$ we obtain that $\mathbf{1}_n \rightarrow 0$. Combining the above we get that

$$\psi(t) = \lim_{n \rightarrow \infty} \psi^n(t) = \int_0^t K(t-s) F(s, \psi(s)) ds, \quad t \leq T,$$

which ends the proof by arbitrariness of T . □

6 Weak uniqueness and the Fourier–Laplace transform

Fix $a \in \mathbb{C}$, $c, T \geq 0$ and $f : [0, T] \rightarrow \mathbb{C}$ continuous. Throughout this section, we fix G_0 a non-decreasing continuous function and $K \in L^1([0, T], \mathbb{R})$ such that (2.5) holds. We let $\psi \in C([0, T], \mathbb{C})$ denote a solution to the Riccati equation (2.13)–(2.14) and X be a non-decreasing continuous weak solution to (1.1). Define the process V^T :

$$V_t^T = V_0^T - \frac{1}{2} \int_0^t (a + \sqrt{c}\psi(T-s))^2 dX_s + \int_0^t (a + \sqrt{c}\psi(T-s)) dW_{X_s} \quad (6.1)$$

$$V_0^T = \int_0^T F(T-s, \psi(T-s)) dG_0(s). \quad (6.2)$$

We note that the Lebesgue-Stieltjes integrals are well-defined since ψ is continuous and (G_0, X) are of locally bounded variation.

A straightforward application of Itô's Lemma yields that the stochastic exponential $\exp(V^T)$ is a local martingale. It is even a true martingale. This is the object of the following lemma, which extends [Abi Jaber et al. \(2019b, Lemma 7.3\)](#).

Lemma 6.1. *Let $g \in L^\infty(\mathbb{R}_+, \mathbb{R})$ and define*

$$U_t = \int_0^t g(s) dW_{X_s}.$$

Then the stochastic exponential $t \mapsto \exp(U_t - \frac{1}{2}\langle U \rangle_t)$ is a martingale. In particular, $\exp(V^T)$ is a martingale.

Proof. Define $M_t = \exp(U_t - \frac{1}{2}\langle U \rangle_t)$. Since M is a nonnegative local martingale, it is a supermartingale by Fatou's lemma, and it suffices to show that $\mathbb{E}[M_T] \geq 1$ for any $T \in \mathbb{R}_+$. To this end, define stopping times $\tau_n = \inf\{t \geq 0 : X_t > n\} \wedge T$. Then $M^{\tau_n} = M_{\tau_n \wedge \cdot}$ is a uniformly integrable martingale for each n by Novikov's condition, and we may define probability measures \mathbb{Q}^n by

$$\frac{d\mathbb{Q}^n}{d\mathbb{P}} = M^{\tau_n}.$$

By Girsanov's theorem, the process

$$W_{cX_t} - \langle W_{cX}, U_{\tau_n \wedge \cdot} \rangle_t = W_{cX_t} - \int_0^t \mathbf{1}_{\{s \leq \tau_n\}} g(s) \sqrt{c} dX_s$$

is a local martingale under \mathbb{Q}^n with quadratic variation cX , and we have

$$X_t = G_0(t) + \int_0^t K(t-s) Z_s^n ds,$$

where Z^n is a continuous semimartingale under \mathbb{Q}^n with characteristics

$$\left(bX - \int_0^\cdot \mathbf{1}_{\{s \leq \tau_n\}} g(s) \sqrt{c} dX_s, cX, 0 \right).$$

Let γ be the constant appearing in (2.5) and let $p > 2$. Observe that the first characteristic of Z^n satisfies a linear growth condition in X , uniformly in (t, ω) . Therefore, due to Lemma 3.2 and Remark 3.3, we have the moment bound

$$\sup_{t \leq T} \mathbb{E}_{\mathbb{Q}^n} [|X_t|^p] \leq \kappa$$

for some constant κ that does not depend on n . For any real-valued function f , write

$$|f|_{C^{0,\alpha}(0,T)} = \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{|t - s|^\alpha}$$

for its α -Hölder semi-norm. We then get

$$\begin{aligned} \mathbb{Q}^n(\tau_n < T) &\leq \mathbb{Q}^n\left(\sup_{t \leq T} X_t > n\right) \\ &\leq \mathbb{Q}^n\left(G_0(T) + |X|_{C^{0,0}(0,T)} > n\right) \\ &\leq \left(\frac{1}{n - G_0(T)}\right)^p \mathbb{E}_{\mathbb{Q}^n} \left[|X|_{C^{0,0}(0,T)}^p\right] \\ &\leq \left(\frac{1}{n - G_0(T)}\right)^p \kappa' \end{aligned}$$

for a constant κ' that does not depend on n , using the fact that G_0 is non-decreasing for the second inequality and Lemma 3.2 with $\alpha = 0$ for the last inequality. We deduce that

$$\mathbb{E}_{\mathbb{P}} [M_T] \geq \mathbb{E}_{\mathbb{P}} [M_T \mathbf{1}_{\{\tau_n = T\}}] = \mathbb{Q}^n(\tau_n = T) \geq 1 - \left(\frac{1}{n - G_0(T)}\right)^p \kappa',$$

and sending n to infinity yields $\mathbb{E}_{\mathbb{P}}[M_T] \geq 1$. This completes the proof. \square

Lemma 6.2. *Assume that the shifted kernels $\Delta_h K$ are in $L^2([0, T], \mathbb{R})$, for all $h > 0$. Set $(G_t)_{t \geq 0}$ and $(g_t)_{t \geq 0}$ as in (2.10)–(2.11). Then, the process $(V_t^T)_{0 \leq t \leq T}$ defined by (6.1)–(6.2) satisfies*

$$V_t^T = \int_0^t f(T-s) dX_s + aW_{X_t} + \int_t^T F(T-s, \psi(T-s)) dG_t(s), \quad (6.3)$$

for all $0 \leq t \leq T$.

Proof. We fix $t \leq T$, $h > 0$ and we define

$$\begin{aligned} X_t^h &= G_0(t) + \int_0^t \Delta_h K(t-s) Z_s ds, \\ g_t^h(s) &= \int_0^t \Delta_h K(s-u) dZ_u, \\ \psi^h(t) &= \int_0^t \Delta_h K(t-s) F(s, \psi(s)) ds, \end{aligned}$$

where we recall that $Z = bX + W_{cX}$ and F is given by (2.14). We stress that the right-hand sides of all three quantities are defined from X and ψ and do not depend on X^h or ψ^h ; X_t^h and g_t^h are well-defined as Itô integrals since $\Delta_h K \in L^2([0, T], \mathbb{R})$.

Step 1. Convergence of X^h, g^h, ψ^h . It follows from the boundedness of ψ , f and $Z(\omega)$, and the L^1 -continuity of the kernel K that

$$\sup_{s \leq T} |\psi^h(s) - \psi(s)| \rightarrow 0, \quad |X_t^h - X_t| \rightarrow 0, \quad \mathbb{P} - a.s. \quad (6.4)$$

as $h \rightarrow 0$. Set

$$G_t^h(s) = G_0(s) + \int_t^{s \vee t} g_t(u) du.$$

By invoking stochastic Fubini's theorem, justified by the L^2 -integrability of $\Delta_h K$, we get, for all $s > t$,

$$G_t^h(s) - G_t(s) = \int_0^s (\Delta_h K(u) - K(u)) (Z_{t \wedge (s-u)} - Z_{t-u}) du.$$

The boundedness of $Z(\omega)$ and the L^1 -continuity of the kernel K , lead to

$$G_t^h(s) \rightarrow G_t(s), \quad \mathbb{P} - a.s. \quad (6.5)$$

as $h \rightarrow 0$, for all $s \in (t, T]$.

Step 2. Proving (6.3). An application of stochastic Fubini's theorem, see [Veraar \(2012, Theorem 2.2\)](#) – justified by the L^2 -integrability of $\Delta_h K$, the boundedness of ψ , f and $X(\omega)$

– yields

$$\begin{aligned}
\int_0^t \psi^h(T-s) dZ_s &= \int_0^t \left(\int_0^{T-s} F(u, \psi(u)) \Delta_h K(T-s-u) du \right) dZ_s \\
&= \int_0^T F(u, \psi(u)) \left(\int_0^{t \wedge (T-u)} \Delta_h K(T-u-s) dZ_s \right) du \\
&= \int_0^{T-t} F(u, \psi(u)) \left(\int_0^t \Delta_h K(T-u-s) dZ_s \right) du \\
&\quad + \int_{T-t}^T F(u, \psi(u)) \left(\int_0^{T-u} \Delta_h K(T-u-s) dZ_s \right) du \\
&= \int_t^T F(T-s, \psi(T-s)) g_t^h(s) ds \\
&\quad + \int_0^t F(T-s, \psi(T-s)) d \left(X_s^h - G_0(s) \right) \\
&= \int_t^T F(T-s, \psi(T-s)) d \left(G_t^h(s) - G_0(s) \right) \\
&\quad + \int_0^t F(T-s, \psi(T-s)) d \left(X_s^h - G_0(s) \right).
\end{aligned}$$

where we used in the fourth identity that $(X_s^h - G_0(s)) = \int_0^s (\int_0^r \Delta_h K(r-u) dZ_u) dr$, due to Lemma 2.1 since $\Delta_h K \in L_{loc}^2$, for $h > 0$. Recalling (6.4)-(6.5) and sending $h \rightarrow 0$ in the previous identity yields, by invoking the dominated convergence for the left-hand side and the Portmanteau theorem on the measures (dX^n, dG_t^n) for the right-hand side (see Daley and Vere-Jones (2003, Theorem A2.3.II)), we obtain that

$$\begin{aligned}
\int_0^t \psi(T-s) dZ_s &= \int_t^T F(T-s, \psi(T-s)) dG_t(s) + \int_0^t F(T-s, \psi(T-s)) dX_s \\
&\quad - \int_0^T F(T-s, \psi(T-s)) dG_0(s).
\end{aligned} \tag{6.6}$$

Using (2.14), we write

$$\begin{aligned}
-\frac{1}{2} \int_0^t (a + \sqrt{c} \psi(T-s))^2 dX_s &= \int_0^t f(T-s) dX_s + b \int_0^t \psi(T-s) dX_s \\
&\quad - \int_0^t F(T-s, \psi(T-s)) dX_s.
\end{aligned}$$

This shows that V^T given by (6.1) can be re-expressed in the form

$$V_t^T = V_0^T + \int_0^t f(T-s) dX_s + \int_0^t \psi(T-s) dZ_s - \int_0^t F(T-s, \psi(T-s)) dX_s + aW_t.$$

Plugging (6.6) in the previous expression and recalling (6.2) yields (6.3). \square

Theorem 6.3. *Fix G_0 continuous and non-decreasing and fix a nonnegative and non-increasing kernel $K \in L^1_{\text{loc}}$ satisfying (2.5). Assume that its shifted kernels $K^n = \Delta_{1/n}K$ satisfy (2.6) and (2.7), for each $n \geq 1$. Let X be a solution to (1.1) for the input (G_0, K) and set $(G_t)_{t \geq 0}$ and $(g_t)_{t \geq 0}$ as in (2.10)–(2.11). Let $a \in \mathbb{C}$, and $f : [0, T] \rightarrow \mathbb{C}$ continuous satisfying (2.12). Then, the Fourier–Laplace transform is given by*

$$\mathbb{E} \left[\exp \left(\int_0^T f(T-s) dX_s + aW_{X_T} \right) \middle| \mathcal{F}_t \right] = \exp(V_t^T), \quad t \leq T, \quad (6.7)$$

where V^T is given by (6.3) and $\psi \in C([0, T], \mathbb{C})$ is a solution to the Riccati–Volterra equation (2.13)–(2.14) such that $\Re(\psi_t) \leq 0$, for all $t \leq T$. In particular, weak uniqueness holds for (1.1).

Proof. The existence of a solution to the Riccati–Volterra equations follows from Theorem 5.4. Lemma 6.2 yields that (6.3) holds, so that the terminal value of V^T reads

$$V_T^T = \int_0^T f(T-s) dX_s + aW_{X_T}.$$

This yields (6.7) by the true martingality of $\exp(V^T)$ obtained in Lemma 6.1. Finally, the law of X is determined by the Laplace transform of the finite-dimensional marginals $(X_{t_1}, \dots, X_{t_m})$:

$$\mathbb{E} \left[\exp \left(\sum_{i=1}^m v_i X_{t_i} \right) \right],$$

with $t_1 < t_2 < \dots < t_m < T$ and $v_i \in \mathbb{C}$ such that $\Re(v_i) \leq 0$, $i = 1, \dots, m$, $m \in \mathbb{N}$. Uniqueness thus follows since these Laplace transforms can be approximated by the quantities $\mathbb{E}[\exp(\int_0^T f(T-s) dX_s)]$ as f ranges through all continuous nonpositive functions and T ranges through \mathbb{R}_+ . \square

7 Application: Hyper-rough Volterra Heston models

In this section, we apply our main result, Theorem 2.5, to the class of hyper-rough Volterra Heston models where the log-price S has the following dynamics

$$\log S_t = \log S_0 - \frac{1}{2}X_t + B_{X_t}, \quad S_0 > 0, \quad (7.1)$$

$$X_t = G_0(t) + \int_0^t K(t-s) (bX_s + W_{cX_s}) ds, \quad (7.2)$$

with

$$B = \rho W + \sqrt{1 - \rho^2} W^\perp, \quad (7.3)$$

and (W, W^\perp) a two dimensional Brownian motion and $\rho \in [-1, 1]$. To ease presentation, we will assume throughout this section that K is proportional to the fractional kernel:

$$K(t) = K_1(t)K_H(t) \quad (7.4)$$

where

$$K_H(t) = \frac{t^{H-1/2}}{\Gamma(H + 1/2)}, \quad t > 0,$$

for some $H \in (-1/2, 1/2]$ and K_1 is a non-singular completely monotone kernel on $[0, \infty)$, e.g. $K_1 \equiv 1$ or $K_1(t) = e^{-\eta t}$, for some $\eta > 0$. Under such specification, the assumptions of Theorem 2.5 needed on the kernel are satisfied due to Examples 2.2 and 2.3.

Furthermore, we assume that

$$G_0(t) = \int_0^t g_0(s) ds, \quad t \geq 0, \quad \text{for some } g_0 \in \mathcal{G}_K, \quad (7.5)$$

recall Example 2.4 for concrete specifications of g_0 .

The following remark shows that X can be thought of as the ‘integrated variance’ process.

Remark 7.1. *Assume that $H \in (0, 1/2]$, then $K \in L_{loc}^2$ and it follows from Lemma 2.1 that $X_t = \int_0^t V_s ds$ where (S, V) is a rough Volterra Heston model in the terminology of Abi Jaber et al. (2019b, Section 7); El Euch and Rosenbaum (2019) satisfying*

$$\begin{aligned} d \log S_t &= -\frac{1}{2} V_t dt + \sqrt{V_t} d\tilde{B}_t, \quad S_0 > 0, \\ V_t &= g_0(t) + \int_0^t K(t-s) b V_s ds + \int_0^t K(t-s) \sqrt{c V_s} d\tilde{W}_s. \end{aligned}$$

If $H \in (-1/2, 0)$, K_H is no longer in L_{loc}^2 . In this case, not only Fubini’s interchange breaks down, but it can also be shown that X is nowhere differentiable almost surely, see Jusselin and Rosenbaum (2018, Proposition 4.6). In this case, one cannot really make sense of the spot variance V and is stuck with the ‘integrated variance’ formulation (7.2), justifying the appellation hyper-rough for such equations.

We are now in place to state the existence and uniqueness of a solution to (7.1)–(7.2) together with the joint Fourier–Laplace transform of $(\log S, X)$. We say that (7.1)–(7.2) admits a weak solution if there exists a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ supporting two continuous and adapted processes (S, X) and a two dimensional Brownian motion (W, W^\perp) with X non-decreasing such that (7.1)–(7.2) hold \mathbb{P} –almost surely.

Theorem 7.2. *Let K be as in (7.4), G_0 as in (7.5) and fix $S_0 > 0$. Then, (7.1)–(7.2) admits a unique weak continuous solution (S, X) such that X is non-decreasing and S is positive. Furthermore, for any $u_1 \in i\mathbb{R}$ and $u_2 \in \mathbb{C}$ satisfying $\Re(u_2) \leq 0$, the joint Fourier–Laplace transform of $(\log S, X)$ is given by*

$$\mathbb{E} \left[\exp(u_1 \log S_T + u_2 X_T) \mid \mathcal{F}_t \right] = \exp \left(u_1 \log S_t + u_2 X_t + \int_t^T F(u_1, u_2, \psi(T-s)) dG_t(s) \right)$$

for all $T \geq 0$, where G_t is given by (2.10) and ψ solves the fractional Riccati–Volterra equation

$$\psi(t) = \int_0^t K(t-s) F(u_1, u_2, \psi(s)) ds, \quad t \geq 0, \quad (7.6)$$

$$F(u_1, u_2, u_3) = \frac{1}{2}(u_1^2 - u_1) + u_2 + \rho\sqrt{c}u_1u_3 + \frac{c}{2}u_3^2. \quad (7.7)$$

Proof. We first note that (G_0, K) satisfy the assumptions of Theorem 2.5 thanks to Examples 2.2–2.3 and Remark 2.6.

Step 1. We prove the existence and uniqueness. An application of Theorem 2.5 yields the existence and uniqueness of a continuous non-decreasing weak solution X to (7.2) on some filtered probability space $(\Omega^X, \mathcal{F}^X, (\mathcal{F}_t^X)_{t \geq 0}, \mathbb{P}^X)$ with a Brownian motion W adapted to \mathcal{F}^X . Up to an eventual extension of the probability space, we let W^\perp denote another Brownian motion independent of \mathcal{F}^X and set $B = \rho\bar{W} + \sqrt{1 - \rho^2}\bar{W}^\perp$ and we let $\mathcal{F} = \mathcal{F}^X \vee \mathcal{F}^\perp$, where \mathcal{F}^\perp is generated by W^\perp . The extended probability space is denoted by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Then,

$$S_t = S_0 \exp \left(B_{X_t} - \frac{1}{2} X_t \right), \quad t \geq 0,$$

is the unique continuous positive solution of (7.1).

Step 2. We derive the Fourier-Laplace transform. It suffices to prove that the Fourier-Laplace transform

$$L_t = \mathbb{E} \left[\exp(u_1 \log S_T + u_2 X_T) \mid \mathcal{F}_t \right] \quad (7.8)$$

can be written as

$$L_t = \exp(u_1 \log S_t + u_2 X_t) \mathbb{E} \left[\exp \left(\int_t^T f(T-s) dX_s + a(W_{X_T} - W_{X_t}) \right) \mid \mathcal{F}_t \right] \quad (7.9)$$

where

$$f(t) = \frac{1}{2}(u_1^2 - u_1) + u_2 - \frac{a^2}{2} \quad \text{and} \quad a = u_1\rho.$$

Indeed, if this the case, the Riccati-Volterra equations (2.13)–(2.14) reduce to (7.6)–(7.7) and the the claimed expression for the Fourier-Laplace transform together with the existence of the corresponding solution ψ follow from Theorem 2.5, since $\Re(a) = 0$ and

$$\Re(f) = \Re(u_2) + \frac{\Re(u_1^2)}{2}(1 - \rho^2) \leq 0,$$

since $\Re(u_2) \leq 0$, $u_1 \in i\mathbb{R}$ and $\rho \in [-1, 1]$. It remains to prove (7.9) by means of a projection argument. For this, we fix $t \leq T$ and we write the variation of (7.1) between t and T , using (7.3) to get

$$\log S_T = \log S_t - \frac{1}{2} \int_t^T dX_s + \rho(W_{X_T} - W_{X_t}) + \sqrt{1 - \rho^2} (W_{X_T}^\perp - W_{X_t}^\perp). \quad (7.10)$$

We then observe that

$$M_t := \mathbb{E} \left[\exp \left(u_1 \sqrt{1 - \rho^2} (W_{X_T}^\perp - W_{X_t}^\perp) \right) \mid \mathcal{F}_t \vee \mathcal{F}^X \right] = \exp \left(\frac{u_1^2}{2} (1 - \rho^2) \int_t^T dX_s \right) \quad (7.11)$$

so that, using successively (7.10), the tower property of the conditional expectation and the fact that X and W are \mathcal{F}^X -measurable, L_t given by (7.8) satisfies

$$\begin{aligned} L_t &= \mathbb{E} \left[\exp(u_1 \log S_T + u_2 X_T) \mid \mathcal{F}_t \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\exp(u_1 \log S_T + u_2 X_T) \mid \mathcal{F}_t \vee \mathcal{F}^X \right] \mid \mathcal{F}_t \right] \\ &= \exp(u_1 \log S_t + u_2 X_t) \mathbb{E} \left[\exp \left((u_2 - \frac{u_1}{2}) \int_t^T dX_s + \rho u_1 (W_{X_T} - W_{X_t}) \right) M_t \mid \mathcal{F}_t \right] \end{aligned}$$

leading to (7.9) due to (7.11). This ends the proof. \square

In particular, for $t = 0$ and $S_0 = 1$, we have $X_0 = 0$, $\log S_0 = 0$ and $dG_0(s) = g_0(s)ds$ so that the unconditional Fourier-Laplace transform reads

$$\mathbb{E} [\exp(u_1 \log S_T + u_2 X_T)] = \exp \left(\int_0^T F(u_1, u_2, \psi(T-s)) g_0(s) ds \right).$$

If in addition $g_0(t) = x_0 + \theta \int_0^t K(s) ds$, for some $x_0, \theta \geq 0$ (recall Example 2.4), then, Fubini's theorem leads to

$$\int_0^T F(u_1, u_2, \psi(T-s)) g_0(s) ds = x_0 \int_0^T F(u_1, u_2, \psi(s)) ds + \theta \int_0^T \psi(s) ds$$

so that

$$\mathbb{E} [\exp(u_1 \log S_T + u_2 X_T)] = \exp \left(x_0 \int_0^T F(u_1, u_2, \psi(s)) ds + \theta \int_0^T \psi(s) ds \right).$$

Remark 7.3. Using Theorem 4.1, one can prove the convergence of the multifactor Markovian approximations designed in [Abi Jaber \(2019\)](#); [Abi Jaber and El Euch \(2019b\)](#) towards the hyper-rough Heston model, where the kernel K_H is approximated by a suitable weighted sum of exponentials $K^n(t) = \sum_{i=1}^n c_i^n e^{-\gamma_i^n t}$. These approximations are therefore still valid for non-positive values of the Hurst index $H \in (-1/2, 0]$, which would allow the simulation of the process X and the numerical approximation of the Riccati–Volterra equations, we refer to the aforementioned articles for more details.

A Catalytic super–Brownian motion and its local occupation time

In this section, we sketch a rigorous derivation of equation (1.7) satisfied by the local occupation time X given by (1.5) formally derived in the introduction. We will make use of the notation $\langle \mu, \phi \rangle$ to denote the quantity $\int_{\mathbb{R}} \mu(dx) \phi(x)$.

We recall that the super–Brownian motion with a single point catalyst \bar{Y} solves the following martingale problem

$$\langle \bar{Y}_t, \phi \rangle = \langle \bar{Y}_0, \phi \rangle + \frac{1}{2} \int_0^t \langle \bar{Y}_s, \Delta \phi \rangle ds + \phi(0) Z_t,$$

where $\Delta = \partial^2/\partial x^2$, ϕ is a suitable test function and Z is a continuous martingale with quadratic variation

$$\langle Z \rangle_t = X_t,$$

where X is the local occupation time defined by (1.5), see [Dawson and Fleischmann \(1994, Theorem 1.2.7\)](#).

In order to make the link with stochastic Volterra equations, we first reformulate the martingale problem in its ‘mild form’.

Lemma A.1. Assume that $\psi \in C^2(\mathbb{R}, \mathbb{R})$ has a Gaussian decay, that is $\sup_{x \in \mathbb{R}} |\psi(x)| e^{cx^2} < \infty$, for some constant c . Then,

$$\langle \bar{Y}_t, \psi \rangle = \langle S_t \bar{Y}_0, \psi \rangle + \int_0^t (S_{t-s} \psi)(0) dZ_s,$$

where

$$(S_t \mu)(x) = \int_{\mathbb{R}} p_t(x-y) \mu(dy) \quad \text{and} \quad p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right), \quad x \geq 0.$$

Sketch of proof. Let $\xi : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable function and set $\phi_t(x) = \xi(t)\phi^0(x)$ for some C^2 function ϕ^0 having a Gaussian decay. An application of Itô's Lemma gives

$$\begin{aligned} d\langle \bar{Y}_t, \phi^0 \rangle \xi(t) &= \xi(t) d\langle \bar{Y}_t, \psi \rangle + \langle \bar{Y}_t, \phi^0 \rangle \xi'(t) dt \\ &= \langle \bar{Y}_t, \frac{1}{2} \Delta \phi_t + \partial_t \phi_t \rangle dt + \phi_t(0) dZ_t. \end{aligned}$$

Thus,

$$\langle \bar{Y}_t, \phi_t \rangle = \langle \bar{Y}_0, \phi_0 \rangle + \int_0^t \langle \bar{Y}_s, \frac{1}{2} \Delta \phi_s + \partial_t \phi_s \rangle ds + \int_0^t \phi_s(0) dZ_s. \quad (\text{A.1})$$

Fix $t \geq 0$ and consider $\phi_s = S_{t-s}\psi$ for all $s \in [0, t]$. Noticing that $\phi_t = \psi$ and $\partial_t \phi_s = -\frac{1}{2} \Delta \phi_s$, the claimed identity follows from (A.1) with this specific test function combined with a density argument. \square

For each $\varepsilon > 0$, let $p_\varepsilon : x \rightarrow (2\pi\varepsilon)^{-1/2} \exp(-x^2/(2\varepsilon))$ be Gaussian density approximations of the dirac mass at 0. It follows from Lemma A.1 that

$$\langle \bar{Y}_t, p^\varepsilon \rangle = \langle S_t \bar{Y}_0, p^\varepsilon \rangle + \int_0^t (S_{t-s} p^\varepsilon)(0) dZ_s.$$

Integrating both sides with respect to time and invoking stochastic Fubini's theorem, see Lemma 2.1, leads to

$$\int_0^t \langle \bar{Y}_s, p^\varepsilon \rangle ds = \int_0^t \langle S_s \bar{Y}_0, p^\varepsilon \rangle ds + \int_0^t (S_{t-s} p^\varepsilon)(0) Z_s ds.$$

Sending $\varepsilon \rightarrow 0$ yields

$$X_t = \lim_{\varepsilon \rightarrow 0} \int_0^t \langle \bar{Y}_s, p^\varepsilon \rangle ds = \int_0^t (S_s \bar{Y}_0)(0) ds + \int_0^t p_{t-s}(0) Z_s ds,$$

showing that X solves (1.7) with the function $g_0(t) = (S_t \bar{Y}_0)(0)$.

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