



# **RADON TRANSFORM ON CERTAIN RIEMANNIAN SPACES AND DIOPHANTINE INTEGRAL GEOMETRY**

Ahmed Abouelaz

## **► To cite this version:**

Ahmed Abouelaz. RADON TRANSFORM ON CERTAIN RIEMANNIAN SPACES AND DIOPHANTINE INTEGRAL GEOMETRY. 2020. <hal-02412693>

**HAL Id: hal-02412693**

**<https://hal.science/hal-02412693v2>**

Preprint submitted on 9 Feb 2020

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RADON TRANSFORM ON  
CERTAIN RIEMANNIAN  
SPACES AND DIOPHANTINE  
INTEGRAL GEOMETRY

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Univh2c. Fac. Sc. Math. Casablanca. Morocco. (2016).

**Dépot légal: 2017MO5284**

**ISBN: 978-9954-9802-0-0**

# Preface

This book contains two parts:

Part 1: Radon transforms on some Riemannian spaces.

Part 2: Diophantine integral geometry in the lattice  $\mathbb{Z}^n$ .

Our work was focused on study of the Radon transform on several spaces: Euclidean space  $\mathbb{R}^n$ , sphere  $\mathcal{S}^d$ , the  $n$ -dimensional torus  $\mathbb{T}^n$ , lattice  $\mathbb{Z}^n$  and Damek-Ricci space ( $NA$ -spaces).

The Paley-Wiener theorems, Paley-Wiener-Schwartz theorems, inversion formulas for the Radon transforms, support theorems for the Radon transforms, Plancherel formulas for the Radon transform, are discussed in details in these spaces.

The complex analog of the Radon transform is known as the Penrose transform. The Penrose transform be considered in the forthcoming book. The Radon transform is widely applicable to tomography, the creation of an image from the projection data associated with cross -selectional scans of an object.

It is known that the Radon transform has many applications precisely in geology, astronomy, medecine and physics.

We study also the discrete Radon transform and the d-plane Radon transform on the lattice  $\mathbb{Z}^n$ . Precisely

We characterize the image of exponential type functions under the discrete Radon transform  $R$  on the lattice  $\mathbb{Z}^n$  of the Euclidean space  $\mathbb{R}^n$ , ( $n \geq 2$ ). We also establish the generalization of Volberg' s uncertainly principle on  $\mathbb{Z}^n$ , which is proved by means of this characterization.

It is noteworthy that this work is the fruit of several papers of the author and also by his collaborators (see the references of this book).

I am grateful to Professor François Rouvière for its advices and suggestions. Many thanks to Professor Abdelbaki Attioui for the effort who provided for the page layout and presentation of this book.

Casablanca, Morocco

*Ahmed ABOUELAZ*



***I dedicate this work***

***to***

*Professor Sigurdur Helgason*  
*Professor François Rouvière*

***to my family***

*Saloua Abouelaz*  
*Farid Naima*

***and to the memory of***

*My father Mohamed Abouelaz*  
*My mother Zaari Alia*  
*Professor André Cerezo.*  
*Professeur Ahmed Intissar*



# Notations and conventions

- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$
- $(-\mathbb{N}) = \{\dots, -3, -2, -1, 0\}$
- $\mathbb{Z} = \mathbb{N} \cup (-\mathbb{N})$
- $\mathbb{R}$  is the real line
- $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$ .
- $\mathbb{C}$  is the complex line, that is:  $\mathbb{C} = \{a + ib \mid (a, b) \in \mathbb{R}^2\}$ .
- $\mathcal{S}(\mathbb{Z}^n)$  is the space consisting of all complex-valued rapidly decreasing  $f$  defined on  $\mathbb{Z}^n$  and called the Schwartz space.
- $\mathbb{C}^{(\mathbb{Z}^n)}$  is the subspace of  $\mathcal{S}(\mathbb{Z}^n)$  consisting of all complex-valued functions  $f$  defined on  $\mathbb{Z}^n$  with finite support.
- $l^p(\mathbb{Z}^n)$ ,  $1 \leq p < +\infty$ , be the space of all complex-valued functions  $f$  on  $\mathbb{Z}^n$  such that  $\sum_{m \in \mathbb{Z}^n} |f(m)|^p < +\infty$ .
- $l^\infty(\mathbb{Z}^n)$  be the space of all complex-valued functions  $f$  on  $\mathbb{Z}^n$  such that  $\sup_{m \in \mathbb{Z}^n} |f(m)| < +\infty$ .
- $W$  is a Lie group and  $\underline{w}$  is its Lie algebra.
- $\delta_{j,k}$  is the Kronecker delta:  $\delta_{j,k} = 0$  for  $j \neq k$  and  $\delta_{j,j} = 1$ .
- $\delta_x$  the delta- measure at the point  $x$ :  $\delta_x(\phi) = \phi(x)$ .
- If  $G$  is a group, we denote by  $\hat{G}$  the dual of  $G$ , i.e the set of all continuous unitary representations of  $G$ .
- $D(W) = C_c^\infty(W)$  and  $D(\mathbb{R}) = C_c^\infty(\mathbb{R})$  are spaces of smooth compactly supported functions.
- $C_c(W)$  is the space of continuous functions on  $W$  with compact support.
- $C_0(W)$  is the space of continuous functions on  $W$  vanishing at infinity.
- $\langle \cdot, \cdot \rangle$  denotes the distributional duality.
- $M(w) = \sup(|w|, |w^{-1}|)$ , where  $|w| = \sup\{|wn| \mid n \in N\}$ ,  $wn$  is the action of the group  $W$  on  $N$ . Only in the Chapter 2,  $N$  is a real vector space of finite dimensional and  $n$  is an element of  $N$ .
- $\mathcal{S}^d$  is the unit sphere of dimensional  $d$  and  $\mathcal{SO}(d+1)$  be the orthogonal special group.
- $V_{n,k} = \mathcal{SO}(n) / \mathcal{SO}(n-k)$  the Steifel variety.

- $GL(n, \mathbb{C})$  the set of all complex matrices of order  $n$  with nonzero determinant.
- $SL(n, \mathbb{C})$  the set of all complex matrices of order  $n$  with determinant equal to 1.
- $SL_*(n, \mathbb{Z})$  the special linear group of degree  $n$  over a ring  $\mathbb{Z}$ , is the set of all  $n \times n$  matrices with determinant equal  $\pm 1$ .
- $d(a)$  the greatest common divisor of the the integers  $a_1, \dots, a_n$ .
- $\mathcal{P}$  the set of all element  $a = (a_1, \dots, a_n) \in \mathbb{Z}^n \setminus \{0\}$  such that  $d(a) = 1$ .
- Let  $G$  be a group and  $K$  is the subgroup of  $G$ ,  $K$  acts on  $G$ . If this action is transitive, then  $G/K$  is homogenous space: example the Steifel variety.
- $R_s$  the spherical Radon transform on the sphere  $\mathcal{S}^d$ .
- The Spherical Fourier transform associated to  $R_s$  is given by

$$\tilde{f}(n) = \int_0^\pi \cos[(n + \lambda)t] R_s f(t) dt, \quad \text{for all } n \in \mathbb{N}.$$

- $B$  being the Euler Beta function.
- $C_n^\lambda(\cos t)$  is the Gegenbauer polynomial.
- $C^\infty(G \parallel K)$  the space of all complex-valued functions defined on  $G = \mathcal{SO}(d+1)$  which are  $K = \mathcal{SO}(d)$  bi-invariants and  $C^\infty$  on  $G$ .
- The Spherical Fourier transform on  $\mathcal{S}^d$  can be written as

$$\tilde{f}(n) = \frac{n!}{(2\lambda)!} B(\lambda + 1/2, 1/2) \int_0^\pi f(a_t) C_n^\lambda(\cos t) (\sin t)^{2\lambda} dt,$$

for all  $f \in C^\infty(G \parallel K)$ ,

- $R_{\alpha, \beta}$  is the generalized the spherical Radon transform on the sphere  $\mathcal{S}^d$ .
- The generalized spherical Fourier transform, associated to  $R_{\alpha, \beta}$ , on compact symmetric space is defined by

$$\tilde{f}(n) = \frac{1}{\Gamma(\alpha + 1)} \int_0^\pi f(t) \gamma_n^{(\alpha, \beta)}(\cos t) \cdot (\sin t/2)^{2\alpha+1} \cos(t/2)^{2\beta+1} dt,$$

- where  $n = 1, 2, \dots$ , and  $\gamma_n^{(\alpha, \beta)}(\cos t) = P_n^{(\alpha, \beta)}(\cos t) / P_n^{(\alpha, \beta)}(1)$   $P_n^{(\alpha, \beta)}(\cos t)$  the Jacobi polynomial with  $\alpha > \beta > -\frac{1}{2}$ .
- We define the classical Radon transform as follows:

$$R_c f(H(\omega, t)) = \int_{H(\omega, t)} f(x) d\mu(x),$$

where  $H(\omega, t) = \{x \in \mathbb{R}^n \mid \omega x = t\}$  is the hyperplane of  $\mathbb{R}^n$ .  $\omega = (\omega_1, \dots, \omega_n) \in \mathcal{S}^{n-1}$  and  $\omega x = x_1 \omega_1 + \dots + x_n \omega_n$ .

- Let  $G$  a Lie group and  $\underline{g}$  its Lie algebra, we denote  $\mathcal{U}(\underline{g})$  the universal enveloping algebra.
- $D^{\underline{g}}(G)$  be the algebra of the invariants differential operators by the left translation.

- Let  $N$  be the real vector space of the finite dimensional,  $W$  a Lie group and  $G = W \odot N$  the semidirect product, we define the representation  $\Pi_\lambda$  of  $G$  as follows

$$\Pi_\lambda(g) \varphi(w) = \lambda(wnw^{-1}) \varphi(wv) \quad \text{for all } (v, w) \in W^2, g = nv, w \in W$$

- $N^*$  is the complexified of the dual  $N'$  (the set of the characters of  $N$ ); then  $\Pi_\lambda = \text{Ind}_N^G \lambda$ .
- $\mathcal{E}'(G)$  the space of all distributions with compact support on  $G = W \odot N$ .
- For  $\lambda \in N^*$  and  $S \in \mathcal{E}'(G)$ , we define the Fourier-Laplace transform  $\mathcal{FS}(\lambda)$  as operator with dense domain on  $H$  by

$$\mathcal{FS}(\lambda) \varphi(w) = \langle S(g), \Pi_\lambda(g) \varphi(w) \rangle, \quad \text{with } \varphi \in H^\infty \subset H \text{ and } g = nw \in G = W \odot N.$$

- $H^\infty$  is the Garding space included in the Hilbert space  $H$ .
- In sections 2.7 and 3.2, we define the classical Fourier transform as follows

$$\widehat{\psi}(\lambda) = \int_{N'} \psi(n) \lambda(-n) dn \quad \text{for all } \psi \in C_c^\infty(N'),$$

where  $C_c^\infty(N')$  is the space of all functions infinitely differential functions on  $N'$  with compact support.

- $\mathcal{F}_I$  is the one-dimensional Fourier transform given by

$$\mathcal{F}_1 f(t) = \int_{\mathbb{R}} f(x) \exp(-2i\pi xt) dx, \quad \text{for all } t \in \mathbb{R}.$$

- The analogue of the Helgason's operator is defined by

$$\mathcal{F}_1 \left( (-\square_y)^k \varphi \right) (\omega, s) = |s|^{2k} \mathcal{F}_1 \varphi(\omega, s) \exp(s\omega y), \quad \text{for all } y \in \mathbb{R}^n$$

and  $(\omega, s) \in S^{n-1} \times \mathbb{R}$ ,  $\varphi$  belongs to  $\mathcal{S}(S^{n-1} \times \mathbb{R})$ .

- We denote  $\mathcal{F}_c$  the classical Fourier transform defined by

$$\mathcal{F}_c f(\lambda) = \int_{\mathbb{R}^n} f(x) \exp(-2i\pi \lambda x) dx \quad \text{for all } (\lambda, f) \in \mathbb{R}^n \times C_c^\infty(\mathbb{R}^n).$$

- We define the Cauchy-Lipshitz's class  $Lip(\alpha, \mathbb{Z}^n, 2)$  by

$$Lip(\alpha, \mathbb{Z}^n, 2) = \left\{ f \in l^1(\mathbb{Z}^n) \mid \|\mathcal{F}f(x+h) - \mathcal{F}f(x)\|_{L^2(\mathbb{T}^n)} = O(|h|^\alpha), h \rightarrow 0 \right\}$$

- For the  $n$ -dimensional torus, we define the Cauchy-Lipshitz's class as follows

$$Lip(\alpha, \mathbb{T}^n, 2) = \left\{ f \in L^2(\mathbb{T}^n) \mid \|f(x+h) - f(x)\|_{L^2(\mathbb{T}^n)} = O(|h|^\alpha), h \rightarrow 0 \right\}.$$

- In section 3.5, we denote  $\widehat{f}(\pi)$  the Fourier operator of the group  $G$ , where  $\pi \in \widehat{G}$  ( $\widehat{G}$  being the unitary dual of  $G$ , i.e the set of all continuous unitary irreducible representations of the group  $G$ ).
- $L$  is the Laplace-Beltrami operator in  $\mathcal{S}^d$ .
- For  $d \in \mathbb{N} \setminus \{0\}$  such that  $1 \leq d \leq n$ , the set consisting of all integers  $d \times n$  matrices is designated by  $M_{d,n}(\mathbb{Z})$ .
- $\mathcal{P}_{d,n}$  is the subset of all elements  $A$  of  $M_{d,n}(\mathbb{Z})$  such that  $A = QD_0V$ , where  $Q \in SL_*(d, \mathbb{Z})$  and  $V \in SL_*(n, \mathbb{Z})$  and  $D_0$  is the matrix given by

$$D_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}$$

(see section 4.19)

- The  $d$ -plane Radon transform on the torus  $\mathbb{T}^n$  is defined by

$$Rf(D(x, A)) = \int_0^1 \cdots \int_0^1 f(x + pr((t_1, \dots, t_d)A)) dt_1 \cdots dt_d.$$

- For  $x \in \mathbb{T}^n$  and  $A \in \mathcal{P}_{d,n}$ , we denote by  $D(x, A)$  the  $d$ -plane in the torus  $\mathbb{T}^n$  given by

$$D(x, A) = \{x + pr((t_1, \dots, t_d)A) \mid (t_1, \dots, t_d) \in \mathbb{R}^d\}.$$

- $\mathcal{P}_k$  is the set defined by

$$\mathcal{P}_k = \{A \in \mathcal{P}_{d,n} \mid Ak = 0\},$$

where  $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ .

- We denote  $\delta(\alpha, \beta)$  the function defined by

$$\delta(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases},$$

where  $(\alpha, \beta) \in \mathbb{Z}^2$ .

- For  $(k, A) \in \mathbb{Z}^n \times \mathcal{P}_{d,n}$ , we denote  $\widehat{Rf}(k, A)$  by

$$\widehat{Rf}(k, A) = \begin{cases} \widehat{f}(k) & \text{if } Ak = 0 \\ 0 & \text{if } Ak \neq 0. \end{cases},$$

- The Damek-Ricci space is a semi-direct product of  $\mathbb{R}^+$  and a two step nilpotent group, i.e  $[\underline{\eta}, [\underline{\eta}, \underline{\eta}]] = 0$  and  $[\underline{\eta}, \underline{\eta}] \neq 0$ , where  $\underline{\eta}$  is the Lie algebra of the nilpotent group  $N$ . The Lie algebra can be decompose in the orthogonal sum of two spaces  $\underline{p}$  and  $\underline{z}$ ,  $\underline{\eta} = \underline{p} \oplus \underline{z}$ , where  $\underline{z}$  is the center of  $\underline{\eta}$ .

- We denote  $k$  the dimension of  $\mathbb{Z}$  and  $m = \dim p = 2m'$ . We designate  $A$  the multiplicative group isomorph to  $\mathbb{R} \setminus \{0\}$ . Then the Damek–Ricci space is denoted by  $S = NA$ .
- Denote  $Q = \frac{1}{2}m + k = \varrho$ . The multiplicative of  $S$  is defined by

$$na.n'a' = n(an'a^{-1})aa', \text{ where } (n, a) \in N \times A \text{ and } (n', a') \in N \times A.$$

- $c(\lambda)$  is the generalized Harish-Chandra function defined by

$$c(\lambda) = \frac{\Gamma(2i\lambda) \Gamma((m+k+1)/2)}{\Gamma(Q/2+i\lambda) \Gamma((m+2)/4+i\lambda)}.$$

- For  $n_1$  fixed in  $N$ , we define the Poisson kernel on  $NA$  by

$$\mathcal{P}(\cdot, n_1) : NA \longrightarrow \mathbb{R} \quad na \longrightarrow P_a(n_1^{-1}n),$$

where, for  $a > 0$ ,  $P_a(n)$  is the function on  $N$  defined by

$$P_a(n) = P_a(X, Z) = a^Q \left( \left( a + \frac{|X|^2}{4} \right) + |Z|^2 \right)^{-Q}.$$

- The kernel  $\mathcal{P}_\lambda$  ( $\lambda \in \mathbb{C}$ ) on  $NA \times N$  is defined by

$$\mathcal{P}_\lambda : NA \times N \longrightarrow \mathbb{R} \quad (na, \bar{n}) \longrightarrow \mathcal{P}_\lambda(na, \bar{n}) = P_a(\bar{n}^{-1}n)^{1/2-(i\lambda)/Q}$$

- For  $f \in D(NA)$ , the Fourier-Helgason transform is defined by

$$\widehat{f}(\lambda, n) = \int_{NA} f(x) \mathcal{P}_\lambda(x, n) dx, \text{ for all } (\lambda, n) \in \mathbb{C} \times N.$$

- The horocyclic Radon transform is

$$\begin{aligned} Rf(\lambda, n) &= \mathcal{F}_1 \widehat{f}(\lambda, n) \\ &= R_n f(\lambda) = e^{\lambda Q/2} \int_N f(n\sigma(n_1 \exp(\lambda H))) dn_1, \end{aligned}$$

where  $\sigma$  is the geodesic inversion defined by

$$\sigma(V, Z, a) = \frac{1}{\left( a + \frac{|V|^2}{4} \right)^2 + |Z|^2} \left[ \left( - \left( a + \frac{|V|^2}{4} \right) + J_Z \right) V, -Z, a \right],$$

see section 5.4.

- In the Chapters 6, 7, 8, 9,10. the Radon transform is denoted by  $R$ .
- For  $(a, k) \in \mathcal{P} \times \mathbb{Z}$ , we denote  $H(a, k)$  the discrete hyperplane of  $\mathbb{Z}^n$  defined by

$$H(a, k) = \{m \in \mathbb{Z}^n \mid am = k\},$$

where  $am = \sum_{i=1}^n a_i m_i$  with  $a = (a_1, \dots, a_n) \in \mathcal{P}$  and  $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$ .

- We define and denote the discrete Radon transform on  $\mathbb{Z}^n$  by

$$Rf(H(a, k)) = \sum_{m \in H(a, k)} f(m), \text{ for all } m \in l^1(\mathbb{Z}^n).$$

- For  $\alpha > 0$ , we define the Fourier-Hermite transform as follows

$$\mathcal{H}_\alpha f(\lambda) = \int_{\mathbb{R}^n} f(x) \exp(-\alpha \|x - \lambda\|^2 - 2i\pi \lambda x) dx$$

We denote the Hermite-Radon transform, by

$$R_{\alpha, t} f(\omega, p) = \int_{x\omega=p} f(x) e_{\alpha, \omega t}(x) d\mu(x)$$

where  $e_{\alpha, \omega t}(x) = \exp(-\alpha \|x - \omega t\|^2)$  and  $(\omega, t, p) \in S^{n-1} \times \mathbb{R} \times \mathbb{R}$ .

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# Introduction

The main purpose of this book is to study several integral transforms, precisely: Radon transforms, discrete Radon transforms on the lattice  $\mathbb{Z}^n$  and the discrete Fourier-Bessel transform on  $\mathbb{Z}^n$ .

Our study was focused on the study of the Radon transform on several spaces such that: Euclidean space  $\mathbb{R}^n$ , compact symmetric spaces  $\mathcal{S}^d = (SO(d+1)/SO(d))$ , the  $n$ -dimensional torus  $\mathbb{T}^n = (\mathbb{R}^n/\mathbb{Z}^n)$ , the lattice  $\mathbb{Z}^n$ , the Damek-Ricci space ( $NA$ -space) and the finite Radon transform.

Note that these transformations have many applications in physic, also in chemistry ... etc.

The Radon transform and discrete Radon transform are quite studied in this book; since these transformations have the great importance in the field of medicine also in geology.

This work is divided in two parts:

Part I: Radon transform on certain Riemannian spaces.

Part II: Study of certain transformations on  $\mathbb{Z}^n$ .

The first part is constituted by five chapters: The **Chp. 1** contains basic concepts of the theory of groups and homogeneous spaces.

In **Chap. 2**, we study in a detailed manner the harmonic analysis of the semi direct product  $G = N \odot W$ ; where  $N$  is a vector space a finite dimension and  $W$  is the connected Lie group. The main results of this chapter are theorems of Paley-Wiener on the group  $G = N \odot W$  and also the Paley-Wiener-Schwartz on  $G$ . Precisely, in this work we characterize the Fourier image of several distribution spaces on some Lie groups and we give applications. The theorems of Paley-Wiener described with respect to a fundamental family of compact and multiplication operators. These objects are constructed via a sub-multiplicative function own continuing. The Paley-Wiener space distributions of compact support becomes the operators of the space on a Sobolev type space, checking some properties. The theorems of Paley-Wiener on the space of indefinitely differentiable functions with compact support (resp the algebra of square integrable functions of compact support) are studied. For

particular cases of Lie groups, simplifications intervenes and the operators studied are then the Hilbert Schmidt operators .

**Chap. 3** contains important and new results for examples: Inversion theorem for the classical Radon transform  $R_c$  (see Theorem 3.3) , Gutzmer formula for the classical Radon transform ( see Theorem 3.7); Hardy's inequality for the operator  $R_c$  (see Theorem 3.8) and finally, we give a characterization of space  $R_c(\mathcal{E}(\mathbb{R}^n)')$ , where  $R_c$  is the classical Radon transform(see Theorem 3.11) Note that the proof of Theorem 3.3 is based on the development of plane waves of Dirac measure; precisely, we use the Galderon identity. The idea is at the origin of Theorem 3.5 (see [56]). The Theorem 3.7 generalizes the Helgason Theorem (see [70, page 116]). Theorem 3.11 gives another statment of Helgason Theorem [71, Theorem 2.6]. In section 3.5, we give the Titchmarsh's Theorem for the classical Radon transform (see Theorem 3.17). We give also certains discrete Titchmarsh' Theorems for the lattice  $\mathbb{Z}^n$  and its dual torus  $\mathbb{T}^n$ .

Let be  $N$  the vector space  $N \simeq \mathbb{R}^n$  and  $W$  is a connected Lie group which acts by a right action on  $W$ . We denote by  $G = N \odot W$  the semidirect product relatively to this action ( see chapter 2).

In the subsection 3.2.1, we establish a local inversion formula for the Radon transform on  $G = N \odot W$ . In addition, we define and prove an inversion formula for the local Radon transform on the group  $\mathcal{S}^3 \odot \mathbb{R}^4$  ( $\mathcal{S}^3 \odot \mathbb{R}^4$  is the semidirect product of the unit sphere and  $\mathbb{R}^4$ ).

We complete this chapter by study the Fourier -Hermite transform and revised Hardy's uncertainty principle; precisely, we study the eigenfunctions of Fourier transform in the goal to establish the revised Hardy uncertainty principle (see Theorem 3.22). We give the inversion theorem and Plancherel formula for this transform;( see Theorems 3.28 and 3.25).

It is clear that the Fourier-Hermite transform generalizes the classical Fourier transform. The Radon-Hermite transform (associated to Fourier-Hermite transform) generalizes the classical Radon transform.

In **Chap. 4**, we studied the integral geometry of the sphere  $\mathcal{S}^d$ . This chapter contain important results such as the geometry approach to the Radon transform of  $\mathcal{S}^d$  and its dual Radon transform. In addition we establish some inversion formulas for this transform also its dual in  $\mathcal{S}^d$ . The Plancherel formula and support theorem are also studied on  $\mathcal{S}^d$ . We finish this chapter by defining and studying the Radon transform on the  $n$ -dimensional torus  $\mathbb{T}^n$ . We give the interesting results for example (range characterization of the image of Radon transform on  $\mathbb{T}^n$ , inversion formula and support theorem for the Radon transform in  $\mathbb{T}^n$ ). We generalize this results to the  $d$ -plane Radon transform on  $\mathbb{T}^n$ .

We study, in **Chap. 5**, the harmonic analysis and integral geometry on Damek-Ricci space  $NA$ . We define and study the Radon transform in  $NA$ -space. The important results are: the inversion formulas and Plancherel formula for the Radon transform and its dual on  $NA$ -space. In addition, we study the spectral theory for the Strichartz 's operator. Precisely, we gener-

alize Strichartz's theorem (see Chap. 5) in the case of Damek-Ricci space  $NA$ . These results generalize the case of a Riemannian symmetric space of the non compact type of rank one.

In **the part II**, we begin by studying the integral geometry in the finite set  $X$ . The analogue definition of finite Radon transform consists the make the average of function  $f$  over the subset of a finite set  $X$  (see [44],[118], [108]). In the sequel, we adopt the definition of [108]) for two reasons:

The Strichartz's definition [108]) is more natural and similar to the classical case. Indeed, in the study (see [108]) the author has used the finite plane geometry to define the finite Radon transform. This definition can be extended (see [108]) to the finite  $k$ -plane transform  $R_k$  with  $k \in \{2, 3, \dots, \text{card}(X)\}$  with  $\text{card}(X)$  is the cardinal of the set  $X$ . More precisely, Let  $X$  be a finite set of points and  $N$  be the cardinal of  $X$ . Let  $Y$  be the set of lines of  $X$ , each line  $y \in Y$  being a subset of  $X$  subject to single axiom "two points determine a unique line", which is equivalent to:

(A) "For any two points  $x_1, x_2 \in X$ , there exists a unique  $y \in Y$  such that  $x_1 \in y$  and  $x_2 \in y$ ".

We say that  $Y$  is simple if for all lines  $y \in Y$ ,  $\text{card}(y) = 2$ .  $Y$  is not simple if there exists  $y_0 \in Y$  such that  $\text{card}(y_0) > 2$ . Let  $l^2(X)$  (resp.  $l^2(Y)$ ) be the space of all complex-valued functions on  $X$  (resp. on  $Y$ ).

The finite Radon transform is defined (see [108]) as the operator  $R$  on  $l^2(X)$  to  $l^2(Y)$  given by the formula

$$Rf(y) = \sum_{x \in y} f(x), y \in Y;$$

and the dual Radon transform of  $R$  is the operator  $R^*$  of  $l^2(Y)$  into  $l^2(X)$  given by

$$R^*F(x) = \sum_{y \ni x} F(y) \text{ for all } F \in l^2(Y) \text{ and } x \in X.$$

In the following, we seek a kernel  $G(y, x)$  which is solution of these equations

$$RG(y_0, ) = \beta_{y_0}(y_0 \in Y)$$

and

$$R^*G(., x_0) = \chi_{x_0}(x_0 \in X),$$

where  $\chi_{x_0}$  is the characteristic function of the set  $\{x_0\}$ , also  $\beta_{y_0}$  is the function defined on  $Y$  such that  $\beta_{y_0}(y) = 0$  if  $y \neq y_0$  and  $\beta_{y_0}(y_0) = 1$ .

Consequently, we calculate explicitly the solution  $G(y, x)$  of the functional equations above, for which we establish the inversion formulas for  $R$  and its dual  $R^*$  (see expression of this kernel in this chapter). We establish then the inversion formulas for  $R$  and  $R^*$ .

In **Chap. 7**, we define and study the Radon transform  $R$  on discrete hyperplane in the lattice  $\mathbb{Z}^n$  ( $n \geq 2$ ) defined by linear diophantine equations. More precisely, we study carefully various natural questions in this context: specific properties of the discrete Radon transform  $R$  and  $R^*$ , inversion formula for  $R$  and also an appropriate support theorem in the discrete case.

The purpose of **Chap. 8** is to extend carefully the discrete Radon transform, studied in the above chapter, to Radon transform  $R$  on the discrete Grassmannian  $\mathbb{G}(d, n)$  ( $n \geq 3$  and  $1 \leq d < n-1$ ) consisting of all discrete  $d$ -planes in the lattice  $\mathbb{Z}^n$  defined by systems of linear diophantine equations.

By analogy with the integral geometry on Grassmann manifolds and projective spaces, which was developed by many authors, (see [71], [72], [73], [74], [75]). Our study in the present work with various natural questions in this context: specific properties of the discrete Radon  $d$ -plane transform  $R$  and its dual  $R^*$ , inversion formula for  $R$  and also an appropriate support theorem for this Radon transform.

**In Chap. 9**, We give a discrete Paley-Wiener theorem relatively at the discrete Radon transform, which characterizes the image of functions on  $\mathbb{Z}^n$  with finite support. Let  $K = \{x_1, \dots, x_n\}$  be a finite set in  $\mathbb{Z}^n$ . We denote by  $\mathbb{C}_K^{(\mathbb{Z}^n)}$  the subspace of  $\mathcal{S}(\mathbb{Z}^n)$  consisting of all complex-valued functions on  $\mathbb{Z}^n$  such that  $\text{supp} f \subset K$ . Let  $\mathbb{G}_K = \{H \in \mathbb{G} \mid H \cap K \neq \emptyset\}$  (where  $\mathbb{G}$  the discrete Grassmannian on  $\mathbb{Z}^n$ ). We denote by  $\mathcal{D}_K(\mathbb{G})$  the subspace of  $\mathcal{S}(\mathbb{G})$  consisting of all complex-valued functions  $F$  on  $\mathbb{G}$  such that  $\text{supp} F \subset \mathbb{G}_K$ . We define  $\mathcal{D}_{*,K}(\mathbb{G})$  as the subspace of  $\mathcal{D}_K(\mathbb{G})$  constituted by the elements  $F$  verifying the moment condition and for each  $m \in \mathbb{Z}^n$ , there exists  $j_m \in \mathbb{N}$  for which  $F(H(a_j, a_j m)) = F(H(a_{j_m}, a_{j_m} m))$  for all  $j \geq j_m$ , where  $a_j = (1, j, j^2, \dots, j^{n-1})$ . we have then

$R$  is a bijection of  $\mathbb{C}_K^{(\mathbb{Z}^n)}$  onto  $\mathcal{D}_{*,K}(\mathbb{G})$  and  $R\left(\mathbb{C}_K^{(\mathbb{Z}^n)}\right) = \mathcal{D}_{*,K}(\mathbb{G})$ .

Let  $\mathcal{S}_*(\mathbb{G})$  denote a subspace of  $\mathcal{S}(\mathbb{G})$  consisting of all  $F \in \mathcal{S}(\mathbb{G})$  such that there exists a  $G \in C^\infty(\mathbb{T}^n)$  for which  $\mathcal{F}_1 F(H(a, \cdot))(\theta) = G(\theta a)$ . We obtain the following result:

$R$  is a bijection of  $\mathcal{S}(\mathbb{Z}^n)$  onto  $\mathcal{S}_*(\mathbb{G})$  and  $R(\mathcal{S}(\mathbb{Z}^n)) = \mathcal{S}_*(\mathbb{G})$ .

We give also a characterization of the image of discrete Hardy space under the discrete Radon transform  $R$  (see Theorem 9.23).

We end this chapter by establishing other inversion formulas for the discrete Radon transform.

**In Chap.10**, we characterize the image of exponential type functions under the discrete Radon transform  $R$  on the lattice  $\mathbb{Z}^n$  of the Euclidean space  $\mathbb{R}^n$  ( $n \geq 2$ ). we also establish the generalization of Volberg's uncertainty principle on  $\mathbb{Z}^n$ , which is proved by means of this characterization. We give also an analogue of Nazarov's uncertainty inequality for  $n$ -dimensional Fourier series from the one for  $n$ -dimensional Fourier transform. Some inequalities are new and better than ones deduced from a classical local uncertainty inequality. We introduce and study the Sobolov's spaces  $l^2(\mathbb{Z}^n, \tau)$  ( $\tau > 0$ ), which allows us to deduce the Gutzmer formula ( see Theorem10.29).

We end this introduction by studying the discrete convolution with Bessel functions, precisely, we define and study of Fourier-Bessel transform. We establish an inversion formula and Plancherel theorem for this transform (see Theorems 10.16, 10.17).



**Part I**  
**RADON TRANSFORMS ON**  
**CERTAIN RIEMANNIAN SPACES**



# Chapter 1

## FUNDAMENTAL CONCEPTS OF GROUP THEORY.

### 1.1 Definition of a group.

A Nonempty set  $G$  is called a group if a product  $x_1x_2$  is defined for every two elements  $x_1, x_2 \in G$ , for which the following conditions holds:

- a)  $x_1x_2 \in G$ , for all  $x_1, x_2 \in G$ ;
- b)  $(x_1x_2)x_3 = x_1(x_2x_3)$ , for all  $x_1, x_2, x_3 \in G$ ;
- c) there exists a unique element  $e$  in  $G$  such that  $ex = xe = x$  for all  $x \in G$ ;  $e$  is called the identity element of the group  $G$ ;
- d) for every element  $x \in G$  there exists a unique element, designated  $x^{-1}$  for which  $xx^{-1} = x^{-1}x = e$ ; the element  $x^{-1}$  is called the inverse of  $x$ . It is clear that  $x$  is the inverse of  $x^{-1}$ , so that  $(x^{-1})^{-1} = x$ .

A group  $G$  is called commutative ( or abelian) if  $x_1x_2 = x_2x_1$  for all every  $x_1, x_2 \in G$  and non commutative in the opposite case.

#### 1.1.1 Examples

- 1) The set of all complex matrices of order  $n$  with nonzero determinant is a group if multiplication is defined as multiplication of matrices; this group is usually denoted as  $GL(n, \mathbb{C})$ . Its identity element is the identity matrix and the inverse element of the matrix  $A$  is the matrix  $A^{-1}$ . The group  $GL(n, \mathbb{R})$  of all real matrices of order  $n$  with nonzero determinant is defined similarly. If  $n \geq 2$  These groups are non commutative.
- 2) Let  $SL(n, \mathbb{C})$  be the set of all complex matrices of order  $n$  with determinant equal to 1. We define a product in  $SL(n, \mathbb{C})$  as the product of matrices. Then  $SL(n, \mathbb{C})$  is a group, since in multiplication of matrices determinants also multiply. The group  $SL(n, \mathbb{R})$  of all real matrices of order  $n$  with determinant equal to 1 is defined similarly.

## Chapter 2

# PALEY WIENER THEOREMS FOR $G = W \odot N$ .

### 2.1 Introduction and notations.

We characterize the Fourier's image of many spaces of distributions on certain Lie groups, we give also some applications. The groups considered are the semidirect product of a real vector space by any connexe Lie group.

In the following we fix a vector space  $N$  and a connexe Lie group  $W$ , this group acts on  $N$  by a left action, and  $G = N \odot W$  their semi-direct product relatively to this action.

Since  $W$  acts on  $N$ , for all element  $w \in W$  we define an endomorphism of  $N$ , denote  $\|w\| = \sup \{|w.n| : |n| \leq 1, n \in N\}$  where  $|\cdot|$  is the Euclidean norm on  $N$ , we also denote  $|\cdot|$  the dual norm on  $N^*$  complexified of the dual  $N'$  (the set of the characters of  $N$ ) and we design by  $M(w)$  the real number defined by  $M(w) = \sup (\|w\|, \|w^{-1}\|)$ , for all  $w \in W$ .

We introduce a Riemannian distance, and with this distance we prove the existence of a fundamental sub-multiplicative function  $\Lambda$  which is continuous and proper function on  $W$ . The topology defined by this function on  $W$  (that is by the balls:  $Q_r = \{w \in W \mid \Lambda(w) \leq r\}$ ,  $r > 0$ ) is equivalent to the initial topology of  $W$ .

We characterize the Fourier transform of the distributions with compact support in  $G$ . They are the families of the operators on a space of the Sobolev type satisfying certain properties. The support of a distribution can be read on these properties.

We characterize also the Fourier transform for the square integrable functions with compact support. When  $G = N \times N$ , the simplifications intervene and therefore we find the classical Paley Wiener theorem.

# Chapter 3

## GUTZMER FORMULA AND RADON TRANSFORM ON $\mathbb{R}^n$ .

### 3.1 Introduction.

In 1917 Johann Radon [97] solved the following problem: find a function  $f$  on the Euclidean plane  $\mathbb{R}^2$  knowing its integrals

$$R_c f(\xi) = \int_{\xi} f,$$

along all lines  $\xi$  in the plane. The operator  $R_c$  is now called the Radon transform. This problem was generalized and studied seriously in  $\mathbb{R}^n$  (and in several varieties) by [71], [72], [60]. The study of the operator  $R_c$  is an important part of the integral geometry. The essential pillars of the integral geometry are

- 1) Inversion formula: knowing  $R_c f$  (and its properties) determine  $f$  (and its properties) ?
- 2) Support theorem of  $R_c$  : knowing the properties the support of  $f$ , can we find the properties the support for  $R_c$  and vice versa ?.

Let  $\mathbb{P}_n$  be the differentiable manifold constituted by all hyperplanes of Euclidean space  $\mathbb{R}^n$ . It is well known that the Euclidean Radon transform associates to a function  $f$  defined and integrable on  $\mathbb{R}^n$ , a function  $R_c f$  on  $\mathbb{P}_n$  given by the formula

$$R_c f(H(\omega, t)) = \int_{H(\omega, t)} f(x) d_{\mu}(x),$$

where  $d_{\mu}(x)$  is the Lebesgue measure on the hyperplane

$$H(\omega, t) = \{x \in \mathbb{R}^n \mid x\omega = t\};$$

where  $x\omega$  denotes the usual inner product of  $x$  and  $\omega$  regarded as two vectors of the Euclidean space  $\mathbb{R}^n$ .  $\omega$  being the element of the unit sphere  $\mathcal{S}^{n-1}$ .



## Chapter 5

# INTEGRAL GEOMETRY AND SPECTRAL THEORY ON NA–SPACE.

### 5.1 Introduction.

We say that  $M$  is a harmonic variety if, for every point  $m \in M$  ( $m$  origin), the Laplacian  $\mathcal{L}$  admits a elementary solution around  $m$ , that is, the solution is function of the geodesic distance  $d(m, x)$  alone.

The question that was put to the mathematicians is, the harmonic varieties are they symmetrical?. The response to this question is given by Ewa Damek and Fulvio Ricci. They gave a great class of harmonic varieties which does not are symmetrical. Precisely, in order to generalize Heisenberg group, Kaplan (see [82]) introduced Lie groups of  $H$ –type. Damek and Ricci ( see [51], [50]) defined and studied a new class of groups called ” extension of Lie group of  $H$ –type”. Damek-Ricci space is a semi-direct product of  $\mathbb{R}^+$  and a two step nilpotent group. More precisely, let  $\underline{\eta}$  be a two-step real nilpotent Lie algebra (i.e.  $[\underline{\eta}, [\underline{\eta}, \underline{\eta}]] = 0$  and  $[\underline{\eta}, \underline{\eta}] \neq 0$ ) endowed with an inner product  $\langle, \rangle$  such that  $\underline{\eta}$  decompose in the orthogonal sum of two spaces  $\underline{p}$  and  $\underline{z}$  ( $\underline{\eta} = \underline{p} \oplus \underline{z}$ , where  $\underline{z}$  is the center of  $\underline{\eta}$ ). We denote by  $k$  the dimension of  $\underline{z}$ . Let  $N$  be the connected and simply connected group of Lie algebra  $\underline{\eta}$ . Since  $\underline{\eta}$  is nilpotent, the exponential map is surjective, we may therefore parametrize  $N$  by  $\underline{p} \oplus \underline{z}$  and write  $(V, Z)$  for  $\exp(V + Z)$  where  $V \in \underline{p}$  and  $Z \in \underline{z}$ . By the Baker- Campbell-Hausdorff formula, the product law in  $N$  is given by the formula

$$(V, Z) \cdot (V', Z') = \left( V + V', Z + Z' + \frac{1}{2} [V, V'] \right),$$

for all  $V, V' \in \underline{p}$  and for all  $Z, Z' \in \underline{z}$ . Let  $dV$  and  $dZ$  be the Lebesgue measures on  $\underline{p}$  and  $\underline{z}$  respectively, the measure  $dVdZ$  is the Haar measure on  $N$  which will be denoted later by  $dn$ . Let  $\langle, \rangle_{\underline{\eta}}$  be an inner product on  $\underline{\eta}$ . The space  $\underline{\eta}$  is called an  $H$ –type algebra if for every  $Z \in \underline{z}$  we have  $J_Z^2 = -|Z|^2 I_{\underline{p}}$ , where  $I_{\underline{p}}$  is identity on  $\underline{p}$  and  $J_Z$  is a map of  $\underline{p}$  into  $\underline{p}$  defined by equality  $\langle J_Z X, Y \rangle_{\underline{\eta}} = \langle [X, Y], Z \rangle_{\underline{\eta}}$ , for all  $X, Y \in \underline{p}$  see [26]. Note that for each unitary  $Z \in \underline{p}$ , the map  $J_Z$  define a complex structure on  $\underline{p}$ , then

**Part II**  
**STUDY OF CERTAIN**  
**TRANSFORMATIONS ON  $\mathbb{Z}^n$ .**



## Chapter 6

# THE FINITE RADON TRANSFORM

### 6.1 Introduction

As we said before, the Radon transform is defined in  $\mathbb{R}^2$  by John Radon [97] and generalized in  $\mathbb{R}^n$  by several authors particularly [62], [74]. The Radon transform in Euclidean space  $\mathbb{R}^n$  associates to a function  $f$  on  $\mathbb{R}^n$  a function  $Rf$  on  $\mathbb{P}^n$  ( $\mathbb{P}^n$  denotes the space of all hyperplanes  $H(\omega; t)$  in  $\mathbb{R}^n$ ) by the formula

$$Rf(\omega, t) = \int_{H(\omega, t)} f(x) d\mu(x),$$

where  $d\mu(x)$  is the Euclidean measure on the hyperplane

$$H(\omega, t) = \{x \in \mathbb{R}^n \mid x\omega = t\},$$

$x\omega$  denotes the usual inner product of  $\omega$  and  $x$  regarded as two vectors of the Euclidean space  $\mathbb{R}^n$ . The vector  $\omega$  being an element of the unit sphere  $\mathcal{S}^{n-1}$ . In the case of the finite set, the analogue of this definition consists the make the average of function  $f$  over the subset of a finite set  $X$  (see [44], [118]). In the sequel, we adopt the definition of [108] for two reasons: The Strichartz's definition [108] is more natural and similar to the classical case. Indeed, in the study (see [108]) the author has used the finite plane geometry to define the finite Radon transform. This definition can be extended (see [108]) to the finite  $k$ -plane transform  $R_k$  with  $k \in \{2, 3, \dots, \text{card}(X)\}$  ( $\text{card}(X)$  is the cardinal of the set  $X$ ).

More precisely, Let  $X$  be a finite set of points and  $N$  be the cardinal of  $X$ . Let  $Y$  be the set of lines of  $X$ , each line  $y \in Y$  being a subset of  $X$  subject to single axiom "two points determine a unique line", which is equivalent to: (A) "For any two points  $x_1, x_2 \in X$ , there exists a unique  $y \in Y$  such that  $x_1 \in y$  and  $x_2 \in y$ ". We say that  $Y$  is simple if for all lines  $y \in Y$ ,  $\text{card}(y) = 2$ .  $Y$  is not simple if there exists  $y_0 \in Y$  such that  $\text{card}(y_0) > 2$ . Let  $l^2(X)$  ( resp.  $l^2(Y)$ ) be the space of all complex-valued functions on  $X$



## Chapter 7

# DIOPHANTINE INTEGRAL GEOMETRY IN $\mathbb{Z}^n$ .

### 7.1 Introduction

Let  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  be a function integrable on each hyperplane in  $\mathbb{R}^n$ . Let  $\mathbb{P}^n$  denote the differentiable manifold  $\mathbb{R}^n \times \mathcal{S}^{n-1}/\pm$  of all hyperplane  $H(\omega, t)$  in  $\mathbb{R}^n$  ( $H(\omega, t) = \{x \in \mathbb{R}^n \mid x\omega = t\}$ ), with  $(\omega, t) \in \mathcal{S}^{n-1} \times \mathbb{R}$ ,  $\mathcal{S}^{n-1}$  being the unit sphere. The Radon transform of  $f$  is defined as the function  $R_c f : \mathbb{P}^n \rightarrow \mathbb{C}$  given by

$$R_c f = \int_{H(\omega, t)} f(x) d\mu(x), \text{ for all } (\omega, t) \in \mathcal{S}^{n-1} \times \mathbb{R}, \quad (7.1)$$

where  $d\mu$  is the Euclidean measure on the hyperplane  $H(\omega, t)$ . In the case of a finite set  $E$ , the analogue of this definition consists in making the average of a function  $f : E \rightarrow \mathbb{C}$  over the non-empty subsets of  $E$ , see [4], [118].

In the case of the lattice  $\mathbb{Z}^n$ , we give in this section an analogue of the definition (7.1), which consists in making the average of a suitable complex-valued function  $f$  on  $\mathbb{Z}^n$  over discrete hyperplane  $H(a, k) = \{x \in \mathbb{Z}^n \mid ax = k\}$  defined by linear diophantine equations, with  $(a, k) \in \mathcal{P} \times \mathbb{Z}$ , where  $\mathcal{P}$  designates the set of all elements  $a = (a_1, \dots, a_n) \in \mathbb{Z}^n \setminus \{0\}$  such that  $d(a) = 1$ ,  $d(a)$  being the greatest common divisor of the integers  $a_1, \dots, a_n$ , and  $ax$  denotes the usual inner product of  $a$  and  $x$  regarded as two vectors of the Euclidean space  $\mathbb{R}^n$ .

Now, we note that one of the difficulties of diophantine integral geometry, relatively to its Euclidean homologue, is the problem of the existence of solutions in  $\mathbb{Z}^n$  for linear diophantine equation  $ax = c$ , with  $a = (a_1, \dots, a_n) \in \mathbb{Z}^n \setminus \{0\}$  and  $c \in \mathbb{Z}$ , therefore the arithmetic of the integers  $a_1, \dots, a_n$  and  $c$  is essential for our study.

We will deepen the study of the discrete Radon transform on  $\mathbb{Z}^n$ , and we generalize it to the discrete grassmannians associated with  $\mathbb{Z}^n$ .



## Chapter 8

# DIOPHANTINE INTEGRAL GEOMETRY ON $G(d, n)$ .

### 8.1 Introductions and preliminaries

In the following section, we extend carefully the discrete Radon transform to the Radon transform  $R$  on the discrete Grassmannian  $G(d, n)$ ,  $n \geq 3$  and  $1 \leq d < n - 1$  consisting of all discrete  $d$ -planes in the lattice  $\mathbb{Z}^n$  defined by systems of linear diophantine equations. By analogy with the integral geometry on Grassmann manifolds and projective spaces, which was developed by many authors, particularly C. A. Berenstein and E. Casadio Tarabulsi [43], F. Gonzalez [66], [65], S. Helgason [73] and B. Rubin [101]. We briefly recall the definition of the classical Radon  $d$ -plane transform  $R_c$  on the Euclidean space  $\mathbb{R}^n$  as well as its dual  $R_c^*$ ,  $d$  being an integer such that  $0 < d < n$ , with  $n \geq 2$ . We denote by  $G(d, n)$  the Grassmann manifold consisting of all affine  $d$ -dimensional planes in  $\mathbb{R}^n$ . The Radon  $d$ -plane transform  $R_c$  is defined by

$$R_c f(\xi) = \int_{\xi} f(x) dm(x), \text{ for all } (f, \xi) \in \mathcal{D}(\mathbb{R}^n) \times G(d, n), \quad (8.1)$$

where  $dm$  is the Euclidean measure on the  $d$ -plane  $\xi$  and  $\mathcal{D}(\mathbb{R}^n)$  denotes the space of all complex-valued  $C^\infty$  functions on  $\mathbb{R}^n$  with compact support. On the other hand, the dual Radon  $d$ -plane transform  $R_c^*$  is given by

$$R_c^* \varphi(x) = \int_{\xi \ni x} \varphi(\xi) d\mu(\xi), \text{ for all } (\varphi, x) \in \mathcal{E}(G(d, n)) \times \mathbb{R}^n, \quad (8.2)$$

where  $\mathcal{E}(G(d, n))$  denotes the space of all complex-valued  $C^\infty$  functions on  $G(d, n)$  and  $d\mu$  is the measure on the set of all  $d$ -planes through  $x$  which is invariant under the group of rotations around  $x$  and such that  $\mu\{\xi \in G(d, n) \mid x \in \xi\} = 1$ .

Now, we note that one of the difficulties to extend the discrete Radon transform, to the Radon transform on the discrete Grassmannian  $\mathbb{G}(d, n)$  consist-



## Chapter 9

# DISCRETE RADON TRANSFORM AND INVERSION FORMULAS

### 9.1 Introduction and preliminaries.

The goal of this section is to reconstruct a function of  $\mathbb{C}(\mathbb{Z}^n)$  from its  $d$ -planes Radon transform. where  $\mathbb{C}(\mathbb{Z}^n)$  is the space of all functions  $f$  defined on  $\mathbb{Z}^n$  with finite support. We establish also a Plancherel formula relatively at discrete Radon transform for the functions  $f$  of  $\mathbb{C}(\mathbb{Z}^n)$ . Afterwards, we characterize some discrete functional spaces by the Radon transform (Paley-Wiener theorem and Paley-Wiener-Schwartz theorem relatively to the discrete Radon transform). Let  $f \in l^1(\mathbb{Z}^n)$ , we define the Strichartz's operator as follows (see [108])

$$\mathcal{R}f(k) = \sum_{j=1}^{\infty} f(kj) \text{ for all } k \in \mathbb{N} \setminus \{0\}. \quad (9.1)$$

The operator  $\mathcal{R}$  can be generalized by

$$\mathcal{R}f(m_1, \dots, m_n) = \sum_{k_1=1}^{\infty} \dots \sum_{k_n=1}^{\infty} f(k_1 m_1, \dots, k_n m_n), \quad (9.2)$$

for all  $m = (m_1, \dots, m_n) \in \mathbb{Z}^n \setminus \{0\}$ , with  $f$  is a function of  $\mathcal{S}(\mathbb{Z}^n)$  such that  $f(0) = 0$ . We denote by  $\mu$  the Mobius function defined on  $\mathbb{N}$  by  $\mu(1) = 1$   $\mu(l) = (-1)^r$  if  $l$  is the product of  $r$  distinct prime numbers,  $\mu(l) = 0$  otherwise. Let  $u = (u_1, \dots, u_n) \in (\mathbb{N} \setminus \{0\})^n$  and  $\hat{u}_j = u_1 u_2 \dots u_{j-1} u_{j+1} \dots u_n$  for all  $j = 1, 2, \dots, n$ .

The function  $T_{(u_1, \dots, u_n)}$  will be defined as follows:  $T_v(m) = (u_1 m_1, \dots, u_n m_n)$  for all  $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$  and  $v = (u_1, \dots, u_n)$ . We denote by  $\mathcal{S}_0(\mathbb{Z}^n)$  the subspace of the Schwartz space  $\mathcal{S}(\mathbb{Z}^n)$  consisting of all complex-valued functions  $f$  defined on  $\mathbb{Z}^n$  such that  $f(0) = 0$ . We design by  $\mathcal{S}_{0,+}(\mathbb{Z}^n)$  the subspace of  $\mathcal{S}_0(\mathbb{Z}^n)$  such that the function  $t \rightarrow f(0, \dots, 0, t)$  ( $t \in \mathbb{Z}$ ) is even. Let  $u = (u_1, \dots, u_n) \in (\mathbb{N} \setminus \{0\})^n$  and  $\hat{u}_j = u_1 u_2 \dots u_{j-1} u_{j+1} \dots u_n$ , we design by  $\hat{u}$  the vector of  $\mathbb{Z}^n$  defined by  $\hat{u} = (\hat{u}_1, \dots, \hat{u}_n) \in \mathbb{Z}^n$ . Recall that  $\mathcal{P}_{s,n}$  is the

# Chapter 10

## STUDY OF CERTAIN DISCRETE TRANSFORMS.

### 10.1 Introduction and preliminaries.

We recall briefly that the uncertainty principle states, roughly speaking, that a non zero function and its Fourier transform cannot both be sharply localized, which can be interpreted topologically by the fact that they cannot have simultaneously their supports in a same too small compact, see the Heisenberg uncertainty principle in [70].

Several versions of the uncertainty principle on have been established by many authors in the last few decades. Among the contributions dealing with this important topic, let us quote principally [35], [96]. On the the other hand, we note that the uncertainty principle is on of the major themes of the classical Fourier analysis as well as its neighbouring parts of the mathematical analysis.

In this section, we characterize the image of exponential type functions under the discrete Radon transform  $R$  on the lattice  $\mathbb{Z}^n$  of the Euclidean space  $\mathbb{R}^n$  ( $n \geq 2$ ). We also establish the generalization of Volberg's uncertainty principle on  $\mathbb{Z}^n$ , which is proved by means of this characterization. Also we shall obtain an analogue of Nazarov's uncertainty inequality for  $n$ - dimensional Fourier series from the one for  $n$ - dimensional Fourier transform. Some inequalities are new and better than ones deduced from a classical local uncertainty inequality (see [29], for more details). However, as for the Nazarov uncertainty principle, it is known only for Fourier transform. In this subsection we shall obtain an analogous uncertainty inequality for  $n$ - dimensional Fourier series. Nazarov's uncertainty inequality is originally appeared in [94] and [80], Jaming extends it to a higher dimensional Fourier transform. The techniques of which we made use essentially the diophantine integral geometry as well as the Fourier analysis.

We consider here the lattice  $\mathbb{Z}^n$  of the Euclidean space  $\mathbb{R}^n$  ( $n \geq 2$ ). For  $a = (a_1, \dots, a_n) \in \mathcal{P} = \{m = (m_1, \dots, m_n) \in \mathbb{Z}^n \setminus \{0\} \mid d(m) = 1\}$ , where  $d(m)$  is the greatest common divisor of the integers  $m_1, \dots, m_n$ , and  $k \in \mathbb{Z}$ , the

## Exercices

1. Let  $\mathbb{Z}^n$  be the lattice of the Euclidean space  $\mathbb{R}^n$ . For  $n \geq 2$ , define the following lattice

$$\frac{\mathbb{Z}^n}{2^n} = \left\{ \frac{m}{2^n} \mid m \in \mathbb{Z}^n \right\}.$$

It is clear that

$$\mathbb{Z}^n \subset \frac{\mathbb{Z}^n}{2} \subset \frac{\mathbb{Z}^n}{2^2} \subset \cdots \subset \frac{\mathbb{Z}^n}{2^n} \subset \cdots$$

Study the diophantine integral geometry of the lattice  $\frac{\mathbb{Z}^n}{2^n}$ ?

2. Can be inverted the transform  $Rf(H(a, k)) = \sum_{am=k, m \in \mathbb{Z}^n} f(m)$ , for  $n > 2$ , where  $(a, k) \in \mathcal{P} \times \mathbb{Z}$ ?
3. Let  $\mathbb{Z}^n + i\mathbb{Z}^n$  be the Gauss 's lattice. Study the diophantine integral geometry associated to the Gauss 's lattice?

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**Remark 3.32** *If  $\alpha = 0$ , we find the Helgason Theorem (see [72, Theorem 2.1 page 10]).*

### 3.13 The d-plane Radon transform on Lie groups of exponential type

In the following, we will define the d-plane Radon transform on solvable Lie groups of exponential type. We establish an inversion theorem and Plancherel formulas of this transform.

#### 3.13.1 Introduction

Among the many works that study of integral geometry on symmetric spaces (see Helgason's works [71]); the case of the integral geometry on solvable Lie groups of exponential type is not however studied.

We give some properties concerning the exponential function. Let  $G$  be the connected Lie group and simply connected and  $\mathfrak{g}$  its Lie algebra. The Lie group is exponential if  $\exp(\mathfrak{g}) = G = E(G)$ . The examples of this Lie groups are: the Lie groups such that: the  $ax + b$  line group i.e the group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{R}) \mid a > 0, b \in \mathbb{R} \right\};$$

this group is the semi-direct product whose the law is given by

$$(b, a)(b', a') = (ab' + b, aa').$$

The Lie group  $GL(n, \mathbb{C})$  is exponential i.e  $\exp(M_{n \times n}(\mathbb{C})) = GL(n, \mathbb{C})$ . The nilpotent groups are exponential. Let  $\mathcal{S}_n$  be the space of the symmetric matrices and  $\mathcal{S}_{n,+}$  the space of symmetric matrices defined positive. the exponential is surjective of  $\mathcal{S}_n$  onto  $\mathcal{S}_{n,+}$ . The Lie group  $SL(n, \mathbb{R})$  is not exponential, more generally, the linear group  $GL(n, \mathbb{R})$  is not exponential, that is  $\exp(\mathfrak{g}) \neq G$ . In the following: for any arbitrary Lie group  $G$  and  $x \in G$ , we define

$$Ind_G(x) = \begin{cases} \min \{k \in \mathbb{N} \mid x^k \in E(G)\} & \text{If the minimum exists} \\ \infty & \text{otherwise} \end{cases}$$

$$Ind_G(G) = \{Ind_G(x) \mid x \in G\}$$

Now, we have the important theorem

$$R_c^* R_c f(x) = \frac{\Omega_d}{\Omega_n} \int_{\mathbb{R}^n} f(x+y) |y|^{d-n} dy$$

$\Omega_n$  is the volume of the unit sphere  $\mathcal{S}^{n-1}$ .

### 3.13.5 The inversion formulas for the $d$ -plane Radon transform on nilpotent groups and certain solvable Lie groups.

Troughout  $G$  would be the solvable connected and simply connected Lie group and  $\underline{g}$  be the Lie algebra of  $G$ .

We will start with the next group

Let  $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a > 0, b \in \mathbb{R} \right\}$ , the  $ax+b$  group  $G$  of affine transformations of real line

$$R_e f(H(t, A(\omega))) = \int_{A(\omega)X=t} f \circ \exp(x_1 X_1 + x_2 X_2) d\mu(x_1, x_2),$$

where  $A(\omega) = \begin{pmatrix} \omega_1 & \omega_2 \\ 0 & 0 \end{pmatrix}$

But,  $A(\omega)X = \begin{pmatrix} \omega_1 & \omega_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \omega_1 + x_2 \omega_2 \\ 0 \end{pmatrix} = x_1 \omega_1 + x_2 \omega_2$ ,  $\omega = (\omega_1, \omega_2) \in \mathcal{S}^1$ . Then,  $H(t, A(\omega)) = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \omega_1 + x_2 \omega_2 = t\}$  is the hyperplane, it follows that

$$\begin{aligned} R_e f(H(t, A(\omega))) &= \int_{x_1 \omega_1 + x_2 \omega_2 = t} f \circ \exp(x_1 X_1 + x_2 X_2) d\mu(x_1, x_2). \\ &= \int_{\underline{g}} f \circ \exp(x_1 X_1 + x_2 X_2) \delta(t - (x_1 \omega_1 + x_2 \omega_2)) dx_1 dx_2 \end{aligned}$$

$f \in \mathcal{D}(G)$  and  $\delta$  Dirak measure. Then

$$f = (4\pi)^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) (-\Delta)^{\frac{1}{2}} R_c^* R_c (f \circ \exp) \circ \text{Log},$$

since here  $\exp(\underline{g}) = G$ . More generally, we consider the group

$$G = \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} = [b, A] \mid \det(A) > 0 \right\},$$

$b$  being a vector that has the same numbers of rows as the matrix  $A$ . The law of  $G$  is  $[b, A] \cdot [b', A'] = [Ab' + b, AA']$ . The Lie algebra of  $G$  is given by

where  $C_{n,d} = (4\pi)^{-\frac{d}{2}} \frac{\Gamma((n-d)/2)}{\Gamma(\frac{n}{2})}$ .

In particular, we are studying Cartan displacement group.  
Let  $G$  be the Cartan's group defined by:

$$G = \left\{ \begin{pmatrix} \cos \theta & \sin \theta & b_1 \\ -\sin \theta & \cos \theta & b_2 \\ 0 & 0 & 1 \end{pmatrix} \mid (b_1, b_2) \in \mathbb{R}^2 \text{ and } \theta \in \mathbb{R} \right\}.$$

Let  $f \in D(\mathcal{E} \cup \Lambda_m(\mathbb{R}))$  ( $m \geq 2$ ), the Radon transform of the Cartan displacement group, can be inverted by

$$f = \frac{1}{2} (4\pi)^{-2} (-\Delta)^2 R_c^* R_c (f \circ \exp) \circ \text{Log}$$

### 3.13.6 Plancherel formulas for the nilpotent groups and affine group of the line for the Radon transform

Plancherel formula for the Radon transform in the affine group

**Theorem 3.44** *Let  $G$  be the affine group and  $f \in \mathcal{D}(G)$ . Then, we have the following formula*

$$\int_G \frac{|f(b, a)|^2}{a^2} da db = C_{2,1}^2 \int_G \frac{\left| (-\Delta)^{\frac{1}{2}} R_c^* R_c (f \circ \exp) (\text{Log}(b, a)) \right|^2}{a^2} da db$$

where  $(b, a) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ , and  $C_{2,1}^2 = \left[ (4\pi)^{-\frac{1}{2}} \Gamma(\frac{1}{2}) \right]^2 = (4\pi)^{-1} \Gamma(\frac{1}{2})^2$

Plancherel formula for the Radon transform in the nilpotent groups

**Theorem 3.45** *Let  $G$  be the nilpotent Lie group connected and simply connected and  $f \in \mathcal{D}(G)$ . Then, we have the following formula*

$$\begin{aligned} \int_G |f(x)|^2 dx &= C_{n,d}^2 \int_G \left| (-\Delta)^{d/2} R_c^* R_c (f \circ \exp) (\text{Log}(x)) \right|^2 d_\mu(x) \\ &= \int_{\hat{G}} \text{Tr}(\pi(f)^* \pi(f)) d\lambda(\pi). \end{aligned}$$

$$\begin{aligned} R_w f(\omega, t) &= \int_{v\omega=t} f(w^1 y w.w) d\mu(y) \\ &= \int_{v\omega=t} w^{-1} f_{w(2)}(y) d\mu(y). \end{aligned}$$

From section 3.2.1 (see formulas (B), (C)), we have

$$f(y_0 w) = \frac{\Gamma(n/2)}{\Gamma(-\frac{n}{2} + \frac{1}{2}) \pi^{\frac{3n-1}{2}}} \int_{\mathcal{S}^{n-1} \times \mathbb{R}} R_w f(\omega, t + (w y_0 w^{-1}) \omega) \frac{1}{t^n} dt d\omega,$$

if  $n$  is even ( $n = \dim N$ ), where  $n = \dim N$

Question:

Let  $G$  be a connected and simply connected real solvable Lie group and  $\underline{g}$  its Lie algebra. Let  $A_2(G)$  be the set defined by

$$A_2(G) = \{x^2 \mid x \in G\}, \text{ where } x^2 = x.x,$$

the point . is the Law of the group  $G$ . The exponential is it surjective of  $\underline{g}$  onto  $A_2(G)$ ?

## BIBLIOGRAPHICAL NOTES

Chapter 3 contains important and new results for example: Inversion theorem for the classical Radon transform  $R_c$  (see Theorem 3.3), Gutzmer formula for the classical Radon transform (see Theorem 3.7); Hardy inequality for the operator  $R_c$  (see Theorem 3.8) and finally characterization of space  $R_c(\mathcal{E}(\mathbb{R}^n)')$  under  $R_c$  (see Theorem 3.11)

Note that the proof of Theorem 3.3 is based on the development of plane waves of Dirac measure, precisely, we use the Galderon identity. The idea is at the origin of Theorem 3.5 (see [56]).

The Theorem 3.7 generalizes the Helgason Theorem (see [71, page 116]. Theorem 3.11 gives another statement of the Helgason Theorem [72, Theorem 2.6].

In section 3.5, we give the Titchmarsh's Theorem for the classical Radon transform (see Theorem 3.17). We give also certain discrete Titchmarsh' Theorems for the lattice  $\mathbb{Z}^n$  and its dual torus  $\mathbb{T}^n$ .

Let be  $N$  the vector space  $N \simeq \mathbb{R}^n$  and  $W$  is a connected Lie group which acts by a right action on  $W$ . We denote by  $G = N \odot W$  the semidirect product relatively to this action ( see chapter 2).

In the subsection 3.2.1, we establish a local inversion formula for the Radon transform on  $G = N \odot W$ . In addition, we give a local Radon transform for

the Radon transform on the group  $\mathcal{S}^3 \odot \mathbb{R}^4$  ( $\mathcal{S}^3 \odot \mathbb{R}^4$  is the semidirect product of the unit sphere and  $\mathbb{R}^4$ ).

We complete this chapter by study the Fourier -Hermite transform and revised Hardy's uncertainty principle; precisely, we study the eigenfunctions of Fourier transform and Radon transform, we establish the inversion theorem and Plancherel formula for this transform; ( see Theorems 3.28 and 3.25).

Note that the Fourier-Hermite transform and the revised Hardy uncertainty principle are defined and studied by Abouelaz and Attioui [13].

It is clear that the Fourier-Hermite transform generalizes the classical Fourier transform. The Radon-Hermite transform (associated to Fourier-Hermite transform) generalizes also the classical Radon transform.

We conclude this **Chapter 3** by defining and studying the Radon transform on the solvable Lie groups( in particular the nilpotent Lie groups). More precisely: we establish the inversion theorems and Plancherel formulas of this transform for the nilpotent Lie groups.



# Chapter 4

## INTEGRAL GEOMETRY ON THE SPHERE AND THE TORUS.

In the following, we will deeply study the integral geometry in the unit sphere and we give certain applications.

### 4.1 Introduction.

Among the many works that study of integral geometry on a symmetric space (See [62], [71], [74], [75],[109]), the case of a symmetric space of compact type is often considered that the non-compact type. The compact case has however been raised on several occasions especially S. Helgason (See [75],[71]), on the other hand, T. Sherman in ( See [102]) has developed a Fourier analysis on the sphere analogous to the abelian Lie groups. Note that Strichartz in [110] has established a local Fourier analysis in  $\mathcal{S}^d$ .

In the following, we define and study a Radon transform on the sphere  $\mathcal{S}^d = SO(d+1)/SO(d)$  of dimension  $d$ . This transform  $R_s$  is obtained by integration a given function over the sphere of dimension  $(d-1)$  passing through a fixed point  $a_\pi = (0, \dots, 0, -1)$  of  $\mathcal{S}^d$  called south pole. Applying the functions on  $\mathcal{S}^d$  that are invariant under rotation around this point, an interesting analogy is obtained with the Abel 's transform of the hyperbolic space  $\mathcal{SO}_0(d, 1)/SO(d)$ (of dimension  $d$ ) which consists of integrate over the horocycles(of dimension  $d-1$ ).

Our transform  $R_s$  on the sphere is thus—like the one of the hyperbolic space—related simply to spherical Fourier transform. This transform can be reversed by means of a differential operator(if  $d$  is odd) or Integro— differential(if  $d$  is even). We adopt the geometric approach, based upon the Radon transform  $R_s$  on the sphere  $\mathcal{S}^d$ , also the dual Radon transform  $(R_s)^*$  on  $\mathcal{S}^d$ . We generalize [17, Theorem 2.2], by replacing in this theorem the assumption  $f$  vanishes in a neighbourhood of the south pole with  $f$  vanishes to second order at the south pole. We give a theorem of transmutation formula of operators. The Paley Wiener theorems on the sphere  $\mathcal{S}^d$  are also studied.