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About the Stein equation for the generalized inverse Gaussian and Kummer distributions

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Abstract

We propose a Stein characterization of the Kummer distribution on $(0, \infty)$. This result follows from our observation that the density of the Kummer distribution satisfies a certain differential equation, leading to a solution of the related Stein equation. A bound is derived for the solution, under a condition on the parameters. The derivation of this bound is carried out using the same framework as in Gaunt 2017 [A Stein characterisation of the generalized hyperbolic distribution. *ESAIM: Probability and Statistics*, **21**, 303–316] in the case of the generalized inverse Gaussian distribution, which we revisit by correcting a minor error in the latter paper.

Keywords : Generalized inverse Gaussian distribution, Kummer distribution, Stein characterization.

1 Introduction

For $a > 0$, $b \in \mathbb{R}$, $c > 0$, the Kummer distribution with parameters a, b, c has density

$$k_{a,b,c}(x) = \frac{1}{\Gamma(a)\psi(a, a - b + 1; c)} x^{a-1} (1+x)^{-a-b} e^{-cx}, \quad (x > 0)$$

where ψ is the confluent hypergeometric function of the second kind.

The generalized inverse Gaussian (hereafter *GIG*) distribution with parameters $p \in \mathbb{R}$, $a > 0$, $b > 0$ has density

$$g_{p,a,b}(x) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1} e^{-\frac{1}{2}(ax+b/x)}, \quad x > 0,$$

where K_p is the modified Bessel function of the third kind.

For details on GIG and Kummer distributions see [4, 5, 11] and references therein, where one can see for instance that these distributions are involved in some characterization problems related to the so-called Matsumoto-Yor property.

In this paper, these two distributions are considered in the context of Stein's method. This method introduced in [9] is a technique used to bound the error in the approximation of the distribution of a random variable of interest by another probability (for instance the normal) distribution. For an overview of Stein's method see [1, 7]. The first steps of this method consist in finding an operator called Stein operator characterizing the targeted distribution, then solving the corresponding so-called *Stein equation*.

One finds in [9] a seminal instance of the method, where Stein showed that a random variable X has a standard normal distribution if and only if for all real-valued absolutely continuous function f such that $\mathbb{E}|f'(Z)| < \infty$ for $Z \sim N(0, 1)$,

$$\mathbb{E}[f'(X) - Xf(X)] = 0.$$

The corresponding *Stein equation* is

$$f'(x) - xf(x) = h(x) - \mathbb{E}h(Z)$$

where h is a bounded function and Z a random variable following the standard normal distribution. The operator $f \mapsto T_f$ defined by $(T_f)(x) = f'(x) - xf(x)$ is the corresponding *Stein operator*.

If a function f_h is a solution of the previous equation, then for any random variable U we have

$$|\mathbb{E}(f'_h(U) - Uf_h(U))| = |\mathbb{E}(h(U)) - \mathbb{E}(h(Z))|.$$

Thus, in order to bound $|\mathbb{E}(h(U)) - \mathbb{E}(h(Z))|$ given h , it's enough to find a solution f_h of the Stein equation and to bound the left-hand side of the previous equation. The problem of solving the Stein equation for other distributions than the standard normal distribution and bounding the solution and its derivatives has been widely studied in the literature (see [3] among many others).

The aim of this paper is to solve the Stein equation and derive a bound of the solution for the Kummer distribution (which is new) and for the generalized inverse Gaussian distribution (which has been done in [2], but there was a little mistake in the bound of the solution).

The idea of this paper emerged by reading the remarkable work by [2] about a Stein characterization of the generalized hyperbolic distribution of which the generalized

inverse Gaussian distribution (GIG) is a limiting case. Among many other results, [2] solved the GIG Stein equation and bounded the solution by using a general result obtained in [8] when the targeted distribution has a density g satisfying

$$(s(x)g(x))' = \tau(x)g(x) \tag{1.1}$$

for some polynomial functions s and τ . Also a bound was obtained for the solution under the condition that the function τ be a decreasing linear function. But since this linearity condition does not hold in the GIG case, the bound given by [2] has to be slightly corrected. This is done in Theorem 3.1 after recalling the general framework of Schoutens [8] and adapting it to the cases where τ is decreasing but not necessarily linear. Indeed, we realized that the procedure adopted in [8] still works, via a slight change, even if τ is not linear.

Observing that the Kummer density also satisfies (1.1), we can use the same methodology (Theorem 4.2) for this distribution. We have to put the restrictions $p \leq -1$ for the GIG density and $1 - b - c \leq 0$ for the Kummer density in order for the corresponding function τ to be decreasing on $(0, \infty)$.

In Section 1 we recall the general framework established by [8] for densities g satisfying (1.1) without the assumption of linearity of τ . We retrieve the Stein operator given in [8] by using the density approach initiated in [10] and further developed in [6].

In Section 2 we show the application of this method to the GIG distribution as mentioned in [2] by giving the right bound for the solution of the Stein equation. Section 3 is devoted to the Stein characterization and the Stein equation related to the Kummer distribution.

2 Stein characterization in the Schoutens framework

Theorem 1 in [8] addressed the problem of establishing a Stein characterization for probability distributions with density g satisfying (1.1) for some polynomial functions s et τ , and proved that a Stein operator in this case is $f \mapsto sf' + \tau f$. We realized (see the following theorem) that the same Stein operator can be arrived at by using the density approach designed in [10] and [6]. The support of the density may be any interval, but here we take this support to be $(0, \infty)$ in the purpose of the application to the GIG and Kummer distributions.

Theorem 2.1 Consider a density g on $(0, \infty)$ such that (1.1) holds for some polynomial functions s and τ . Then a positive random variable X has density g if and only if for any differentiable function f such that $\lim_{x \rightarrow 0} s(x)g(x)f(x) = \lim_{x \rightarrow \infty} s(x)g(x)f(x) = 0$,

$$\mathbb{E}[s(X)f'(X) + \tau(X)f(X)] = 0.$$

Proof: We use Corollary 2.1 of [6]. According to this corollary, a Stein operator related to the density function g is

$$T_g f(x) = \frac{1}{g(x)}(fg)'(x).$$

Applying this operator to sf , we have

$$\begin{aligned} T_g(sf)(x) &= \frac{1}{g(x)}(sfg)'(x) \\ &= \frac{1}{g(x)}(f'(x)s(x)g(x) + f(x)(sg)'(x)) \end{aligned}$$

which, by (1.1), reads

$$\begin{aligned} T_g(sf)(x) &= \frac{1}{g(x)}(f'(x)s(x)g(x) + f(x)\tau(x)g(x)) \\ &= f'(x)s(x) + f(x)\tau(x). \end{aligned}$$

□

Theorem 2.1 shows that the Stein equation related to any density g satisfying (1.1) enjoys the tractable form

$$s(x)f'(x) + \tau(x)f(x) = h(x) - \mathbb{E}h(W) \quad (2.1)$$

where W is random variable with density g . Schoutens [8] found a solution to the Stein equation (2.1) and established a bound for the solution, under the condition that the function τ be a decreasing linear function (which is the case for the so-called Pearson and Ord classes of distributions considered in [8]).

The following result comes from Proposition 1 in [8]. We again take the support of the density function to be $(0, \infty)$.

Proposition 2.1 Consider a density function $g > 0$ on $(0, \infty)$ satisfying Equation (1.1), for some polynomial functions s and τ . Then a solution of the Stein equation (2.1) is

$$\begin{aligned} f_h(x) &= \frac{1}{s(x)g(x)} \int_0^x g(t) [h(t) - \mathbb{E}h(W)] dt \\ &= \frac{-1}{s(x)g(x)} \int_x^{+\infty} g(t) [h(t) - \mathbb{E}h(W)] dt. \end{aligned} \quad (2.2)$$

Remark 2.1 *The proof of this proposition is given in [8] just by calculating the derivative of the function f_h defined by (2.2) and checking directly that f_h satisfies (2.1). Our following proposition complements this result.*

Proposition 2.2 *Under the notation and assumptions of Proposition 2.1,*

- *The solutions of the Stein equation (2.1) are of the form*

$$\begin{aligned} f_h(x) &= \frac{1}{s(x)g(x)} \int_0^x g(t) [h(t) - \mathbb{E}h(W)] dt + \frac{C}{s(x)g(x)} \\ &= \frac{-1}{s(x)g(x)} \int_x^{+\infty} g(t) [h(t) - \mathbb{E}h(W)] dt + \frac{C}{s(x)g(x)} \end{aligned} \quad (2.3)$$

where C is constant.

- *Suppose $\lim_{x \rightarrow 0} s(x)g(x) = 0$. For the solution to be bounded, it is necessary that $C = 0$ in (2.3).*

Proof:

Multiplying both sides of (2.1) by $g(x)$ we have

$$s(x)g(x)f'(x) + \tau(x)g(x)f(x) = g(x)(h(x) - \mathbb{E}h(W))$$

which, by (1.1), can be written

$$s(x)g(x)f'(x) + (sg)'(x)f(x) = g(x)(h(x) - \mathbb{E}h(W)),$$

i.e.

$$(sgf)'(x) = g(x)(h(x) - \mathbb{E}h(W)).$$

As a consequence, there exists a constant C such that

$$s(x)g(x)f(x) = \int_0^x g(t) [h(t) - \mathbb{E}h(W)] dt + C \quad (2.4)$$

which implies (2.3).

Suppose f is bounded. Since $\lim_{x \rightarrow 0} s(x)g(x) = 0$, letting x tend to 0 in (2.4) yields $C = 0$.

The second expression for f_h follows from the fact that, since W has density g ,

$$\int_0^{+\infty} g(t) [h(t) - \mathbb{E}h(W)] dt = 0.$$

□

The following proposition proves that the solution given by (2.2) is bounded indeed if h is bounded, and thus is the unique bounded solution to the Stein equation associated to the density g . A bound is provided.

Proposition 2.3 Consider a density function $g > 0$ on $(0, \infty)$ satisfying Equation (1.1), where s and τ are polynomial functions such that $s > 0$ on $(0, \infty)$ and τ is decreasing and has a unique zero α on $(0, \infty)$. Assume that $\lim_{x \rightarrow 0} s(x)g(x) = \lim_{x \rightarrow \infty} s(x)g(x) = 0$. If h is a bounded continuous function, then

$$\|f_h\| \leq M \|h(\cdot) - \mathbb{E}h(Z)\| \quad (2.5)$$

where

$$M = \max \left(\frac{1}{s(\alpha)g(\alpha)} \int_0^\alpha g(t)dt, \frac{1}{s(\alpha)g(\alpha)} \int_\alpha^{+\infty} g(t)dt \right)$$

and $\|f_h\| = \sup_{x>0} |f_h(x)|$.

Remark 2.2 This result is a reformulation of Lemma 1 in [8] without the assumption that τ is linear. With this assumption, [8] established the same bound with $\alpha = \mathbb{E}(X)$ (for a random variable X with density g), which is not true if τ is not linear. The proof given below follows the lines of that of [8] where we observed that the assumption of linearity of τ was used nowhere except to state that its only zero is $\alpha = \mathbb{E}(X)$.

The proof of Proposition 2.3 uses the following lemma :

Lemma 2.1 Under the assumptions of Proposition 2.3,

$$\int_0^x g(t)dt \leq \frac{s(x)g(x)}{\tau(x)} \quad \text{for } x < \alpha$$

and

$$\int_x^{+\infty} g(t)dt \leq \frac{-s(x)g(x)}{\tau(x)} \quad \text{for } x > \alpha.$$

Proof: Suppose $x < \alpha$. Since τ is positive and decreasing on $(0, \alpha)$, we have $\frac{\tau(t)}{\tau(x)} \geq 1$ for all $t \leq x$. Therefore

$$\begin{aligned} \int_0^x g(t)dt &\leq \int_0^x g(t) \frac{\tau(t)}{\tau(x)} dt \\ &= \frac{1}{\tau(x)} \int_0^x \tau(t)g(t)dt \\ &= \frac{s(x)g(x)}{\tau(x)} \end{aligned}$$

because of (1.1) and as $\lim_{t \rightarrow 0} s(t)g(t) = 0$.

For $x > \alpha$, since τ is negative and decreasing on (α, ∞) , we have $\frac{\tau(t)}{\tau(x)} \geq 1$ for all $t \geq x$. As a consequence,

$$\begin{aligned} \int_x^{+\infty} g(t)dt &\leq \int_x^{+\infty} g(t) \frac{\tau(t)}{\tau(x)} dt \\ &= \frac{-s(x)g(x)}{\tau(x)} \end{aligned}$$

since $\lim_{t \rightarrow \infty} s(t)g(t) = 0$. □

Now, let us prove (2.5).

Proof:

For $x < \alpha$,

$$\begin{aligned} |f_h(x)| &= \left| \frac{1}{s(x)g(x)} \int_0^x g(t) [h(t) - \mathbb{E}h(W)] dt \right| \\ &\leq \frac{1}{s(x)g(x)} \int_0^x g(t) |h(t) - \mathbb{E}h(W)| dt \\ &\leq \|h(\cdot) - \mathbb{E}h(W)\| \frac{1}{s(x)g(x)} \int_0^x g(t) dt. \end{aligned}$$

Let $l(x) = \frac{1}{s(x)g(x)} \int_0^x g(t) dt$. Then l is differentiable on $(0, \infty)$ and

$$\begin{aligned} l'(x) &= \frac{-(s(x)g(x))'}{(s(x)g(x))^2} \int_0^x g(t) dt + \frac{1}{s(x)} \\ &= \frac{-\tau(x)}{s^2(x)g(x)} \int_0^x g(t) dt + \frac{1}{s(x)} \end{aligned}$$

Using Lemma 2.1, we conclude that $l'(x) \geq 0$. Then $l(x) \leq l(\alpha)$.

For $x > \alpha$,

$$|f_h(x)| = \|h(x) - \mathbb{E}h(W)\| \frac{1}{s(x)g(x)} \int_x^{+\infty} g(t) dt.$$

Let $u(x) = \frac{1}{s(x)g(x)} \int_x^{+\infty} g(t) dt$. The function u is differentiable on $(0, \infty)$ and

$$u'(x) = \frac{-\tau(x)}{s^2(x)g(x)} \int_x^{+\infty} g(t) dt - \frac{1}{s(x)}$$

By Lemma 2.1, we conclude that $u'(x) \leq 0$. Then $u(x) \leq u(\alpha)$. □

In the two next sections we apply the previous results to the GIG and Kummer distributions.

3 About the Stein equation of the generalized inverse Gaussian distribution

Recall that the density of the GIG distribution with parameters $p \in \mathbb{R}$, $a > 0$, $b > 0$ is

$$g_{p,a,b}(x) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1} e^{-\frac{1}{2}(ax+b/x)}, \quad x > 0, \quad (3.1)$$

where K_p is the modified Bessel function of the third kind.

Let

$$s(x) = x^2 \quad \text{and} \quad \tau(x) = \frac{b}{2} + (p+1)x - \frac{a}{2}x^2. \quad (3.2)$$

Then, as observed by [2], the GIG density $g_{p,a,b}$ satisfies

$$(s(x)g_{p,a,b}(x))' = \tau(x)g_{p,a,b}(x).$$

This enables us to apply Theorem 2.1 to retrieve the following Stein characterization of the GIG distribution given in [4] and [2]:

Proposition 3.1 *A random variable X follows the GIG distribution with density $g_{p,a,b}$ if and only if, for all real-valued and differentiable function f such that*

$$\lim_{x \rightarrow \infty} g_{p,a,b}(x)f(x) = \lim_{x \rightarrow 0} g_{p,a,b}(x)f(x) = 0, \text{ we have:}$$

$$\mathbb{E} \left[X^2 f'(X) + \left(\frac{b}{2} + (p+1)X - \frac{a}{2}X^2 \right) f(X) \right] = 0.$$

The corresponding Stein equation is

$$x^2 f'(x) + \left(\frac{b}{2} + (p+1)x - \frac{a}{2}x^2 \right) f(x) = h(x) - \mathbb{E}h(W) \quad (3.3)$$

where h is a bounded function and W a random variable following the GIG distribution with parameters p, a, b .

We apply Proposition 2.2 and Proposition 2.3 to solve Equation (3.3) and bound the solution. Let us check that the assumptions of these propositions are true in the GIG case.

Firstly, we note that, by (3.1),

$$s(x)g_{p,a,b}(x) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p+1} e^{-\frac{1}{2}(ax+b/x)}, \quad x > 0,$$

which shows that $\lim_{x \rightarrow \infty} s(x)g_{p,a,b}(x) = \lim_{x \rightarrow 0} s(x)g_{p,a,b}(x) = 0$.

Secondly, observe that if $p \leq -1$, then the function τ defined by (3.2) is decreasing on $(0, \infty)$ and that its only zero on $(0, \infty)$ is $\alpha = \frac{p+1 + \sqrt{(p+1)^2 + ab}}{a}$.

Thus, by using Proposition 2.2 and Proposition 2.3, we obtain the following theorem.

Theorem 3.1 *The GIG Stein equation (3.3) has solution*

$$\begin{aligned} f_h(x) &= \frac{1}{s(x)g_{p,a,b}(x)} \int_0^x g_{p,a,b}(t) [h(t) - \mathbb{E}h(W)] dt \\ &= \frac{-1}{s(x)g_{p,a,b}(x)} \int_x^{+\infty} g_{p,a,b}(t) [h(t) - \mathbb{E}h(W)] dt \end{aligned} \quad (3.4)$$

where W follows the GIG distribution with parameters p, a, b .

If h is a bounded continuous function and $p \leq -1$, then the function defined by (3.4) is the unique bounded solution of (3.3) and

$$\|f_h\| \leq M \|h(\cdot) - \mathbb{E}h(W)\|$$

where

$$\alpha = \frac{p+1 + \sqrt{(p+1)^2 + ab}}{a},$$

$$M = \max \left(\frac{1}{s(\alpha)g_{p,a,b}(\alpha)} \int_0^\alpha g_{p,a,b}(t) dt; \frac{1}{s(\alpha)g_{p,a,b}(\alpha)} \int_\alpha^{+\infty} g_{p,a,b}(t) dt \right).$$

Remark 3.1 *This result was claimed by Gaunt (see [2]) with $\alpha = \mathbb{E}(X)$ by applying Proposition 1 of [8]. The only slight mistake is that τ is not a polynomial function of degree one as in [8].*

4 About the Stein equation related to the Kummer distribution

Recall that for $a > 0$, $b \in \mathbb{R}$, $c > 0$, the Kummer distribution $K(a, b, c)$ has density

$$k_{a,b,c}(x) = \frac{1}{\Gamma(a)\psi(a, a-b+1; c)} x^{a-1} (1+x)^{-a-b} e^{-cx}, \quad (x > 0)$$

where ψ is the confluent hypergeometric function of second kind. Let

$$s(x) = x(1+x) \quad \text{and} \quad \tau(x) = (1-b)x - cx(1+x) + a. \quad (4.1)$$

Then

$$(s(x)k_{a,b,c}(x))' = \tau(x)k_{a,b,c}(x).$$

Then we can use Theorem 2.1 to obtain the following Stein characterization of the Kummer distribution:

Theorem 4.1 *A random variable X follows the $K(p, a, b)$ distribution if and only if, for all differentiable function f ,*

$$\mathbb{E}[X(X+1)(f'(X) + ((1-b)X - cX(1+X) + a)f(X))] = 0.$$

The corresponding Stein equation is

$$x(x+1)f'(x) + [(1-b)x - cx(1+x) + a]f(x) = h(x) - \mathbb{E}h(W) \quad (4.2)$$

where W has density $k_{a,b,c}$.

We have

$$s(x)k_{a,b,c}(x) = \frac{1}{\Gamma(a)\psi(a, a-b+1; c)} x^a(1+x)^{1-a-b} e^{-cx}, \quad x > 0$$

and we see that $\lim_{x \rightarrow \infty} s(x)k_{a,b,c}(x) = \lim_{x \rightarrow 0} s(x)k_{a,b,c}(x) = 0$.

Note also that if $1-b-c \leq 0$, then the function τ defined by (4.1) is decreasing on $(0, \infty)$ and its only zero on this interval is $\alpha = \frac{1-b-c + \sqrt{(1-b-c)^2 + 4ac}}{2c}$.

Then we use again Proposition 2.2 and Proposition 2.3 to obtain the following result:

Theorem 4.2 *The Kummer Stein equation (4.2) has solution*

$$\begin{aligned} f_h(x) &= \frac{1}{s(x)k_{a,b,c}(x)} \int_0^x k_{a,b,c}(t) [h(t) - \mathbb{E}h(W)] dt \\ &= \frac{-1}{s(x)k_{a,b,c}(x)} \int_x^{+\infty} k_{a,b,c}(t) [h(t) - \mathbb{E}h(W)] dt \end{aligned} \quad (4.3)$$

where $W \sim K(a, b, c)$.

If h is a bounded continuous function and $1-b-c \leq 0$, then f_h defined by (4.3) is the unique bounded solution of (4.2) and

$$\|f_h\| \leq M \|h(\cdot) - \mathbb{E}h(W)\|$$

where

$$\alpha = \frac{1-b-c + \sqrt{(1-b-c)^2 + 4ac}}{2c},$$

$$M = \max \left(\frac{1}{s(\alpha)k_{a,b,c}(\alpha)} \int_0^\alpha k_{a,b,c}(t) dt; \frac{1}{s(\alpha)k_{a,b,c}(\alpha)} \int_\alpha^{+\infty} k_{a,b,c}(t) dt \right).$$

Remark 4.1 *These results could be used in future work to provide rates of convergence in limit problems related to the GIG and Kummer distributions.*

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