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# How many weights can a linear code have?

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**Abstract** We study the combinatorial function  $L(k, q)$ , the maximum number of nonzero weights a linear code of dimension  $k$  over  $\mathbb{F}_q$  can have. We determine it completely for  $q = 2$ , and for  $k = 2$ , and provide upper and lower bounds in the general case when both  $k$  and  $q$  are  $\geq 3$ . A refinement  $L(n, k, q)$ , as well as nonlinear analogues  $N(M, q)$  and  $N(n, M, q)$ , are also introduced and studied.

**Keywords** Linear codes · Hamming weight · Perfect difference sets

**Mathematics Subject Classification** 94B05 · 05B10

## 1 Introduction

There are several problems in extremal combinatorics on distances in codes. For instance, the famous paper [5] derives an upper bound on the size of a code  $C$  over  $\mathbb{F}_q$  with exactly  $s$  distinct distances:

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$$|C| \leq \sum_{j=0}^s \binom{n}{j} (q-1)^j. \quad (1)$$

In the same spirit, other authors have given upper bounds on the size of codes with one or several forbidden distances [6].

In this note, we tackle a related but distinctly different problem: how many distinct weights can a linear code of given dimension over a given finite field have? In other words, we study the combinatorial function  $L(k, q)$ , the maximum number of nonzero weights a code of dimension  $k$  over  $\mathbb{F}_q$  may have. While an upper bound is easy to prove (Proposition 2), its tightness is nontrivial<sup>1</sup> and we only manage to establish it in some special cases like  $k = 2$  or  $q = 2$  (Cf. Theorems 1 and 2). Numerical experiments with very long random codes suggest it is tight for all  $k$ 's and  $q$ 's. We leave the question as an open problem. We can also study the more refined function  $L(n, k, q)$ , the maximum number of nonzero weights an  $[n, k]_q$  code may have. This latter function is related to both  $L(k, q)$  and Eq. (1) above. The nonlinear counterpart of  $L(k, q)$  denoted by  $N(M, q)$ , can be determined explicitly (Theorem 6). The nonlinear counterpart of  $L(n, k, q)$  denoted by  $N(n, M, q)$ , can also be studied. The rate of convergence of  $N(n, M, q)$  towards  $N(M, q)$  requires perfect difference sets [3] and primes in short intervals [2] for its careful study.

The material is organized as follows. Section 2 collects the necessary notations and definitions. Section 3 studies upper bounds in the linear code case. Section 4 derives lower bounds in that situation. Section 5 introduces and investigates the function  $L(n, k, q)$ . Section 6 tackles the nonlinear analogues of  $L(k, q)$  and  $L(n, k, q)$ , denoted by  $N(M, q)$ , and  $N(n, M, q)$ , respectively. Section 7 concludes the article. An appendix collects some numerical values, which comfort the Conjecture that Proposition 2 is tight.

## 2 Definitions and notation

Let  $q$  be a prime power, and  $\mathbb{F}_q$  denote the finite field of order  $q$ . By a **code** of length  $n$  over  $\mathbb{F}_q$ , we shall mean a proper subset of  $\mathbb{F}_q^n$ . This code is **linear** if it is a  $\mathbb{F}_q$ -vector subspace of  $\mathbb{F}_q^n$ . The **dimension** of a code, denoted by  $k$ , is equal to its dimension as a vector space. The parameters of such a code are written compactly as  $[n, k]_q$ . The **Hamming weight** of  $x \in \mathbb{F}_q^n$ , denoted by  $w(x)$ , is the number of indices  $i$  where  $x_i \neq 0$ . The **Hamming distance** between  $x \in \mathbb{F}_q^n$ , and  $y \in \mathbb{F}_q^n$ , denoted by  $d(x, y)$ , is defined by  $d(x, y) = w(x - y)$ . For a given prime power  $q$  and given values of  $k$ , let  $L(k, q)$  denote the largest possible number of nonzero weights a  $q$ -ary code can have. If  $C(n)$  is a family of codes of parameters  $[n, k_n]_q$ , the **rate**  $R$  is defined as

$$R = \limsup_{n \rightarrow \infty} \frac{k_n}{n}.$$

Recall that the  $q$ -ary **entropy function**  $H_q(\cdot)$  is defined for  $0 < y < 1$ , by

$$H_q(y) = y \log_q(q-1) - y \log_q(y) - (1-y) \log_q(1-y).$$

## 3 Upper bounds

The following monotonicity properties of  $L(k, q)$  are given without proof.

<sup>1</sup> After submission of this article, a proof was found in [1].

**Proposition 1** For all nonnegative integers  $k, m$  and all prime powers  $q$  we have:

$$\begin{aligned} L(k, q) &\leq L(k + 1, q), \\ L(k, q) &\leq L(k, q^m). \end{aligned}$$

The next result is trivial but crucial.

**Proposition 2** For all prime powers  $q$ , and all integers  $k \geq 1$ , we have

$$L(k, q) \leq \frac{q^k - 1}{q - 1}.$$

*Proof* The total numbers of nonzero codewords of a code of dimension  $k$  over  $\mathbb{F}_q$  is  $q^k - 1$ , and all the nonzero multiples of a given codeword share the same weight. □

This bound is met with equality if  $q = 2$ .

**Theorem 1** For all integers  $k \geq 1$ , we have

$$L(k, 2) = 2^k - 1.$$

*Proof* Denote by  $G_k$  the generator matrix of an  $[n, k]_q$  with  $L(k, 2)$  weights  $w_1 < w_2 < \dots < w_{L(k,2)}$ . Define  $H_{k+1}$  a matrix obtained from  $G_k$  by adding a  $k$  by  $t$  block of zeros, and by  $G_{k+1}$  the matrix obtained by  $H_{k+1}$  by adding an a row with first  $n$  coordinates zero and last  $t$  coordinates = 1. The code spanned by the rows of  $G_{k+1}$  has all these weights plus the  $L(k, 2) + 1$  new weights  $t < t + w_1 < \dots < t + w_{L(k,2)}$ . The two sets of weights will have void intersection if  $w_{L(k,2)} < t$ . This makes  $2L(k, 2) + 1$  weights altogether. Note that the rank of  $G_{k+1}$  is  $k + 1$ . Thus we have proved that  $L(k + 1, 2) \geq 2L(k, 2) + 1$ , which implies by induction, starting from  $L(1, 2) = 1$ , the lower bound  $L(k, 2) \geq 2^k - 1$ . The result follows. □

*Remark* We are now ready to given an alternative proof of Theorem 1. we can exhibit a linear code  $C$  with dimension  $k$  over  $\mathbb{F}_2$  with  $2^k - 1$  nonzero weights. Let the generator matrix of  $C$  be

$$\left( \begin{array}{cccc|cccc|cccc|cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \end{array} \right),$$

$\underbrace{\qquad\qquad\qquad}_{a_1} \quad \underbrace{\qquad\qquad\qquad}_{a_2} \quad \underbrace{\qquad\qquad\qquad}_{a_3} \quad \underbrace{\qquad\qquad\qquad}_{a_4} \quad \underbrace{\qquad\qquad\qquad}_{a_k}$

where  $a_1 = 1, a_2 = 2, a_3 = 2^2, a_4 = 2^3, \dots, a_k = 2^{k-1}$ . Since  $a_{j_1} + a_{j_2} + \dots + a_{j_t} = (\dots 010 \dots 010 \dots 010 \dots)_2$  in base 2, and the coordinates of 1's are  $j_1, j_2, \dots, j_t$ , respectively.

Thus, we obtain all integers of  $k$  bits as possible weights that is the set  $\{1, 2, 3, \dots, 2^k - 1\}$  of cardinality  $2^k - 1$  in all.

The bound in Proposition 2 is also tight when  $k = 2$ .

**Theorem 2** For all prime powers  $q$ , we have  $L(2, q) = q + 1$ .

*Proof* Let  $\{u, v\}$  be a basis of a code  $C$  candidate to have  $q + 1$  weights. Denote by  $S, T$  the supports of  $u, v$  respectively. Let  $|S \setminus T| = a, |T \setminus S| = b$ . On the intersection  $S \cap T$  assume  $v$  is the all-one vector. Denote by  $\omega$  a primitive root of  $\mathbb{F}_q$ . Assume  $|S \cap T| = \binom{q}{2}$  and that  $u$  restricted to  $S \cap T$  is

$$(1, \omega, \omega, \omega^2, \omega^2, \omega^2, \dots, \omega^{q-2}, \dots, \omega^{q-2}),$$

where  $\omega^i$  occurs  $i + 1$  times. With these conventions, we see that the weights of  $C$  are

- $w(u) = a + \binom{q}{2}$ ,
- $w(v) = b + \binom{q}{2}$ ,
- $w(u - xv) = a + b + \binom{q}{2} - i$  if  $x = \omega^{i-1}$  for  $i = 1, 2, \dots, q - 1$ .

Assume  $a < b$ . The above weights will be pairwise different if  $a + b + \binom{q}{2} - (q - 1) > b + \binom{q}{2}$ , that is if  $a \geq q$ . Thus, under these conditions,  $C$  counts  $2 + q - 1 = q + 1$  nonzero weights. □

*Remark* The shortest  $[n, 2]_q$  code with  $L(2, q)$  nonzero weights obtained by this construction has  $n = \binom{q}{2} + 2q + 1$ .

### 4 Lower bounds

The easiest lower bound is

**Proposition 3** For all prime powers  $q$ , and all integers  $k \geq 1$ , we have  $L(k, q) \geq k$ .

*Proof* Consider the code  $\mathbb{F}_q^k$ , of length and dimension  $k$ . □

This can be improved to a bound that is exponential in  $k$ .

**Proposition 4** For all prime powers  $q$ , and all integers  $k \geq 1$ , we have

$$L(k + 1, q) \geq 2L(k, q) + 1.$$

In particular, for all integers  $k \geq 2$ , we have

$$L(k, q) \geq 2^{k-2}q + 2^{k-2} + 1.$$

*Proof* Same argument as in the first proof of Theorem 1. The second assertion follows by iterating this bound starting from  $L(2, q) = q + 1$ . □

An asymptotic version of the preceding results is as follows. Define

$$\lambda(q) = \limsup_{n \rightarrow \infty} \frac{1}{k} \log_q(L(k, q)).$$

**Theorem 3** For all prime powers  $q$  we have

$$\log_q 2 \leq \lambda(q) \leq 1.$$

In particular  $\lambda(2) = 1$ .

*Proof* The first inequality comes from Proposition 4. The second one comes Proposition 2. □

*Remark* Since we conjecture that the bound of Proposition 2 is tight, it is natural to conjecture that  $\lambda(q) = 1$  for all prime powers  $q$ .

### 5 Refinements and asymptotics

A more complex function is  $L(n, k, q)$  the largest number of nonzero weights an  $[n, k]_q$ -code can have. This function is related to  $L(k, q)$  in several ways. The following monotonicity properties of  $L(n, k, q)$  are given without proof.

**Proposition 5** *For all nonnegative integers  $k, m$  and all prime powers  $q$  we have:*

$$L(n, k, q) \leq L(n, k + 1, q),$$

$$L(n, k, q) \leq L(n, k, q^m).$$

The three following lemmas are useful for the proof of Theorem 4.

**Lemma 1** *For all prime powers  $q$ , and all nonnegative integers  $n, k$  we have  $L(n, k, q) \leq L(k, q)$ .*

*Proof* Immediate from the definitions. □

The new function is also monotone in  $n$ .

**Lemma 2** *For all prime powers  $q$ , and all nonnegative integers  $n, k$  we have  $L(n, k, q) \leq L(n + 1, k, q)$ .*

*Proof* If  $C$  is an  $[n, k]_q$  code with  $L(n, k, q)$  nonzero weights, then  $C$  extended by a constant zero coordinate is an  $[n + 1, k]_q$ -code with the same number of nonzero weights. □

**Lemma 3** *For all prime powers  $q$ , and all nonnegative integers  $n, k$  we have  $L(n, k, q) \leq n$ .*

*Proof* Note that, by definition of the Hamming weight, a code of length  $n$  can have at most  $n$  distinct weights. □

We now connect the new function  $L(n, k, q)$  with  $L(k, q)$ .

**Theorem 4** *For all prime powers  $q$ , and all nonnegative integers  $k$  we have*

$$\lim_{n \rightarrow \infty} L(n, k, q) = L(k, q).$$

*More precisely, there is an integer  $n_0 \geq L(k, q)$ , such that for all  $n \geq n_0$  we have  $L(n, k, q) = L(k, q)$ .*

*Proof* By Lemmas 1 and 2, the sequence  $n \mapsto L(n, k, q)$  is increasing and bounded. Hence, being integral, it converges stably to a limit which can be no other than  $L(k, q)$ . Let  $n_0$  be such that  $L(n_0, k, q) = L(k, q)$ . By Lemma 3, we see that  $n_0 \geq L(k, q)$ . □

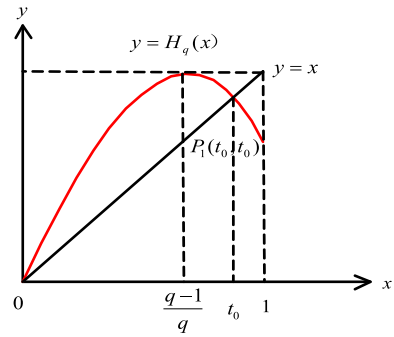
*Remark* The computations of the Appendix suggest that such an  $n_0$  can be very large. If Proposition 2 is tight then, by Theorem 4  $n_0 \geq \frac{q^k - 1}{q - 1}$ . In the special case  $q = 2$ , the second proof of Theorem 1 shows that  $n_0 = 2^k - 1$ .

There is a link to Delsarte’s bound (Eq. (1)) quoted in the Introduction.

**Proposition 6** *For all prime powers  $q$ , and all integers  $n \geq k \geq 1$ , we have*

$$q^k \leq \sum_{i=0}^{L(n,k,q)} \binom{n}{i} (q - 1)^i.$$

**Fig. 1** Definition of  $t(q)$



Further

$$L(k, q) \leq \frac{\sum_{i=0}^{L(n,k,q)} \binom{n}{i} (q-1)^i - 1}{q-1}.$$

*Proof* The first assertion is a direct application of Eq. (1) in Introduction ([5, Theorem 4.1]) with  $|C| = q^k$ , and  $s = L(n, k, q)$ . Combining this result with Proposition 2 gives the second assertion.  $\square$

We give an asymptotic version of the preceding results. Let

$$\mathcal{L}(R) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log_q(L(n, \lfloor Rn \rfloor, q)).$$

**Theorem 5** *If  $C_n$  is a family of codes of rate  $R$  then*

$$\mathcal{L}(R) \leq R\lambda(q) \leq H_q(\mathcal{L}(R)).$$

*In particular  $\mathcal{L}(R) \leq t(q)$ , where  $t(q)$  is the unique solution in the range  $(0, \frac{q-1}{q})$  of  $H_q(x) = x$ . See Fig. 1.*

*Proof* The first inequality follows by Lemma 1, upon observing that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} (L(k, q)) = R\lambda(q).$$

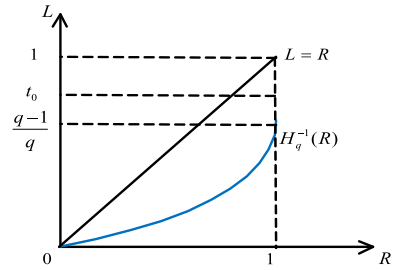
The second inequality comes from the second assertion of Proposition 6, after using standard entropic estimates [7]. The second assertion is obtained by combining the first and second inequality.  $\square$

Define the domain  $\mathcal{D}$  as the set of points in the plane  $(R, \mathcal{L})$  that are realized by a family of codes. By the preceding result, this domain is contained in the domain of boundaries given by, counterclockwise, in Fig. 2 by

1. the straight line  $\mathcal{L} = R$  from  $R = 0$  till  $R = t(q)$ ,
2. the horizontal line  $\mathcal{L} = t(q)$  from  $R = t(q)$  till  $R = 1$ ,
3. the vertical line  $R = 1$  from  $\mathcal{L} = t(q)$  till  $\mathcal{L} = \frac{q-1}{q}$ ,
4. the curve  $\mathcal{L} = H_q^{-1}(R)$ , from  $R = 1$  till  $R = 0$ .

Determining the domain  $\mathcal{D}$  explicitly, in the same way as the domain of packing and covering codes in [4] is a challenging open problem.

**Fig. 2** Boundaries of domain  $\mathcal{D}$



### 6 Nonlinear codes

*Warning* In this section only  $q$  is an arbitrary integer  $> 1$ .

The nonlinear analogue of the function  $L(k, q)$  is the function  $N(M, q)$  which is the largest number of distances between two codewords of an unrestricted code of size  $M$  over some finite alphabet  $A_q$  of size  $q$ . This function is completely determined in the following Theorem.

**Theorem 6** For all integers  $M \geq 2$ , we have

$$N(M, q) = \binom{M}{2}.$$

*Proof* By definition we have immediately  $N(M, q) \leq \binom{M}{2}$ . By an inductive process, we construct a code  $C_M$  with  $\binom{M}{2}$  distances. To simplify matters take  $q = 2$ . We search for codes in a special form where nonzero codewords are of the form  $(1, 1, \dots, 1, 0, \dots, 0)$ , that is a run of ones followed by a run of zeros. Thus the distance between two such codewords is equal to the difference of their weights. For  $M = 2$ , we may take the length 1 code  $\{0, 1\}$ . Assume  $C_M$  is constructed with codewords of successive weights  $w_0 = 0 < w_1 < \dots < w_{M-1}$ . We construct a code  $C_{M+1}$  by adding a tail of zeros to  $C_M$  on the right, of length to be specified later, and by adding a new codeword of weight  $w_M$ . The new distances are  $M$  in number, given by  $w_M, w_M - w_1, \dots, w_M - w_{M-1}$ . These distances are pairwise distinct because  $(w_M - w_i) - (w_M - w_j) = w_j - w_i$ . To make sure they are distinct from the distances in  $C_M$ , we must check that

$$(w_M - w_i) \neq w_j - w_k,$$

with  $i, j, k$  distinct nonnegative integers  $\leq M - 1$ . This is enforced if we take  $w_M$  large enough. This condition on  $w_M$ , in turn, will determine how long the tail must be. Since  $\binom{M+1}{2} - \binom{M}{2} = M$ , we are done. □

The nonlinear analogue of the function  $L(n, k, q)$  is the function  $N(n, M, q)$  which is the largest number of distances between two codewords of an unrestricted code of size  $M$  and length  $n$  over some alphabet  $A_q$ , of size  $q$ .

The analogue of Theorem 4 in this context is as follows. The proof is similar and omitted.

**Theorem 7** For all integers  $q > 1$ , and all nonnegative integers  $M$  we have

$$\lim_{n \rightarrow \infty} N(n, M, q) = N(M, q).$$

More precisely, there is an integer  $n_0 \geq N(M, q)$ , such that for all  $n \geq n_0$  we have  $N(n, M, q) = N(M, q)$ .



Denote by  $N_0(M, q)$  the smallest integer  $n$  such that  $N(n, M, q) = N(M, q)$ .

**Proposition 7** *If  $M - 1$  is a power of a prime, then  $N_0(M, q) \leq 2N(M, q) + 1$ .*

*Proof* Assume  $M = s + 1$ , where  $s$  is a power of a prime. We know there is a Singer difference set [9]  $S = \{v_0, v_1, \dots, v_{s+1}\}$ , with parameters  $(s^2 + s + 1, s + 1, 1)$ . Consider the  $s + 1$  by  $s^2 + s + 1$  matrix with rows  $g_i$ , when  $g_i$  contains  $v_i$  consecutive ones to the left and zeros elsewhere. The Hamming distance from  $g_i$  to  $g_j$  is  $|v_i - v_j|$ . The code formed by the  $M$  rows of this matrix has length  $s^2 + s + 1 = M^2 - M + 1 = 2\binom{M}{2} + 1$  and  $\binom{M}{2}$  distances, by the design property. Hence, in this case,  $n_0 \leq 2\binom{M}{2} + 1$ . For instance, if  $s = 2$ , we have  $S = \{1, 2, 4\}$ , and the code is  $\{1000000, 1100000, 1111000\}$ . See [3, p. 264] for details on, and examples of Singer difference sets.  $\square$

Denote, for any integer  $t$ , by  $pp(t)$  the smallest prime power  $\geq t$ .

**Corollary 1** *For all integers  $M > 1$ , we have*

$$N_0(M, q) \leq 2N(pp(M - 1) + 1, q) + 1 \leq 2N(2M, q) \sim 8\binom{M}{2}.$$

*Proof* We claim that  $N_0(M, q)$  is a nondecreasing function of  $M$ . The first inequality will follow by the previous theorem, since  $M \leq pp(M - 1) + 1$ . To prove the claim note that, if we have a set of  $M + 1$  vectors of length  $N_0(M + 1, q)$ , with  $\binom{M+1}{2}$  distances, removing any vector will result into a set of  $M$  vectors with  $\binom{M+1}{2} - M = \binom{M}{2}$  distances. Hence  $N_0(M, q) \leq N_0(M + 1, q)$ . The second inequality follows by the crude bound  $pp(x) \leq 2x$ , valid for any positive integer  $x$ .  $\square$

*Remark* It is possible to reduce the upper bound on  $pp(x)$  to  $pp(x) \leq x + x^a$ , with  $a = 0.525$ , building on recent estimates on the existence of primes in short intervals [2]. This sharpens the upper bound on  $N_0(M, q)$  to  $2N(M + O(M^a), q) + 1 \sim 2\binom{M}{2}$ , for  $M \rightarrow \infty$ .

## 7 Conclusion and open problems

In this note, we have studied a problem of extremal combinatorics: maximizing the number of distinct nonzero weights a linear code can have. We conjecture, based on extensive numerical calculations on very long codes, that the bound of Proposition 2 is tight but cannot prove it. A proof was found later in [1]. A recursive approach in the manner of the proof of Theorem 6 would require to produce  $q^k$  new weights to go from  $L(k, q)$  to  $L(k + 1, q)$ . But a code achieving  $L(k, q)$  has only  $\frac{q^k - 1}{q - 1} < q^k$  distinct weights. Thus establishing the tightness of Proposition 1 is the main open problem of this note. Sharpening the upper bound on  $N_0(M, q)$  of Corollary 1 is also a challenging question. Determining explicitly the domain  $\mathcal{D}$  of Sect. 5 seems to require better lower bounds on  $L(n, kq)$  that those at our disposal.

## Appendix: numerical examples

We provide lower bounds on  $L(k, q)$  by computing the number of weights in long random codes produced by the computer package Magma [8]. We give some numerical examples in Table 1 about the lower bound of Proposition 4.

When  $n$  is in the millions, we can find linear  $[n, k]_q$ -codes that meet the upper bound in Proposition 2: see Table 2.

**Table 1** Proposition 4

$k$	3	4	4	6	6	10	10	12	12	12
$q$	3	5	8	9	13	16	25	29	49	121
$L(k, q) \geq$	11	29	41	177	241	4609	6913	31, 745	52, 225	125, 953

**Table 2**  $n = 6, 000, 000$

$k$	3	3	3	3	3	3	3	4	4	4	5	5
$q$	3	4	5	7	8	9	11	3	4	5	3	4
$L(k, q) =$	13	21	31	57	73	91	133	40	85	156	121	341

### References

1. Albersson T., Neri A.: Maximum weight spectrum codes. <https://arxiv.org/pdf/1803.04020.pdf>.
2. Baker R.C., Harman G., Pintz J.: The difference between consecutive primes, II. Proc. Lond. Math. Soc. **83**(3), 532–562 (2001).
3. Beth T., Jungnickel D., Lenz H.: Design theory. BI-Institut, Mannheim, Wien, Zurich (1985).
4. Cohen G., Honkala I., Litsyn S., Solé P.: Long packing and covering codes. IEEE Trans. Inf. Theory **43**, 1617–1619 (1997).
5. Delsarte P.: Four fundamentals parameters of a code and their combinatorial significance. Inf. Control **23**, 407–438 (1973).
6. Enomoto H., Frankl P., Ito N., Nomura K.: Codes with given distances. Graph Comb. **3**, 25–38 (1987).
7. Huffman W.C., Pless V.: Fundamentals of Error Correcting Codes. Cambridge University Press, Cambridge (2003).
8. <http://magma.maths.usyd.edu.au/magma/>. Accessed 28 Jan 2018
9. Singer J.: A theorem in finite geometry and some applications to number theory. Trans. Am. Math Soc. **43**, 377–385 (1938).