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STOCHASTIC HOMOGENIZATION OF NONCONVEX INTEGRALS IN THE SPACE OF FUNCTIONS OF BOUNDED DEFORMATION

OMAR ANZA HAFSA AND JEAN-PHILIPPE MANDALLENNA

ABSTRACT. We study stochastic homogenization by Γ -convergence of nonconvex integrals of the calculus of variations in the space of functions of bounded deformation.

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1. INTRODUCTION

The space of functions of bounded deformation has been introduced by [TS78, Suq78, MSC79, Suq79] to study variational problems of plasticity theory (see [Tem80, Tem83]). This space is made of vectorial L^1 -functions u whose the symmetric part of the distributional derivative, i.e. $Eu := \frac{1}{2}(Du + Du^T)$, is a vectorial Radon measure. For such functions u we have $Eu = \mathcal{E}u dx + E^s u$ with $\mathcal{E}u := \frac{1}{2}(\nabla u + \nabla u^T)$ the symmetrized gradient of u , where $(\mathcal{E}u, E^s u)$ is the Lebesgue decomposition of Eu with respect to the Lebesgue measure dx . In the context of the hyperelastic-plastic theory, at the macroscopic scale, the energy of deformation of a hyperelastic-plastic material occupying in a reference configuration a bounded open set O is of the form

$$\int_O W_{\text{macro}}(Eu) \tag{1.1}$$

Key words and phrases. Stochastic homogenization, Γ -convergence, nonconvex integrand, space of functions of bounded deformation.

where W_{macro} is the energy density of the hyperelastic-plastic material at the macroscopic scale. From the point of view of homogenization, an important problem is to look for an effective formula for W_{macro} which takes the heterogeneities of the material at the microscopic scale into account. To do this, a classical procedure consists of considering periodic or stochastic energy integrals on regular deformations representing the material at small scales $\varepsilon > 0$, i.e.

$$\int_O W\left(\frac{x}{\varepsilon}, \mathcal{E}u, \omega\right) dx \quad (1.2)$$

where $W(\frac{\cdot}{\varepsilon}, \xi, \omega)$ is the energy density of the material at the scale ε , and to pass to the limit, in the sense of Γ -convergence of De Giorgi, as ε tends to 0. So, under suitable assumptions on W , the problem is to know whether the Γ -limit of (1.2) is of type (1.1) and to find the formula of the energy density W_{macro} which will depend on W . In the periodic and convex case, i.e. when $W(\frac{\cdot}{\varepsilon}, \xi)$ is convex with respect to ξ , this Γ -convergence problem was solved by Bouchitté in [Bou87, Theorem 3.2] (see also Ansini and Ebobisse [AE01, Theorem 5.1]). The object of the present paper is to deal with the stochastic and nonconvex case.

The plan of the paper is as follows. The main result of the paper is stated in Sect. 2 (see Theorem 2.1). In Sect. 3 we give auxiliary results needed to prove Theorem 2.1. A key tool in the proof Theorem 2.1 is the one of subadditive process: this is recalled in §3.1. The properties of the homogenized density, which is defined as the almost sure limit of a subadditive process, are established in §3.2. In §3.3 we recall some properties of the functions of bounded deformation that we use in the proof of the lower bound and the upper bound. To establish the upper bound we also need a relaxation theorem in the space of functions of bounded deformation and the use of the Vitali envelope of a set function: these are recalled in §3.4 and §3.5 respectively. Finally, Theorem 2.1 is proved in Sect. 4. Its proof is divided into two propositions: the lower bound (see Proposition 4.1) and the upper bound (see Proposition 4.2).

2. MAIN RESULT

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\tau_z\}_{z \in \mathbb{Z}^N})$ be a dynamical system, let $N \in \mathbb{N}^*$, let $O \subset \mathbb{R}^N$ is a bounded open set, let $\mathcal{O}(O)$ be the class of open subsets of O and let $\text{BD}(O)$ be the space of functions of bounded deformation on O , i.e.

$$\text{BD}(O) := \left\{ u \in L^1(O; \mathbb{R}^N) : Eu := \frac{1}{2} (Du + Du^T) \in M(O) \right\},$$

where $M(O)$ is the space of $N \times N$ matrix-valued bounded Radon measures on O and Du is the distributional derivative of u . For each $u \in \text{BD}(O)$, $Eu = \mathcal{E}u dx + E^s u$ with $\mathcal{E}u(x) := \frac{1}{2}(\nabla u(x) + \nabla u(x)^T)$ the symmetrized gradient, where $(\mathcal{E}u, E^s u)$ is the Lebesgue decomposition of Eu with respect to the Lebesgue measure on O that we denote by dx . Let $\text{LD}(O) \subset \text{BD}(O)$ be given by

$$\text{LD}(O) := \left\{ u \in L^1(O; \mathbb{R}^N) : E^s u = 0 \right\}.$$

In this paper we are concerned with stochastic integrals $I_\varepsilon : \text{BD}(O) \times \Omega \rightarrow [0, \infty]$, depending on a parameter $\varepsilon > 0$, defined by

$$I_\varepsilon(u, \omega) := \begin{cases} \int_O W\left(\frac{x}{\varepsilon}, \mathcal{E}u(x), \omega\right) dx & \text{if } u \in \text{LD}(O) \\ \infty & \text{if } u \in \text{BD}(O) \setminus \text{LD}(O) \end{cases}$$

where $W : \mathbb{R}^N \times \mathbb{R}_{\text{sym}}^{N \times N} \times \Omega \rightarrow [0, \infty[$ is a Borel measurable stochastic integrand¹ satisfying the following conditions:

(C₁) W is $\{\tau_z\}_{z \in \mathbb{Z}^N}$ -covariant, i.e.

$$W(x + z, \xi, \omega) = W(x, \xi, \tau_z(\omega))$$

for $x \in \mathbb{R}^N$, all $\xi \in \mathbb{R}_{\text{sym}}^{N \times N}$, all $z \in \mathbb{Z}^N$ and all $\omega \in \Omega$;

(C₂) W has 1-growth, i.e. there exist $\alpha, \beta > 0$ such that for every $\omega \in \Omega$, one has

$$\alpha|\xi| \leq W(x, \xi, \omega) \leq \beta(1 + |\xi|) \quad (2.1)$$

for all $x \in \mathbb{R}^N$ and all $\xi \in \mathbb{R}_{\text{sym}}^{N \times N}$ with $\mathbb{R}_{\text{sym}}^{N \times N}$ denoting the space of $N \times N$ symmetric real matrices;

(C₃) W is Lipschitz continuous, i.e. there exists $C > 0$ such that for every $\omega \in \Omega$, one has

$$|W(x, \xi, \omega) - W(x, \zeta, \omega)| \leq C|\xi - \zeta|$$

for all $x \in \mathbb{R}^N$ and all $\xi \in \mathbb{R}_{\text{sym}}^{N \times N}$;

(C₄) W is symmetric quasiconvex, i.e. for every $\omega \in \Omega$, one has

$$W(x, \xi, \omega) = \inf \left\{ \int_{]0,1[^N} W(x, \xi + \mathcal{E}\phi(y), \omega) dy : \phi \in C_c^1(]0,1[^N; \mathbb{R}^N) \right\}$$

for all $x \in \mathbb{R}^N$ and all $\xi \in \mathbb{R}_{\text{sym}}^{N \times N}$.

The object of the paper is to compute the almost sure Γ -limit of $\{I_\varepsilon\}_{\varepsilon > 0}$ as $\varepsilon \rightarrow 0$ with respect to the strong convergence of $L^1(O; \mathbb{R}^N)$. By the almost sure $\Gamma(L^1)$ -limit of $\{I_\varepsilon\}_{\varepsilon > 0}$ as $\varepsilon \rightarrow 0$ we mean a functional $I_{\text{hom}} : \text{BD}(O) \times \Omega \rightarrow [0, \infty]$ such that for \mathbb{P} -a.e. $\omega \in \Omega$, one has:

Γ -lim: for every $u \in \text{BD}(O)$, $\Gamma(L^1)$ - $\underline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(u, \omega) \geq I_{\text{hom}}(u, \omega)$ with

$$\Gamma(L^1)\text{-}\underline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(u, \omega) := \inf \left\{ \underline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(u_\varepsilon, \omega) : u_\varepsilon \rightarrow u \text{ in } L^1(O; \mathbb{R}^N) \right\},$$

or equivalently, for every $u \in \text{BD}(O)$ and every $\{u_\varepsilon\}_{\varepsilon > 0} \subset \text{LD}(O)$ such that $u_\varepsilon \rightarrow u$ in $L^1(O; \mathbb{R}^N)$,

$$\underline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(u_\varepsilon, \omega) \geq I_{\text{hom}}(u, \omega);$$

¹By a Borel measurable stochastic integrand $W : \mathbb{R}^N \times \mathbb{R}_{\text{sym}}^{N \times N} \times \Omega \rightarrow [0, \infty[$ we mean that W is $(\mathcal{B}(\mathbb{R}^N) \otimes \mathcal{B}(\mathbb{R}_{\text{sym}}^{N \times N}) \otimes \mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable, where $\mathcal{B}(\mathbb{R}^N)$, $\mathcal{B}(\mathbb{R}_{\text{sym}}^{N \times N})$ and $\mathcal{B}(\mathbb{R})$ denote the Borel σ -algebra on \mathbb{R}^N , $\mathbb{R}_{\text{sym}}^{N \times N}$ and \mathbb{R} respectively.

Γ - $\overline{\text{lim}}$: for every $u \in \text{BD}(O)$, $\Gamma(L^1)$ - $\overline{\text{lim}}_{\varepsilon \rightarrow 0} I_\varepsilon(u, \omega) \leq I_{\text{hom}}(u, \omega)$ with

$$\Gamma(L^1)\text{-}\overline{\text{lim}}_{\varepsilon \rightarrow 0} I_\varepsilon(u, \omega) := \inf \left\{ \overline{\text{lim}}_{\varepsilon \rightarrow 0} I_\varepsilon(u_\varepsilon, \omega) : u_\varepsilon \rightarrow u \text{ in } L^1(O; \mathbb{R}^N) \right\},$$

or equivalently, for every $u \in \text{BD}(O)$ there exists $\{u_\varepsilon\}_{\varepsilon > 0} \subset \text{LD}(O)$ such that $u_\varepsilon \rightarrow u$ in $L^1(O; \mathbb{R}^N)$ and

$$\overline{\text{lim}}_{\varepsilon \rightarrow 0} I_\varepsilon(u_\varepsilon, \omega) \leq I_{\text{hom}}(u, \omega).$$

We then write $\Gamma(L^1)\text{-}\lim_{\varepsilon \rightarrow 0} I_\varepsilon = I_{\text{hom}}$. (For more details on the theory of Γ -convergence we refer to [DM93].) The main result of the paper is the following.

Theorem 2.1. *Assume that (C_1) , (C_2) , (C_3) and (C_4) hold. Then, $\Gamma(L^1)\text{-}\lim_{\varepsilon \rightarrow 0} I_\varepsilon = I_{\text{hom}}$ with $I_{\text{hom}} : \text{BD}(O) \times \Omega \rightarrow [0, \infty]$ given by*

$$I_{\text{hom}}(u, \omega) := \int_O W_{\text{hom}}(\mathcal{E}u(x), \omega) dx + \int_O W_{\text{hom}}^\infty \left(\frac{dE^s u}{d|E^s u|}(x), \omega \right) d|E^s u|(x),$$

where $W_{\text{hom}}, W_{\text{hom}}^\infty : \mathbb{R}^{N \times N}_{\text{sym}} \times \Omega \rightarrow [0, \infty[$ are defined by:

$$W_{\text{hom}}(\xi, \omega) := \inf_{k \in \mathbb{N}^*} \frac{1}{k^N} \mathbb{E}^{\mathcal{I}} \left[\inf \left\{ \int_{]0, k[^N} W(x, \xi + \mathcal{E}v(x), \cdot) dx : v \in \text{LD}_0(]0, k[^N) \right\} \right] (\omega);$$

$$W_{\text{hom}}^\infty(\xi, \omega) := \overline{\text{lim}}_{\varepsilon \rightarrow 0} \frac{W_{\text{hom}}(t\xi, \omega)}{t},$$

where $\mathbb{E}^{\mathcal{I}}$ denotes the conditional expectation over \mathcal{I} with respect to \mathbb{P} , with \mathcal{I} being the σ -algebra of invariant sets with respect to $(\Omega, \mathcal{F}, \mathbb{P}, \{\tau_z\}_{z \in \mathbb{Z}^N})$. If in addition $(\Omega, \mathcal{F}, \mathbb{P}, \{\tau_z\}_{z \in \mathbb{Z}^N})$ is ergodic, then W_{hom} is deterministic and is given by

$$W_{\text{hom}}(\xi) := \inf_{k \in \mathbb{N}^*} \frac{1}{k^N} \mathbb{E} \left[\inf \left\{ \int_{]0, k[^N} W(y, \xi + \mathcal{E}v(y), \cdot) dy : v \in \text{LD}_0(]0, k[^N) \right\} \right],$$

where \mathbb{E} denotes the expectation with respect to \mathbb{P} .

Periodic homogenization by Γ -convergence for nonconvex Hencky plasticity functionals has been recently studied by Jesenko and Schmidt (see [JS18]). Analogue results of Theorem 2.1 in the space of functions of bounded variation were obtained by De Arcangelis and Gargiulo in the periodic case (see [DAG95]) and by Abddaimi, Licht and Michaille in the stochastic case (see [AML97]).

3. AUXILIARY RESULTS

3.1. A subadditive theorem. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{\tau_z\}_{z \in \mathbb{Z}^N}$ be satisfying the following three properties:

- $\tau_z : \Omega \rightarrow \Omega$ is \mathcal{F} -measurable for all $z \in \mathbb{Z}^N$;
- $\tau_z \circ \tau_{z'} = \tau_{z+z'}$ and $\tau_{-z} = \tau_z^{-1}$ for all $z, z' \in \mathbb{Z}^N$;
- $\mathbb{P}(\tau_z(A)) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$ and all $z \in \mathbb{Z}^N$.

Definition 3.1. Such a $\{\tau_z\}_{z \in \mathbb{Z}^N}$ is said to be a group of \mathbb{P} -preserving transformation on $(\Omega, \mathcal{F}, \mathbb{P})$ and the quadruplet $(\Omega, \mathcal{F}, \mathbb{P}, \{\tau_z\}_{z \in \mathbb{Z}^N})$ is called a measurable dynamical system.

Let $\mathcal{I} := \{A \in \mathcal{F} : \mathbb{P}(\tau_z(A) \Delta A) = 0 \text{ for all } z \in \mathbb{Z}^N\}$ be the σ -algebra of invariant sets with respect to $(\Omega, \mathcal{F}, \mathbb{P}, \{\tau_z\}_{z \in \mathbb{Z}^N})$.

Definition 3.2. When $\mathbb{P}(A) \in \{0, 1\}$ for all $A \in \mathcal{I}$, the measurable dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\tau_z\}_{z \in \mathbb{Z}^N})$ is said to be ergodic.

In what follows, we assume that $(\Omega, \mathcal{F}, \mathbb{P}, \{\tau_z\}_{z \in \mathbb{Z}^N})$ is a measurable dynamical system and we denote the class of bounded Borel subsets of \mathbb{R}^N by $\mathcal{B}_b(\mathbb{R}^N)$.

Definition 3.3. We say that $\mathcal{S} : \mathcal{B}_b(\mathbb{R}^N) \rightarrow L^1(\Omega, \mathcal{F}, \mathbb{P})$ is a subadditive process if \mathcal{S} is subadditive, i.e.

$$\mathcal{S}(B \cup B') \leq \mathcal{S}(B) + \mathcal{S}(B')$$

for all $B, B' \in \mathcal{B}_b(\mathbb{R}^N)$ such that $B \cap B' = \emptyset$, and $\{\tau_z\}_{z \in \mathbb{Z}^N}$ -covariant, i.e.

$$\mathcal{S}(B + z) = \mathcal{S}(B) \circ \tau_z$$

for all $B \in \mathcal{B}_b(\mathbb{R}^N)$ and all $z \in \mathbb{Z}^N$.

Definition 3.4. We say that $\{Q_\varepsilon\}_{\varepsilon > 0} \subset \mathcal{B}_b(\mathbb{R}^N)$ is regular if there exist $\{I_\varepsilon\}_{\varepsilon > 0}$ and $C > 0$ such that every I_ε is an interval² in \mathbb{Z}^N , $I_\varepsilon \subset I_{\varepsilon'}$ whenever $\varepsilon' < \varepsilon$ and $|I_\varepsilon| \leq C|Q_\varepsilon|$ for all $\varepsilon > 0$.

The following theorem, which is an extension of Akcoglu-Krengel's subadditive theorem (see [AK81, Kre85]), was proved by Licht and Michaille in [LM02, Theorem 4.1].

Theorem 3.5. Let $\mathcal{S} : \mathcal{B}_b(\mathbb{R}^N) \rightarrow L^1(\Omega, \mathcal{F}, \mathbb{P})$ be a subadditive process such that:

$$(S_1) \quad \gamma(\mathcal{S}) := \inf \left\{ \int_{\Omega} \frac{\mathcal{S}(I)(\omega)}{|I|} d\mathbb{P}(\omega) : I \text{ is an interval in } \mathbb{Z}^N \right\} > -\infty;$$

(S₂) there exists $h \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $|\mathcal{S}(Q)| \leq h$ for all $Q \in \mathcal{B}_b(\mathbb{R}^N)$ such that Q is convex and $Q \subset [0, 1]^N$.

Then, there exists $\Omega' \in \mathcal{F}$ with $\mathbb{P}(\Omega') = 1$ such that for every $\omega \in \Omega'$, one has

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{S}(Q_\varepsilon)(\omega)}{|Q_\varepsilon|} = \inf_{k \in \mathbb{N}^*} \frac{1}{k^N} \mathbb{E}^{\mathcal{I}} [\mathcal{S}([0, k]^N)](\omega)$$

for all $\{Q_\varepsilon\}_{\varepsilon > 0} \subset \mathcal{B}_b(\mathbb{R}^N)$ such that Q_ε is convex for all $\varepsilon > 0$, $\lim_{\varepsilon \rightarrow 0} \text{diam}(Q_\varepsilon) = \infty$ and $\{Q_\varepsilon\}_{\varepsilon > 0}$ is regular, where $\mathbb{E}^{\mathcal{I}}$ denotes the conditional expectation over \mathcal{I} with respect to \mathbb{P} . If in addition $(\Omega, \mathcal{F}, \mathbb{P}, \{\tau_z\}_{z \in \mathbb{Z}^N})$ is ergodic, then for every $\omega \in \Omega'$, one has

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{S}(Q_\varepsilon)(\omega)}{|Q_\varepsilon|} = \inf_{k \in \mathbb{N}^*} \frac{1}{k^N} \mathbb{E} [\mathcal{S}([0, k]^N)],$$

where $\mathbb{E}[\mathcal{S}([0, k]^N)]$ denotes the expectation of $\mathcal{S}([0, k]^N)$ with respect to \mathbb{P} .

Remark 3.6. For any cube Q in \mathbb{R}^N , $\{Q_\varepsilon\}_{\varepsilon > 0}$ defined by $Q_\varepsilon := \frac{1}{\varepsilon}Q$ is regular. Moreover, every Q_ε is convex and $\lim_{\varepsilon \rightarrow 0} \text{diam}(Q_\varepsilon) = \infty$.

²By an interval I in \mathbb{Z}^N we mean that $I = \prod_{i=1}^N [a_i, b_i[$ with $a_i, b_i \in \mathbb{Z}$.

3.2. Definition and properties of the homogenized density. Let $\xi \in \mathbb{R}_{\text{sym}}^{N \times N}$ and let $\mathcal{S}^\xi : \mathcal{B}_b(\mathbb{R}^N) \rightarrow L^1(\Omega, \mathcal{F}, \mathbb{P})$ be defined by

$$\mathcal{S}^\xi(B)(\omega) := \inf \left\{ \int_{\mathring{B}} W(x, \xi + \mathcal{E}v(x), \omega) dx : v \in \text{LD}_0(\mathring{B}) \right\} \quad (3.1)$$

with $W : \mathbb{R}^N \times \mathbb{R}_{\text{sym}}^{N \times N} \times \Omega \rightarrow [0, \infty[$ satisfying (C_1) and (C_2) , where $(\Omega, \mathcal{F}, \mathbb{P}, \{\tau_z\}_{z \in \mathbb{Z}^N})$ is a dynamical system. Then, it is easily seen that \mathcal{S}^ξ is a subadditive process satisfying (S_1) and (S_2) of Theorem 3.5 and, according to Remark 3.6, the following proposition is a straightforward consequence of Theorem 3.5, the Lipschitz continuity of W , i.e. (C_3) , and the fact that $\mathbb{Q}_{\text{sym}}^{N \times N}$ is dense in $\mathbb{R}_{\text{sym}}^{N \times N}$, where $\mathbb{Q}_{\text{sym}}^{N \times N}$ denotes the space of $N \times N$ symmetric rational matrices.

Proposition 3.7. *Assume that (C_1) , (C_2) and (C_3) hold. Then, there exists $\widehat{\Omega} \in \mathcal{F}$ with $\mathbb{P}(\widehat{\Omega}) = 1$ such that for every $\omega \in \widehat{\Omega}$, one has*

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{S}^\xi \left(\frac{1}{\varepsilon} Q \right) (\omega)}{\left| \frac{1}{\varepsilon} Q \right|} = \inf_{k \in \mathbb{N}^*} \frac{1}{k^N} \mathbb{E}^{\mathcal{I}} \left[\mathcal{S}^\xi([0, k[{}^N]) (\omega) \right]$$

for all $\xi \in \mathbb{R}_{\text{sym}}^{N \times N}$ and all cube Q in \mathbb{R}^N . If in addition $(\Omega, \mathcal{F}, \mathbb{P}, \{\tau_z\}_{z \in \mathbb{Z}^N})$ is ergodic, then for every $\omega \in \widehat{\Omega}$, one has

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{S}^\xi \left(\frac{1}{\varepsilon} Q \right) (\omega)}{\left| \frac{1}{\varepsilon} Q \right|} = \inf_{k \in \mathbb{N}^*} \frac{1}{k^N} \mathbb{E} \left[\mathcal{S}^\xi([0, k[{}^N]) \right].$$

Definition 3.8. According to Proposition 3.7 we define $W_{\text{hom}} : \mathbb{R}_{\text{sym}}^{N \times N} \times \Omega \rightarrow [0, \infty[$ by

$$\begin{aligned} W_{\text{hom}}(\xi, \omega) &:= \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{S}^\xi \left(\frac{1}{\varepsilon} Q \right) (\omega)}{\left| \frac{1}{\varepsilon} Q \right|} \\ &= \inf_{k \in \mathbb{N}^*} \frac{1}{k^N} \mathbb{E}^{\mathcal{I}} \left[\inf \left\{ \int_{]0, k[{}^N} W(x, \xi + \mathcal{E}v(x), \cdot) dx : v \in \text{LD}_0(]0, k[{}^N) \right\} \right] (\omega). \end{aligned}$$

When $(\Omega, \mathcal{F}, \mathbb{P}, \{\tau_z\}_{z \in \mathbb{Z}^N})$ is ergodic, W_{hom} is deterministic, i.e. $W_{\text{hom}} : \mathbb{R}_{\text{sym}}^{N \times N} \rightarrow [0, \infty[$ is given by

$$W_{\text{hom}}(\xi) = \inf_{k \in \mathbb{N}^*} \frac{1}{k^N} \mathbb{E} \left[\inf \left\{ \int_{]0, k[{}^N} W(x, \xi + \mathcal{E}v(x), \cdot) dx : v \in \text{LD}_0(]0, k[{}^N) \right\} \right].$$

Finally, here are some properties of W_{hom} that will be useful in the proof of Proposition 4.2.

Proposition 3.9. *Assume that (C_1) , (C_2) , (C_3) and (C_4) hold. Then, W_{hom} has 1-growth, is Lipschitz continuous and symmetric quasiconvex.*

Proof of Proposition 3.9. It is easily seen that W_{hom} has 1-growth and is Lipschitz continuous. We only prove that W_{hom} is symmetric quasiconvex. Let $\omega \in \widehat{\Omega}$ (where $\widehat{\Omega}$ is given by Proposition 3.7). From Proposition 3.7 (and Definition 3.8) we have

$$W_{\text{hom}}(\zeta, \omega) = \lim_{\varepsilon \rightarrow 0} W_{\text{hom}}^\varepsilon(\zeta, \omega) \text{ for all } \zeta \in \mathbb{R}_{\text{sym}}^{N \times N}, \quad (3.2)$$

where $W_{\text{hom}}^\varepsilon(\cdot, \omega) : \mathbb{R}_{\text{sym}}^{N \times N} \rightarrow [0, \infty[$ is given by

$$W_{\text{hom}}^\varepsilon(\zeta, \omega) := \frac{1}{|\frac{1}{\varepsilon}Y|} \inf \left\{ \int_{\frac{1}{\varepsilon}Y} W(x, \zeta + \mathcal{E}v(x), \omega) dx : v \in \text{LD}_0 \left(\frac{1}{\varepsilon}Y \right) \right\}$$

with $Y :=]0, 1[^N$. Fix $\xi \in \mathbb{R}_{\text{sym}}^{N \times N}$ and $\phi \in C_c^1(Y; \mathbb{R}^N)$. We have to prove that

$$W_{\text{hom}}(\xi, \omega) \leq \int_Y W_{\text{hom}}(\xi + \mathcal{E}\phi(y), \omega) dy. \quad (3.3)$$

As $W_{\text{hom}}^\varepsilon(\xi, \omega) \leq \beta(1 + |\xi|)$ for all $\varepsilon > 0$, according to (3.2) and Lebesgue's dominated convergence theorem, to establish (3.3) it suffices to show that for every $\varepsilon > 0$, one has

$$W_{\text{hom}}^\varepsilon(\xi, \omega) \leq \int_Y W_{\text{hom}}^\varepsilon(\xi + \mathcal{E}\phi(y), \omega) dy. \quad (3.4)$$

Fix $\varepsilon > 0$. By using the symmetric quasiconvexity of W and Fubini's theorem, we have

$$\begin{aligned} W_{\text{hom}}^\varepsilon(\xi, \omega) &\leq \frac{1}{|\frac{1}{\varepsilon}Y|} \inf \left\{ \int_{\frac{1}{\varepsilon}Y} \int_Y W(x, \xi + \mathcal{E}v(x) + \mathcal{E}\phi(y), \omega) dy dx : v \in \text{LD}_0 \left(\frac{1}{\varepsilon}Y \right) \right\} \\ &= \inf \left\{ \int_Y \frac{1}{|\frac{1}{\varepsilon}Y|} \int_{\frac{1}{\varepsilon}Y} W(x, \xi + \mathcal{E}\phi(y) + \mathcal{E}v(x), \omega) dx dy : v \in \text{LD}_0 \left(\frac{1}{\varepsilon}Y \right) \right\}. \end{aligned} \quad (3.5)$$

On the other hand, fix any $\delta > 0$. By Castaing's selection measurable theorem we can assert that there exists a measurable map

$$\begin{aligned} Y &\rightarrow \text{LD}_0 \left(\frac{1}{\varepsilon}Y \right) \\ y &\mapsto v_y \end{aligned}$$

such that for a.e. $y \in Y$, one has

$$\frac{1}{|\frac{1}{\varepsilon}Y|} \int_{\frac{1}{\varepsilon}Y} W(x, \xi + \mathcal{E}\phi(y) + \mathcal{E}v_y(x), \omega) dx \leq W_{\text{hom}}^\varepsilon(\xi + \mathcal{E}\phi(y), \omega) + \delta \quad (3.6)$$

From (3.5) and (3.6) we deduce that

$$W_{\text{hom}}^\varepsilon(\xi, \omega) \leq \int_Y W_{\text{hom}}^\varepsilon(\xi + \mathcal{E}\phi(y), \omega) dy + \delta,$$

and (3.4) follows by letting $\delta \rightarrow 0$. ■

3.3. Some properties of functions of bounded deformation. Here we recall some properties of functions of bounded deformation that we use in the proof of Theorem 2.1. (For more details on the space of bounded deformation, we refer to [ACDM97, DPR19] and the references therein.)

Let $O \subset \mathbb{R}^N$, let $M(O)$ be the space of $N \times N$ matrix-valued bounded Radon measures on O and let $\text{BD}(O)$ the space of functions of bounded deformation on O , i.e.

$$\text{BD}(O) := \left\{ u \in L^1(O; \mathbb{R}^N) : Eu := \frac{1}{2} (Du + Du^T) \in M(O) \right\},$$

Du denotes the distributional derivative of u . For each $u \in \text{BD}(O)$ we have

$$Eu = \mathcal{E}u dx + E^s u,$$

where $(\mathcal{E}u = \frac{dEu}{dx}, E^s u)$ is the Lebesgue decomposition of Eu with respect to the Lebesgue measure on O that we denote by dx . Moreover, $\mathcal{E}u$ is the approximate symmetrized gradient of u , i.e.

Theorem 3.10. *For dx -a.e. $x_0 \in O$, $\mathcal{E}u(x_0) = \frac{1}{2}(\nabla u(x_0) + \nabla u(x_0)^T)$ and*

$$\lim_{\rho \rightarrow 0} \frac{1}{|Q_\rho(x_0)|} \int_{Q_\rho(x_0)} \frac{|u - u_{x_0}|}{\rho} dx = 0,$$

where u_{x_0} is the affine function defined by $u_{x_0}(x) := u(x_0) + \nabla u(x_0)(x - x_0)$ and $Q_\rho(x_0) := x_0 + \rho Q$ with Q the unit cell centered at the origin.

and the analogue of Alberti's rank-one theorem holds, i.e.

Theorem 3.11. *Let $u \in \text{BD}(O)$. Then, for $|E^s u|$ -a.e. $x_0 \in O$ there exist $a(x_0) \in \mathbb{R}^N$ and $b(x_0) \in \mathbb{R}^N$ with $|a(x_0)| = |b(x_0)| = 1$ such that*

$$\frac{dEu}{d|Eu|}(x_0) = a(x_0) \odot b(x_0). \quad (3.7)$$

Theorem 3.11, which will be used in the proof of Proposition 4.1, has recently been established by De Philippis and Rindler in [DPR16]. The following two lemmas will be also useful in the proof of Proposition 4.1.

Lemma 3.12. *Let $u \in \text{BD}(O)$, let $x_0 \in \text{supp}(|E^s u|)$ be such that (3.7) holds and let Q be the unit cube centered at the origin whose the sides are either orthogonal or parallel to $a(x_0)$. Then:*

$$\begin{aligned} \lim_{\rho \rightarrow 0} \frac{|Eu|(Q_\rho(x_0))}{\rho^N} &= \infty; \\ \overline{\lim}_{\rho \rightarrow 0} \frac{|Eu|(Q_{\delta\rho}(x_0))}{|Eu|(Q_\rho(x_0))} &\geq \delta^N \text{ for all } \delta \in]0, 1[. \end{aligned} \quad (3.8)$$

Lemma 3.13. *Let $u \in \text{BD}(O)$, let $x_0 \in \text{supp}(|E^s u|)$ be such that (3.7) holds and, for each $\rho > 0$, let $v_\rho \in \text{BD}(Q)$ be defined by*

$$v_\rho(x) := \frac{\rho^{N-1}}{|Eu|(Q_\rho(x_0))} \left(u(x_0 + \rho x) - \frac{1}{\rho^N} \int_{Q_\rho(x_0)} u(y) dy \right) + R_\rho(y), \quad (3.9)$$

where $R_\rho : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a rigid deformation³. Then:

(i) $E v_\rho(Q) = \frac{Eu(Q_\rho(x_0))}{|Eu|(Q_\rho(x_0))}$ and

$$\lim_{\rho \rightarrow 0} E v_\rho(Q) = \lim_{\rho \rightarrow 0} \frac{Eu(Q_\rho(x_0))}{|Eu|(Q_\rho(x_0))} = a(x_0) \odot b(x_0); \quad (3.10)$$

³By a rigid deformation we mean a map $R : \mathbb{R}^N \rightarrow \mathbb{R}^N$ defined by $R(x) = Sx + \sigma$ for all $x \in \mathbb{R}^N$, where S is a $N \times N$ skew-symmetric matrix and $\sigma \in \mathbb{R}^N$.

- (ii) up to a subsequence, $v_\rho \rightarrow v$ in $L^1(Q; \mathbb{R}^N)$ and $Ev_\rho \rightharpoonup Ev$ weakly in $M(O)$ with $v \in \text{BD}(Q)$ defined by

$$v(x) := \bar{v}(\langle b(x_0), x \rangle) a(x_0) + c(a(x_0) \odot b(x_0))x + R(y), \quad (3.11)$$

where $\bar{v} :]-\frac{1}{2}, \frac{1}{2}[\rightarrow \mathbb{R}$ is bounded and increasing, $c > 0$ and $R : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a rigid deformation. Moreover, for a.e. $\delta \in]0, 1[$, one has

$$\lim_{\rho \rightarrow 0} Ev_\rho(\delta Q) = Ev(\delta Q). \quad (3.12)$$

For a proof of Lemmas 3.12 and 3.13 we refer to [KR19, DPR17].

3.4. A relaxation theorem in the space of functions of bounded deformation. The following result has been recently established by Kosiba and Rindler (see [KR19, Theorem 1.3] and also [Rin11, BFT00, ARPR17]).

Theorem 3.14. *Let $O \subset \mathbb{R}^N$ be a bounded open set, let $V : \mathbb{R}_{\text{sym}}^{N \times N} \rightarrow [0, \infty[$ be a continuous and symmetric quasiconvex integrand having 1-growth, let $J : \text{BD}(O) \rightarrow [0, \infty]$ defined by*

$$J(u) := \begin{cases} \int_O V(\mathcal{E}u(x))dx & \text{if } u \in \text{LD}(O) \\ \infty & \text{if } u \in \text{BD}(O) \setminus \text{LD}(O) \end{cases}$$

and let $\bar{J} : \text{BD}(O) \rightarrow [0, \infty]$ be the L^1 -lower semicontinuous envelope of J , i.e.

$$\bar{J}(u) := \inf \left\{ \liminf_{n \rightarrow \infty} J(u_n) : u_n \rightarrow u \text{ in } L^1(O; \mathbb{R}^N) \right\}.$$

Then, for every $u \in \text{BD}(O)$, one has

$$\bar{J}(u) = \int_O V(\mathcal{E}u(x))dx + \int_O V^\infty \left(\frac{dE^s u}{d|E^s u|}(x) \right) d|E^s u|(x)$$

with $V^\infty : \mathbb{R}_{\text{sym}}^{N \times N} \rightarrow [0, \infty[$ given by $V^\infty(\xi) := \overline{\lim}_{\varepsilon \rightarrow 0} \frac{V(t\xi)}{t}$.

3.5. Integral representation of the Vitali envelope of a set function. What follows was first developed in [BFM98, BB00] (see also [AHM16, AHM17, AHCM17]). Let $O \subset \mathbb{R}^N$ be a bounded open set and let $\mathcal{O}(O)$ be the class of open subsets of O . We begin with the concept of the Vitali envelope of a set function.

For each $\delta > 0$ and each $A \in \mathcal{O}(O)$, denote the class of countable families $\{Q_i = Q_{\rho_i}(x_i)\}_{i \in I}$ (where $Q_{\rho_i}(x_i) := x_i + \rho_i Q$ where Q is the unit cell centered at the origin) of disjoint open cubes of A with $x_i \in A$, $\rho_i > 0$ and $\text{diam}(Q_i) \in]0, \delta[$ such that $|A \setminus \cup_{i \in I} Q_i| = 0$ by $\mathcal{V}_\delta(A)$.

Definition 3.15. Given $\mathcal{S} : \mathcal{O}(O) \rightarrow [0, \infty]$, for each $\delta > 0$ we define $\mathcal{S}^\delta : \mathcal{O}(O) \rightarrow [0, \infty]$ by

$$\mathcal{S}^\delta(A) := \inf \left\{ \sum_{i \in I} \mathcal{S}(Q_i) : \{Q_i\}_{i \in I} \in \mathcal{V}_\delta(A) \right\}. \quad (3.13)$$

By the Vitali envelope of \mathcal{S} we call the set function $\mathcal{S}^* : \mathcal{O}(O) \rightarrow [-\infty, \infty]$ defined by

$$\mathcal{S}^*(A) := \sup_{\delta > 0} \mathcal{S}^\delta(A) = \lim_{\delta \rightarrow 0} \mathcal{S}^\delta(A). \quad (3.14)$$

The interest of Definition 3.15 comes from the following integral representation result. (For a proof we refer to [AHCM17, §A.4].)

Theorem 3.16. *Let $\mathcal{S} : \mathcal{O}(O) \rightarrow [0, \infty]$ be a set function satisfying the following two conditions:*

- (a) *there exists a finite Radon measure ν on Ω which is absolutely continuous with respect to dx such that $\mathcal{S}(A) \leq \nu(A)$ for all $A \in \mathcal{O}(O)$;*
- (b) *\mathcal{S} is subadditive, i.e., $\mathcal{S}(A) \leq \mathcal{S}(B) + \mathcal{S}(C)$ for all $A, B, C \in \mathcal{O}(O)$ with $B, C \subset A$, $B \cap C = \emptyset$ and $|A \setminus B \cup C| = 0$.*

Then $\lim_{\rho \rightarrow 0} \frac{\mathcal{S}(Q_\rho(\cdot))}{|Q_\rho(\cdot)|} \in L^1(\Omega)$ and for every $A \in \mathcal{O}(O)$, one has

$$\mathcal{S}^*(A) = \int_A \lim_{\rho \rightarrow 0} \frac{\mathcal{S}(Q_\rho(x))}{|Q_\rho(x)|} dx.$$

4. PROOF OF THE HOMOGENIZATION THEOREM

Theorem 2.1 is a direct consequence of the following two propositions (see Proposition 4.1 in §4.1 and Proposition 4.2 in §4.2).

4.1. The lower bound. Here we establish that $\Gamma(L^1)\text{-}\underline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon \geq I_{\text{hom}}$.

Proposition 4.1 (lower bound). *Under the assumption of Theorem 2.1, for \mathbb{P} -a.e. $\omega \in \Omega$, one has*

$$\underline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(u_\varepsilon, \omega) \geq I_{\text{hom}}(u, \omega) \quad (4.1)$$

for all $u \in \text{BD}(O)$ and all $\{u_\varepsilon\}_{\varepsilon > 0} \subset \text{LD}(O)$ such that $u_\varepsilon \rightarrow u$ in $L^1(O; \mathbb{R}^N)$.

Proof of Proposition 4.1. The proof of this proposition follows the same method as in [AML97, Theorem 3.1]. Let $\omega \in \widehat{\Omega}$ where $\widehat{\Omega} \in \mathcal{F}$ is given by Proposition 3.7. Let $u \in \text{BD}(O)$ and let $\{u_\varepsilon\}_{\varepsilon > 0} \subset \text{LD}(O)$ be such that $u_\varepsilon \rightarrow u$ in $L^1(O; \mathbb{R}^N)$. Without loss of generality we can assume that

$$\underline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(u_\varepsilon, \omega) = \lim_{\varepsilon \rightarrow 0} I_\varepsilon(u_\varepsilon, \omega) < \infty, \text{ and so } \sup_{\varepsilon > 0} I_\varepsilon(u_\varepsilon, \omega) < \infty. \quad (4.2)$$

For each $\varepsilon > 0$, we define the (positive) Radon measure μ_ε on O by

$$\mu_\varepsilon := W \left(\frac{\cdot}{\varepsilon}, \mathcal{E}u(\cdot), \omega \right) dx.$$

From (4.2) we see that $\sup_{\varepsilon > 0} \mu_\varepsilon(O) < \infty$, and so there exists a (positive) Radon measure μ on O such that (up to a subsequence) $\mu_\varepsilon \rightarrow \mu$ weakly. By Lebesgue's decomposition theorem, we have $\mu = \mu^a + \mu^s$ where μ^a and μ^s are (positive) Radon measures on O such that $\mu^a \ll dx$ and $\mu^s \perp dx$. Thus, to prove (4.1) it suffices to show that:

$$\mu^a \geq W_{\text{hom}}(\mathcal{E}u(\cdot), \omega) dx; \quad (4.3)$$

$$\mu^s \geq W_{\text{hom}}^\infty \left(\frac{dE^s u}{d|E^s u|}(\cdot), \omega \right) |E^s u|. \quad (4.4)$$

Proof of (4.3). It suffices to prove that

$$\lim_{\rho \rightarrow 0} \frac{\mu(Q_\rho(x_0))}{|Q_\rho(x_0)|} \geq W_{\text{hom}}(\mathcal{E}u(x_0), \omega) \quad (4.5)$$

for dx -a.a. $x_0 \in O$ with $Q_\rho(x_0) := x_0 + \rho Q$ where Q is the unit cell centered at the origin. As $\mu(O) < \infty$ without loss of generality we can assume that $\mu(\partial Q_\rho(x_0)) = 0$ for all $\rho > 0$, and so to prove (4.5) it is sufficient to establish that

$$\lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\mu_\varepsilon(Q_\rho(x_0))}{|Q_\rho(x_0)|} \geq W_{\text{hom}}(\mathcal{E}u(x_0), \omega). \quad (4.6)$$

Fix any $\varepsilon > 0$, any $\rho > 0$, any $s \in]0, 1[$ and any $\delta \in]0, 1[$. Fix any $q \in \mathbb{N}^*$ and consider $\{Q_i\}_{i \in \{0, \dots, q\}} \subset Q_{\delta\rho}(x_0)$ given by

$$Q_i := \begin{cases} Q_{s\delta\rho}(x_0) & \text{if } i = 0 \\ Q_{s\delta\rho + \frac{i}{q}\delta\rho(1-s)}(x_0) & \text{if } i \in \{1, \dots, q\}. \end{cases}$$

For every $i \in \{1, \dots, q\}$, consider a Uryshon function $\varphi_i \in C^\infty(O)$ for the pair $(O \setminus Q_i, \overline{Q_{i-1}})^4$ such that

$$\|\nabla \varphi_i\|_{L^\infty(O; \mathbb{R}^N)} \leq \frac{q}{\delta\rho(1-s)}$$

and define $u_\varepsilon^i \in u_{x_0} + \text{LD}_0(Q_{\delta\rho}(x_0))$ by

$$u_\varepsilon^i := u_{x_0} + \varphi_i(u_\varepsilon - u_{x_0})$$

with $u_{x_0}(x) := u(x_0) + \nabla u(x_0)(x - x_0)$. Fix any $i \in \{1, \dots, q\}$. We then have

$$\mathcal{E}u_\varepsilon^i = \begin{cases} \mathcal{E}u_\varepsilon & \text{in } Q_{i-1} \\ \mathcal{E}u(x_0) + \varphi_i \mathcal{E}(u_\varepsilon - u_{x_0}) + \nabla \varphi_i \odot (u_\varepsilon - u_{x_0}) & \text{in } Q_i \setminus Q_{i-1} \\ \mathcal{E}u(x_0) & \text{in } Q_{\delta\rho}(x_0) \setminus Q_i, \end{cases}$$

and so

$$\begin{aligned} \int_{Q_{\delta\rho}(x_0)} W\left(\frac{x}{\varepsilon}, \mathcal{E}u_\varepsilon^i, \omega\right) dx &= \int_{Q_{i-1}} W\left(\frac{x}{\varepsilon}, \mathcal{E}u_\varepsilon, \omega\right) dx + \int_{Q_i \setminus Q_{i-1}} W\left(\frac{x}{\varepsilon}, \mathcal{E}u_\varepsilon^i, \omega\right) dx \\ &\quad + \int_{Q_{\delta\rho}(x_0) \setminus Q_i} W\left(\frac{x}{\varepsilon}, \mathcal{E}u(x_0), \omega\right) dx. \end{aligned}$$

⁴By a Uryshon function from O to \mathbb{R} for the pair $(O \setminus V, K)$, where $K \subset V \subset O$ with K compact and V open, we mean $\varphi \in C^\infty(O)$ such that $\varphi(x) \in [0, 1]$ for all $x \in O$, $\varphi(x) = 0$ for all $x \in O \setminus V$ and $\varphi(x) = 1$ for all $x \in K$.

Taking the right inequality in (2.1) into account, we see that:

$$\begin{aligned}
\frac{1}{|Q_\rho(x_0)|} \int_{Q_{i-1}} W\left(\frac{x}{\varepsilon}, \mathcal{E}u_\varepsilon, \omega\right) dx &\leq \frac{1}{|Q_\rho(x_0)|} \int_{Q_{\delta\rho}(x_0)} W\left(\frac{x}{\varepsilon}, \mathcal{E}u_\varepsilon, \omega\right) dx \\
&\leq \frac{\mu_\varepsilon(Q_\rho(x_0))}{|Q_\rho(x_0)|}; \\
\frac{1}{|Q_\rho(x_0)|} \int_{Q_i \setminus Q_{i-1}} W\left(\frac{x}{\varepsilon}, \mathcal{E}u_\varepsilon^i, \omega\right) dx &\leq c\delta^N(1-s)^N + \frac{\beta}{|Q_\rho(x_0)|} \int_{Q_i \setminus Q_{i-1}} |\mathcal{E}(u_\varepsilon - u_{x_0})| dx \\
&\quad + \frac{\beta}{\delta(1-s)} \frac{1}{|Q_\rho(x_0)|} \int_{Q_i \setminus Q_{i-1}} \frac{|u_\varepsilon - u_{x_0}|}{\rho} dx; \\
\frac{1}{|Q_\rho(x_0)|} \int_{Q_{\delta\rho}(x_0) \setminus Q_i} W\left(\frac{x}{\varepsilon}, \mathcal{E}u(x_0), \omega\right) dx &\leq c\delta^N(1-s)^N,
\end{aligned}$$

where $c := \beta(1 + |\mathcal{E}u(x_0)|)$. Consider $\mathcal{S}^{\mathcal{E}u(x_0)}(\cdot)(\omega)$ defined by (3.1) with $\xi = \mathcal{E}u(x_0)$. From the above we deduce that

$$\begin{aligned}
\delta^N \frac{\mathcal{S}^{\mathcal{E}u(x_0)}\left(\frac{1}{\varepsilon}Q_{\delta\rho}(x_0)\right)(\omega)}{\left|\frac{1}{\varepsilon}Q_{\delta\rho}(x_0)\right|} &\leq \frac{1}{|Q_\rho(x_0)|} \int_{Q_{\delta\rho}(x_0)} W\left(\frac{x}{\varepsilon}, \mathcal{E}u_\varepsilon^i, \omega\right) dx \\
&\leq \frac{\mu_\varepsilon(Q_\rho(x_0))}{|Q_\rho(x_0)|} + 2c\delta^N(1-s)^N \\
&\quad + \frac{\beta}{|Q_\rho(x_0)|} \int_{Q_i \setminus Q_{i-1}} |\mathcal{E}(u_\varepsilon - u_{x_0})| dx \\
&\quad + \frac{\beta q}{\delta(1-s)} \frac{1}{|Q_\rho(x_0)|} \int_{Q_i \setminus Q_{i-1}} \frac{|u_\varepsilon - u_{x_0}|}{\rho} dx,
\end{aligned}$$

and averaging these inequalities, we obtain

$$\begin{aligned}
\delta^N \frac{\mathcal{S}^{\mathcal{E}u(x_0)}\left(\frac{1}{\varepsilon}Q_{\delta\rho}(x_0)\right)(\omega)}{\left|\frac{1}{\varepsilon}Q_{\delta\rho}(x_0)\right|} &\leq \frac{\mu_\varepsilon(Q_\rho(x_0))}{|Q_\rho(x_0)|} + 2c\delta^N(1-s)^N \\
&\quad + \frac{1}{q} \frac{\beta}{|Q_\rho(x_0)|} \int_{Q_{\delta\rho}(x_0) \setminus Q_{s\delta\rho}(x_0)} |\mathcal{E}(u_\varepsilon - u_{x_0})| dx \\
&\quad + \frac{\beta q}{\delta(1-s)} \frac{1}{|Q_\rho(x_0)|} \int_{Q_\rho(x_0)} \frac{|u_\varepsilon - u_{x_0}|}{\rho} dx.
\end{aligned}$$

Taking Proposition 3.7 (and Definition 3.8) and Theorem 3.10, letting $q \rightarrow \infty$ and then $\varepsilon \rightarrow 0$ and $\rho \rightarrow 0$, we conclude that

$$\delta^N W_{\text{hom}}(\mathcal{E}u(x_0), \omega) \leq \lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\mu_\varepsilon(Q_\rho(x_0))}{|Q_\rho(x_0)|} + 2c\delta^N(1-s)^N,$$

(4.6) follows by letting $s \rightarrow 1$ and $\delta \rightarrow 1$.

Proof of (4.4). It suffices to prove that

$$\lim_{\rho \rightarrow 0} \frac{\mu(Q_\rho(x_0))}{|Eu|(Q_\rho(x_0))} \geq W_{\text{hom}}^\infty(a(x_0) \odot b(x_0), \omega) \tag{4.7}$$

for $|E^s u|$ -a.a. $x_0 \in O$ such that (3.7) holds with $Q_\rho(x_0) := x_0 + \rho Q$ where Q is the unit cube centered at the origin whose the sides are either orthogonal or parallel to $b(x_0)$. Fix such a x_0 . As $\mu(O) < \infty$, without loss of generality we can assume that $\mu(\partial Q_\rho(x_0)) = 0$ for all $\rho > 0$, and so to prove (4.7) it is sufficient to establish that

$$\lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\mu_\varepsilon(Q_\rho(x_0))}{|Eu|(Q_\rho(x_0))} \geq W_{\text{hom}}^\infty(a(x_0) \odot b(x_0), \omega) \quad (4.8)$$

For each $\rho > 0$ and each $\varepsilon > 0$, let $v_\rho \in \text{BD}(Q)$ be given by (3.9), let $v_{\rho,\varepsilon} \in \text{LD}(Q)$ be defined by

$$v_{\rho,\varepsilon}(x) := \frac{\rho^{N-1}}{|Eu|(Q_\rho(x_0))} \left(u_\varepsilon(x_0 + \rho x) - \frac{1}{\rho^N} \int_{Q_\rho(x_0)} u_\varepsilon(y) dy \right) + R_\rho(y)$$

and set

$$t_\rho := \frac{|Eu|(Q_\rho(x_0))}{\rho^N}.$$

Then, as $u_\varepsilon \rightarrow u$ in $L^1(O; \mathbb{R}^N)$ and by using Lemma 3.12, we have:

$$\lim_{\varepsilon \rightarrow 0} \|v_{\rho,\varepsilon} - v_\rho\|_{L^1(O; \mathbb{R}^N)} = 0 \text{ for all } \rho > 0; \quad (4.9)$$

$$\lim_{\rho \rightarrow 0} t_\rho = \infty. \quad (4.10)$$

From Lemma 3.13(ii), up to a subsequence, we have

$$\kappa_\rho := \|v_\rho - v\|_{L^1(Q; \mathbb{R}^N)}^{\frac{1}{2}} \rightarrow 0 \text{ as } \rho \rightarrow 0 \quad (4.11)$$

with $v \in \text{BD}(Q)$ given by (3.11). Fix any $\delta \in]0, 1[$ such that (3.12) holds. Fix any $\rho > 0$ and any $\varepsilon > 0$. Let $u_{\rho,\varepsilon} \in \text{LD}(Q)$ be defined by

$$u_{\rho,\varepsilon} := t_\rho v_{\rho,\varepsilon}. \quad (4.12)$$

First of all, it is easy to see that

$$\frac{1}{|Eu|(Q_\rho(x_0))} \int_{Q_{\delta\rho}(x_0)} W\left(\frac{x}{\varepsilon}, \mathcal{E}u_\varepsilon, \omega\right) dx = \frac{1}{t_\rho} \int_{\delta Q} W\left(\frac{x_0 + \rho x}{\varepsilon}, \mathcal{E}u_{\rho,\varepsilon}, \omega\right) dx. \quad (4.13)$$

On the other hand, fix any $q \in \mathbb{N}^*$ and consider $\{Q_i\}_{i \in \{0, \dots, q\}} \subset \delta Q$ given by

$$Q_i := \begin{cases} (1 - \kappa_\rho)\delta Q & \text{if } i = 0 \\ \left(1 - \kappa_\rho + i \frac{\kappa_\rho}{q}\right) \delta Q & \text{if } i \in \{1, \dots, q\}. \end{cases}$$

For every $i \in \{1, \dots, q\}$, consider a Uryshon function $\varphi_i \in C^\infty(O)$ for the pair $(O \setminus Q_i, \overline{Q_{i-1}})$ such that

$$\|\nabla \varphi_i\|_{L^\infty(O; \mathbb{R}^N)} \leq \frac{q}{\kappa_\rho} \quad (4.14)$$

and define $u_{\rho,\varepsilon}^i \in t_\rho \Theta_\delta + \text{LD}_0(\delta Q)$ by

$$u_{\rho,\varepsilon}^i := t_\rho \Theta_\delta + \varphi_i(u_{\rho,\varepsilon} - t_\rho \Theta_\delta)$$

where Θ_δ is the affine function defined by

$$\Theta_\delta(x) := \frac{Ev(\delta Q)}{\delta^N}x + \frac{\Psi((\frac{\delta}{2})^-) + \Psi((-\frac{\delta}{2})^+)}{2}a(x_0) + R(y)$$

with $\Psi :]-\frac{1}{2}, \frac{1}{2}[\rightarrow \mathbb{R}$ given by $\Psi(r) := \bar{v}(r) + cr$, where $\bar{v} :]-\frac{1}{2}, \frac{1}{2}[\rightarrow \mathbb{R}$, $c > 0$ and R (a rigid deformation) are given by Lemma 3.13(ii). (Note that the trace of v and Θ_δ are equal on the faces of δQ orthogonal to $b(x_0)$.) Fix any $i \in \{1, \dots, q\}$. We then have

$$\mathcal{E}u_{\rho,\varepsilon}^i = \begin{cases} \mathcal{E}u_{\rho,\varepsilon} & \text{in } Q_{i-1} \\ \frac{t_\rho}{\delta^N}Ev(\delta Q) + \varphi_i\mathcal{E}(u_{\rho,\varepsilon} - t_\rho\Theta_\delta) + \nabla\varphi_i \odot (u_{\rho,\varepsilon} - t_\rho\Theta_\delta) & \text{in } Q_i \setminus Q_{i-1} \\ \frac{t_\rho}{\delta^N}Ev(\delta Q) & \text{in } \delta Q \setminus Q_i. \end{cases}$$

Hence

$$\begin{aligned} \frac{1}{t_\rho} \int_{\delta Q} W\left(\frac{x_0 + \rho x}{\varepsilon}, \mathcal{E}u_{\rho,\varepsilon}^i, \omega\right) dx &\leq \frac{1}{t_\rho} \int_{\delta Q} W\left(\frac{x_0 + \rho x}{\varepsilon}, \mathcal{E}u_{\rho,\varepsilon}, \omega\right) dx \\ &+ \frac{1}{t_\rho} \int_{Q_i \setminus Q_{i-1}} W\left(\frac{x_0 + \rho x}{\varepsilon}, \mathcal{E}u_{\rho,\varepsilon}^i, \omega\right) dx \\ &+ \frac{1}{t_\rho} \int_{\delta Q \setminus Q_i} W\left(\frac{x_0 + \rho x}{\varepsilon}, \frac{t_\rho}{\delta^N}Ev(\delta Q), \omega\right) dx. \end{aligned} \quad (4.15)$$

Moreover, taking (4.12) and (4.14) into account, from the right inequality in (2.1) we see that:

$$\frac{1}{t_\rho} \int_{\delta Q \setminus Q_i} W\left(\frac{x_0 + \rho x}{\varepsilon}, \frac{t_\rho}{\delta^N}Ev(\delta Q), \omega\right) dx \leq \Delta_1(\rho, \delta); \quad (4.16)$$

$$\begin{aligned} \frac{1}{t_\rho} \int_{Q_i \setminus Q_{i-1}} W\left(\frac{x_0 + \rho x}{\varepsilon}, \mathcal{E}u_{\rho,\varepsilon}^i, \omega\right) dx &\leq \Delta_1(\rho, \delta) \\ &+ \frac{\beta}{t_\rho} \int_{Q_i \setminus Q_{i-1}} |\mathcal{E}(u_{\rho,\varepsilon} - t_\rho\Theta_\delta)| dx \\ &+ \frac{\beta q}{\kappa_\rho} \int_{Q_i \setminus Q_{i-1}} |v_{\rho,\varepsilon} - \Theta_\delta| dx \end{aligned} \quad (4.17)$$

with $\Delta_1(\rho, \delta) := \frac{\beta\delta^N}{t_\rho} + \frac{\beta(1-(1-\kappa_\rho)^N)}{\delta^N}|Ev(\delta Q)|$. Note that by (4.10) and (4.11) we have

$$\Delta_1(\rho, \delta) \rightarrow 0 \text{ as } \rho \rightarrow 0. \quad (4.18)$$

Set $\xi_{\rho,\delta} := \frac{t_\rho}{\delta^N} Ev(\delta Q) \in \mathbb{R}_{\text{sym}}^{N \times N}$ and consider $\mathcal{S}^{\xi_{\rho,\delta}}(\cdot)(\omega)$ defined by (3.1) with $\xi = \xi_{\rho,\delta}$. From (4.13), (4.15), (4.16) and (4.17) we deduce that

$$\begin{aligned} \frac{1}{|Eu|(Q_\rho(x_0))} \int_{Q_{\delta\rho}(x_0)} W\left(\frac{x}{\varepsilon}, \mathcal{E}u_\varepsilon, \omega\right) dx &\geq \frac{\delta^N \mathcal{S}^{\xi_{\rho,\delta}}\left(\frac{1}{\varepsilon}Q_\rho(x_0)\right)(\omega)}{t_\rho \left|\frac{1}{\varepsilon}Q_\rho(x_0)\right|} - 2\Delta_1(\rho, \delta) \\ &\quad - \frac{\beta}{t_\rho} \int_{Q_i \setminus Q_{i-1}} |\mathcal{E}(u_{\rho,\varepsilon} - t_\rho \Theta_\delta)| dx \\ &\quad - \frac{\beta q}{\kappa_\rho} \int_{Q_i \setminus Q_{i-1}} |v_{\rho,\varepsilon} - \Theta_\delta| dx \end{aligned}$$

for all $i \in \{1, \dots, q\}$, and averaging these inequalities, it follows that

$$\begin{aligned} \frac{1}{|Eu|(Q_\rho(x_0))} \int_{Q_{\delta\rho}(x_0)} W\left(\frac{x}{\varepsilon}, \mathcal{E}u_\varepsilon, \omega\right) dx &\geq \frac{\delta^N \mathcal{S}^{\xi_{\rho,\delta}}\left(\frac{1}{\varepsilon}Q_\rho(x_0)\right)(\omega)}{t_\rho \left|\frac{1}{\varepsilon}Q_\rho(x_0)\right|} - 2\Delta_1(\rho, \delta) \\ &\quad - \frac{1}{q} \frac{\beta}{t_\rho} \int_{\delta Q \setminus (1-\kappa_\rho)\delta Q} |\mathcal{E}(u_{\rho,\varepsilon} - t_\rho \Theta_\delta)| dx \\ &\quad - \frac{\beta}{\kappa_\rho} \int_{\delta Q \setminus (1-\kappa_\rho)\delta Q} |v_{\rho,\varepsilon} - \Theta_\delta| dx. \end{aligned}$$

But

$$\frac{\beta}{\kappa_\rho} \int_{\delta Q \setminus (1-\kappa_\rho)\delta Q} |v_{\rho,\varepsilon} - \Theta_\delta| dx \leq \frac{\beta}{\kappa_\rho} \|v_{\rho,\varepsilon} - v\|_{L^1(Q; \mathbb{R}^N)} + \frac{\beta}{\kappa_\rho} \int_{\delta Q \setminus (1-\kappa_\rho)\delta Q} |v - \Theta_\delta| dx,$$

and so, for every $\varepsilon > 0$ and every $q \in \mathbb{N}^*$, one has

$$\begin{aligned} \frac{1}{|Eu|(Q_\rho(x_0))} \int_{Q_{\delta\rho}(x_0)} W\left(\frac{x}{\varepsilon}, \mathcal{E}u_\varepsilon, \omega\right) dx &\geq \frac{\delta^N \mathcal{S}^{\xi_{\rho,\delta}}\left(\frac{1}{\varepsilon}Q_\rho(x_0)\right)(\omega)}{t_\rho \left|\frac{1}{\varepsilon}Q_\rho(x_0)\right|} - 2\Delta_1(\rho, \delta) \\ &\quad - \frac{1}{q} \frac{\beta}{t_\rho} \int_{\delta Q \setminus (1-\kappa_\rho)\delta Q} |\mathcal{E}(u_{\rho,\varepsilon} - t_\rho \Theta_\delta)| dx \\ &\quad - \frac{\beta}{\kappa_\rho} \|v_{\rho,\varepsilon} - v\|_{L^1(Q; \mathbb{R}^N)} \\ &\quad - \frac{\beta}{\kappa_\rho} \int_{\delta Q \setminus (1-\kappa_\rho)\delta Q} |v - \Theta_\delta| dx. \end{aligned}$$

Taking Proposition 3.7 (and Definition 3.8) and (4.9) into account and recalling that $\kappa_\rho = \|v_\rho - v\|_{L^1(Q; \mathbb{R}^N)}^{\frac{1}{2}}$ and $\xi_{\rho, \delta} = \frac{t_\rho}{\delta^N} Ev(\delta Q)$, letting $q \rightarrow \infty$ and then $\varepsilon \rightarrow 0$, we obtain

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \frac{\mu_\varepsilon(Q_\rho(x_0))}{|Eu|(Q_\rho(x_0))} &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|Eu|(Q_\rho(x_0))} \int_{Q_\rho(x_0)} W\left(\frac{x}{\varepsilon}, \mathcal{E}u_\varepsilon, \omega\right) dx \\
&\geq \overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{|Eu|(Q_\rho(x_0))} \int_{Q_{\delta\rho}(x_0)} W\left(\frac{x}{\varepsilon}, \mathcal{E}u_\varepsilon, \omega\right) dx \\
&\geq \frac{\delta^N}{t_\rho} W_{\text{hom}}\left(\frac{t_\rho}{\delta^N} Ev(\delta Q), \omega\right) - 2\Delta_1(\rho, \delta) - \beta\kappa_\rho \\
&\quad - \frac{\beta}{\kappa_\rho} \int_{\delta Q \setminus (1-\kappa_\rho)\delta Q} |v - \Theta_\delta| dx.
\end{aligned} \tag{4.19}$$

As W_{hom} is Lipschitz continuous (see Proposition 3.9) we have

$$\frac{\delta^N}{t_\rho} W_{\text{hom}}\left(\frac{t_\rho}{\delta^N} Ev_\rho(\delta Q), \omega\right) \leq C |Ev_\rho(\delta Q) - Ev(\delta Q)| + \frac{\delta^N}{t_\rho} W_{\text{hom}}\left(\frac{t_\rho}{\delta^N} Ev(\delta Q), \omega\right)$$

and noticing that $Ev_\rho(\delta Q) = \frac{Eu(Q_{\delta\rho}(x_0))}{|Eu|(Q_\rho(x_0))}$ we see that

$$\begin{aligned}
\frac{\delta^N}{t_\rho} W_{\text{hom}}\left(\frac{t_\rho}{\delta^N} a(x_0) \odot b(x_0), \omega\right) &\leq C \left| a(x_0) \odot b(x_0) - \frac{Eu(Q_\rho(x_0))}{|Eu|(Q_\rho(x_0))} \right| \\
&\quad + C \left| \frac{Eu(Q_\rho(x_0))}{|Eu|(Q_\rho(x_0))} - \frac{Eu(Q_{\delta\rho}(x_0))}{|Eu|(Q_\rho(x_0))} \right| \\
&\quad + \frac{\delta^N}{t_\rho} W_{\text{hom}}\left(\frac{t_\rho}{\delta^N} Ev_\rho(\delta Q), \omega\right) \\
&\leq C \left| a(x_0) \odot b(x_0) - \frac{Eu(Q_\rho(x_0))}{|Eu|(Q_\rho(x_0))} \right| + C \left| 1 - \frac{|Eu|(Q_{\delta\rho}(x_0))}{|Eu|(Q_\rho(x_0))} \right| \\
&\quad + \frac{\delta^N}{t_\rho} W_{\text{hom}}\left(\frac{t_\rho}{\delta^N} Ev_\rho(\delta Q), \omega\right),
\end{aligned}$$

where $C > 0$ is the Lipschitz constant. Hence

$$\begin{aligned}
\frac{\delta^N}{t_\rho} W_{\text{hom}}\left(\frac{t_\rho}{\delta^N} Ev(\delta Q), \omega\right) &\geq \frac{\delta^N}{t_\rho} W_{\text{hom}}\left(\frac{t_\rho}{\delta^N} a(x_0) \odot b(x_0), \omega\right) - C |Ev_\rho(\delta Q) - Ev(\delta Q)| \\
&\quad - C \left| a(x_0) \odot b(x_0) - \frac{Eu(Q_\rho(x_0))}{|Eu|(Q_\rho(x_0))} \right| \\
&\quad - C \left| 1 - \frac{|Eu|(Q_{\delta\rho}(x_0))}{|Eu|(Q_\rho(x_0))} \right|.
\end{aligned} \tag{4.20}$$

As the trace of v and Θ_δ are equal on the faces of δQ orthogonal to $b(x_0)$, from Poincaré's inequality we can assert that

$$\int_{\delta Q \setminus (1-\kappa_\rho)\delta Q} |v - \Theta_\delta| dx \leq C' \kappa_\rho \Delta_2(\rho, \delta), \tag{4.21}$$

where $C' > 0$ does not depend on ρ and $\Delta_2(\rho, \delta) := \int_{\delta Q \setminus (1-\kappa_\rho)\delta Q} \left| Ev - \frac{Ev(\delta Q)}{\delta^N} \right| dx$. Note that by (4.11) we have

$$\Delta_2(\rho, \delta) \rightarrow 0 \text{ as } \rho \rightarrow 0. \quad (4.22)$$

From (4.19), (4.20) and (4.21) we deduce that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{\mu_\varepsilon(Q_\rho(x_0))}{|Eu|(Q_\rho(x_0))} &\geq \frac{\delta^N}{t_\rho} W_{\text{hom}} \left(\frac{t_\rho}{\delta^N} a(x_0) \odot b(x_0), \omega \right) - C |Ev_\rho(\delta Q) - Ev(\delta Q)| \\ &\quad - C \left| a(x_0) \odot b(x_0) - \frac{Eu(Q_\rho(x_0))}{|Eu|(Q_\rho(x_0))} \right| - C \left| 1 - \frac{|Eu|(Q_{\delta\rho}(x_0))}{|Eu|(Q_\rho(x_0))} \right| \\ &\quad - 2\Delta_1(\rho, \delta) - \beta\kappa_\rho - \beta C' \Delta_2(\rho, \delta). \end{aligned}$$

Letting $\rho \rightarrow 0$ and taking (3.8), (3.10), (3.12), (4.10), (4.11), (4.18) and (4.22) into account, we conclude that

$$\lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\mu_\varepsilon(Q_\rho(x_0))}{|Eu|(Q_\rho(x_0))} \geq W_{\text{hom}}^\infty(a(x_0) \odot b(x_0), \omega) - C(1 - \delta^N),$$

and (4.8) follows by letting $\delta \rightarrow 1$. ■

4.2. The upper bound. Here we establish that $\Gamma(L^1)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon \leq I_{\text{hom}}$.

Proposition 4.2 (upper bound). *Under the assumption of Theorem 2.1, for \mathbb{P} -a.e. $\omega \in \Omega$, one has*

$$\Gamma(L^1)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(u, \omega) \leq I_{\text{hom}}(u, \omega) \quad (4.23)$$

for all $u \in \text{BD}(O)$.

Proof of Proposition 4.2. In what follows, for each $\varepsilon > 0$, we consider $\mathcal{I}_\varepsilon : \text{BD}(O) \times \mathcal{O}(O) \times \Omega \rightarrow [0, \infty]$ defined by

$$\mathcal{I}_\varepsilon(u, A, \omega) := \begin{cases} \int_A W\left(\frac{x}{\varepsilon}, \mathcal{E}u(x), \omega\right) dx & \text{if } u \in \text{LD}(O) \\ \infty & \text{if } u \in \text{BD}(O) \setminus \text{LD}(O). \end{cases} \quad (4.24)$$

(Then $I_\varepsilon = \mathcal{I}_\varepsilon(\cdot, O, \cdot)$ for all $\varepsilon > 0$.)

Let $\omega \in \widehat{\Omega}$ where $\widehat{\Omega} \in \mathcal{F}$ is given by Proposition 3.7. The proof is divided into two steps.

Step 1: establishing the upper bound on $\text{LD}(O)$. We prove that (4.23) holds on $\text{LD}(O)$, i.e. for every $u \in \text{LD}(O)$, one has

$$\Gamma(L^1)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(u, \omega) \leq \int_O W_{\text{hom}}(\mathcal{E}u, \omega) dx.$$

For this we proceed into four substeps. Fix $u \in \text{LD}(O)$.

Substep 1-1: using the Vitali envelope. Consider the set function $\mathcal{S}_u : \mathcal{O}(O) \rightarrow [0, \infty]$ defined by

$$\mathcal{S}_u(A) := \overline{\lim}_{\varepsilon \rightarrow 0} \inf \left\{ \mathcal{I}_\varepsilon(v, A, \omega) : v \in u + \text{LD}_0(A) \right\}. \quad (4.25)$$

For each $\delta > 0$ and each $A \in \mathcal{O}(O)$, denote the class of countable families $\{Q_i = Q_{\rho_i}(x_i)\}_{i \in I}$ (where $Q_{\rho_i}(x_i) := x_i + \rho_i Q$ where Q is the unit cell centered at the origin) of disjoint open

cubes of A with $x_i \in A$, $\rho_i > 0$ and $\text{diam}(Q_i) \in]0, \delta[$ such that $|A \setminus \cup_{i \in I} Q_i| = 0$ by $\mathcal{V}_\delta(A)$, consider $\mathcal{S}_u^\delta : \mathcal{O}(O) \rightarrow [0, \infty]$ given by

$$\mathcal{S}_u^\delta(A) := \inf \left\{ \sum_{i \in I} \mathcal{S}_u(Q_i) : \{Q_i\}_{i \in I} \in \mathcal{V}_\delta(A) \right\},$$

and define $\mathcal{S}_u^* : \mathcal{O}(O) \rightarrow [0, \infty]$ by

$$\mathcal{S}_u^*(A) := \sup_{\delta > 0} \mathcal{S}_u^\delta(A) = \lim_{\delta \rightarrow 0} \mathcal{S}_u^\delta(A).$$

The set function \mathcal{S}_u^* is called the Vitali envelope of \mathcal{S}_u (see §3.5). We prove that for every $A \in \mathcal{O}(O)$, one has

$$\Gamma(L^1)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon(u, A, \omega) \leq \mathcal{S}_u^*(A). \quad (4.26)$$

Fix $A \in \mathcal{O}(O)$ such that $\mathcal{S}_u^*(A) < \infty$. Fix any $\delta > 0$. By definition of $\mathcal{S}_u^\delta(A)$, there exists $\{Q_i\}_{i \in I} \in \mathcal{V}_\delta(A)$ such that

$$\sum_{i \in I} m_u(Q_i) \leq \mathcal{S}_u^\delta(A) + \frac{\delta}{2}. \quad (4.27)$$

Fix any $\varepsilon > 0$ and define $\mathcal{S}_{u,\varepsilon} : \mathcal{O}(O) \rightarrow [0, \infty]$ by

$$\mathcal{S}_{u,\varepsilon}(A) := \inf \left\{ \mathcal{I}_\varepsilon(v, A, \omega) : v \in u + \text{LD}_0(A) \right\}. \quad (4.28)$$

(Thus $\mathcal{S}_u = \overline{\lim}_{\varepsilon \rightarrow 0} \mathcal{S}_{u,\varepsilon}$.) Given any $i \in I$, by definition of $\mathcal{S}_{u,\varepsilon}(Q_i)$, there exists $v_\varepsilon^i \in u + \text{LD}_0(Q_i)$ such that

$$\mathcal{I}_\varepsilon(v_\varepsilon^i, Q_i, \omega) \leq \mathcal{S}_{u,\varepsilon}(Q_i) + \frac{\delta |Q_i|}{2|A|}. \quad (4.29)$$

Define $u_\varepsilon^\delta \in u + \text{LD}_0(A)$ by

$$u_\varepsilon^\delta := \begin{cases} u & \text{in } O \setminus A \\ v_\varepsilon^i & \text{in } Q_i. \end{cases}$$

From (4.29) we see that

$$\mathcal{I}_\varepsilon(u_\varepsilon^\delta, A, \omega) \leq \sum_{i \in I} \mathcal{S}_{u,\varepsilon}(Q_i) + \frac{\delta}{2},$$

hence $\overline{\lim}_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon(u_\varepsilon^\delta, A, \omega) \leq \mathcal{S}_u^\delta(A) + \delta$ by using (4.27), and consequently

$$\overline{\lim}_{\varepsilon \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon(u_\varepsilon^\delta, A, \omega) \leq \mathcal{S}_u^*(A). \quad (4.30)$$

On the other hand, we have

$$\|u_\varepsilon^\delta - u\|_{L^1(O; \mathbb{R}^N)} = \int_A |u_\varepsilon^\delta - u| dx = \sum_{i \in I} \int_{Q_i} |v_\varepsilon^i - u| dx,$$

and so, as $\text{diam}(Q_i) \in]0, \delta[$ for all $i \in I$, from Poincaré's inequality we can assert that

$$\begin{aligned} \|u_\varepsilon^\delta - u\|_{L^1(O; \mathbb{R}^N)} &\leq C\delta \sum_{i \in I} \int_{Q_i} |\mathcal{E}v_\varepsilon^i - \mathcal{E}u| dx \\ &\leq C\delta \left(\sum_{i \in I} \int_{Q_i} |\mathcal{E}v_\varepsilon^i| dx + \int_A |\mathcal{E}u| dx \right), \end{aligned} \quad (4.31)$$

where $C > 0$ is independent of δ , ε and i . Taking the left inequality in (2.1), (4.29) and (4.27) into account, from (4.31) we deduce that

$$\overline{\lim}_{\varepsilon \rightarrow 0} \|u_\varepsilon^\delta - u\|_{L^1(O; \mathbb{R}^N)} \leq C\delta \left(\frac{1}{\alpha} (\mathcal{S}_u^\delta(A) + \varepsilon) + \int_A |\mathcal{E}u| dx \right)$$

which gives

$$\overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \|u_\varepsilon^\delta - u\|_{L^1(O; \mathbb{R}^N)} = 0 \quad (4.32)$$

because $\lim_{\delta \rightarrow 0} m_u^\delta(A) = m_u^*(A) < \infty$. According to (4.30) and (4.32), by diagonalization there exists a mapping $\varepsilon \mapsto \delta_\varepsilon$, with $\delta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, such that:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \|w_\varepsilon - u\|_{L^1(O; \mathbb{R}^N)} &= 0; \\ \overline{\lim}_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon(w_\varepsilon, A, \omega) &\leq \mathcal{S}_u^*(A) \end{aligned}$$

with $w_\varepsilon := u_{\delta_\varepsilon}^\varepsilon$, and (4.26) follows because $\Gamma(L^1)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon(u, A, \omega) \leq \overline{\lim}_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon(w_\varepsilon, A, \omega)$.

Substep 1-2: differentiation with respect to dx . We prove that

$$\Gamma(L^1)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(u, \omega) \leq \int_O \lim_{\rho \rightarrow 0} \frac{\mathcal{S}_u(Q_\rho(x))}{|Q_\rho(x)|} dx \quad (4.33)$$

with \mathcal{S}_u given by (4.25) and $Q_\rho(x) := x + \rho Q$ where Q is the unit cell centered at the origin. Recalling that $I_\varepsilon = \mathcal{I}_\varepsilon(\cdot, O, \cdot)$ for all $\varepsilon > 0$, from (4.26) we have

$$\Gamma(L^1)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(u, \omega) \leq \mathcal{S}_u^*(O).$$

But, it is clear that $\mathcal{S}_u \leq (\beta + |\mathcal{E}u|)dx$, i.e. the assumption (a) of Theorem 3.16 is satisfied, and it is easily seen that \mathcal{S}_u is subadditive in the sense of the assumption (b) of Theorem 3.16. Consequently (4.33) follows from Theorem 3.16.

Substep 1-3: using approximate differentiability. We prove that for dx -a.e. $x_0 \in O$, one has

$$\lim_{\rho \rightarrow 0} \frac{\mathcal{S}_u(Q_\rho(x_0))}{|Q_\rho(x_0)|} \leq \lim_{\rho \rightarrow 0} \frac{\mathcal{S}_{u_{x_0}}(Q_\rho(x_0))}{|Q_\rho(x_0)|} \quad (4.34)$$

with $u_{x_0}(x) := u(x_0) + \nabla u(x_0)(x - x_0)$. Fix any $\delta > 0$. Fix any $\varepsilon > 0$, any $s \in]0, 1[$ and any $\rho \in]0, \delta[$. By definition of $\mathcal{S}_{u_{x_0}, \varepsilon}(Q_{s\rho}(x_0))$ in (4.28) there exists $v \in u_{x_0} + \text{LD}_0(Q_{s\rho}(x_0))$ such that

$$\int_{Q_{s\rho}(x_0)} W\left(\frac{x}{\varepsilon}, \mathcal{E}v, \omega\right) dx \leq \mathcal{S}_{u_{x_0}, \varepsilon}(Q_{s\rho}(x_0)) + \delta |Q_{s\rho}(x_0)|. \quad (4.35)$$

Consider a Uryshon function $\varphi \in C^\infty(O)$ for the pair $(O \setminus Q_\rho(x_0), \overline{Q}_{s\rho}(x_0))$ such that

$$\|\nabla \varphi\|_{L^\infty(O; \mathbb{R}^N)} \leq \frac{1}{\rho(1-s)}. \quad (4.36)$$

Define $w \in \text{LD}(Q_\rho(x_0))$ by

$$w := u + \varphi(u_{x_0} - u).$$

Then $w \in u + \text{LD}_0(Q_\rho(x_0))$ and

$$\mathcal{E}w = \begin{cases} \mathcal{E}u(x_0) & \text{in } Q_{s\rho}(x_0) \\ \nabla \varphi \odot (u_{x_0} - u) + \varphi \mathcal{E}u(x_0) + (1 - \varphi) \mathcal{E}u & \text{in } Q_\rho(x_0) \setminus Q_{s\rho}(x_0). \end{cases}$$

Taking (4.35), the right inequality in (2.1) and (4.36) into account we see that

$$\begin{aligned}
\frac{\mathcal{S}_{u,\varepsilon}(Q_\rho(x_0))}{|Q_{s\rho}(x_0)|} &\leq \frac{1}{|Q_{s\rho}(x_0)|} \int_{Q_\rho(x_0)} W\left(\frac{x}{\varepsilon}, \mathcal{E}w, \omega\right) dx \\
&= \frac{1}{|Q_{s\rho}(x_0)|} \int_{Q_{s\rho}(x_0)} W\left(\frac{x}{\varepsilon}, \mathcal{E}w, \omega\right) dx \\
&\quad + \frac{1}{|Q_{s\rho}(x_0)|} \int_{Q_\rho(x_0) \setminus Q_{s\rho}(x_0)} W\left(\frac{x}{\varepsilon}, \mathcal{E}w, \omega\right) dx \\
&\leq \frac{\mathcal{S}_{u_{x_0},\varepsilon}(Q_{s\rho}(x_0))}{|Q_{s\rho}(x_0)|} + \delta \\
&\quad + \beta \left(\frac{1}{(1-s)s^N} \frac{1}{|Q_\rho(x_0)|} \int_{Q_\rho(x_0)} \frac{|u - u_{x_0}|}{\rho} dx + \frac{\Delta(\rho, s)}{|Q_{s\rho}(x_0)|} \right)
\end{aligned}$$

with $\Delta(\rho, s) := |Q_\rho(x_0) \setminus Q_{s\rho}(x_0)| |\mathcal{E}u(x_0)| + \int_{Q_\rho(x_0) \setminus Q_{s\rho}(x_0)} |\mathcal{E}u| dx$. Noticing that $|Q_\rho(x_0)| \geq |Q_{s\rho}(x_0)|$ and letting $\varepsilon \rightarrow 0$, we obtain

$$\begin{aligned}
\frac{\mathcal{S}_u(Q_\rho(x_0))}{|Q_\rho(x_0)|} &\leq \frac{\mathcal{S}_{u_{x_0}}(Q_{s\rho}(x_0))}{|Q_{s\rho}(x_0)|} + \delta \\
&\quad + \beta \left(\frac{1}{(1-s)} \frac{|Q_\rho(x_0)|}{|Q_{s\rho}(x_0)|} \frac{1}{|Q_\rho(x_0)|} \int_{Q_\rho(x_0)} \frac{|u - u_{x_0}|}{\rho} dx + \frac{\Delta(\rho, s)}{|Q_{s\rho}(x_0)|} \right).
\end{aligned}$$

On the other hand, without loss of generality we can assert that

$$\lim_{\rho \rightarrow 0} \frac{1}{|Q_\rho(x_0)|} \int_{Q_\rho(x_0)} \left| |\mathcal{E}u(x)| - |\mathcal{E}u(x_0)| \right| dx = 0.$$

But

$$\begin{aligned}
\frac{\Delta(\rho, s)}{|Q_{s\rho}(x_0)|} &\leq 2 \left(\frac{1}{s^N} - 1 \right) |\mathcal{E}u(x_0)| \\
&\quad + \frac{1}{s^N} \frac{1}{|Q_\rho(x_0)|} \int_{Q_\rho(x_0)} \left| |\mathcal{E}u(x)| - |\mathcal{E}u(x_0)| \right| dx,
\end{aligned}$$

and so

$$\overline{\lim}_{\rho \rightarrow 0} \frac{\Delta(\rho, s)}{|Q_{s\rho}(x_0)|} \leq 2 \left(\frac{1}{s^N} - 1 \right) |\mathcal{E}u(x_0)|. \tag{4.37}$$

Letting $\rho \rightarrow 0$ in (4.37) and using Theorem 3.10 and (4.37) we deduce that

$$\begin{aligned}
\lim_{\rho \rightarrow 0} \frac{\mathcal{S}_u(Q_\rho(x_0))}{|Q_\rho(x_0)|} &\leq \lim_{\rho \rightarrow 0} \frac{\mathcal{S}_{u_{x_0}}(Q_{s\rho}(x_0))}{|Q_{s\rho}(x_0)|} + \delta + 2 \left(\frac{1}{s^N} - 1 \right) |\mathcal{E}u(x_0)| \\
&= \lim_{\rho \rightarrow 0} \frac{\mathcal{S}_{u_{x_0}}(Q_\rho(x_0))}{|Q_\rho(x_0)|} + \delta + 2 \left(\frac{1}{s^N} - 1 \right) |\mathcal{E}u(x_0)|.
\end{aligned}$$

Letting $s \rightarrow 1$ we conclude that

$$\lim_{\rho \rightarrow 0} \frac{\mathcal{S}_u(Q_\rho(x_0))}{|Q_\rho(x_0)|} \leq \lim_{\rho \rightarrow 0} \frac{\mathcal{S}_{u_{x_0}}(Q_\rho(x_0))}{|Q_\rho(x_0)|} + \delta$$

and (4.34) follows by letting $\delta \rightarrow 0$.

Substep 1-4: end of step 1. Combining (4.33) with (4.34) we deduce that

$$\Gamma(L^1)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(u, \omega) \leq \int_O \lim_{\rho \rightarrow 0} \frac{\mathcal{S}_{u_x}(Q_\rho(x))}{|Q_\rho(x)|} dx.$$

On the other hand, taking (4.24) and (4.25) into account, we see that

$$\lim_{\rho \rightarrow 0} \frac{\mathcal{S}_{u_x}(Q_\rho(x))}{|Q_\rho(x)|} = \lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{\mathcal{S}^{\mathcal{E}u(x)}\left(\frac{1}{\varepsilon}Q_\rho(x)\right)(\omega)}{\left|\frac{1}{\varepsilon}Q_\rho(x)\right|}$$

with $\mathcal{S}^{\mathcal{E}u(x)}\left(\frac{1}{\varepsilon}Q_\rho(x)\right)(\omega)$ given by (3.1) with $\xi = \mathcal{E}u(x)$. But, by Proposition 3.7 (and Definition 3.8) we have

$$\overline{\lim}_{\varepsilon \rightarrow 0} \frac{\mathcal{S}^{\mathcal{E}u(x)}\left(\frac{1}{\varepsilon}Q_\rho(x)\right)(\omega)}{\left|\frac{1}{\varepsilon}Q_\rho(x)\right|} = W_{\text{hom}}(\mathcal{E}u(x), \omega)$$

for all $\rho > 0$, hence

$$\Gamma(L^1)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(u, \omega) \leq \int_O W_{\text{hom}}(\mathcal{E}u(x), \omega) dx,$$

which completes the proof of Step 1.

Step 2: using a relaxation theorem. From Step 1 we have

$$\Gamma(L^1)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(u, \omega) \leq J_{\text{hom}}(u)$$

for all $u \in \text{BD}(O)$ with $J_{\text{hom}} : \text{BD}(O) \rightarrow [0, \infty]$ given by

$$J_{\text{hom}}(u) := \begin{cases} \int_O W_{\text{hom}}(\mathcal{E}u(x), \omega) dx & \text{if } u \in \text{LD}(O) \\ \infty & \text{if } u \in \text{BD}(O) \setminus \text{LD}(O). \end{cases}$$

As $\Gamma(L^1)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(\cdot, \omega)$ is L^1 -lower semicontinuous we deduce that

$$\Gamma(L^1)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(u, \omega) \leq \overline{J}_{\text{hom}}(u)$$

for all $u \in \text{BD}(O)$, where $\overline{J}_{\text{hom}}$ is the L^1 -lower semicontinuous envelope of J_{hom} . But, by Proposition 3.9 can assert that W_{hom} is continuous and symmetric quasiconvex and has 1-growth, hence by Theorem 3.14 we have

$$\overline{J}_{\text{hom}}(u) = \int_O W_{\text{hom}}(\mathcal{E}u(x), \omega) dx + \int_O W_{\text{hom}}^\infty\left(\frac{dE^s u}{d|E^s u|}(x), \omega\right) d|E^s u|(x)$$

for all $u \in \text{BD}(O)$, and (4.23) follows. ■

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