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BOUNDS ON THE MINIMUM DISTANCE OF ALGEBRAIC GEOMETRY CODES DEFINED OVER SOME FAMILIES OF SURFACES

YVES AUBRY, ELENA BERARDINI, FABIEN HERBAUT AND MARC PERRET

Abstract. We prove lower bounds for the minimum distance of algebraic geometry codes over surfaces whose canonical divisor is either nef or anti-strictly nef and over surfaces without irreducible curves of small genus. These lower bounds are sharpened for surfaces whose arithmetic Picard number equals one, surfaces without curves with small self-intersection and fibered surfaces. What characterizes the bounds we obtain is that they involve a measure of closeness of the divisor defining the code to the ample cone in the Néron-Severi group.

Contents

1. Introduction

After the construction of Goppa codes over algebraic curves ([10]) and their successful use by Tsfaman, Vlăduţ and Zink in beating the Gilbert-Varshamov bound ([21]), algebraic geometry codes over curves have been largely studied. Even though the same construction holds on varieties of higher dimension, the literature is less abundant in this context. However one can consult [17] for a survey of Little and [14] for an extensive use of intersection theory involving the Seshadri constant proposed by S. H. Hansen. Some work has also been undertaken in the direction of surfaces. Rational surfaces yielding to good codes were constructed by Couvreur in [8] from some blow-ups of the plane and by

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Blache *et al.* in [6] from Del Pezzo surfaces. Codes from cubic surfaces where studied by Voloch and Zarzar in [24], from toric surfaces by J. P. Hansen in [13], from Hirzebruch surfaces by Nardi in [18], from ruled surfaces by one of the authors in [1] and from abelian surfaces by Haloui in [12] in the specific case of simple Jacobians of genus 2 curves, and by the authors in [5] for general abelian surfaces. Following the classification of algebraic surfaces according to their Kodaira dimension $\kappa \in \{-1, 0, 1, 2\}$, this means that up to now most of the work on algebraic geometry codes over surfaces was devoted to surfaces of Kodaira dimension -1 (this is the case of rational and ruled surfaces) and 0 (this is for instance the case of abelian surfaces). As far as we know, no author have paid attention to other surfaces, except Zarzar ([25]) and Little and Schenck ([16]) who studied surfaces whose arithmetic Picard number is one.

The aim of this paper is to provide a study of the minimum distance $d(X, G, S)$ of the algebraic geometry code $\mathcal{C}(X, G, S)$ constructed from an algebraic surface X, a set S of rational points on X and a rational effective divisor G on X avoiding S.

We prove in Section 3 lower bounds for the minimum distance $d(X, G, S)$ under some specific assumptions on the *geometry of the surface* itself. Two quite wide families of surfaces are studied. The first one is that of surfaces whose canonical divisor is either nef or anti-strictly nef. The second one consists of surfaces which do not contain irreducible curves of low genus. We obtain the following theorem, where we denote, as in the whole paper, by $\mathbb{F}_q$ the finite field with q elements and where we set $m := \lfloor 2\sqrt{q} \rfloor$.

Theorem. *(Theorem 3.3 and Theorem 3.5) Let X be an absolutely irreducible smooth projective algebraic surface defined over $\mathbb{F}_q$ whose canonical divisor is denoted by K_X. Consider a set S of rational points on X, a rational effective divisor G avoiding S and an ample divisor H on X. We set*

$$\alpha(G, H) := \frac{(G.H)^2}{2H^2} + \frac{1}{2} G.K_X \text{ and}$$

$$d^*(X, G, S, H) := \sharp S - G.H(q + 1 + m) - m\alpha(G, H).$$

1) *(i) If K_X is nef, then*

$$d(X, G, S) \geq d^*(X, G, S, H).$$

(ii) If $-K_X$ is strictly nef, then

$$d(X, G, S) \geq d^*(X, G, S, H) + \frac{m}{2} G.(H + K_X).$$

2) *If there exists an integer $\ell > 0$ such that any $\mathbb{F}_q$-irreducible curve lying on X and defined over $\mathbb{F}_q$ has arithmetic genus strictly greater than ℓ, then*

$$d(X, G, S) \geq d^*(X, G, S, H) + \left(G.H - \frac{\alpha(G, H)}{\ell} \right) (q + 1 + m).$$

Inside both families, adding some extra geometric assumptions on the surface yields in Section 4 to some improvements for these lower bounds. This is the case for surfaces whose arithmetic Picard number is one, for which we obtain general bounds (see Remark 4.4). This is also the case for surfaces without irreducible curves defined over $\mathbb{F}_q$ with small self-intersection, so as for fibered surfaces with nef canonical divisor. Theorems 4.8 and 4.9 (that hold for fibered surfaces) improve the bounds of Theorems 3.3 and 3.5 (that hold for the whole wide family). Indeed the bound on the minimum distance $d(X, G, S)$ is increased by the non-negative *defect* $\delta(B) = q + 1 + mg_B - \sharp B(\mathbb{F}_q)$ of the base curve B. Finally in Section 5 we specify our bounds to the case of surfaces of degree $d \geq 3$ embedded in $\mathbb{P}^3$.

Let us emphasize that most of the lower bounds we obtain depend on the *geometry of the divisor* G through its position in the Néron-Severi group $\mathrm{NS}(X)$. Precisely these bounds depend on the choice of an auxiliary ample divisor H through the quantities $G.H$ and $\alpha(G, H) = \frac{(G.H)^2}{2H^2} + \frac{1}{2}G.K_X$. If we denote by π_G the virtual arithmetic genus of G, then $\alpha(G, H)$ is greater than $\pi_G - 1$, with equality if and only if G is ample and H is numerically proportional to G. So an interesting perspective could be to say that our bounds depend on $\alpha(G, H) - \pi_G + 1$, which will be interpreted in Remark 3.1 as a measure of closeness of G to the ample cone in $\mathrm{NS}(X)$. If G is ample, the best choice for H will be G itself (or any divisor numerically proportional to G), for then $\alpha(G, H) = \pi_G - 1$. In case G is not ample, the closer $\pi_G - 1$ and $\alpha(G, H)$ are, the better the lower bounds are.

2. Background

Codes from algebraic surfaces are defined in the same way as on algebraic curves: we evaluate some functions with prescribed poles on some sets of rational points. Whereas the key tool for the study of the minimum distance in the 1-dimensional case is the mere fact that a function has as many zeroes as poles, in the 2-dimensional case most of the proofs rest on intersection theory.

We recall in this section the few results on intersection theory we need. Following the authors cited in the Introduction we define the algebraic geometry code. We recall quickly how the dimension of the code can be lower bounded under the assumption of the injectivity of the evaluation map. Thus we prove a lemma that will be used in the course of the paper to bound below the minimum distance of the code for several families of surfaces. Finally, we recall some results on the number of rational points on curves over finite fields.

2.1. **Intersection theory.**

Intersection theory has almost become a mainstream tool to study codes over surfaces (see [1], [14], [24], [25], [16], [5]) and it is also central in our proofs. We do not recall here the classical definitions of the different equivalent classes of divisors and we refer the reader to [15, §V] for a presentation.

We denote by $\mathrm{NS}(X)$ the *arithmetic* Néron-Severi group of a smooth surface X defined over $\mathbb{F}_q$ whose rank is called the arithmetic Picard number of X, or Picard number for short. Recall that a divisor D on X is said to be *nef* (respectively *strictly nef*) if $D.C \geq 0$ (respectively $D.C > 0$) for any absolutely irreducible curve C on X. A divisor D is said to be *anti-ample* if $-D$ is ample, *anti-nef* if $-D$ is nef and *anti-strictly nef* if $-D$ is strictly nef. Let us emphasize three classical results we will use in this paper.

The first one is (a generalisation of) the adjunction formula (see [15, §V, Exercise 1.3]). For any $\mathbb{F}_q$-irreducible curve D on X of arithmetic genus π_D, we have

$$D.(D + K_X) = 2\pi_D - 2 \tag{1}$$

where K_X is the canonical divisor of X. This formula allows to define the virtual arithmetic genus of any divisor D on X.

The second one is the corollary of the Hodge index theorem stating that if H and D are two divisors on X with H ample, then

$$H^2D^2 \leq (H.D)^2, \tag{2}$$

where equality holds if and only if H and D are numerically proportional.

The last one is a simple outcome of Bézout's theorem in projective spaces (and the trivial part of the Nakai-Moishezon criterion). It ensures that for any ample divisor H on X and for any irreducible curve C on X, we have $H^2 > 0$ and $H.C > 0$.

2.2. **Algebraic geometry codes.**

2.2.1. *Definition of AG codes.* We study, as in the non-exhaustive list of papers [1], [24], [8], [14], [25], [12], [16] and [5], the generalisation of Goppa algebraic geometry codes from curves to surfaces. In the whole paper we consider an absolutely irreducible smooth projective algebraic surface X defined over $\mathbb{F}_q$ and a set S of rational points on X. Given a rational divisor G on X avoiding S, the algebraic geometry code, or AG codes for short, is defined by evaluating the elements of the Riemann-Roch space $L(G)$ at the points of S. Precisely we define the linear code $\mathcal{C}(X, G, S)$ as the image of the evaluation map $\mathrm{ev} : L(G) \longrightarrow \mathbb{F}_q^{\sharp S}$.

2.2.2. *Length and dimension of AG codes.* From the very definition, the length of the code is $\sharp S$. As soon as the morphism ev is injective - see (7) for a sufficient condition - the dimension of the code equals $\ell(G) = \dim_{\mathbb{F}_q} L(G)$ which can be easily bounded from below using standard algebraic geometry tools as follows. By Riemann-Roch theorem (see [15, V, §1]), we have

$$\ell(G) - s(G) + \ell(K_X - G) = \frac{1}{2}G.(G - K_X) + 1 + p_a(X)$$

where $p_a(X)$ is the arithmetic genus of X, and where the so-called *superabundance* $s(G)$ of G in X, being in fact itself a dimension of some vector space, is non-negative.

Now, under the assumption that

$$K_X.H < G.H, \tag{3}$$

for some ample divisor H, we have from [15, V, Lemma 1.7] that $\ell(K_X - G) = 0$. Thus, if the evaluation map ev is injective and under assumption (3), then we get the lower bound

$$\dim \mathcal{C}(X, G, S) = \ell(G) \geq \frac{1}{2}G.(G - K_X) + 1 + p_a(X) \tag{4}$$

for the dimension of the code $\mathcal{C}(X, G, S)$.

2.2.3. *Toward the minimum distance of AG codes.* It follows that the difficulty lies in the estimation of the minimum distance $d(X, G, S)$ of the code. For any non-zero $f \in L(G)$, we introduce as in [12] the number $N(f)$ of zero coordinates of the codeword $\mathrm{ev}(f)$. The Hamming weight $w(\mathrm{ev}(f))$ of the codeword $\mathrm{ev}(f)$ satisfies

$$w(\mathrm{ev}(f)) \geq \sharp S - N(f), \tag{5}$$

from which it follows that

$$d(X, G, S) \geq \sharp S - \max_{f \in L(G)\setminus\{0\}} N(f). \tag{6}$$

We also deduce from (5) that

$$\text{ev is injective if } \max_{f \in L(G)\setminus\{0\}} N(f) < \sharp S. \tag{7}$$

We now broadly follow the way of [12]. We associate to any non-zero function $f \in L(G)$ the rational effective divisor

$$D_f := G + (f) = \sum_{i=1}^{k} n_i D_i \geq 0, \tag{8}$$

where (f) is the principal divisor defined by f, the n_i are positive integers and each D_i is a reduced $\mathbb{F}_q$-irreducible curve.

Note that in this setting, the integer k and the curves D_i's depend on $f \in L(G)$. Several lower bounds for the minimum distance $d(X, G, S)$ in this paper will follow from the key lemma below.

Lemma 2.1. *Let X be a smooth projective surface defined over $\mathbb{F}_q$, S be a set of rational points on X and G be a rational effective divisor on X avoiding S. Set $m = \lfloor 2\sqrt{q} \rfloor$ and keep the notations introduced in (8). If there exist real numbers a, b_1, b_2, c, such that for any non-zero $f \in L(G)$ the three following assumptions are satisfied*

(1) $k \leq a$

(2) $\sum_{i=1}^{k} \pi_{D_i} \leq b_1 + kb_2$ *and*
(3) for any $1 \leq i \leq k$ *we have* $\sharp D_i(\mathbb{F}_q) \leq c + m\pi_{D_i}$
then the minimum distance $d(X, G, S)$ *of* $\mathcal{C}(X, G, S)$ *satisfies*

$$d(X, G, S) \geq \sharp S - a(c + mb_2) - mb_1.$$

Proof. Let us write the principal divisor $(f) = (f)_0 - (f)_\infty$ as the difference of its effective divisor of zeroes minus its effective divisor of poles. Since G is effective and f belongs to $L(G)$, we have $(f)_\infty \leq G$. Hence, formula (8) reads $G + (f)_0 - (f)_\infty = \sum_{i=1}^{k} n_i D_i$, that is

$$(f)_0 = \sum_{i=1}^{k} n_i D_i + (f)_\infty - G \leq \sum_{i=1}^{k} n_i D_i.$$

This means that any $\mathbb{F}_q$-rational point of $(f)_0$ lies in some D_i so

$$N(f) \leq \sum_{i=1}^{k} \sharp D_i(\mathbb{F}_q). \tag{9}$$

Then it follows successively from the assumptions of the lemma that

$$N(f) \leq \sum_{i=1}^{k} (c + m\pi_{D_i}) \leq kc + m(b_1 + kb_2) \leq mb_1 + a(c + mb_2).$$

Finally Lemma 2.1 follows from (6). □

2.3. **Two upper bounds for the number of rational points on curves.** We manage to fulfill assumption (3) in Lemma 2.1 using the bounds on the number of rational points given in Theorem 2.2 and Proposition 2.3 below. Point (2) of Theorem 2.2 appears in the proof of Theorem 3.3 of Little and Schenck in [16] within a more restrictive context, whereas point (1) follows from [2]. We state a general theorem and give here the full proof for the sake of completeness following [16].

Theorem 2.2 (Aubry-Perret [2] and Little-Schenck [16])**.** *Let* D *be an* $\mathbb{F}_q$*-irreducible curve of arithmetic genus* π_D *lying on a smooth projective algebraic surface. Then,*

(1) we have $\sharp D(\mathbb{F}_q) \leq q + 1 + m\pi_D$.
(2) (Little-Schenck) If moreover D *is* not *absolutely irreducible, we have*

$$\sharp D(\mathbb{F}_q) \leq \pi_D + 1.$$

Proof. We first prove the second item, following the proof of [16, Th. 3.3]. Since D is $\mathbb{F}_q$-irreducible, the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ acts transitively on the set of its $\bar{r} \geq 1$ absolutely irreducible components $D_1, \ldots, D_{\bar{r}}$. Since a $\mathbb{F}_q$-rational point on D is stable under the action of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, it lies in the intersection $\cap_{1 \leq i \leq \bar{r}} D_i$. Under the assumption that D is not absolutely irreducible, that is $\bar{r} \geq 2$, it follows that $\sharp D(\mathbb{F}_q) \leq \sharp(D_i \cap D_j)(\overline{\mathbb{F}}_q) \leq D_i.D_j$ for every couple (i, j) with $i \neq j$.

As a divisor, D can be written over $\overline{\mathbb{F}}_q$ as $D = \sum_{i=1}^{\bar{r}} a_i D_i$. By transitivity of the Galois action, we have $a_1 = \cdots = a_{\bar{r}} = a$. Now since D can be assumed to be reduced, we have $a = 1$, so that finally $D = \sum_{i=1}^{\bar{r}} D_i$. Using the adjonction formula (1) for D and each D_i, and taking into account that $\pi_{D_i} \geq 0$ for any i, we get

$$\begin{aligned} 2\pi_D - 2 &= (K_X + D).D \\ &= \sum_{i=1}^{\bar{r}} (K_X + D_i).D_i + \sum_{i \neq j} D_i.D_j \\ &= \sum_{i=1}^{\bar{r}} (2\pi_{D_i} - 2) + \sum_{i \neq j} D_i.D_j \\ &\geq -2\bar{r} + \sum_{i \neq j} D_i.D_j. \end{aligned}$$

Since there are $\bar{r}(\bar{r}-1)$ pairs (i,j) with $i \neq j$, we deduce that for at least one such pair (i_0, j_0), we have

$$D_{i_0}.D_{j_0} \leq \frac{2(\pi_D - 1 + \bar{r})}{\bar{r}(\bar{r}-1)}.$$

It is then easily checked that the left hand of the former inequality is a decreasing function of $\bar{r} \geq 2$, so that we obtain

$$\sharp D(\mathbb{F}_q) \leq D_{i_0}.D_{j_0} \leq \frac{2(\pi_D - 1 + 2)}{2(2-1)} = \pi_D + 1$$

and the second item is proved.

The first item follows from Aubry-Perret's bound in [2] in case D is absolutely irreducible, that is in case $\bar{r} = 1$, and from the second item in case D is not absolutely irreducible since $\pi_D + 1 \leq q + 1 + m\pi_D$. □

The following bound will be useful in Subsection 4.3 for the study of codes from fibered surfaces.

Proposition 2.3 (Aubry-Perret [4]). *Let C be a smooth projective absolutely irreducible curve of genus g_C over $\mathbb{F}_q$ and D be an $\mathbb{F}_q$-irreducible curve having $\bar{r}$ absolutely irreducible components $\overline{D}_1, \ldots, \overline{D}_{\bar{r}}$. Suppose there exists a regular map $D \to C$ in which no $\overline{\mathbb{F}}_q$-irreducible component does map to a point. Then*

$$|\sharp D(\mathbb{F}_q) - \sharp C(\mathbb{F}_q)| \leq (\overline{r} - 1)q + m(\pi_D - g_C).$$

Proof. Since C is smooth and no geometric component of D does map to a point, the map $D \to C$ is flat. Hence by [4, Th.14] we have

$$|\sharp D(\mathbb{F}_q) - \sharp C(\mathbb{F}_q)| \leq (\overline{r} - 1)(q-1) + m\left(\sum_{i=1}^{\overline{r}} g_{\overline{D}_i} - g_C\right) + \Delta_D$$

where $\Delta_D = \sharp\tilde{D}(\overline{\mathbb{F}}_q) - \sharp D(\overline{\mathbb{F}}_q)$ with $\tilde{D}$ the normalization of D. The result follows from [4, Lemma 2] where it is proved that $m\sum_{i=1}^{\bar{r}} g_{\overline{D}_i} + \Delta_D - \bar{r} + 1 \leq m\pi_D$. □

3. The minimum distance of codes from some families of algebraic surfaces

We are unfortunately unable to fulfil simultaneously assumptions (1) and (2) of Lemma 2.1 for general surfaces. So we focus on two families of algebraic surfaces where we do succeed. To begin with, let us fix some common notations.

We consider a rational effective divisor G on the surface X avoiding a set S of rational points on X and H a fixed auxiliary ample divisor on X. We study, in accordance to Section 2.2, the evaluation code $\mathcal{C}(X, G, S)$ and we denote by $d(X, G, S)$ its minimum distance.

The role of the auxiliary ample divisor H is to enable us to establish some explicit lower bounds for $d(X, G, S)$, depending on H through a *decreasing* function of the quantity

$$\alpha(G, H) := \frac{(G.H)^2}{2H^2} + \frac{1}{2}G.K_X \tag{10}$$

that will appear in most of our statements. It is worth noticing that we have, by (2) and (1),

$$\alpha(G, H) \geq \frac{G^2H^2}{2H^2} + \frac{1}{2}G.K_X = \frac{G^2}{2} + \frac{1}{2}G.K_X = \pi_G - 1,$$

with equality if and only if G and H are numerically proportional. In case G is ample the best choice for H will be G itself. In case G is not ample, then we will get some lower bound for the minimum distance depending on the choice of H. The closer to $\pi_G - 1$ this constant $\alpha(G, H)$ will be, the better the lower bound for $d(X, G, S)$ will be.

Remark 3.1. The geometric meaning of $\alpha(G, H) - \pi_G + 1$ is the following. The intersection product induces a non-degenerate bilinear pairing on $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, and the class of G can be uniquely written

$$G = \frac{G.H}{H^2}H + \left(G - \frac{G.H}{H^2}H\right)$$

as the sum of a multiple of H with $p^{\perp}(G) := \left(G - \frac{G.H}{H^2}H\right)$ which is orthogonal to H. Now, the Hodge index theorem states that the intersection product is definite negative on the orthogonal of H, so its opposite induces a norm $\|.\|_H$ on $H^{\perp}$. There $\alpha(G, H) - \pi_G + 1 = -\frac{1}{2}(G - \frac{G.H}{H^2}H)^2$ is nothing but $\frac{1}{2}\|p^{\perp}(G)\|_H^2$.

3.1. Surfaces whose canonical divisor is either nef or anti-strictly nef.

We study in this section codes defined over surfaces such that either the canonical divisor K_X is nef, or its opposite $-K_X$ is strictly nef. This family is quite large. It contains, for instance:

- surfaces whose canonical divisor K_X is anti-ample.
- Minimal surfaces of Kodaira dimension 0, for which the canonical divisor is numerically zero, hence nef. These are abelian surfaces, $K3$ surfaces, Enriques surfaces and hyperelliptic or quasi-hyperelliptic surfaces (see [7]).
- Minimal surfaces of Kodaira dimension 2. These are the so called minimal surfaces of general type. For instance, surfaces in $\mathbb{P}^3$ of degree $d \geq 4$, without curves C with $C^2 = -1$, are minimal of general type.
- Surfaces whose arithmetic Picard number is one.
- Surfaces of degree 3 embedded in $\mathbb{P}^3$.

The main theorem of this section (Theorem 3.3) rests mainly on the next lemma, fulfilling assumptions (1) and (2) of Lemma 2.1.

Lemma 3.2. *Let $D = \sum_{i=1}^{k} n_i D_i$ be the decomposition as a sum of $\mathbb{F}_q$-irreducible and reduced curves of an effective divisor D linearly equivalent to G. Assume that H is an ample divisor on X. Then we have:*

(1) $k \leq G.H$;

(2) (i) if K_X is nef, then $\sum_{i=1}^{k} \pi_{D_i} \leq \alpha(G,H) + k$;

(ii) if $-K_X$ is strictly nef, then $\sum_{i=1}^{k} \pi_{D_i} \leq \alpha(G,H) - \frac{1}{2}G.K_X + \frac{1}{2}k$.

Proof. Using that D is numerically equivalent to G, that $n_i > 0$ and $D_i.H > 0$ for every $i = 1, \ldots, k$ since H is ample, we prove item (1):

$$G.H = D.H = \sum_{i=1}^{k} n_i D_i.H \geq \sum_{i=1}^{k} D_i.H \geq k.$$

Now we apply inequality (2) to H and D_i for every i to get $D_i^2 H^2 \leq (D_i.H)^2$. We thus have, together with adjunction formula (1) and inequality $H^2 > 0$,

$$\pi_{D_i} - 1 \leq (D_i.H)^2/2H^2 + D_i.K_X/2. \tag{11}$$

To prove point (i) of item (2) we sum from $i = 1$ to k and thus obtain

$$\begin{aligned} \sum_{i=1}^{k} \pi_{D_i} - k &\leq \frac{1}{2H^2} \sum_{i=1}^{k} (D_i.H)^2 + \frac{1}{2} \sum_{i=1}^{k} D_i.K_X \\ &\leq \frac{1}{2H^2} \left(\sum_{i=1}^{k} n_i D_i.H \right)^2 + \frac{1}{2} \sum_{i=1}^{k} n_i D_i.K_X \\ &\leq \frac{(G.H)^2}{2H^2} + \frac{G.K_X}{2} \\ &= \alpha(G,H), \end{aligned} \tag{12}$$

where we use the positivity of the coefficients n_i, the numeric equivalence between D and $\sum_{i=1}^{k} n_i D_i$, the amplitude of H and the hypothesis taken on K_X.

Under the hypotheses of point (ii) we have $D_i.K_X \leq -1$, so replacing in the first line of (12) gives $\sum_{i=1}^{k} \pi_{D_i} - k \leq \frac{1}{2H^2} \sum_{i=1}^{k} (D_i.H)^2 - \frac{k}{2}$. We conclude in the same way. □

Theorem 3.3. *Let G be a rational effective divisor on a surface X avoiding a set S of rational points and H be any ample divisor. We set*

$$d^*(X, G, S, H) := \sharp S - G.H(q + 1 + m) - m\alpha(G, H). \tag{13}$$

(i) If K_X is nef, then

$$d(X, G, S) \geq d^*(X, G, S, H).$$

(ii) If $-K_X$ is strictly nef, then

$$d(X, G, S) \geq d^*(X, G, S, H) + \frac{m}{2} G.(H + K_X).$$

Proof. The theorem follows from Lemma 2.1 for which assumption (1) and (2) hold from Lemma 3.2 and assumption (3) holds from Theorem 2.2. □

3.2. **Surfaces without irreducible curves of small genus.** We consider in this section surfaces X with the property that there exists an integer $\ell \geq 1$ such that any $\mathbb{F}_q$-irreducible curve D lying on X and defined over $\mathbb{F}_q$ has arithmetic genus $\pi_D \geq \ell + 1$. It turns out that under this hypothesis, we can fulfil assumptions (1) and (2) of Lemma 2.1 without any hypothesis on K_X contrary to the setting of Section 3.1.

Examples of surfaces with this property do exist. For instance:

- simple abelian surfaces satisfy this property for $\ell = 1$ (see [5] for abelian surfaces with this property for $\ell = 2$).
- Fibered surfaces on a smooth base curve B of genus $g_B \geq 1$ and generic fiber of arithmetic genus $\pi_0 \geq 1$, and whose singular fibers are $\mathbb{F}_q$-irreducible, do satisfy this property for $\ell = \min(g_B, \pi_0) - 1$.
- Smooth surfaces in $\mathbb{P}^3$ of degree d whose arithmetic Picard group is generated by the class of an hyperplane section do satisfy this property for $\ell = \frac{(d-1)(d-2)}{2} - 1$ (see Lemma 5.2).

Lemma 3.4. *Let X be a surface without $\mathbb{F}_q$-irreducible curves of arithmetic genus less than or equal to ℓ for ℓ a positive integer. Consider a rational effective divisor G and an ample divisor H on X. Let $D = \sum_{i=1}^{k} n_i D_i$ be the decomposition as a sum of $\mathbb{F}_q$-irreducible and reduced curves of an effective divisor D linearly equivalent to G. Then we have*

(1) $k \leq \frac{\alpha(G,H)}{\ell}$;

(2) $\sum_{i=1}^{k} \pi_{D_i} \leq \alpha(G, H) + k$.

In case X falls in both families of Section 3.1 and this Section 3.2, the present new bound of the first item for k is better than the one of Lemma 3.2 if and only if $\alpha(G, H) < \ell G.H$. In the general setting, this inequality sometimes holds true,

sometimes not. Let us consider for instance $H = K_X$ and $G = rK_X$, where K_X is supposed to be ample. In this setting $\alpha(rK_X, K_X) = r(r+1)K_X^2/2$ and thus the inequality holds if and only if $r < 2\ell - 1$.

Proof. By assumption, we have $0 \leq \ell \leq \pi_{D_i} - 1$ and $n_i \geq 1$ for any $1 \leq i \leq k$, hence using adjunction formula (1), we have

$$2\ell k \leq 2\sum_{i=1}^{k}(\pi_{D_i} - 1) \leq 2\sum_{i=1}^{k} n_i(\pi_{D_i} - 1) = \sum_{i=1}^{k} n_i D_i^2 + \sum_{i=1}^{k} n_i D_i.K_X.$$

Moreover using (2) and (8), we get

$$2\ell k \leq \sum_{i=1}^{k} n_i \frac{(D_i.H)^2}{H^2} + (\sum_{i=1}^{k} n_i D_i).K_X \leq \sum_{i=1}^{k} n_i^2 \frac{(D_i.H)^2}{H^2} + G.K_X.$$

Since H is ample, we obtain

$$2\ell k \leq \sum_{i,j=1}^{k} n_i n_j \frac{(D_i.H)(D_j.H)}{H^2} + G.K_X = \frac{(\sum_{i,=1}^{k} n_i D_i.H)^2}{H^2} + G.K_X.$$

By (8) and by definition of $\alpha(G, H)$, we conclude that

$$2\ell k \leq \frac{(G.H)^2}{H^2} + G.K_X = 2\alpha(G, H),$$

proving both items of Lemma 3.4. □

Theorem 3.5. *Let X be a surface without $\mathbb{F}_q$-irreducible curves of arithmetic genus less than or equal to ℓ for ℓ a positive integer. Consider an effective divisor G on X avoiding a finite set S of rational points, and H an ample divisor. Then we have*

$$d(X, G, S) \geq d^*(X, G, S, H) + \left(G.H - \frac{\alpha(G, H)}{\ell}\right)(q + 1 + m).$$

Proof. The theorem follows from Lemma 2.1, for which items (1) and (2) hold from Lemma 3.4 and item (3) holds from Theorem 2.2. □

4. Three improvements

We manage to obtain better parameters for the conditions (1), (2) or (3) of Lemma 2.1 in three cases: surfaces of arithmetic Picard number one, surfaces which do not contain $\mathbb{F}_q$-irreducible curves of small self-intersection and whose canonical divisor is either nef or anti-nef, and fibered surfaces with nef canonical divisor.

4.1. **Surfaces with Picard number one.** As mentioned in the introduction, the case of surfaces X whose arithmetic Picard number equals one has already attracted some interest (see [25], [24], [16], [6]). We prove in this subsection Lemma 4.1 and Theorem 4.3 which improve, under this rank one assumption, the bounds of Lemma 3.2 and Theorem 3.3. These new bounds depend on the sign of $3H^2 + H.K_X$, where H is the ample generator of $\mathrm{NS}(X)$.

Lemma 4.1. *Let X be a smooth projective surface of arithmetic Picard number one. Let H be the ample generator of $\mathrm{NS}(X)$. Let $G = rH$ for r a positive integer. For any non-zero function $f \in L(rH)$ consider the decomposition $D_f = \sum_{i=1}^{k} n_i D_i$ into $\mathbb{F}_q$-irreducible and reduced curves D_i with positive integer coefficients n_i as in (8). Then the sum of the arithmetic genera of the curves D_i satisfies:*

(i) $\sum_{i=1}^{k} \pi_{D_i} \leq (k-1)\pi_H + \pi_{(r-k+1)H}$ *if* $3H^2 + H.K_X \geq 0$;
(ii) $\sum_{i=1}^{k} \pi_{D_i} \leq H^2(r-k)^2/2 + H^2(r-2k) + k$ *if* $3H^2 + H.K_X < 0$.

Remark 4.2. Note that the condition $3H^2 + H.K_X \geq 0$ is satisfied as soon as $H.K_X \geq 0$. It is also satisfied in the special case where $K_X = -H$ which corresponds to Del Pezzo surfaces.

Proof. In order to prove the first item, we consider a non-zero function $f \in L(rH)$ and we keep the notations already introduced in (8), namely $D_f = \sum_{i=1}^{k} n_i D_i$. As $\mathrm{NS}(X) = \mathbb{Z}H$, for all i we have $D_i = a_i H$ and we know by Lemma 2.2 in [25] that $k \leq r$. Intersecting with the ample divisor H enables to prove that for all i we have $a_i \geq 1$ and that $\sum_{i=1}^{k} n_i a_i = r$. Thus to get an upper bound for $\sum_{i=1}^{k} \pi_{D_i} = \sum_{i=1}^{k} \pi_{a_i H}$, we are reduced to bounding $\left(\sum_{i=1}^{k} a_i^2\right) H^2/2 + \left(\sum_{i=1}^{k} a_i\right) H.K_X/2 + k$ under the constraint $\sum_{i=1}^{k} a_i n_i = r$. Our strategy is based on the two following arguments.

First, the condition $3H^2 + H.K_X \geq 0$ guarantees that $a \mapsto \pi_{aH}$ is an increasing sequence. Indeed, for integers $a' > a \geq 1$ we have $\pi_{a'H} \geq \pi_{aH}$ if and only if $(a+a')H^2 \geq -H.K_X$, which is true under the condition above because $a + a' \geq 3$. As a consequence, if we fix an index i between 1 and k and if we consider that the product $n_i a_i$ is constant, then the value of $\pi_{a_i H}$ is maximum when a_i is, that is when $a_i = n_i a_i$ and $n_i = 1$.

Secondly, assume that all the n_i equal 1 and that $\sum_{i=1}^{k} a_i = r$. We are now reduced to bounding $\sum_{i=1}^{k} a_i^2$. We can prove that the maximum is reached when all the a_i equal 1 except one which equals $r - k + 1$. Otherwise, suppose for example that $2 \leq a_1 \leq a_2$. Then $a_1^2 + a_2^2 < (a_1 - 1)^2 + (a_2 + 1)^2$ and $\sum_{i=1}^{k} a_i^2$ is not maximum, and the first item is thus proved.

For the second item, using the adjonction formula we get

$$\sum_{i=1}^{k} \pi_{D_i} - k \leq \frac{1}{2H^2} \sum_{i=1}^{k} (D_i.H)^2 + \frac{1}{2} \sum_{i=1}^{k} D_i.K_X.$$

Again as $\mathrm{NS}(X) = \mathbb{Z}H$, for all i we have $D_i = a_i H$. Thus we get

$$\sum_{i=1}^{k} \pi_{D_i} - k \leq \frac{1}{2H^2} \sum_{i=1}^{k} a_i^2 (H^2)^2 + \frac{1}{2} \sum_{i=1}^{k} a_i H.K_X.$$

Now using that $H.K_X \leq -3H^2$ by hypothesis, that $\sum_{i=1}^{k} a_i \geq k$ since every a_i is positive and that since $\sum_{i=1}^{k} a_i \leq r$ we can prove again that $\sum_{i=1}^{k} a_i^2 \leq (r-k+1)^2 + (k-1)$, we get

$$\sum_{i=1}^{k} \pi_{D_i} - k \leq \frac{H^2}{2}((r-k+1)^2 + (k-1)) - \frac{3H^2}{2}k.$$

Some easy calculation shows that this is equivalent to our second statement. □

Theorem 4.3. *Let X be a smooth projective surface of arithmetic Picard number one. Let H be the ample generator of $\mathrm{NS}(X)$ and S a finite set of rational points avoiding H. For any positive integer r, the minimum distance $d(X, rH, S)$ of the code $\mathcal{C}(X, rH, S)$ satisfies:*

(i) if $3H^2 + H.K_X \geq 0$, then

$$d(X, rH, S) \geq \begin{cases} \sharp S - (q + 1 + m\pi_{rH}) & \text{if } r > 2(q+1+m)/mH^2, \\ \sharp S - r(q + 1 + m\pi_H) & \text{otherwise.} \end{cases}$$

(ii) If $3H^2 + H.K_X < 0$, then

$$d(X, rH, S) \geq \begin{cases} \sharp S - (q+1+m) - mH^2(r^2-3)/2 & \text{if } r > 2(q+1+m)/mH^2 - 3, \\ \sharp S - r(q + 1 + m - mH^2) & \text{otherwise.} \end{cases}$$

Proof. For any non-zero $f \in L(rH)$, we have by (9) and by point (1) of Theorem 2.2

$$N(f) \leq k(q+1) + m \sum_{i=1}^{k} \pi_{D_i}.$$

We apply Lemma 4.1 to bound $\sum_{i=1}^{k} \pi_{D_i}$. We get in the first case $N(f) \leq \phi(k)$ where $\phi(k) := m\pi_{(r-k+1)H} + k(q + 1 + m\pi_H) - m\pi_H$. Remark that $\pi_{(r-k+1)H}$ is quadratic in k and so $\phi(k)$ is a quadratic function with positive leading coefficient. In [24, Lemma 2.2] Voloch and Zarzar proved that if X has arithmetic Picard number one then $k \leq r$. Thus $\phi(k)$ attends its maximum for $k = 1$ or for $k = r$ and $N(f) \leq \max\{\phi(1), \phi(r)\}$. A simple calculus shows that $\phi(1) - \phi(r) > 0$ if and only if $r > 2(q+1+m)/mH^2$. Since we have $d(X, rH, S) \geq \sharp S - \max_{f \in L(G)\setminus\{0\}} N(f)$, part (i) of the theorem is proved.

The treatment of part (ii) is the same, except that we use Lemma 4.1 to bound $\sum_{i=1}^{k} \pi_{D_i}$.

□

Remark 4.4. Little and Schenck have given bounds in [16, §3] for the minimum distance of codes from algebraic surfaces of Picard number one. In particular, they obtain (if we keep the notations of Theorem 4.3): $d(X, rH, S) \geq \sharp S-(q+1+m\pi_H)$ for $r = 1$ ([16, Th. 3.3]) and $d(X, rH, S) \geq \sharp S - r(q + 1 + m\pi_H)$ for $r > 1$ and q large ([16, Th. 3.5]). Comparing their bounds with Theorem 4.3, one can see that when $3H^2 + H.K_X \geq 0$ we get the same bound for $r = 1$ and also for $r > 1$ without any hypothesis on q. Moreover, when $3H^2 + H.K_X < 0$, our bounds improve the ones given by Little and Schenck, again without the hypothesis of q large when $r > 1$.

4.2. **Surfaces without irreducible curves defined over $\mathbb{F}_q$ with small self-intersection and whose canonical divisor is either nef or anti-nef.** We consider in this section surfaces X such that there exists some integer $\beta \geq 0$ for which any $\mathbb{F}_q$-irreducible curve D lying on X and defined over $\mathbb{F}_q$ has self-intersection $D^2 \geq \beta$. We prove in this case Lemma 4.5 below, from which we can tackle assumption (1) in Lemma 2.1 in case $\beta > 0$. Unfortunately, Lemma 4.5 enables to fulfil assumption (2) of Lemma 2.1 only in case the intersection of the canonical divisor with $\mathbb{F}_q$-irreducible curves has constant sign, that is for surfaces of Section 3.1. The lower bound for the minimum distance we get is better than the one given in Theorem 3.3.

Let us propose some examples of surfaces with this property:

- abelian surfaces satisfy this property for $\beta = 0$ and even for $\beta = 2$ in the case of simple abelian surfaces.
- Surfaces whose arithmetic Picard number is one. Indeed, let D be a curve defined over $\mathbb{F}_q$ on X, and $\mathrm{NS}(X) = \mathbb{Z}H$ with H ample. Then we have that $D = aH$ in $\mathrm{NS}(X)$ for some $a \in \mathbb{Z}$. Since H is ample we get $1 \leq D.H = aH^2$ hence $a \geq 1$ and $D^2 = a^2H^2 \geq H^2$.
- Surfaces whose canonical divisor is anti-nef and without irreducible curves of arithmetic genus less or equal to ℓ. Indeed the adjunction formula gives $D^2 = 2\pi_D - 2 - D.K_X \geq 2\pi_D - 2 \geq 2\ell$.

Lemma 4.5. *Let X be a surface on which any $\mathbb{F}_q$-irreducible curve has self-intersection at least $\beta \geq 0$. Assume that G is a rational effective divisor on X and that H is an ample divisor. Let $D = \sum_{i=1}^{k} n_i D_i$ be the decomposition as a sum of $\mathbb{F}_q$-irreducible and reduced curves of an effective divisor D linearly equivalent to G. Then we have*

(a) if $\beta > 0$ then $k \leq \frac{G.H}{\sqrt{\beta H^2}}$;

(b) $\sum_{i=1}^{k}(2\pi_{D_i} - 2 - D_i.K_X) \leq \varphi(k)$, with

$$\varphi(k) := (k-1)\beta + \left(\frac{G.H}{\sqrt{H^2}} - (k-1)\sqrt{\beta}\right)^2. \tag{14}$$

Proof. Since by hypothesis we have $\sqrt{\beta} \leq \sqrt{D_i^2}$, we deduce that $k\sqrt{\beta} \leq \sum_{i=1}^{k} n_i\sqrt{D_i^2}$. By (2), we get $k\sqrt{\beta} \leq \sum_{i=1}^{k} n_i \frac{D_i.H}{\sqrt{H^2}} = \frac{G.H}{\sqrt{H^2}}$, from which the first item follows.

Setting $x_i \coloneqq \sqrt{2\pi_{D_i} - 2 - D_i.K_X}$, we have by adjunction formula $x_i = \sqrt{D_i^2} \geq \sqrt{\beta}$. Moreover the previous inequalities ensure that $\sum_{i=1}^k x_i \leq \sum_{i=1}^k n_i\sqrt{D_i^2} \leq \frac{G.H}{\sqrt{H^2}}$. Then, the maximum of $\sum_{i=1}^k (2\pi_{D_i} - 2 - D_i.K_X) = \sum_{i=1}^k x_i^2$ under the conditions $x_i \geq \sqrt{\beta}$ and $\sum_{i=1}^k x_i \leq \frac{G.H}{\sqrt{H^2}}$ is reached when each but one x_i equals the minimum $\sqrt{\beta}$, and only one is equal to $\frac{G.H}{\sqrt{H^2}} - (k-1)\sqrt{\beta}$, which concludes the proof. □

Theorem 4.6. *Let X be a surface on which any $\mathbb{F}_q$-irreducible curve has self-intersection at least β. Consider an effective divisor G on X avoiding a set S of rational points and an ample divisor H. Suppose $\beta > 0$. Then*

$$d(X,G,S) \geq \begin{cases} \sharp S - \max\{\psi(1), \psi(\frac{G.H}{\sqrt{\beta H^2}})\} - \frac{m}{2}\frac{G.H}{\sqrt{2\beta H^2}} & \text{if } K_X \text{ is nef,} \\ \sharp S - \max\{\psi(1), \psi(\frac{G.H}{\sqrt{\beta H^2}})\} & \text{if } -K_X \text{ is nef} \end{cases}$$

with

$$\psi(k) \coloneqq \frac{m}{2}\varphi(k) + k(q+1+m),$$

where $\varphi(k)$ is given by equation (14).

Proof. For any non-zero $f \in L(G)$, we have by (9) and by point (1) of Theorem 2.2 that $N(f) \leq k(q+1) + m\sum_{i=1}^k \pi_{D_i}$. Lemma 4.5 implies that $N(f) \leq k(q+1) + \frac{m}{2}\big(2k + \varphi(k) + \sum_{i=1}^k D_i.K_X\big)$. In case K_X is nef, we have $\sum_{i=1}^k D_i.K_X \leq \sum_{i=1}^k n_i D_i.K_X = G.K_X$, and in case $-K_X$ is nef, we get $\sum_{i=1}^k D_i.K_X \leq 0$, and the theorem follows. □

4.3. **Fibered surfaces with nef canonical divisor.** We consider in this subsection AG codes from fibered surfaces whose canonical divisor is nef. We adopt the vocabulary of [20, III, §8] and we refer the reader to this text for the basic notions we recall here. A fibered surface is a surjective morphism $\pi : X \to B$ from a smooth projective surface X to a smooth absolutely irreducible curve B. We denote by π_0 the common arithmetic genus of the fibers and by g_B the genus of the base curve B. Here are some non-trivial examples of fibered surfaces:

- ruled surfaces ($\pi_0 = 0$) and elliptic surfaces ($\pi_0 = 1$).
- For any $d \geq 1$, the dimension of the space of degree d homogeneous polynomials in three variables is $\binom{d+2}{2}$, hence the space $\mathcal{P}_d$ of plane curves of degree d is $\mathcal{P}_d = \mathbb{P}^{\binom{d+2}{2}-1}$. Thus, any curve B drawn in $\mathcal{P}_d$ gives rise to a fibered surface, whose fibers are plane curves of degree d, that is with $\pi_0 = \frac{(d-1)(d-2)}{2}$. The locus of singular curves being a subvariety of $\mathcal{P}_d$, choosing B not contained in this singular locus yields to a fibered surface with smooth generic fiber. As the locus of reducible curves has high codimension in $\mathcal{P}_d$, choosing B avoiding this locus yields to fibered surfaces without reducible fibers.

On a fibered surface, every divisor can be uniquely written as a sum of *horizontal* curves (that is mapped onto B by π) and *fibral* curves (that is mapped on a point by π).

Lemma 4.7. *Let $\pi : X \to B$ be a fibered surface. Let G be an effective divisor and H be an ample divisor on X. For any effective divisor D linearly equivalent to G, consider its decomposition $D = \sum_{i=1}^{k} n_i D_i$ as a sum of reduced $\mathbb{F}_q$-irreducible curves. Denote by $\overline{r}_i$ the number of absolutely irreducible components of D_i. Then, we have*

$$\sum_{i=1}^{k} \overline{r}_i \leq G.H.$$

Proof. Write $D = \sum_{i=1}^{k} n_i D_i = \sum_{i=1}^{k} n_i \sum_{j=1}^{\overline{r}_i} D_{i,j}$ where the $D_{i,j}$ are the absolutely irreducible components of D_i.

Using that $n_i > 0$, that D is numerically equivalent to G and that $D_{i,j}.H > 0$, we get

$$\sum_{i=1}^{k} \overline{r}_i \leq \sum_{i=1}^{k} \sum_{j=1}^{\overline{r}_i} D_{i,j}.H \leq \sum_{i=1}^{k} n_i \sum_{j=1}^{\overline{r}_i} D_{i,j}.H = \sum_{i=1}^{k} n_i D_i.H = G.H,$$

which proves the lemma. □

The next theorem involves the *defect* $\delta(B)$ of a smooth absolutely irreducible curve B defined over $\mathbb{F}_q$ of genus g_B, which is defined by

$$\delta(B) \coloneqq q + 1 + m g_B - \sharp B(\mathbb{F}_q).$$

By the Serre-Weil theorem this defect is a non-negative number. The so-called maximal curves have defect 0, and the smaller the number of rational points B have, the greater the defect is.

Theorem 4.8. *Let $\pi : X \to B$ be a fibered surface whose canonical divisor K_X is nef. Assume that G is an effective divisor on X having at least one horizontal component and avoiding a set S of rational points, and that H is an ample divisor. Then the minimum distance of $\mathcal{C}(X, G, S)$ satisfies*

$$d(X, G, S) \geq d^*(X, G, S, H) + \delta(B)$$

where $d^*(X, G, S, H)$ is given by formula (13).

Recall that the general bound we obtain in Theorem 3.3 in Section 3 for surfaces with nef canonical divisor is $d(X, G, S) \geq d^*(X, G, S, H)$, thus the bound from Theorem 4.8 is always equal or better. Actually Theorem 4.8 is surprising, since it says that the lower bound for the minimum distance is all the more large because the defect $\delta(B)$ is. Consequently it looks like considering fibered surfaces on curves with few rational points and large genus could lead to potentially good codes.

Proof. Recall that for any non-zero $f \in L(G)$, we have $d(X,G,S) \geq \sharp S - N(f)$, and that $N(f) \leq \sum_{i=1}^{k} \sharp D_i(\mathbb{F}_q)$ if we use the notation $D_f := G+(f) = \sum_{i=1}^{k} n_i D_i$ introduced in (8). We again denote by $\overline{r}_i$ the number of absolutely irreducible components of D_i. In order to introduce the $\mathbb{F}_q$-irreducible components of D_f, write $k = h + v$, where h (respectively v) is the number of horizontal curves denoted by $H_1, \ldots, H_h$, (respectively fibral curves denoted by $F_1, \ldots, F_v$). Then we get $N(f) \leq \sum_{i=1}^{h} \sharp H_i(\mathbb{F}_q) + \sum_{i=1}^{v} \sharp F_i(\mathbb{F}_q)$. Since B is a smooth curve, the morphisms $H_i \to B$ are flat. We can thus apply Proposition 2.3 to horizontal curves and Theorem 2.2 to fibral curves and we obtain
(15)

$$\begin{aligned} N(f) &\leq h(\sharp B(\mathbb{F}_q) - mg_B) + m\sum_{i=1}^{h} \pi_{H_i} + q\sum_{i=1}^{h}(\overline{r}_i - 1) + qv + v + m\sum_{i=1}^{v} \pi_{F_i} \\ &= h(\sharp B(\mathbb{F}_q) - mg_B - q) + m\sum_{i=1}^{k} \pi_{D_i} + q\sum_{i=1}^{k} \overline{r}_i + v, \end{aligned}$$

where we used the fact that $v \leq \sum_{i=h+1}^{k} \overline{r}_i$.

Since the canonical divisor of the fibered surface is assumed to be nef, Lemma 3.1 gives a bound for $\sum_{i=1}^{k} \pi_{D_i}$. Then set $v = k - h$ and use Lemma 4.7 with (15) to obtain

$$\begin{aligned} N(f) &\leq h(\sharp B(\mathbb{F}_q) - mg_B - q) + m\alpha(G,H) + mk + qG.H + v \\ &= h(\sharp B(\mathbb{F}_q) - mg_B - q - 1) + m\alpha(G,H) + mk + qG.H + k \\ &= -h\delta(B) + m\alpha(G,H) + mk + qG.H + k. \end{aligned}$$

Now, $D_f.F \equiv G.F > 0$ since F is a generic fiber and G is assumed to have at least one horizontal component. Thus, D_f has also at least one horizontal component, that is $h \geq 1$. Moreover, we deduce $k \leq G.H$ from Lemma 4.1. As the defect $\delta(B)$ is non-negative it follows that

$$N(f) \leq -\delta(B) + G.H(q + 1 + m) + m\alpha(G,H)$$

and the theorem is proved. □

Now, we hypothesise that every singular fiber is $\mathbb{F}_q$-irreducible. We consider the case where π_0 and g_B are both at least 2. Set $\ell = \min(\pi_0, g_B) - 1 \geq 1$. We recall again that every divisor on X can be uniquely written as a sum of horizontal and fibral curves. If we denote by H an horizontal curve and by V a fibral curve defined over $\mathbb{F}_q$, we have that $\pi_H \geq g_B$ and $\pi_V = \pi_0$. Therefore, in this setting, X contains no $\mathbb{F}_q$-irreducible curves defined over $\mathbb{F}_q$ of arithmetic genus smaller than or equal to ℓ. Thus Lemma 3.4 applies for the number k of $\mathbb{F}_q$-irreducible components of D_f and gives $k \leq \alpha(G,H)/\ell$. Considering this new bound for k in the proof of Theorem 4.8, we get instead the following result.

Theorem 4.9. *Let $\pi : X \to B$ be a fibered surface whose canonical divisor K_X is nef. We consider an effective divisor G on X having at least one horizontal component and avoiding a set S of rational points, and H an ample divisor on X. We denote by g_B the genus of B and by π_0 the arithmetic genus of the fibers and we set $\ell = \min(\pi_0, g_B) - 1$. Suppose that every singular fiber is $\mathbb{F}_q$-irreducible and that $\ell \geq 1$. Then the minimum distance of $\mathcal{C}(X, G, S)$ satisfies*

$$d(X, G, S) \geq d^*(X, G, S, H) + \left(G.H - \frac{\alpha(G, H)}{\ell} \right) (q + 1 + m) + \delta(B),$$

where $d^*(X, G, S, H)$ is given by formula (13).

Naturally this bound is better than the one in Theorem 4.8 if and only if $\alpha(G, H) < \ell G.H$. Furthermore it improves the bound of Theorem 3.5 by the addition of the term $\delta(B)$.

5. An example: surfaces in $\mathbb{P}^3$

This section is devoted to the study of the minimum distance of AG codes over a surface X of degree $d \geq 3$ embedded in $\mathbb{P}^3$.

We consider the class L of an hyperplane section of X. So L is ample, $L^2 = d$ and the canonical divisor of X is $K_X = (d-4)L$ (see [19, p.212]). In this setting, we fix an effective divisor G and an ample divisor H. We apply our former theorems to this context to give bounds on the minimum distance of the code $\mathcal{C}(X, G, S)$.

We recall that cubic surfaces are considered by Voloch and Zarzar in [24] and [25] to provide computationally good codes. In Section 4 of [16] Little and Schenck propose theoretical and experimental results for surfaces in $\mathbb{P}^3$ always in the prospect of finding good codes. We also contribute to this study with a view to bounding the minimum distance according to the geometry of the surface.

Proposition 5.1. *Let X be a surface of degree $d \geq 3$ embedded in $\mathbb{P}^3$. Consider a rational effective divisor G avoiding a set S of rational points on X and an ample divisor H on X. Then the minimum distance of the code $\mathcal{C}(X, G, S)$ satisfies*

(1) if X is a cubic surface, then

$$d(X, G, S) \geq d^*(X, G, S, H) + \frac{1}{2} m(G.H - G.L).$$

(2) If X has degree $d \geq 4$ then

$$d(X, G, S) \geq d^*(X, G, S, H),$$

where

$$d^*(X, G, S, H) = \sharp S - G.H(q + 1 + m) - m\alpha(G, H)$$

is the function defined in (13).

Proof. Since $K_X = (d-4)L$ we have for cubic surfaces that $K_X = -L$ and thus the canonical divisor is anti-ample, while for surfaces of degree $d \geq 4$ the canonical divisor ample or the zero divisor, thus is nef. Hence we can apply Theorem 3.3 from which the proposition follows. □

5.1. **Surfaces in $\mathbb{P}^3$ without irreducible curves of low genus.** In the complex setting, the Noether-Lefschetz theorem asserts that a general surface X of degree $d \geq 4$ in $\mathbb{P}^3$ is such that $\mathrm{Pic}(X) = \mathbb{Z}L$, where L is the class of an hyperplane section (see [11]). Here, general means outside a countable union of proper subvarieties of the projective space parametrizing the surfaces of degree d in $\mathbb{P}^3$. Even if we do not know an analog of this statement in our context, it suggests us the strong assumptions we take in this subsection, namely in Lemma 5.2 and Proposition 5.3.

Lemma 5.2. *Let X be a surface of degree $d \geq 4$ in $\mathbb{P}^3$ of arithmetic Picard number one. Suppose that $\mathrm{NS}(X)$ is generated by the class of an hyperplane section L. Consider an $\mathbb{F}_q$-irreducible curve D on X of arithmetic genus π_D. Then*

$$\pi_D \geq (d-1)(d-2)/2.$$

Proof. Let a be the integer such that $D = aL$ in $\mathrm{NS}(X)$. Since D is an $\mathbb{F}_q$-irreducible curve and L is ample, we must have $a > 0$. Then, using the adjonction formula, we get

$$\begin{aligned} 2\pi_D - 2 = D^2 + D.K &= a^2L^2 + aL.(d-4)L \\ &= a^2 d + ad(d-4) \geq d + d(d-4), \end{aligned}$$

and thus $\pi_D \geq (d-1)(d-2)/2$. □

By the previous lemma it is straightforward that in our context X does not contain any $\mathbb{F}_q$-irreducible curves of arithmetic genus smaller than or equal to ℓ for $\ell = (d-1)(d-2)/2 - 1 = d(d-3)/2$. This allows us to apply Theorem 3.5, and get the following proposition.

Proposition 5.3. *Let X be a degree $d \geq 4$ surface in $\mathbb{P}^3$ of arithmetic Picard number one whose Néron-Severi group $\mathrm{NS}(X)$ is generated by the class of an hyperplane section L. Assume that $G = rL$ for a positive integer r and that S is a set of rational points avoiding L. Then the minimum distance of the code $\mathcal{C}(X, rL, S)$ satisfies*

$$d(X, rL, S) \geq d^*(X, rL, S, L) + rd\left(1 - \frac{r+d-4}{d(d-3)}\right)(q+1+m)$$

where

$$d^*(X, rL, S, L) = \sharp S - rd(q+1+m) - mrd(r+d-4)/2.$$

5.2. **Surfaces in $\mathbb{P}^3$ of arithmetic Picard number one.** In this subsection we suppose that the arithmetic Picard number of X is one, but we do not take the assumption that the Néron-Severi group is generated by an hyperplane section. Also in this case we can apply Theorem 4.3 which brings us to the following proposition.

Proposition 5.4. *Let X be a surface of degree $d \geq 4$ in $\mathbb{P}^3$. Assume that $\mathrm{NS}(X) = \mathbb{Z}H$ for an ample divisor H. We then consider $L = hH$, the class of an hyperplane section of X, and $G = rH$. Let S be a set of rational points on X avoiding H.*

Then the minimum distance of the code $\mathcal{C}(X, rH, S)$ satisfies

$$d(X, rH, S) \geq \begin{cases} \sharp S - (q+1+m) - rH^2(r + h(d-4))/2 \text{ if } r > 2(q+1+m)/mH^2, \\ \sharp S - r(q+1+m) - rH^2(1 + h(d-4))/2 \text{ otherwise.} \end{cases}$$

Proof. Since we have $3H^2 + H.K_X = 3H^2 + H.(d-4)L = 3H^2 + h(d-4)H^2 = H^2(3 + h(d-4)) \geq 0$, we can apply point (i) of Theorem 4.3 from which the proposition follows. □

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