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Synchronization of interconnected linear systems via dynamic saturation redesign

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Abstract: In this paper we explore the use of dynamic saturations on the interconnections of linear systems over a network, to reduce the effect of impulsive perturbations corrupting the exchanged information. First, we show that the proposed redesign preserves the synchronization property of the network. Second, simulation results show that dynamic saturations are a promising tool to reduce the impact of impulsive perturbations affecting the communication among the agents.

1. INTRODUCTION

Networks of dynamical systems and synchronization are omnipresent in our daily lives. Power networks (Dörfler et al. (2013)), flow networks (Bürger and De Persis (2015)), robot or vehicle fleets (Olfati-Saber (2006)), sensor networks (Sivrikaya and Yener (2004)), and social networks (Mirtabatabaei and Bullo (2012)) are some of the most popular examples that can be framed in this same context. From the seminal works on consensus (Moreau (2004)) and synchronization of identical linear systems (Scardovi and Sepulchre (2008)), in the last decade researchers have considered many aspects of networks control such as heterogeneous networks (Wieland et al. (2011)), switching networks (Lu et al. (2011)), and high-order systems (Seo et al. (2009)). In more recent years the activity has focused on the more challenging problem of nonlinear networks. Different approaches such as passivity (Arcak (2007)), dissipativity (Stan and Sepulchre (2007)) and high gain (Panteley and Loría (2017)) have been considered and the literature on the subject is now vast. Indeed, most of the results of the linear framework have been extended succesfully to the nonlinear context (Abdessameud et al. (2015), (Isidori et al., 2014)).

These different approaches have in common the possibility to rewrite the synchronization has a stabilization problem of a certain set. For instance, the synchronization error with respect to the origin in Scardovi and Sepulchre (2008), or passivity with respect to a desired target set in Arcak (2007). While the stability of the aforementioned set and the impact of noise (Wells et al. (2015)), perturbations (He et al. (2017)), and *hybrid* phenomena (such as switching topology Casadei et al. (2018) and open networks (Hendrickx and Martin (2017)) have been considered, novel approaches to the control design that allow mitigating the effect of these perturbations over the networks have still to be explored. Some authors have adapted classical H_∞ techniques to the case of networks of identical linear systems (Huang and Feng (2008), Dal Col et al. (2018)). In the context of heterogeneous networks, Khong et al. (2016) studied the problem of synchronization by means of integral quadratic constraint (IQC). However, these techniques are often hard to extend to the nonlinear framework.

Recent works concerning the design of observers have suggested that the use of nonlinear functions such as saturations and deadzones might help in reducing the effect of perturbations affecting the measurements. Both in the context of linear (Alessandri and Zaccarian (2018), Cocetti et al. (2018)) and nonlinear systems (Astolfi et al. (2017); Cocetti et al. (2019)), the improvements obtained are significant. Inspired by these results and their applicability to the nonlinear framework, in this paper we investigate the use of saturations in networks with the aim of reducing the effect of impulsive perturbations acting over the systems communication lines. The perturbations may represent external/exogenous attacks to the network, commutations in the topology or the emergence of new nodes. First, we prove that the nominal behavior of the networks, namely the synchronization, is not compromised. Then, with the aid of simulation results, we show that the saturations helps in reducing the propagation of impulsive perturbations over the network. The paper is organized as follows. In Section 2, we briefly review some basic results in the literature of synchronization. In Section 3, we formulate the problem at stake. In Section 4, the main result of the paper is stated. Finally, in Section 5, simulation results show the effectiveness of the proposed approach.

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Notation $\mathbb{R}$ is the set of real numbers. On $\mathbb{R}^n$, we define $\mathbb{1}_n = [1, \cdots, 1]^\top$. On $\mathbb{R}^n$, we define the row-vector basis $\mathtt{b}_1 = [1, 0, \cdots, 0]$, $\mathtt{b}_i = [0, \cdots, 0, 1, 0, \cdots, 0]$ with all zeros except 1 in the i-th position, $\mathtt{b}_n = [0, \cdots, 0, 1]$, so that $I_n = \mathrm{col}(\mathtt{b}_1^\top, \ldots, \mathtt{b}_n^\top)$. Let $(x, y) := [x^\top, y^\top]^\top$ for any column vectors x and y. Given a symmetric matrix $P \in \mathbb{R}^{n\times n}$, let us denote by $\lambda_m(P)$ and $\lambda_M(P)$ the minimum and maximum eigenvalues of P, respectively. Given $y \in \mathbb{R}$ and $\sigma \in \mathbb{R}_{\geq 0}$, we define $\mathrm{sat}_\sigma(y) := \max\{-\sigma, \min\{\sigma, y\}\}$ and $\mathrm{dz}_\sigma(y) := y - \mathrm{sat}_\sigma(y)$.

Graph theory In a general framework, a communication graph is described by a triplet $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A\}$ in which $\mathcal{V}$ is a set of n *nodes* $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is a set of *edges* e_{jk} that models the interconnection between nodes with the flow of information from node j to node k *weighted* by the (k, j)-th entry $a_{kj} \geq 0$ of the *adjacency matrix* $A \in \mathbb{R}^{N\times N}$. We denote by $L \in \mathbb{R}^{n\times n}$ the *Laplacian matrix* of the graph, defined as

$$\ell_{kj} = -a_{kj} \text{ for } k \neq j, \qquad \ell_{kj} = \sum_{i=1}^{n} a_{ki} \text{ for } k = j\,.$$

For a time-invariant graph, the following result holds, see Godsil and Royle (2001).

Lemma 1. A time-invariant graph, is connected if and only if L has only one trivial eigenvalue $\lambda_1(L) = 0$ and all other eigenvalues $\lambda_2(L), \ldots, \lambda_n(L)$ have positive real parts.

2. PRELIMINARIES

Consider a network of n identical agents described by

$$\begin{aligned} \dot{x}_j &= Ax_j + u_j \\ y_j &= Cx_j + Dw_j \end{aligned} \tag{1}$$

for $j = 1, \ldots, n$, where $x_j \in \mathbb{R}^d$, is the state, $y_j \in \Re$ is the output, $w_j \in \mathbb{R}$ is an impulsive disturbance acting over the output through the gain $D \in \mathbb{R}$, (A, C) is a detectable pair matrices, and $u_j \in \mathbb{R}^d$ is the diffusive coupling control input to be defined.

Agents (1) are connected according to a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A\}$, fulfilling the following assumption.

Assumption 1. The graph $\mathcal{G}$ contains at least a spanning tree and its Laplacian L is diagonalizable. As a consequence (see Godsil and Royle (2001)), there exists $\mu > 0$ such that, for all $i = 2, \ldots, n$ the following holds:

$$\mathrm{Re}\lambda_i(L) \geq \mu\,.$$

In the unperturbed case, i.e. when $w_j = 0$ for all $j = 1, \ldots, n$, it is well known (see, e.g., Scardovi and Sepulchre (2008) and references therein) that by assigning the input of all the agents $j = 1, \ldots, n$ as

$$u_j = -K \sum_{i=1}^{n} \ell_{ji}\, y_i, \tag{2}$$

where ℓ_{ki} denotes the (k, i) entry of the Laplacian matrix L, synchronization, namely asymptotic convergence to zero of $x_i - x_k$ for all $i, k \in \{1, \ldots, n\}$, is ensured for suitable selections of K (a possible one is described in Theorem 1 below).

To characterize the collective behavior, and as a guideline for selecting K, it is customary to write in compact form the network interconnection of (1) via (2) as

$$\dot{\mathbf{x}} = [(I_n \otimes A) - (L \otimes KC)]\,\mathbf{x} \tag{3}$$

where $\mathbf{x} := (x_1, \ldots, x_n)$. Following the same approach by Fax and Murray (2004), Seo et al. (2012), Isidori et al. (2014), the compact form (3) can be conveniently manipulated by introducing the transformation $T \in \mathbb{R}^{n\times n}$ defined as

$$T := \begin{pmatrix} 1 & 0_{1\times(n-1)} \\ \mathbb{1}_{n-1} & I_{n-1} \end{pmatrix},\ T^{-1} = \begin{pmatrix} 1 & 0_{1\times(n-1)} \\ -\mathbb{1}_{n-1} & I_{n-1} \end{pmatrix}, \tag{4}$$

which (from Assumption 1) satisfies

$$\tilde{L} = T^{-1}LT = \begin{pmatrix} 0 & L_{12} \\ 0_{(n-1)\times 1} & L_{22} \end{pmatrix}, \tag{5}$$

where $\mathrm{eig}\{L_{22}\} = \{\lambda_2, \ldots, \lambda_n\}$.

Consider then the change of variables $\tilde{\mathbf{x}} = (T^{-1} \otimes I_d)\mathbf{x}$, with T as in (4). From the structure of T, one immediately realizes that

$$\tilde{\mathbf{x}} = (x_1, x_2 - x_1, \ldots, x_n - x_1)$$

and the system in the new coordinates reads

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= (T^{-1} \otimes I_d)\,[(I_n \otimes A) - (L \otimes KC)]\,(T \otimes I_d)\tilde{\mathbf{x}} \\ &= \Big[(I_n \otimes A) - (\tilde{L} \otimes KC)\Big]\,\tilde{\mathbf{x}}\,. \end{aligned}$$

Moreover, introducing $e_k = x_k - x_1$, for $k = 2, 3, \ldots, n$, and

$$\boldsymbol{e} := (e_2, e_3, \ldots, e_n)\,,$$

yielding $\tilde{\mathbf{x}} = \mathrm{col}(x_1, \boldsymbol{e})$, it is readily seen that the system above exhibits a triangular structure of the form

$$\dot{x}_1 = Ax_1 - (L_{12} \otimes KC)\boldsymbol{e} \tag{6a}$$

$$\dot{\boldsymbol{e}} = \big((I_{n-1} \otimes A) - (L_{22} \otimes KC)\big)\boldsymbol{e}. \tag{6b}$$

With this triangular structure, it is well known that assessing global exponential stability of $\boldsymbol{e}$ (namely exponential convergence to zero of the error coordinates) guarantees asymptotic synchronization of the agents to the *perturbed* trajectory of x_1 (which may be bounded or unbounded).

Theorem 1. Under Assumption 1, the following dual algebraic Riccati equation admits a solution $P = P^T > 0$

$$PA^\top + AP - 2\mu PC^\top CP + aI = 0$$

for any $a > 0$ and μ given in Assumption 1. Moreover, selecting $K = PC^\top$ in (2) ensures that the $\boldsymbol{e}$-dynamics in (6b) are globally exponentially stable.

Proof. The proof follows from the construction in (Isidori et al., 2014, App. A). In particular, consider the invertible complex matrix $M \in \mathbb{C}^{n\times n}$ such that $ML_{22}M^{-1} = \mathrm{diag}(\lambda_2, \ldots, \lambda_n)$. Then, define the quadratic function

$$V(\boldsymbol{e}) := \boldsymbol{e}^\top(M^*M \otimes P^{-1})\boldsymbol{e} = \boldsymbol{e}^\top(H_R \otimes P^{-1})\boldsymbol{e}, \tag{7}$$

where M^* is the conjugate transpose of M and $H_R = \mathrm{Re}(M^*M)$ is Hermitian and positive definite by construction. From the property of the Kronecker product, we get

$$\underline{\alpha}^2\|\boldsymbol{e}\|^2 \leq V(\boldsymbol{e}) \leq \bar{\alpha}^2\|\boldsymbol{e}\|^2, \tag{8}$$

with the positive scalars $\underline{\alpha}^2 := \lambda_m(H_R)\lambda_M(P)^{-1} > 0$ and $\bar{\alpha}^2 := \lambda_M(H_R)\lambda_m(P)^{-1} > 0$. Moreover, from the calculations in (Isidori et al., 2014, page 2689), there exists $\alpha > 0$, possibly depending on P, such that, for all $i = 2, \ldots, n$,

$$P^{-1}(A - \lambda_i KC) + (A - \lambda_i KC)^* P^{-1} < -\alpha I\,.$$

Then, according to (Isidori et al., 2014, eqn. (36)), we get

$$\begin{aligned}\dot{V} &= 2\boldsymbol{e}^\top (H_R \otimes P^{-1})\Big(((I_{n-1}\otimes A) - (L_{22}\otimes KC))\boldsymbol{e}\Big)\\ &\le -\alpha \boldsymbol{e}^\top (H_R \otimes I_d)\boldsymbol{e}\,, \end{aligned} \tag{9}$$

which concludes the proof. □

3. PROBLEM FORMULATION

In this paper we are interested in studying the case in which impulsive perturbations act on the communication among agents (1), namely $w_j \neq 0$ for some j. As mentioned in the introduction, w_j can arise from hybrid phenomena (such as a switching topology) or *glitches* in the communication among agents. By changing coordinates according to (4), the network of (1), (2), reads

$$\dot{x}_1 = Ax_1 - (L_{12}\otimes KC)\boldsymbol{e} + (N_{12}\otimes KD)\boldsymbol{w} \tag{10a}$$

$$\dot{\boldsymbol{e}} = \Big((I_{n-1}\otimes A) - (L_{22}\otimes KC)\Big)\boldsymbol{e} + (N_{22}\otimes KD)\boldsymbol{w} \tag{10b}$$

where N_{12}, N_{22} are matrices of appropriate dimensions satisfying

$$T^{-1}L = \begin{pmatrix} N_{12} \\ N_{22}\end{pmatrix}\,,$$

and $\boldsymbol{w} := (w_1,\dots,w_n)$. The forthcoming results follows from Theorem 1 for the perturbed system (10).

Corollary 1. Consider the perturbed $\boldsymbol{e}$-dynamics in (10b). If K is selected according to Theorem 1, there exist $c,\gamma > 0$ such that the following bound holds

$$\langle \nabla V(\boldsymbol{e}), \dot{\boldsymbol{e}}\rangle \le -c\|\boldsymbol{e}\|^2 + \gamma\|\boldsymbol{w}\|^2 \tag{11}$$

for all $\boldsymbol{e}\in\mathbb{R}^{(n-1)\times d}$, $x_1\in\mathbb{R}^d$, $\boldsymbol{w}\in\mathbb{R}^n$.

Proof. Following the steps of the proof of Theorem 1, we obtain, along (10b),

$$\begin{aligned}\dot{V} &= 2\boldsymbol{e}^\top (H_R\otimes P^{-1})\\ &\qquad \Big(((I_{n-1}\otimes A) - (L_{22}\otimes KC))\boldsymbol{e} + (N_{22}\otimes KD)\boldsymbol{w}\Big)\\ &\le -\alpha\boldsymbol{e}^\top(H_R\otimes I_d)\boldsymbol{e} + \|\boldsymbol{e}\|\|H_R\|\|P^{-1}\|\|N_{22}\|\|K\|\boldsymbol{w}.\end{aligned}$$

Then, (11) follows from the Young inequality, with simple computations. □

Corollary 1 guarantees that a bounded exogenous perturbation $\boldsymbol{w}$ always produces a bounded synchronization error $\boldsymbol{e}$, namely practical synchronization is guaranteed. If now we are interested in mitigating the effect of $\boldsymbol{w}$ over the network, one could think of designing K to both guarantee synchronization and attenutation of perturbations (for instance, exploiting H_∞ methods as in Khong et al. (2016) or Dal Col et al. (2018)). However, it is worth pointing out that both α and γ in (11) are functions of the *degree of freedom* K and structural limitations arise when a control input is restricted within the class of linear regulator, see Seron et al. (2012).

In this paper we propose a redesign technique that maintains the design of K proposed in Theorem 1 and adds a dynamic saturation effect over the diffusive coupling (2). First we show that the proposed technique still guarantees synchronization of the unperturbed network. Then, simulation results show that the impact of the disturbance $\boldsymbol{w}$ over the network is significantly reduced.

4. MAIN RESULT

In dynamic saturation redesign we introduce a saturation in the communication among agents. In particular, we replace the linear selection in (2) by

$$u_j = -K\,\mathrm{sat}_{\sqrt{\sigma_j}}\left(\sum_{i=1}^n \ell_{ji}y_i\right) \tag{12}$$

where the saturation level $\sigma_j \in \mathbb{R}$ obeys the following dynamic depending on design parameters $\theta, r\in\mathbb{R}$

$$\dot{\sigma}_j = -\theta\sigma_j + r\left(\sum_{i=1}^n \ell_{ji}y_i\right)^2. \tag{13}$$

Note that the σ_j dynamics in (13) can further expressed as

$$\begin{aligned}\dot{\sigma}_j &= -\theta\sigma_j + r\left((\mathtt{b}_j L\mathbf{y})^2\right.\\ &= -\theta\sigma_j + r\left((\mathtt{b}_j L\otimes C)\mathbf{x}\right)^2\\ &= -\theta\sigma_j + (\mathtt{b}_j L\otimes C)\mathbf{x}r((\mathtt{b}_j L\otimes C)\mathbf{x})^\top\\ &= -\theta\sigma_j + (\mathtt{b}_j L\otimes C)\mathbf{x}r\mathbf{x}^\top(L^\top\mathtt{b}_j^\top\otimes C^\top)).\end{aligned}$$

As a consequence, the network (1) interconnected via (12) and (13) can be written in the following compact form

$$\begin{aligned}\dot{\mathbf{x}} &= (I_n\otimes A)\mathbf{x} - (I_n\otimes K)\,\mathbf{sat}_{\sqrt{\boldsymbol{\sigma}}}\left((L\otimes C)\mathbf{x}\right)\\ \dot{\boldsymbol{\sigma}} &= \Theta\boldsymbol{\sigma} + (L\otimes C)\mathbf{x}R\mathbf{x}^\top(L^\top\otimes C^\top)\end{aligned} \tag{14}$$

with $\Theta := \mathrm{diag}\{-\theta,\dots,-\theta\}$, $R := \mathrm{diag}\{r,\dots,r\}$, and $\mathbf{sat}_{\sqrt{\boldsymbol{\sigma}}}(\cdot)$ defined as

$$\mathbf{sat}_{\sqrt{\boldsymbol{\sigma}}}\left((L\otimes C)\mathbf{x}\right) := \begin{pmatrix}\mathrm{sat}_{\sqrt{\sigma_1}}\left((\mathtt{b}_1 L\otimes C)\mathbf{x}\right)\\ \mathrm{sat}_{\sqrt{\sigma_2}}\left((\mathtt{b}_2 L\otimes C)\mathbf{x}\right)\\ \vdots\\ \mathrm{sat}_{\sqrt{\sigma_n}}\left((\mathtt{b}_n L\otimes C)\mathbf{x}\right)\end{pmatrix}.$$

Theorem 2. Consider the network of n agents (14), under Assumption 1 with K designed according to Theorem 1. Then, for any θ there exists $r^\star > 0$ such that for any $r > r^\star$ the set

$$\mathbf{X} = \{(x_1,x_2,\dots,x_n):\ x_1 = x_2 = \cdots = x_N\} \tag{15}$$

is globally exponentially stable.

Proof. We start by suitably rewriting the $\mathbf{x}$-dynamics in (14) as

$$\begin{aligned}\dot{\mathbf{x}} &= (I_n\otimes A)\mathbf{x} - (I_n\otimes K)(L\otimes C)\mathbf{x}\\ &\quad + (I_n\otimes K)(L\otimes C)\mathbf{x} - (I_n\otimes K)\mathbf{sat}_{\sqrt{\boldsymbol{\sigma}}}\left((L\otimes C)\mathbf{x}\right)\\ &= \left[(I_n\otimes A) - (L\otimes KC)\right]\mathbf{x} + (I_n\otimes K)\mathbf{dz}_{\sqrt{\boldsymbol{\sigma}}}\left((L\otimes C)\mathbf{x}\right),\end{aligned}$$

where $\mathbf{dz}_{\sqrt{\boldsymbol{\sigma}}}(\cdot)$ is defined as

$$\mathbf{dz}_{\sqrt{\boldsymbol{\sigma}}}\left((L\otimes C)\mathbf{x}\right) = \begin{pmatrix}\mathrm{dz}_{\sqrt{\sigma_1}}\left((\mathtt{b}_1 L\otimes C)\mathbf{x}\right)\\ \mathrm{dz}_{\sqrt{\sigma_2}}\left((\mathtt{b}_2 L\otimes C)\mathbf{x}\right)\\ \vdots\\ \mathrm{dz}_{\sqrt{\sigma_n}}\left((\mathtt{b}_n L\otimes C)\mathbf{x}\right)\end{pmatrix}.$$

We now apply the change of coordinates $\tilde{\mathbf{x}} = (T^{-1}\otimes I_d)\mathbf{x}$ with T as in (4) to obtain

$$\begin{aligned}\dot{\tilde{\mathbf{x}}} &= [(I_n\otimes A) - (\tilde{L}\otimes KC)]\tilde{\mathbf{x}}\\ &\qquad + (T^{-1}\otimes K)\mathbf{dz}_{\sqrt{\boldsymbol{\sigma}}}\left((LT\otimes C)\tilde{\mathbf{x}}\right)\\ \dot{\boldsymbol{\sigma}} &= \Theta\boldsymbol{\sigma} + (LT\otimes C)\tilde{\mathbf{x}}R\tilde{\mathbf{x}}^\top(T^\top L^\top\otimes C^\top)\end{aligned} \tag{16}$$

Again, just as in (6), we want to exploit the structure of $\tilde{\mathbf{x}}$ in order to express in a more suitable form the dynamics (16). To this end, recall that we have

$$\boldsymbol{e} = (e_2, e_3, \dots, e_n)\,, \qquad \tilde{\mathbf{x}} = (x_1, \boldsymbol{e}). \tag{17}$$

By definition of L and the structure of T in (4), we have

$$LT = \begin{bmatrix} 0 & M_1 \\ 0 & M_2 \\ \vdots & \vdots \\ 0 & M_n \end{bmatrix}, \tag{18}$$

where $M \in \mathbb{R}^{n\times(n-1)}$. By using the structure in (18), we also obtain $(LT \otimes C)\tilde{\mathbf{x}} = (M \otimes C)\boldsymbol{e}$. As a consequence, in order to simplify the forthcoming computations, let us introduce the following notation

$$\begin{aligned} \zeta &:= (LT \otimes C)\tilde{\mathbf{x}} = (M \otimes C)\boldsymbol{e}, \\ \zeta_i &:= \mathtt{b}_i(LT \otimes C)\tilde{\mathbf{x}} = (M_i \otimes C)\boldsymbol{e} \ \ \forall i = 1, \dots, n. \end{aligned} \tag{19}$$

First of all, note that we can take advantage of (18) and (19), to get

$$\mathbf{dz}_{\sqrt{\boldsymbol{\sigma}}}((LT \otimes C)\tilde{\mathbf{x}}) = \mathbf{dz}_{\sqrt{\boldsymbol{\sigma}}}(\zeta) = \begin{pmatrix} \mathrm{dz}_{\sqrt{\sigma_1}}(\zeta_1) \\ \vdots \\ \mathrm{dz}_{\sqrt{\sigma_n}}(\zeta_n) \end{pmatrix}. \tag{20}$$

As a consequence, by using (6b), (17), (19) and (20), dynamics (16) reads

$$\dot{x}_1 = Ax_1 - (L_{12} \otimes KC)\boldsymbol{e} + K\,\mathrm{dz}_{\sqrt{\sigma_1}}(\zeta_1) \tag{21a}$$

$$\begin{aligned} \dot{e}_i &= (A - \mathtt{b}_i L_{22} KC)e_i + K[\mathrm{dz}_{\sqrt{\sigma_i}}(\zeta_i) - \mathrm{dz}_{\sqrt{\sigma_1}}(\zeta_1)] \\ &= Ae_i + K[\mathrm{sat}_{\sqrt{\sigma_i}}(\zeta_i) - \mathrm{sat}_{\sqrt{\sigma_1}}(\zeta_1)],\ i = 2, \dots, n, \end{aligned} \tag{21b}$$

$$\dot{\sigma}_i = -\theta\sigma_i + r\zeta_i^2, \qquad i = 1, \dots, n, \tag{21c}$$

Moreover, according to (4), (17) (19), we have the following identities

$$\begin{aligned} \mathtt{b}_1(T^{-1} \otimes K)\mathbf{dz}_{\sqrt{\boldsymbol{\sigma}}}(\zeta) &= K\,\mathrm{dz}_{\sqrt{\sigma_1}}(\zeta_1), \\ \mathtt{b}_i(T^{-1} \otimes K)\mathbf{dz}_{\sqrt{\boldsymbol{\sigma}}}(\zeta) &= K[\mathrm{dz}_{\sqrt{\sigma_i}}(\zeta_i) - \mathrm{dz}_{\sqrt{\sigma_1}}(\zeta_1)], \end{aligned}$$

for all $i = 2, \dots, n$. Then the $\boldsymbol{e}$-dynamics (21b) can be written in compact form, by also using (20), as

$$\begin{aligned} \dot{\boldsymbol{e}} &= ((I_{n-1} \otimes A) - (L_{22} \otimes KC))\boldsymbol{e} + K_{\boldsymbol{e}}\mathbf{dz}_{\sqrt{\boldsymbol{\sigma}}}(\zeta) \\ &= (I_{n-1} \otimes A)\boldsymbol{e} + K_{\boldsymbol{e}}\mathbf{sat}_{\sqrt{\boldsymbol{\sigma}}}(\zeta) \end{aligned} \tag{22}$$

where

$$K_{\boldsymbol{e}} := \mathrm{col}(\mathtt{b}_2, \dots, \mathtt{b}_n)(T^{-1} \otimes K).$$

As a consequence, similarly to the proof of Theorem 1, we consider a Lyapunov function for the $\boldsymbol{e}, \boldsymbol{\sigma}$ dynamics which is independent of x_1. In particular, consider the Lyapunov function

$$W(\boldsymbol{e}, \boldsymbol{\sigma}) = V(\boldsymbol{e}) + \sum_{i=1}^{n} \left(r^{-2}\sigma_i + \max\{\zeta_i^2 - \sigma_i, 0\} \right) \tag{23}$$

with V defined as in (7). From (7) and (8) we have that there exist $\overline{w} > \underline{w} > 0$ satisfying

$$\underline{w}(\|\boldsymbol{e}\|^2 + \|\boldsymbol{\sigma}\|) \le W(\boldsymbol{e}, \boldsymbol{\sigma}) \le \overline{w}(\|\boldsymbol{e}\|^2 + \|\boldsymbol{\sigma}\|).$$

Now let $\mathcal{I} \subset \{1, \dots, n\}$ be the subset of indexes i for which $\sigma_i \ge \zeta_i^2$ and $\mathcal{J} = \{1, \dots, n\} \setminus \mathcal{I}$ be the set of indexes j for which $\sigma_j < \zeta_j^2$. Then we have

$$\begin{aligned} \sigma_i \ge \zeta_i^2 \quad &\Rightarrow \quad \mathrm{dz}_{\sqrt{\sigma_i}}(\zeta_i) = 0, \qquad \forall i \in \mathcal{I} \\ \sigma_j < \zeta_j^2 \quad &\Rightarrow \quad \mathrm{dz}_{\sqrt{\sigma_j}}(\zeta_j) \ne 0, \qquad \forall j \in \mathcal{J}. \end{aligned} \tag{24}$$

Now, we compute the time derivative of W defined in (23). We obtain, using the definitions in (19),

$$\dot{W} = \dot{V} + \sum_{i\in\mathcal{I}} r^{-2}\dot{\sigma}_i + \sum_{j\in\mathcal{J}} (r^{-2} - 1)\dot{\sigma}_j + \sum_{j\in\mathcal{J}} 2\zeta_j(M_j \otimes C)\dot{\boldsymbol{e}}$$

and therefore, using (21c), (22) and also (9),

$$\begin{aligned} \dot{W} \le& -\alpha\boldsymbol{e}^\top(H_R \otimes I_d)\boldsymbol{e} + 2\boldsymbol{e}^\top(H_R \otimes P^{-1})K_{\boldsymbol{e}}\mathbf{dz}_{\sqrt{\boldsymbol{\sigma}}}(\zeta) \\ &+ r^{-2}\sum_{i\in\mathcal{I}} \left(-\theta\sigma_i + r\zeta_i^2\right) + (r^{-2} - 1)\sum_{j\in\mathcal{J}} \left(-\theta\sigma_j + r\zeta_j^2\right) \\ &+ \sum_{j\in\mathcal{J}} 2\zeta_j(M_j \otimes CA)\boldsymbol{e} \\ &+ \sum_{j\in\mathcal{J}} 2\zeta_j(M_j \otimes C)K_{\boldsymbol{e}}\mathbf{sat}_{\sqrt{\boldsymbol{\sigma}}}(\zeta). \end{aligned} \tag{25}$$

Let us now introduce the following positive scalars α_ℓ and κ, independent of r, to simplify notation:

$$\begin{aligned} &\alpha(H_R \otimes I_d) \ge \alpha_\ell I > 0, \\ &\kappa := \max_{j\in\{1,\dots,n\}} \{\|(H_R \otimes P^{-1})K_{\boldsymbol{e}}\|^2, \|(M_j \otimes CA)\|^2, \\ &\qquad\qquad \|(M_j \otimes C)K_{\boldsymbol{e}}\|^2\|(M \otimes C)\|^2\}, \end{aligned}$$

and exploit the following bounds, resulting from the properties in (24) and definition (19):

$$\begin{aligned} \|\mathbf{dz}_{\sqrt{\boldsymbol{\sigma}}}(\zeta)\|^2 &\le \sum_{j\in\mathcal{J}} \zeta_j^2, \\ \|\mathbf{sat}_{\sqrt{\boldsymbol{\sigma}}}(\zeta)\| &\le \|\zeta\| \le \|(M \otimes C)\|\|\boldsymbol{e}\|. \end{aligned} \tag{26}$$

Then we may refine the upper bound (25) for $\dot{W}$ (for reading convenience we preserve the position of each term in (25)), where we impose $r > 1$ (so that $1 - r^{-2} > 0$) and apply standard Young inequalities [1] in lines 1, 3 and 4, and use the properties of sets $\mathcal{I}, \mathcal{J}$ in (24):

$$\begin{aligned} \dot{W} \le& -\alpha_\ell\|\boldsymbol{e}\|^2 + \nu\|\boldsymbol{e}\|^2 + \frac{\kappa}{\nu}\sum_{j\in\mathcal{J}} \zeta_j^2 \\ &- r^{-2}\theta\sum_{i\in\mathcal{I}} \sigma_i + \frac{n}{r}\|\zeta\|^2 - \left(1 - \frac{1}{r^2}\right)\sum_{j\in\mathcal{J}} \left(\left(\frac{r}{2} - \theta\right)\sigma_j + \frac{r}{2}\zeta_j^2\right) \\ &+ \sum_{j\in\mathcal{J}} \nu\|\boldsymbol{e}\|^2 + \frac{\kappa}{\nu}\zeta_j^2 \\ &+ \sum_{j\in\mathcal{J}} \nu\|\boldsymbol{e}\|^2 + \frac{\kappa}{\nu}\zeta_j^2. \end{aligned} \tag{27}$$

Finally, combining the different terms in (27) and using again the second bound in (26), we obtain

$$\begin{aligned} \dot{W} \le& -\|\boldsymbol{e}\|^2\left(\alpha_\ell - \nu - \frac{n}{r}\|(M \otimes C)\|^2 - \sum_{j\in\mathcal{J}} 2\nu\right) \\ &- \left(\frac{r - r^{-1}}{2} - \frac{\kappa}{\nu} - 2\frac{\kappa}{\nu}\right)\sum_{j\in\mathcal{J}} \zeta_j^2 \\ &- r^{-2}\theta\sum_{i\in\mathcal{I}} \sigma_i - \left(1 - \frac{1}{r^2}\right)\left(\frac{r}{2} - \theta\right)\sum_{j\in\mathcal{J}} \sigma_j. \end{aligned} \tag{28}$$

Since κ is indepedent of r and ν, for any $\theta > 0$ we can first select $r \ge r^* \ge \nu^{-1}$ and ν small enough so that

$$\alpha_\ell - \nu - n\nu\|(M \otimes C)\|^2 - \sum_{j\in\mathcal{J}} 2\nu > 0,$$

and finally fix $r \ge r^* > 0$ large enough so that

$$\frac{r - r^{-1}}{2} - \frac{\kappa}{\nu} - 2\frac{\kappa}{\nu} > 0, \quad \left(1 - \frac{1}{r^2}\right)\left(\frac{r}{2} - \theta\right) > 0.$$

By recalling that $\mathcal{I} \cup \mathcal{J} = \{1, \dots, n\}$, from (28) there exists a small enough $\varepsilon > 0$ satisfying

$$\dot{W} \le -\varepsilon(\|\boldsymbol{e}\|^2 + \|\boldsymbol{\sigma}\|).$$

The last inequality, combined with the definition of W and $(\boldsymbol{e}, \boldsymbol{\sigma})$, completes the proof. □

[1] The Young inequality is the well-knonw upper bound $2ab \le \nu a^2 + b^2/\nu$, holding for any $a, b \in \mathbb{R}$ and any $\nu > 0$.

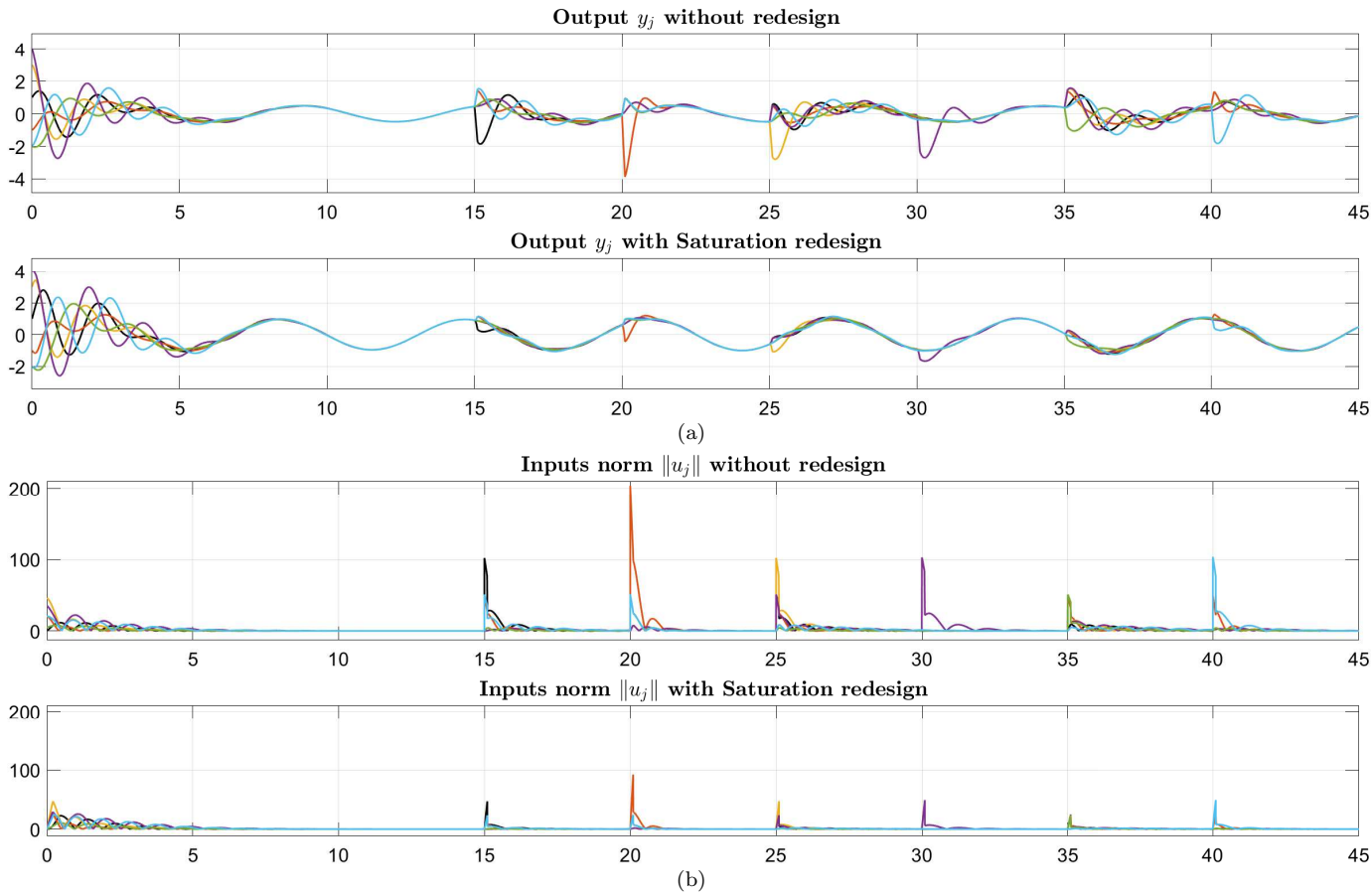

Fig. 1. In subfigure 1a, synchronization without coupling redesign (upper plot) and with coupling redesign (lower plot). The effect of a perturbation over the output of an agent has a variable impact depending on what output is affected. With the dynamic saturation design, this impact is highly reduced independently of the affected node. In the subfigure 1b, norm $\|u_j\|$ of of each input without redesign (upper plot) and with saturation redesign (lower plot). Every time an impulsive perturbation occurs, the classic design provides and input about twice as large than the saturation redesign.

5. SIMULATION RESULTS

We consider a network of $n = 6$ linear oscillators, whose dynamics are

$$\begin{aligned} \dot{x}_{j_1} &= x_{j_2} + u_{j_1}, \, \dot{x}_{j_2} = -x_{j_1} + u_{j_2} \\ y_j &= x_{j_1} + w_j \end{aligned} \tag{29}$$

for $j = 1, \ldots, 6$, exchanging their output information y_j with their neighbors with control inputs $u_j = (u_{j_1}, u_{j_2})$ defined according to (2) and K chosen according to Theorem 1. An impulsive perturbation δ_j acting over the communication is considered. In particular, we consider the case where the perturbation δ_j acts on only one agent at a time. The same network is then considered with input designed according to (12)-(13), with $\theta = 1$ and $r = 10$. The network considered in this example is described by the Laplacian matrix

$$L = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & -2 \end{pmatrix}. \tag{30}$$

Simulation results are shown in Figure 1, both for the linear design (upper plot) and the redesign proposed in Section 4 (lower plot): after reaching synchronization, the output of one of the oscillators is perturbed at $t = 15\,\mathrm{s}$. Then, each oscillator's output is perturbed with an interval of $5\,\mathrm{s}$. First, in Figure 1a we can observe that the redesign improves the performances of the network importantly. While the nominal linear design fails to return rapidly to synchronization, the redesign helps to make the network more robust. Second, in Figure 1b we can clearly observe that the norm of the inputs u_j without redesign and with saturation redesign is significantly different: thanks to the dynamic saturations not only the impact of the impulsive perturbations is highly reduced, but also the control effort is reduced to less than one half. This fact is a direct consequence of the evolution of the saturation levels σ_j in Figure 2.

6. CONCLUSIONS

In this paper, a preliminary analysis of the use of dynamic saturations to reduce the effect of impulsive perturbations in networks has been developed. In the context of linear systems, we have shown that it is possible to redesign the classical linear diffusive coupling by adding a dynamic saturation, without compromising the synchronization property. Then, with the aid of simulations, we have verified that the dynamic saturations reduce the impact of impulsive perturbations over the communication among

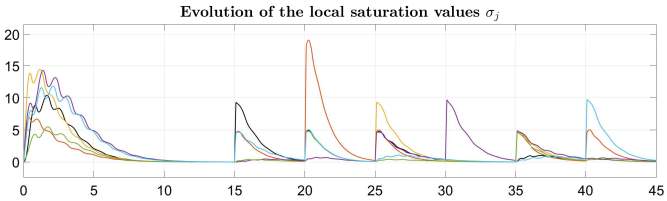

Fig. 2. The evolution of the local saturation values σ_j.

agents. In the near future, we would like to consider the possibility of decentralizing design of the dynamic saturations, namely each agent may have its own design parameters. More importantly, we aim to quantifying analytically the gain in terms of *performances* that we obtain with respect to the standard linear design. Another remarkable aspect of this approach is its applicability to the nonlinear framework. With respect to this, the research activity has already started and the first promising results are under investigation.

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