

Emulation-based output regulation of linear networked control systems subject to scheduling and uncertain transmission intervals

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Abstract: We investigate output regulation for linear networked control systems using the emulation approach. We consider a regulator solving the problem for linear systems in the absence of the network, by following the Francis-Wonham framework. Next, the network is taken into consideration, in particular the induced time-varying inter-transmission intervals and scheduling constraints, and we model the overall system dynamics as a hybrid system. We show that only practical regulation can be achieved in general since the steady-state control input steering the output to zero cannot be generated in the networked context (in general). Explicit upper bounds on the output asymptotic gain and on the maximum allowable transmission interval guaranteeing boundedness of the closed-loop trajectories are provided, which are shown to be related.

1. INTRODUCTION

By output regulation, we refer to the problem in which we aim to control a given output of a plant to asymptotically track a prescribed trajectory and/or to asymptotically reject undesired disturbances, while keeping all the trajectories of the system bounded. In such framework, we suppose to know the model generating the disturbances and references, denoted in the following as exosignals. For linear systems, an elegant solution was proposed by Francis, Wonham, see Francis and Wonham (1976) with the clear delineation of the *internal model principle*. The objective of this work is to investigate whether output regulation can be achieved in the context of linear networked control systems (NCS), namely when the regulator and the linear plant communicate over a multi-purpose network, which may be shared with other tasks. The challenge is that the communication channel induces undesired phenomena, such as uncertain transmission intervals, scheduling, network delays and packet loss, which may destroy the desired regulation properties, if ignored.

To this date, works on NCS mainly focused on modelling, stabilization and observer design (see, e.g., Cloosterman et al. (2010), Heemels et al. (2010), Nešić and Teel (2004) Carnevale et al. (2007), Postoyan and Nešić (2012)), while only few works have considered the out-

put regulation problem in the networked context. Some solutions have been proposed to address this problem for discrete-time linear systems and sampled-data linear systems (see, e.g. Sureshbabu and Rugh (1997), Lawrence and Medina (2001), Fujioka and Hara (2006), García-Sandoval et al. (2007), Antunes et al. (2014)). These works do not take into account the inter-sampling behavior when the plant has continuous-time dynamics, and are not suited to analyze some network phenomena, such as time-varying sampling, scheduling and packet loss. Other researchers studied the related, though different, problem of output tracking for nonlinear NCSs (see, e.g., Postoyan et al. (2014), van de Wouw et al. (2010), Gao and Chen (2008)). The problem of output regulation for minimum phase single-input single-output nonlinear systems when only the measurement is sampled has been addressed in Astolfi et al. (2018), while in Liu and Huang (2017), it is shown that practical output regulation in presence of event-triggered policies can be achieved for the same class of systems. Nevertheless, output regulation of NCS remains a largely open problem, as results adapted to network effects, such as uncertain time-varying sampling, scheduling, network delays, quantization, are missing to the best of our knowledge.

A typical approach for controller design for NCS is using emulation, where the design of the regulator is performed by assuming that the network is absent. As next step, network phenomena are taken into account in the closed-loop model and their effects on the stability and perfor-

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mance of the overall system are analyzed. In this work, we follow this approach to address the output regulation of linear NCS, as we believe that a full understanding of such framework is missing in the literature. The analysis follows the ideas of emulation approach for NCS presented in Nešić and Teel (2004); Carnevale et al. (2007) for point stabilization, by means of the hybrid formalism of Goebel et al. (2012). Note that, the input-to-state stability (ISS) properties of the unperturbed closed-loop systems, see, e.g., Cloosterman et al. (2010) or Heemels et al. (2010) allow to conclude only boundedness of the closed-loop trajectories in presence of exosignals. Therefore, a thinner analysis is needed to conclude whether asymptotic output regulation can be achieved.

The main contributions of the current paper are as follows. First a hybrid model for the closed-loop regulated system including the effects of time-varying sampling and scheduling is constructed, although we believe that it is possible to include transmission delays in the analysis by suitably combining the presented results with those in Heemels et al. (2010). Secondly, an explicit bound for the maximum allowable transmission interval (MATI) guaranteeing boundedness of the trajectories of the closed-loop system, in presence of the exosignals, is derived. Next, a performance analysis is provided. We show that, with such architecture, asymptotic output regulation cannot be achieved in general, since the emulated version of the control is not able to generate the right steady-state input able to steer the output error constantly to zero². In other words, the internal model property of the regulator is destroyed. The characteristics of the plant, exosignals and the network are related to the ultimate bound of the regulated errors. This bound can be used for both analysis and network design and represents the main novelty of this work. Finally, we highlight two particular situations in which asymptotic output regulation is still preserved, namely the case of constant exosignals and the case in which the plant input is directly connected to the controller (in other words, only the plant output is transmitted over the network, like in Astolfi et al. (2018)). This finding is consistent also with the work in Postoyan et al. (2014) on the tracking control of NCS, and suggests that different architectures need to be used if asymptotic output regulation is sought.

The paper is organized as follows. The problem is formally stated in Section 2. The main results are given in Section 3 and a numerical example is provided in Section 4. Section 5 contains the conclusions. The proofs of the main results are given in the Appendix.

Notation Let $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_{\geq 0} := [0, \infty)$, $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. The notation $|x|$ stands for the standard Euclidean norm of $x \in \mathbb{R}^n$, and $\|x\|_{\mathcal{A}} := \inf_{a \in \mathcal{A}} |x - a|$ denotes the distance from x to a set $\mathcal{A} \subseteq \mathbb{R}^n$. Given $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, we denote $(x, y) = (x^T, y^T)^T$. We consider hybrid systems of the form Goebel et al. (2012) $\dot{x} = F(x)$, $x \in \mathcal{C}$, $x^+ = G(x)$, $x \in \mathcal{D}$, where $x \in \mathbb{R}^{n_x}$ is the state, \mathcal{C} is the flow set, F is the flow map, \mathcal{D} is the jump set and G is the jump map. We refer to Goebel et al. (2012) for the definitions of solutions to the hybrid system, maximal solutions and pre-compact

solutions. We say that a compact set $\mathcal{A} \subset \mathbb{R}^{n_x}$ is *uniformly globally exponentially stable* (UGES) for the hybrid system if all maximal solutions are complete and there exist $M, \lambda > 0$ such that, for any $x(0, 0) \in \mathbb{R}^{n_x}$, $\|x(t, j)\|_{\mathcal{A}} \leq M \exp(-\lambda(t + j))\|x(0, 0)\|_{\mathcal{A}}$ for all $(t, j) \in \text{dom } x$.

2. PROBLEM FORMULATION

2.1 Preliminaries on linear output regulation

We recall in this section the basic ingredients of linear output regulation in continuous-time, see Francis and Wonham (1976) or (Byrnes et al., 2012, Section 1.3) for more details. Consider a system of the form

$$\begin{aligned} \dot{x} &= Ax + Bu + Pw \\ y &= Cx + Qw, \end{aligned} \quad (1)$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input and $y \in \mathbb{R}^{n_y}$ is the measured output that we aim to regulate to zero, with $n_x, n_u, n_y \in \mathbb{N}$ and $n_u \geq n_y$. The exogenous variable $w \in \mathbb{R}^{2n_\omega+1}$, $n_\omega \in \mathbb{N}_0$, is assumed to be generated by an exosystem of the form

$$\dot{w} = Sw, \quad S = \text{blkdiag}(S_0, S_1, \dots, S_{n_\omega}) \quad (2)$$

where $S_0 = 0$, $S_i = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix}$, $\omega_1 < \dots < \omega_{n_\omega} \in \mathbb{R}$.

Matrix S is neutrally stable by construction. The variable w is not directly measurable for feedback design, but the matrix S is supposed to be perfectly known. We consider the next definition of output regulation.

Definition 1. For system (1), the problem of output regulation is solved if there exists a controller processing the regulated output y satisfying the following requirements:

- (i) *Internal stability requirement:* when $w = 0$, any trajectory of the closed-loop system converges exponentially to zero.
- (ii) *Output regulation requirement:* when $w \neq 0$, the trajectories of the closed-loop are bounded for all $t \geq 0$ and the output y asymptotically converges to zero.

As shown in (Byrnes et al., 2012, Section 1.3), the output regulation problem for linear system (1) can be solved if and only if there exist matrices Π, Ψ being the solution to the regulator equations

$$\begin{aligned} \Pi S &= A\Pi + B\Psi + P \\ 0 &= C\Pi + Q, \end{aligned} \quad (3)$$

where the matrix Π uniquely defines the state steady-state $x = \Pi w$ on which the regulated output y is zero, and Ψ defines the input steady-state $u = \Psi w$, denoted in the following as *friend* (see Isidori and Marconi (2012)), which renders the given manifold $x = \Pi w$ positively invariant. Equation (3) always admits a solution under the following assumption.

Assumption 1. The pair (A, B) is stabilizable, the pair (A, C) is detectable, and the matrix $\begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix}$ is full rank for any λ in the spectrum of S .

Under Assumption 1, the output regulation problem is solved by a controller of the form

$$\dot{\xi} = F\xi + G y, \quad u = K\xi, \quad (4)$$

where $\xi \in \mathbb{R}^{n_\xi}$ is the state of the regulator, with $n_\xi \in \mathbb{N}$, and F, G, K are of the form

² This is derived from the *regulator equations*, see Francis and Wonham (1976).

$$F = \begin{pmatrix} \Phi & 0 \\ M & N \end{pmatrix}, \quad G = \begin{pmatrix} \Gamma \\ H \end{pmatrix}, \quad K = (K_1 \ K_2) \quad (5)$$

with (Φ, Γ) being a controllable pair satisfying $\sigma(\Phi) = \sigma(S)$, and such that the closed-loop matrix

$$\mathbf{A} := \begin{pmatrix} A & BK \\ GC & F \end{pmatrix} \quad (6)$$

is Hurwitz. In the rest of the paper, we assume that regulator (4) has been designed as described above so that items (i) and (ii) of Definition 1 hold. We refer to (Byrnes et al., 2012, Section 1.3) for more details.

2.2 NCS model

We investigate the scenario where a network is used to ensure the communication between plant (1) and regulator (4), as depicted in Figure 1. In this work, we focus on the effects of sampling and scheduling, and ignore delays, packet loss and quantization. See the remark at the end of this subsection. In particular, we describe the model used to characterize the channel limitations and the transmission protocol by following the framework used in Nešić and Teel (2004); Carnevale et al. (2007); Heemels et al. (2010). This framework allows to take into account a large class of network scheduling protocols, such as round robin (RR), try-once-discard (TOD), as well as the simplest case of sampled-data systems. Afterwards, we derive a hybrid model for the resulting NCS.

Transmission over the network occur at times t_i , $i \in \mathbb{Z}_{\geq 0}$, satisfying $0 < \epsilon \leq t_{i+1} - t_i \leq \tau^*$ where τ^* is the *maximum allowable transmission interval* (MATI) and ϵ is the lower bound on the minimum achievable transmission interval given by hardware constraints. The inter-transmission intervals $t_{i+1} - t_i$ may be time-varying and uncertain. Note that ϵ can be arbitrarily small and prevents Zeno solutions (see Goebel et al. (2012)). We model this transmission policy using the variable τ , whose dynamics is given by

$$\dot{\tau} = 1 \quad \tau \in [0, \tau^*], \quad \tau^+ = 0 \quad \tau \in [\epsilon, \tau^*]. \quad (7)$$

The sensors and actuators are grouped into $\ell \in \mathbb{N}$ nodes, which are connected to the network. Hence, at each transmission instant a single node is allowed to transmit its data over the channel by the *scheduling rule*. The networked versions of y and u are denoted respectively as \hat{y} and \hat{u} and these correspond to the most recently transmitted output and input values. For sake of simplicity, in this work we suppose that the variables \hat{u}, \hat{y} are generated by zero-order hold devices between two successive transmission instants in (8), though more complex holding functions could be taken into account, see Nešić and Teel (2004). The associated dynamics are given by

$$\begin{cases} \dot{\hat{u}} = 0 \\ \dot{\hat{y}} = 0 \end{cases} \tau \in [0, \tau^*], \quad \begin{cases} \hat{u}^+ = u + h_u(\kappa, e_u, e_y) \\ \hat{y}^+ = y + h_y(\kappa, e_u, e_y) \end{cases} \tau \in [\epsilon, \tau^*], \quad (8)$$

where $e := (e_y, e_u) \in \mathbb{R}^{n_e}$, $n_e = n_y + n_u$, $e_y := \hat{y} - y$, $e_u := \hat{u} - u$, and where $\kappa \in \mathbb{N}_0$ is a counter variable needed to keep track of the number of transmissions, whose dynamics is given by

$$\dot{\kappa} = 0 \quad \tau \in [0, \tau^*], \quad \kappa^+ = p(\kappa) \quad \tau \in [\epsilon, \tau^*]. \quad (9)$$

We suppose that κ lives in a given compact set $[0, \kappa^*]$, for some $\kappa^* \in \mathbb{N}_0$, hence $p(\kappa) \in [0, \kappa^*]$, for all $\kappa \in \mathbb{N}_0$. Note that usually the jump map of κ is taken as $\kappa + 1$,

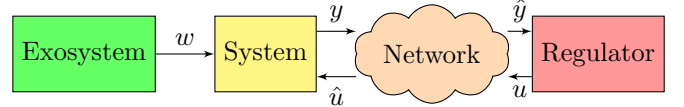


Fig. 1. Output regulation for NCS.

see, e.g., Nešić and Teel (2004); Postoyan et al. (2014). We proceed differently here to ensure that κ remains in a compact set, which is useful to endow for the forthcoming stability results with some nominal robustness according to Goebel et al. (2012). The functions $h := (h_u, h_y)$ in (8) and p in (9) model the scheduling mechanism, which grants access to the network to a single node at each transmissions instant. *Static* and *dynamic* algorithms can be described by appropriate selection of h, p, κ^* , in view of Nešić and Teel (2004). For instance, in the case of RR protocol, each node transmits in an ℓ -cyclic manner. In this case, we select $h(\kappa, e) := (I - \Delta(\kappa))e$, with $\Delta(\kappa) := \text{diag}(\delta_1(\kappa), \dots, \delta_{n_e}(\kappa))$,

$$\delta_i(\kappa) := \begin{cases} 1, & \text{if } i = \kappa \\ 0, & \text{otherwise,} \end{cases} \quad p(\kappa) = \begin{cases} \kappa + 1, & \text{if } \kappa < \kappa^* \\ 0, & \text{otherwise,} \end{cases} \quad (10)$$

and $\kappa^* = \ell - 1$, see Nešić and Teel (2004). Another example is the TOD protocol, which grants access to the node with the biggest local network-induced error, see Walsh et al. (2002); Nešić and Teel (2004). In this case, we select $h(e) = (I - \Upsilon(e))e$, $\Upsilon(e) := \text{diag}(\psi_1(e), \dots, \psi_{n_e}(e))$,

$$\psi_i(e) := \begin{cases} 1, & \text{if } i = \min(\arg \max_i |e_i|) \\ 0, & \text{otherwise,} \end{cases}$$

and $\kappa^* = 0$, $p = 0$. The simple case in which inputs and outputs values are simultaneously transmitted at each transmission instant, namely the case of *sampled-data systems*, is described by $p = 0$, $h = 0$ and $\kappa^* = 0$.

Now, the dynamics of the plant in (1), with u replaced by its networked version \hat{u} , is described by

$$\dot{x} = Ax + B\hat{u} + Pw \quad \tau \in [0, \tau^*], \quad x^+ = x \quad \tau \in [\epsilon, \tau^*]. \quad (11)$$

Similarly, controller (4) no longer receives y but its networked version \hat{y} . As a consequence, we study the interconnection of system (11) with the following emulated regulator

$$\begin{cases} \dot{\xi} = F\xi + G\hat{y} \\ u = K\xi \end{cases} \tau \in [0, \tau^*], \quad \xi^+ = \xi \quad \tau \in [\epsilon, \tau^*], \quad (12)$$

By denoting $\chi := (x, \xi) \in \mathbb{R}^{n_\chi}$, with $n_\chi = n_x + n_\xi$, we derive the following hybrid model of the overall NCS closed-loop system

$$\begin{cases} \dot{w} = Sw \\ \dot{\chi} = \mathbf{A}\chi + \mathbf{B}e + \mathbf{P}w \\ \dot{e} = \mathbf{M}\chi + \mathbf{N}e + \mathbf{R}w \\ \dot{\kappa} = 0 \\ \dot{\tau} = 1 \\ y = \mathbf{C}\chi + \mathbf{Q}w \end{cases} \left. \begin{cases} w^+ = w \\ \chi^+ = \chi \\ e^+ = h(e, \kappa) \\ \kappa^+ = p(\kappa) \\ \tau^+ = 0 \end{cases} \right\} q \in \mathcal{C}, \quad \left. \begin{cases} w^+ = w \\ \chi^+ = \chi \\ e^+ = h(e, \kappa) \\ \kappa^+ = p(\kappa) \\ \tau^+ = 0 \end{cases} \right\} q \in \mathcal{D}, \quad (13)$$

where $q := (\chi, e, \kappa, \tau)$, $\mathcal{C} := \mathbb{R}^{n_\chi} \times \mathbb{R}^{n_e} \times [0, \kappa^*] \times [0, \tau^*]$, $\mathcal{D} := \mathbb{R}^{n_\chi} \times \mathbb{R}^{n_e} \times [0, \kappa^*] \times [\epsilon, \tau^*]$, the matrix \mathbf{A} is defined in (6), $\mathbf{C} := (C \ 0)$, $\mathbf{Q} := Q$, and the matrices $\mathbf{B}, \mathbf{M}, \mathbf{N}, \mathbf{P}, \mathbf{R}$ are given by

$$\begin{aligned} \mathbf{B} &:= \begin{pmatrix} 0 & B \\ G & 0 \end{pmatrix}, \quad \mathbf{P} := \begin{pmatrix} P \\ GQ \end{pmatrix}, \quad \mathbf{M} := \begin{pmatrix} -C & 0 \\ 0 & -K \end{pmatrix} \mathbf{A}, \\ \mathbf{N} &:= \begin{pmatrix} -C & 0 \\ 0 & -K \end{pmatrix} \mathbf{B}, \quad \mathbf{R} := \begin{pmatrix} -CP - QS \\ -KGG \end{pmatrix} \end{aligned} \quad (14)$$

Remark. Delays and packet loss effects can be similarly handled in the hybrid context by following Heemels et al. (2010), Nešić and Teel (2004).

2.3 Problem Statement

The objective of this work is to investigate whether it is possible to preserve both requirements of Definition 1 for system (13), namely internal stability and output regulation, under suitable conditions on the network. As far as the first requirement is concerned, note that without appropriate conditions on the scheduling protocol and on τ^* , Assumption 1 is not sufficient, in general, to guarantee the stability of the trajectories of closed-loop system (13) (see for instance Carnevale et al. (2007) in case of asymptotic stabilization). Concerning the second requirement, we will show that asymptotic output regulation can be preserved only in some particular cases. As a consequence, in all those cases in which we are not able to guarantee convergence to zero of the output, we aim to provide an error bound, depending on plant, controller and network properties.

3. STABILITY AND PERFORMANCE GUARANTEES

We present now the main results of this paper. First, we formulate an assumption on the scheduling protocol. Then, we derive the main theorems and we analyse a certain number of special cases where asymptotic output regulation can be achieved. Finally, we briefly discuss how the main result can be used to compute bounds on the MATI and on the asymptotic gain of the output.

3.1 Assumption on the scheduling protocol

We adopt the following assumption on the scheduling protocol, similar to Nešić and Teel (2004); Heemels et al. (2010).

Assumption 2. The function h is continuous and there exist $W : \mathbb{N}_0 \times \mathbb{R}^{n_e} \rightarrow \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument, $\lambda \in [0, 1)$, $\underline{a}, \bar{a}, \bar{b} > 0$ such that $\underline{a}|e| \leq W(\kappa, e) \leq \bar{a}|e|$, $W(p(\kappa), h(\kappa, e)) \leq \lambda W(\kappa, e)$ and $\left| \frac{\partial W(\kappa, e)}{\partial e} \right| \leq \bar{b}$ for all $\kappa \in [0, \kappa^*]$ and almost all $e \in \mathbb{R}^{n_e}$.

Assumption 2 mainly states that the protocol (8), (9) is UGES, see Nešić and Teel (2004). Examples of protocols verifying Assumption 2 are the RR protocol, TOD protocol. We refer to Nešić and Teel (2004) for more details.

3.2 Main Result

Before introducing the main result of this paper, we state the following supporting lemma.

Lemma 1. There exist $\mathbf{X} = \mathbf{X}^T > 0$ and $\rho, c > 0$ satisfying

$$\begin{bmatrix} \mathbf{A}^T \mathbf{X} + \mathbf{X} \mathbf{A} + 2\rho^2 \mathbf{X} + \frac{2\bar{b}^2}{c^2} \mathbf{M}^T \mathbf{M} & \mathbf{X} \mathbf{B} \\ \mathbf{B}^T \mathbf{X} & -c^2 \underline{a}^2 \mathbf{I} \end{bmatrix} < 0. \quad (15)$$

We impose that the maximum allowed transmission interval (MATI) τ^* is less than $\mathcal{T}(c, \rho, \lambda)$, where

$$\mathcal{T}(c, \rho, \lambda) := \begin{cases} \frac{1}{Lr} \arctan \left(\frac{r(1-\lambda)}{2 \frac{\lambda}{1+\lambda} (\frac{\gamma}{L} - 1) + 1 + \lambda} \right) & \gamma > L \\ \frac{1}{L} \frac{1-\lambda}{1+\lambda} & \gamma = L \\ \frac{1}{Lr} \operatorname{arctanh} \left(\frac{r(1-\lambda)}{2 \frac{\lambda}{1+\lambda} (\frac{\gamma}{L} - 1) + 1 + \lambda} \right) & \gamma < L \end{cases} \quad (16)$$

with $r := \sqrt{\left| \left(\frac{\gamma}{L} \right)^2 - 1 \right|}$, $\gamma := c^2$, $L := |\mathbf{N}| \bar{b} / \underline{a} + \rho^2$, with c, ρ satisfying Lemma 1 and λ satisfying Assumption 2. We are now ready to state the main result.

Theorem 1. Suppose that Assumptions 1 and 2 are verified with $\lambda \in (0, 1)$ and moreover $\tau^* < \mathcal{T}(c, \rho, \lambda)$, with \mathcal{T} defined in (16). Then, the following properties hold:

- (1) Consider system (13) with flow set characterized by $(w, q) \in \{0\} \times \mathcal{C}$ and jump set $(w, q) \in \{0\} \times \mathcal{D}$. Then, the set $\{0\} \times \{0\} \times \{0\} \times [0, \kappa^*] \times [0, \tau^*]$ is UGES.
- (2) Consider system (13) with flow set $\mathcal{W} \times \mathcal{C}$ and jump set $\mathcal{W} \times \mathcal{D}$, for some given compact set $\mathcal{W} \subset \mathbb{R}^{n_w}$. Then, all the maximal solutions q to (13) are pre-compact and $\lim_{t+j \rightarrow \infty} \|q(t, j)\|_{\mathcal{A}} = 0$ where $\mathcal{A} := \{q : |y| \leq \theta \bar{w}\}$ with $\bar{w} := \max_{v \in \mathcal{W}} |v|$ and $\theta := \frac{\bar{b} |\Psi S|}{\mu \rho c}$, where Ψ is defined in (3) and $\mu > 0$ is the largest real number satisfying $\mu^2 \mathbf{C}^T \mathbf{C} \leq \mathbf{X}$.

Item 1) of Theorem 1 ensures that the *internal stability requirement* of Definition 1 is guaranteed, namely that the origin of the closed-loop system is exponentially stable when $w = 0$, even in presence of sampling and scheduling. This result is a consequence of the fact that, in this case, output regulation reduces to a stabilization problem via emulation approaches, thus boiling down in the framework considered for instance in Carnevale et al. (2007).

Item 2) of Theorem 1 states that, when $w \neq 0$, the trajectories of the resulting NCS closed-loop system are bounded, but in general, the output y is not asymptotically converging to zero, namely the *output regulation requirement* in Definition 1 is not ensured. This comes from the fact that, in steady-state, the networked version of u is not able to provide the right *friend* Ψw , but only an approximation of it. As a consequence, the steady-state behaviour of the x -component in (13) will not coincide exactly with Πw defined in (3), and y cannot converge to zero. In Section 3.4, however, we discuss some special cases where asymptotic regulation can still be achieved. Note that this analysis is consistent with Postoyan et al. (2014), where it is shown that asymptotic tracking control is in general not achievable in presence of network phenomena, due to the sampling and hold of the feedforward action.

In item 2) of Theorem 1, we also provide an explicit expression of the asymptotic gain θ relating the impact of w on y . The parameter θ is directly proportional to the norm of the *friend* Ψ and to the frequencies of the exosignal w (namely to S), and inversely proportional to the parameters ρ, c which uniquely define the MATI τ^* via (16). Further comments about the relation between θ and τ^* are given in Section 3.5.

3.3 Sampled-data case

Theorem 1 requires that $\lambda \in (0, 1)$ in Assumption 2. When $\lambda = 0$, which corresponds to the sampled-data case (or in other words, in absence of scheduling), the analysis needs to be slightly modified. First, we revisit Lemma 1 as follows.

Lemma 2. *There exist $\mathbf{X} = \mathbf{X}^T > 0$, $\mathbf{H} = \mathbf{H}^T > 0$ and $\rho, c > 0$ satisfying*

$$\begin{bmatrix} \mathbf{A}^T \mathbf{X} + \mathbf{X} \mathbf{A} + 2\rho^2 \mathbf{X} + \frac{2}{c^2} \mathbf{M}^T \mathbf{H} \mathbf{M} & \mathbf{X} \mathbf{B} \\ \mathbf{B}^T \mathbf{X} & -c^2 \mathbf{H} \end{bmatrix} < 0. \quad (17)$$

We have the next result for the sampled-data case, the proof of which is given in the Appendix.

Proposition 1. *Suppose Assumption 1 is verified and $\tau^* < \mathcal{T}(c, \rho, 0)$, with \mathcal{T} defined in (16) where $\gamma := c^2$ and $L > 0$ is the smallest real number satisfying $(L - \rho^2) \mathbf{H} \geq \mathbf{H} |\mathbf{N}|$, with ρ, c and \mathbf{H} according to Lemma 2. Then, items 1) and 2) of Theorem 1 hold for system (13) with $p = 0$, $h = 0$ and $\kappa^* = 0$, where in particular $\theta := \frac{\sqrt{|S^T \Psi^T \mathbf{H} \Psi S|}}{\mu \rho c}$.*

3.4 Asymptotic regulation

We discuss here two particular cases in which the exact output regulation requirement of Definition 1 can be still achieved for the NCS system (13).

Constant exosignals. When w is constant signal, i.e. $S = S_0 = 0$ in (2), then $\theta = 0$ in Theorem 1 and Proposition 1 and the output regulation requirement of Definition 1 holds. In other words, the internal model property of the integral action (see Francis and Wonham (1976); Byrnes et al. (2012)) is preserved in case of zero-sampling holder and scheduling of the measured output and the control input. We expect that similar results may hold also for general classes of nonlinear systems, see Astolfi and Praly (2017); Astolfi et al. (2018).

No network between the controller and the actuators. When only the output y is transmitted over the network, namely when we interconnect the system (1) with the regulator (12), system (13) is defined with $e := e_y$, $\mathbf{B} := (0, G^T)^T$, $\mathbf{M} := -\mathbf{C} \mathbf{A}$, $\mathbf{N} := -\mathbf{C} \mathbf{B}$, $\mathbf{R} := -\mathbf{C} \mathbf{P} - \mathbf{Q} \mathbf{S}$. The following result can be stated.

Proposition 2. *Suppose Assumptions 1 and 2 are verified with $\lambda \in (0, 1)$. Let $\tau^* < \mathcal{T}(c, 0, \lambda)$, with \mathcal{T} defined as in Theorem 1, with c coming from Lemma 1. Then item 1) and 2) of Theorem 1 are verified where in particular $\mathcal{A} := \{q : y = 0\}$, hence items 1) and 2) of Definition 1 hold.*

Remark. The emulation approach considered in this work is different from the approach followed in many other papers, such as Lawrence and Medina (2001); García-Sandoval et al. (2007) and references therein, where ripple-free regulators composed by a discrete-time regulator and a continuous-time modulator are used to asymptotically guarantee zero inter-sampling behaviour. Most likely, the approach followed in Lawrence and Medina (2001), García-Sandoval et al. (2007), allows to design a regulator with larger MATI, though its use is limited to the case of periodic sampling, namely when $\epsilon = \tau^*$, with no scheduling.

3.5 Network design for practical regulation

A consequence of Theorem 1 (and Proposition 1) is that, in general, asymptotic regulation cannot be guaranteed. Therefore, a question of particular interest is how to design the communication network in order to guarantee given performance in terms of the asymptotic regulated output bound θ . To this end, we can exploit Theorem 1 as follows.

Suppose, in particular, that we want to ensure $\theta < \theta_{\max}$ in Theorem 1 for some fixed $\theta_{\max} > 0$. By recalling that $\mu^2 \mathbf{C}^T \mathbf{C} \leq \mathbf{X}$, compute $c^* > 0$ as the smallest value for c satisfying

$$\begin{bmatrix} \mathbf{A}^T \mathbf{X} + \mathbf{X} \mathbf{A} + 2 \frac{\bar{b}^2}{c^2} \left(\frac{|\Psi S|^2}{\theta_{\max}^2} \mathbf{C}^T \mathbf{C} + \mathbf{M}^T \mathbf{M} \right) & \mathbf{X} \mathbf{B} \\ \mathbf{B}^T \mathbf{X} & -c^2 \underline{a}^2 \mathbf{I} \end{bmatrix} < 0 \quad (18)$$

and select $\rho(c) = \bar{b} |\Psi S| / (\mu c \theta_{\max})$. Finally, compute $\tau_{\max}^* = \mathcal{T}^*(c, \rho(c), \lambda)$ with (16), which is the maximum allowable transmission interval for which, given the plant dynamics and emulated regulator, the maximum gain θ_{\max} from w to y is satisfied. The same reasoning can be followed in the case $\lambda = 0$ where, in place of (15), we use (17). In view of the proposed procedure, performances in terms of θ can be improved by choosing a smaller MATI τ^* . This is confirmed in Section 4 via a numerical example. In conclusion, the result of Theorem 1 can be used as a design tool (with potentially some conservatism) to select the communication network parameter MATI with a performance based design trade-off.

4. NUMERICAL EXAMPLE

Consider the output regulation problem described in Section 2.1 with $n_x = 3$, $n_u = 2$, $n_y = 1$, $n_\omega = 1$,

$$\mathbf{A} = \begin{pmatrix} 0.1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \mathbf{P} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$\mathbf{C} = (100)$, $\mathbf{Q} = (10)$, \mathbf{S} is of the form (2) with $\omega_1 = 0.1$ and S_0 is not present and therefore no integral action is needed. We design controller (4) with (5), and we compute Ψ solution to (3). We select $n_\xi = 5$, $\Phi = \mathbf{S}$, $\Gamma = (0, 1)^T$, $\mathbf{H} = (1, 0.1, 0.1)^T$,

$$\mathbf{M} = \begin{pmatrix} 0 & -0.2 \\ 0 & -0.2 \\ 0 & 0 \end{pmatrix} \quad \mathbf{N} = \begin{pmatrix} -1.9 & 0 & 0 \\ -0.1 & -1.1 & 0 \\ -0.1 & 1 & -1 \end{pmatrix} \quad \mathbf{K}_1 = \begin{pmatrix} 0 & -0.2 \\ 0 & -0.2 \end{pmatrix}$$

$$\mathbf{K}_2 = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -0.1 & 0 \end{pmatrix} \quad \Psi = \begin{pmatrix} -0.193 & 0.901 \\ 1.580 & -1.733 \end{pmatrix}.$$

It can be verified that the matrix \mathbf{A} is Hurwitz. Finally, we suppose that the exogenous signal w ranges in the compact set $\mathcal{W} = \{w \in \mathbb{R}^2 : |w| \leq \bar{w}\}$, with $\bar{w} = 1$. We consider, in this numerical example, the case in which there are $\ell = 3$ nodes corresponding to the 2 inputs and the output, respectively, with the RR protocol (10) and with aperiodic sampling with $\epsilon = \tau^*/2$. From Nešić and Teel (2004), we know that Assumption 2 is satisfied with $\lambda = \sqrt{(\ell - 1)/\ell}$, $\bar{a} = \bar{b} = \sqrt{\ell}$, $\underline{a} = 1$. By following Section 3.5, we derive the MATI τ^* to satisfy given gain θ_{\max} for the regulated output. We used the inequality (18) to obtain c, ρ and the definition (16) to compute τ^* . We compare those value by running a simulation in Matlab/Simulink for the same

θ_{\max}	c	ρ	τ^* ($\cdot 10^{-2}$)	θ_{sim} ($\cdot 10^{-4}$)
1	2.581	0.182	2.25	3.47
0.5	2.676	0.336	2.11	3.31
0.1	4.337	0.667	0.94	0.86
0.05	6.039	0.668	0.51	0.44

Table 1. Given bound of θ_{\max} and bound θ_{sim} obtained from simulation.

values of MATI τ^* . It can be also verified that values of τ^* larger than 0.35 leads to instability. The asymptotic bounds of the output found in the simulations, denoted as θ_{sim} , are given in Table 1. As expected by the discussion in Section 3.5, performances in terms of output bounds can be improved by choosing a smaller MATI. On the other hand, the specifications on the network can be less stringent if the performance specification is loosened.

5. CONCLUSION

We investigated the problem of output regulation for linear NCSs in presence of uncertain transmission intervals and scheduling between the plant, sensors, actuators and controller. We have shown that in general internal stability and boundedness of the regulated output error is guaranteed, provided that the network and the scheduling protocol satisfy certain properties. In other words, the presence of network effects may destroy the internal model property of the regulator. Specific cases where asymptotic output regulation is achieved are discussed, namely when the regulator is directly connected with the actuators, or when the exogenous signals are constants. For the general case we have provided a performance analysis in which an expression for the gain from external disturbances to output error is given. Future works include the analysis of other network phenomena (such as delay or quantization) and the set up of new control strategies able to guarantee asymptotic regulation in such framework by means of *smart actuators*.

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Appendix A. PROOFS

A.1 Proof of Lemma 1

Since \mathbf{A} is Hurwitz, there always exist a symmetric and positive definite matrix \mathbf{X} and a sufficiently small $\rho > 0$ such that $\mathbf{A}^T \mathbf{X} + \mathbf{X} \mathbf{A} + 2\rho^2 \mathbf{X} < 0$. The desired result is obtained by the Schur's complement of (15) and by selecting $c > 0$ large enough.

A.2 Proof of Lemma 2

Since \mathbf{A} is Hurwitz, there always exist a symmetric and positive definite matrix \mathbf{X} and a sufficiently small $\rho > 0$ such that $\mathbf{A}^T \mathbf{X} + \mathbf{X} \mathbf{A} + 2\rho^2 \mathbf{X} < 0$. The desired result is obtained by the Schur's complement of (15) and by selecting any $\mathbf{H} = \mathbf{H}^T > 0$ and $c > 0$ large enough. Note that when $\mathbf{H} = I$, inequality (17) coincides with (15), with $\underline{a} = \bar{b} = 1$.

A.3 Proof of Theorem 1

The proof of this theorem follows the main arguments used in the proof of Theorem 1 in Postoyan et al. (2014), though three main points characterize the framework considered here: the definition of a change of coordinates for system (13) where the regulated output does not depend on the exogenous signal w ; the definition of a Lyapunov function which is slightly different from the one proposed in Carnevale et al. (2007), since it is adapted for the use of inequality (15); the use of the regulator equations (3) to give an explicit expression of the asymptotic bound θ showing its explicit dependency on the MATI via the parameters $c, \rho > 0$, on the properties of the network via \bar{b} coming from Assumption 2, on the friend Ψ , and on the frequency of w contained in S .

First, let $\mathbf{\Pi} = (\mathbf{\Pi}_x, \mathbf{\Pi}_\xi) \in \mathbb{R}^{n \times n}$ be computed as the unique solution³ to the Sylvester equation $\mathbf{\Pi}S = \mathbf{A}\mathbf{\Pi} + \mathbf{P}$. Standard results in output regulation theory⁴ can be used to show that $\mathbf{\Pi}_x = \mathbf{\Pi}$, with $\mathbf{\Pi}$ defined in (3), and, moreover, $K\mathbf{\Pi}_\xi = \Psi$. Consider the change of coordinates $\tilde{\chi} = \chi - \mathbf{\Pi}w$ by which system (13) is transformed into

$$\left. \begin{aligned} \dot{w} &= Sw \\ \dot{\tilde{\chi}} &= \mathbf{A}\tilde{\chi} + \mathbf{B}e \\ \dot{e} &= \mathbf{M}\tilde{\chi} + \mathbf{N}e + \mathbf{L}w \\ \dot{\kappa} &= 0 \\ \dot{\tau} &= 1 \end{aligned} \right\} \kappa \in [0, \kappa^*], \tau \in [0, \tau^*],$$

$$\left. \begin{aligned} w^+ &= w \\ \tilde{\chi}^+ &= \tilde{\chi} \\ e^+ &= h(e, \kappa) \\ \kappa^+ &= p(\kappa) \\ \tau^+ &= 0 \\ y &= \mathbf{C}z \end{aligned} \right\} \kappa \in [0, \kappa^*], \tau \in [0, \tau^*], \quad (\text{A.1})$$

where $\mathbf{L} := \mathbf{M}\mathbf{\Pi} + \mathbf{R} = \begin{pmatrix} 0 \\ -K\mathbf{\Pi}_\xi S \end{pmatrix} = \begin{pmatrix} 0 \\ -\Psi S \end{pmatrix}$. The last equality is a direct consequence of the definitions of the

³ Uniqueness of $\mathbf{\Pi}$ is a direct consequence of the fact that the matrix \mathbf{A} defined in (6) is Hurwitz and therefore $\sigma(\mathbf{A}) \cap \sigma(S) = \emptyset$.

⁴ See (Byrnes et al., 2012, Section 1.3).

matrices \mathbf{M} , \mathbf{R} and $\mathbf{\Pi}_\xi$. We write system (A.1) in the compact form

$$\left. \begin{aligned} \dot{w} &= Sw \\ \dot{\eta} &= \mathbf{F}(\eta, w) \end{aligned} \right\} \eta \in \mathcal{C}_\eta, \quad \left. \begin{aligned} w^+ &= w \\ \eta^+ &= \mathbf{G}(\eta) \end{aligned} \right\} \eta \in \mathcal{D}_\eta, \quad (\text{A.2})$$

where $\eta := (\tilde{\chi}, e, \kappa, \tau)$. Note that in view of (Goebel et al., 2012, Theorem 6.30), of Assumption 2, and of the definition of \mathcal{C}_η , \mathcal{D}_η , system (A.2) is well-posed.

Let $\phi : [0, \mathcal{T}] \rightarrow \mathbb{R}$, with \mathcal{T} defined in (16), be the solution to

$$\dot{\phi} = -2L\phi - \gamma(\phi^2 + 1) \quad \phi(0) = \lambda^{-1} \quad (\text{A.3})$$

where $\lambda \in (0, 1)$ is defined in Assumption 2, and $\gamma = c^2$. We have the following result according to Carnevale et al. (2007).

Claim 1. $\phi(\tau) \in [\lambda, \lambda^{-1}]$ for all $\tau \in [0, \mathcal{T}]$.

Consider now the following function as also employed in Carnevale et al. (2007):

$$U(\eta) := V(\tilde{\chi}) + \phi(\tau)W^2(\kappa, e) \quad (\text{A.4})$$

where $V(z) := \tilde{\chi}^T \mathbf{X} \tilde{\chi}$, with \mathbf{X} satisfying (15), and W defined in Assumption 2. By definition of V, W , and as a consequence of Claim 1, there exist $\bar{\varrho}, \underline{\varrho} > 0$ such that

$$\underline{\varrho}|\tilde{\chi}, e|^2 \leq U(\eta) \leq \bar{\varrho}|\tilde{\chi}, e|^2 \quad (\text{A.5})$$

for any $\eta \in \mathcal{C}_\eta \cup \mathcal{D}_\eta$. Let $\eta \in \mathcal{C}_\eta$. By using Assumption 2, Claim 1 and the definition of \mathbf{G} according to (A.1) and (A.2), we have that

$$\begin{aligned} U(\mathbf{G}(\eta)) &= V(\tilde{\chi}) + \phi(0)W^2(p(\kappa), h(\kappa, e)) \\ &\leq V(\tilde{\chi}) + \lambda W^2(\kappa, e) \leq U(\eta) \end{aligned} \quad (\text{A.6})$$

for all $(w, \eta) \in \mathcal{W} \times \mathcal{D}_\eta$. Hence, we have shown that U does not increase at jumps. Next, by using the definitions of \mathbf{F} according to (A.1) and (A.3), we compute

$$\begin{aligned} U^\circ(\eta, \mathbf{F}(\eta, w)) &= \tilde{\chi}^T (\mathbf{X} \mathbf{A} + \mathbf{A}^T \mathbf{X}) \tilde{\chi} + \tilde{\chi}^T \mathbf{X} \mathbf{B} e + e^T \mathbf{B}^T \mathbf{X} \tilde{\chi} \\ &\quad + 2\phi(\tau)W(\kappa, e) \frac{\partial W(\kappa, e)}{\partial e} (\mathbf{M}\tilde{\chi} + \mathbf{N}e + \mathbf{L}w) \\ &\quad - [2L\phi(\tau) + \gamma(\phi^2(\tau) + 1)]W^2(\kappa, e) \end{aligned} \quad (\text{A.7})$$

for all $(w, \eta) \in \mathcal{W} \times \mathcal{C}_\eta$. By using Young's inequality and Assumption 2, we have

$$\begin{aligned} 2\phi(\tau)W(\kappa, e) \frac{\partial W(\kappa, e)}{\partial e} (\mathbf{M}\tilde{\chi} + \mathbf{L}w) \\ \leq \gamma\phi^2(\tau)W^2(\kappa, e) + \frac{2\bar{b}^2}{\gamma} (\tilde{\chi}^T \mathbf{M}^T \mathbf{M} \tilde{\chi} + w^T \mathbf{L}^T \mathbf{L} w) \end{aligned}$$

and

$$\left| \frac{\partial W(\kappa, e)}{\partial e} \mathbf{N}e \right| \leq |\mathbf{N}| \frac{\bar{b}}{\underline{a}} W(\kappa, e)$$

in view of Assumption 2. As a consequence, by definition of L and γ , and by using the first inequality in Assumption 2, we derive from (A.7)

$$\begin{aligned} U^\circ(\eta, \mathbf{F}(\eta, w)) &\leq \tilde{\chi}^T (\mathbf{X} \mathbf{A} + \mathbf{A}^T \mathbf{X}) \tilde{\chi} + 2\tilde{\chi}^T \mathbf{X} \mathbf{B} e - 2\rho^2 \phi(\tau)W^2(\kappa, e) \\ &\quad - \gamma \underline{a}^2 |e|^2 + \frac{2\bar{b}^2}{c^2} (\tilde{\chi}^T \mathbf{M}^T \mathbf{M} \tilde{\chi} + w^T \mathbf{L}^T \mathbf{L} w) \end{aligned} \quad (\text{A.8})$$

for all $(w, \eta) \in \mathcal{W} \times \mathcal{C}_\eta$. Therefore, by applying (15) to (A.8), and by recalling the definition of U in (A.4), we directly obtain

$$U^\circ(\eta, \mathbf{F}(\eta, w)) \leq -2\rho^2 U(\eta) + \frac{2\bar{b}^2}{c^2} w^T \mathbf{L}^T \mathbf{L} w \quad (\text{A.9})$$

for all $(w, \eta) \in \mathcal{W} \times \mathcal{C}_\eta$.

When $w = 0$, inequality (A.9) reads

$$U^\circ(\eta, f) \leq -2\rho^2 U(\eta). \quad (\text{A.10})$$

As a consequence, item (1) of Theorem 1 holds in view of (A.5), (A.6), (A.10), and by invoking the same arguments as in the proof of Theorem 1 in Carnevale et al. (2007). Note that maximal solutions are complete as all the conditions of (Goebel et al., 2012, Proposition 6.10) are verified.

When $w \neq 0$ and $w \in \mathcal{W} \subset \mathbb{R}^{n_w}$, we may upper bound the term in (A.9) $w^T \mathbf{L}^T \mathbf{L} w \leq \sup_{w \in \mathcal{W}} |w|^2 |\Psi S|^2$, where we used the definition of \mathbf{L} in (A.1). As a consequence, inequalities (A.6) and (A.9) give

$$\begin{aligned} U^\circ(\eta, \mathbf{F}(\eta, w)) &\leq -2\rho^2(U(\eta) - \mu^2 \theta^2 \bar{w}^2) \\ U(\mathbf{G}(\eta)) - U(\eta) &\leq 0 \end{aligned} \quad (\text{A.11})$$

for all $(w, \eta) \in \mathcal{W} \times \mathcal{C}_\eta$ and $(w, \eta) \in \mathcal{W} \times \mathcal{D}_\eta$ respectively, with θ, μ and \bar{w} defined in the statement of the theorem. Let (w, η) be a solution to system (A.2). By definition of the Clarkes derivative (see Section II) and page 100 in Teel and Praly (2000), it holds that, for all j and for almost all $t \in I^j$ (where $I^j = \{t : (t, j) \in \text{dom}(w, \eta)\}$)

$$\begin{aligned} \dot{U}(\eta(t, j)) &\leq U^\circ(\eta(t, j), \mathbf{F}(\eta(t, j), w(t, j))) \\ &\leq -2\rho^2(U(\eta(t, j))) \end{aligned}$$

as $\eta(t, j) \in \mathcal{C}_\eta$ for all $(t, j) \in \text{dom}(w, \eta)$. From previous inequality, and by recalling that for all $j \in J$ (where $J = \{j : (t, j) \in \text{dom}(w, \eta)\}$) the second inequality (A.11) holds, we obtain

$$U(\eta(t, j)) \leq \exp(-2\rho^2 t) U(\eta(0, 0)) + \mu^2 \theta^2 \bar{w}^2, \quad (\text{A.12})$$

for $(t, j) \in \text{dom}(w, \eta)$. As a consequence, since system (A.2) is well-posed (Goebel et al., 2012, Theorem 6.8), we can apply (Goebel et al., 2012, Proposition 6.10) to conclude that all maximal solutions are complete. Therefore, in view of (A.12) and (A.5) are compact and converge to the set $\mathcal{A}_q := \{(w, \eta) : U(\eta) \leq \mu^2 \theta^2 \bar{w}^2\}$. Since $y = \mathbf{C}\tilde{\chi}$, by using the inequality $\mu^2 y^T \mathbf{C}^T \mathbf{C} y \leq \tilde{\chi}^T \mathbf{X} \tilde{\chi} \leq U(\eta)$ we have that $\mathcal{A} \supseteq \mathcal{A}_q$ concluding the proof.

A.4 Sketch of the proof of Proposition 1

First of all, note that when $h = 0$, $p = 0$, $\kappa^* = 0$, Assumption 2 is verified with $W(\kappa, e) := \sqrt{e^T \mathbf{H} e}$ and $\lambda = 0$. As a consequence, the proof follows the same arguments of the proof of Theorem 1, which are here omitted for sake of brevity, where in particular we use the inequality (17) instead of (15). Note that by taking $\mathbf{H} = I$, we obtain $\underline{a} = \bar{a} = \bar{b} = 1$ in Assumption 2, and the inequality (17) reduces to (15).

A.5 Sketch of the proof of Proposition 2

By applying the change of coordinates $\tilde{\chi} = \chi - \mathbf{\Pi}w$ we obtain system (A.1) where in particular $\mathbf{L} := 0$ (this can be easily shown by removing the last n_u rows from the definition of \mathbf{L} in (A.1)). As a consequence, since the $(\tilde{\chi}, e)$ -flow map and jump maps are independent of w , the proof reduces to show UGAS of the set $\mathcal{W} \times \{0\} \times \{0\} \times [0, \kappa^*] \times [0, \tau^*]$ of system (A.1). This can be shown by using the inequality (15) with $\rho = 0$ and by following the same arguments of the proof of Theorem 1 in Carnevale et al.

(2007) and by properly recalling the definition of $\gamma = c^2$. The proof concludes by noting that solutions in that set satisfy $y = 0$ in view of the definition of system (A.1).