Parametrix for wave equations on a rough background III : space-time regularity of the phase

Jeremie Szeftel

Parametrix for wave equations on a rough background III: space-time regularity of the phase

Jérémie Szeftel

Laboratoire Jacques-Louis Lions,
Université Pierre et Marie Curie, France
jeremie.szeftel@upmc.fr

Abstract. This is the third of a sequence of four papers [21], [22], [23], [24] dedicated to the construction and the control of a parametrix to the homogeneous wave equation $\Box_{\mathbf{g}}\phi = 0$, where $\mathbf{g}$ is a rough metric satisfying the Einstein vacuum equations. Controlling such a parametrix as well as its error term when one only assumes L^2 bounds on the curvature tensor $\mathbf{R}$ of $\mathbf{g}$ is a major step of the proof of the bounded L^2 curvature conjecture proposed in [12], and solved by S. Klainerman, I. Rodnianski and the author in [17]. On a more general level, this sequence of papers deals with the control of the eikonal equation on a rough background, and with the derivation of L^2 bounds for Fourier integral operators on manifolds with rough phases and symbols, and as such is also of independent interest.

Contents

1 Introduction

We consider the Einstein vacuum equations,

$$\mathbf{Ric}_{\alpha\beta} = 0 \tag{1.1}$$

where $\mathbf{Ric}_{\alpha\beta}$ denotes the Ricci curvature tensor of a four dimensional Lorentzian space time $(\mathcal{M}, \mathbf{g})$[1]. (1.1) corresponds to an evolution problem. An initial data set consists of

[1]Our convention is that $\mathbf{g}$ has signature $(-,+,+,+)$. Furthermore, the curvature tensor $\mathbf{R}$ is defined for all vectorfields X, Y, Z by $\mathbf{R}(X,Y)Z = \mathbf{D}_X\mathbf{D}_Y Z - \mathbf{D}_Y\mathbf{D}_X Z - \mathbf{D}_{[X,Y]}Z$. In a coordinates system, we have $\mathbf{R}\left(\frac{\partial}{\partial x^\gamma}, \frac{\partial}{\partial x^\mu}\right)\frac{\partial}{\partial x^\beta} = \mathbf{R}^\alpha{}_{\beta\gamma\mu}\frac{\partial}{\partial x^\alpha}$, and the Ricci tensor $\mathbf{Ric}$ is given by $\mathbf{Ric}_{\beta\mu} = \mathbf{R}^\alpha{}_{\beta\alpha\mu}$.

a three dimensional manifold Σ_0 together with a Riemannian metric g and a symmetric 2-tensor k on Σ_0. For a given initial data set (Σ_0, g, k), the Cauchy problem consists in finding a metric $\mathbf{g}$ satisfying (1.1) and an embedding of Σ_0 in $\mathcal{M}$ such that the metric induced by $\mathbf{g}$ on Σ_0 coincides with g and the 2-tensor k is the second fundamental form of the embedding. (1.1) is an overdetermined system and the initial data set (Σ_0, g, k) must satisfy the constraint equations

$$\begin{cases} \nabla^j k_{ij} - \nabla_i \mathrm{Tr} k = 0, \\ R - |k|^2 + (\mathrm{Tr} k)^2 = 0, \end{cases} \tag{1.2}$$

where the covariant derivative ∇ is defined with respect to the metric g, R is the scalar curvature of g, and $\mathrm{Tr}k$ is the trace of k with respect to the metric g.

One of the fundamental problems in general relativity is to study the long term regularity and asymptotic properties of the Cauchy developments of general, asymptotically flat, initial data sets (Σ_0, g, k). As far as local regularity is concerned it is natural to ask what are the minimal regularity properties of the initial data which guarantee the existence and uniqueness of local developments. In [17], we obtain the following result which solves the bounded L^2 curvature conjecture proposed in [12]:

Theorem 1.1 (Theorem 2.10 in [17]) *Let $(\mathcal{M}, \mathbf{g})$ be an asymptotically flat solution to the Einstein vacuum equations (1.1) together with a maximal foliation[2] by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t. Let $r_{vol}(\Sigma_t, 1)$ the volume radius on scales ≤ 1 of Σ_t[3]. Assume that the initial slice (Σ_0, g, k) is such that:*

$$\|R\|_{L^2(\Sigma_0)} \leq \varepsilon, \ \|k\|_{L^2(\Sigma_0)} + \|\nabla k\|_{L^2(\Sigma_0)} \leq \varepsilon \ \text{and} \ r_{vol}(\Sigma_0, 1) \geq \frac{1}{2}$$

where R denotes the curvature tensor of g. Then, there exists a small universal constant $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then the following control holds on $0 \leq t \leq 1$:

$$\|\mathbf{R}\|_{L^\infty_{[0,1]} L^2(\Sigma_t)} \lesssim \varepsilon, \ \|k\|_{L^\infty_{[0,1]} L^2(\Sigma_t)} + \|\nabla k\|_{L^\infty_{[0,1]} L^2(\Sigma_t)} \lesssim \varepsilon \ \text{and} \ \inf_{0 \leq t \leq 1} r_{vol}(\Sigma_t, 1) \geq \frac{1}{4}.$$

Remark 1.2 *While the first nontrivial improvements for well posedness for quasilinear hyperbolic systems (in spacetime dimensions greater than $1+1$), based on Strichartz estimates, were obtained in [3], [2], [26], [27], [9], [15], [19], Theorem 1.1, is the first result in which the full nonlinear structure of the quasilinear system, not just its principal part, plays a crucial role. We note that though the result is not optimal with respect to the standard scaling of the Einstein equations, it is nevertheless critical with respect to its causal geometry, i.e. L^2 bounds on the curvature is the minimum requirement necessary to obtain lower bounds on the radius of injectivity of null hypersurfaces. We refer the reader to section 1 in [17] for more motivations and historical perspectives concerning Theorem 1.1.*

Remark 1.3 *The regularity assumptions on Σ_0 in Theorem 1.1 - i.e. R and ∇k bounded in $L^2(\Sigma_0)$ - correspond to an initial data set $(g, k) \in H^2_{loc}(\Sigma_0) \times H^1_{loc}(\Sigma_0)$.*

[2] See section 2.1 for a definition.

[3] See Remark 1.5 below for a definition.

Remark 1.4 *In [17], our main result is stated for corresponding large data. We then reduce the proof to the small data statement of Theorem 1.1 relying on a truncation and rescaling procedure, the control of the harmonic radius of Σ_0 based on Cheeger-Gromov convergence of Riemannian manifolds together with the assumption on the lower bound of the volume radius of Σ_0, and the gluing procedure in [6], [5]. We refer the reader to section 2.3 in [17] for the details.*

Remark 1.5 *We recall for the convenience of the reader the definition of the volume radius of the Riemannian manifold Σ_t. Let $B_r(p)$ denote the geodesic ball of center p and radius r. The volume radius $r_{vol}(p,r)$ at a point $p \in \Sigma_t$ and scales $\leq r$ is defined by*

$$r_{vol}(p,r) = \inf_{r' \leq r} \frac{|B_{r'}(p)|}{(r')^3},$$

with $|B_r|$ the volume of B_r relative to the metric g_t on Σ_t. The volume radius $r_{vol}(\Sigma_t, r)$ of Σ_t on scales $\leq r$ is the infimum of $r_{vol}(p,r)$ over all points $p \in \Sigma_t$.

Remark 1.6 *In [17], as is it the case in this paper, the timelike scalar function t appearing in Theorem 1.1 is normalized at spacelike infinity by $n \to 1$, where n denotes the lapse of the time foliation (see (2.3) for a definition of n).*

The proof of Theorem 1.1, obtained in the sequence of papers [17], [21], [22], [23], [24], [25], relies on the following ingredients[4]:

A *Provide a system of coordinates relative to which (1.1) exhibits a null structure.*

B *Prove appropriate bilinear estimates for solutions to $\Box_{\mathbf{g}}\phi = 0$, on a fixed Einstein vacuum background[5].*

C *Construct a parametrix for solutions to the homogeneous wave equation $\Box_{\mathbf{g}}\phi = 0$ on a fixed Einstein vacuum background, and obtain control of the parametrix and of its error term only using the fact that the curvature tensor is bounded in L^2.*

Steps **A** and **B** are carried out in [17]. In particular, the proof of the bilinear estimates rests on a representation formula for the solutions of the wave equation using the following plane wave parametrix[6]:

$$Sf(t,x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t,x,\omega)} f(\lambda\omega)\lambda^2 d\lambda d\omega,\ (t,x) \in \mathcal{M} \tag{1.3}$$

where $u(\cdot,.,\omega)$ is a solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ on $\mathcal{M}$ such that $u(0,x,\omega) \sim x.\omega$ when $|x| \to +\infty$ on Σ_0[7]. Therefore, in order to complete the proof of the bounded L^2 curvature conjecture, we need to carry out step **C** with the parametrix defined in (1.3).

[4]We also need trilinear estimates and an $L^4(\mathcal{M})$ Strichartz estimate (see the introduction in [17]).

[5]Note that the first bilinear estimate of this type was obtained in [13].

[6](1.3) actually corresponds to a half-wave parametrix. The full parametrix corresponds to the sum of two half-parametrix. See [22] for the construction of the full parametrix.

[7]The asymptotic behavior for $u(0,x,\omega)$ when $|x| \to +\infty$ will be used in [22] to generate with the parametrix any initial data set for the wave equation.

Remark 1.7 *Note that the parametrix* (1.3) *is invariantly defined*[8]*, i.e. without reference to any coordinate system. This is crucial since coordinate systems consistent with* L^2 *bounds on the curvature would not be regular enough to control a parametrix.*

Remark 1.8 *In addition to their relevance to the resolution of the bounded* L^2 *curvature conjecture, the methods and results of step* **C** *are also of independent interest. Indeed, they deal on the one hand with the control of the eikonal equation* $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ *at a critical level*[9]*, and on the other hand with the derivation of* L^2 *bounds for Fourier integral operators with significantly lower differentiability assumptions both for the corresponding phase and symbol compared to classical methods (see for example [20] and references therein).*

In view of the energy estimates for the wave equation, it suffices to control the parametrix at $t = 0$ (i.e. restricted to Σ_0)

$$Sf(0,x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(0,x,\omega)} f(\lambda\omega)\lambda^2 d\lambda d\omega,\ x \in \Sigma_0 \tag{1.4}$$

and the error term

$$Ef(t,x) = \Box_{\mathbf{g}} Sf(t,x) = i \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t,x,\omega)} \Box_{\mathbf{g}} u(t,x,\omega) f(\lambda\omega)\lambda^3 d\lambda d\omega,\ (t,x) \in \mathcal{M}. \tag{1.5}$$

This requires the following ingredients, the two first being related to the control of the parametrix restricted to Σ_0 (1.4), and the two others being related to the control of the error term (1.5):

C1 *Make an appropriate choice for the equation satisfied by* $u(0,x,\omega)$ *on* Σ_0*, and control the geometry of the foliation generated by the level surfaces of* $u(0,x,\omega)$ *on* Σ_0*.*

C2 *Prove that the parametrix at* $t = 0$ *given by* (1.4) *is bounded in* $\mathcal{L}(L^2(\mathbb{R}^3), L^2(\Sigma_0))$ *using the estimates for* $u(0,x,\omega)$ *obtained in* **C1***.*

C3 *Control the geometry of the foliation generated by the level hypersurfaces of* u *on* $\mathcal{M}$*.*

C4 *Prove that the error term* (1.5) *satisfies the estimate* $\|Ef\|_{L^2(\mathcal{M})} \leq C\|\lambda f\|_{L^2(\mathbb{R}^3)}$ *using the estimates for* u *and* $\Box_{\mathbf{g}} u$ *proved in* **C3***.*

Step **C1** has been carried out in [21] and step **C2** has been carried out in [22]. In the present paper, we focus on step **C3**. This step was initiated in the sequence of papers [14], [10], [11] where the authors prove in particular the estimate $\Box_{\mathbf{g}} u \in L^\infty(\mathcal{M})$ using a geodesic foliation. In view of achieving step **C4**, we actually need to work in a time

[8] Our choice is reminiscent of the one used in [19] in the context of $H^{2+\epsilon}$ solutions of quasilinear wave equations. Note however that the construction in that paper is coordinate dependent.

[9] As we will see in this paper, we need at least L^2 bounds on the curvature to obtain a lower bound on the radius of injectivity of the null level hypersurfaces of the solution u of the eikonal equation, which in turn is necessary to control the local regularity of u.

foliation[10]. We start by reproving the estimates obtained in [14], [10], [11] in the case of a time foliation. We also obtain new estimates which will be crucial for the proof of step **C4**. Let us mention in particular

- a lower bound for the radius of injectivity of the null level hypersurfaces of u,
- the control of the second fundamental form k,
- the control of the null lapse associated to u,
- a second order derivative of $\Box_{\mathbf{g}} u$ requires an estimate,
- the control of the regularity of the u-foliation on $\mathcal{M}$ with respect to the parameter $\omega \in \mathbb{S}^2$, which requires estimates for first and second order derivatives with respect to ω of various geometric quantities related to u.

The difficulty will be to obtain the aforementioned estimates when assuming only L^2 bounds on the curvature tensor $\mathbf{R}$. Indeed, this level of regularity for $\mathbf{R}$ is critical for the control of the eikonal equation. In turn, at numerous places in this paper, we will encounter log-divergences which have to be tackled by ad-hoc techniques taking full advantage of the structure of the Einstein equations. More precisely, we will use the regularity obtained in Step C1, together with null transport equations tied to the eikonal equation, elliptic systems of Hodge type, the geometric Littlewood-Paley theory of [10], sharp trace theorems, and an extensive use of the structure of the Einstein equations, to propagate the regularity on Σ_0 to the space-time, thus achieving Step **C3**.

The rest of the paper is as follows. In section 2, we state our main result. In section 3, we derive embeddings with respect to the foliation generated by t and u on $\mathcal{M}$ which are consistent with the level of regularity we are considering. In section 4, we investigate the regularity with respect to (t, x) of the foliation generated by u on $\mathcal{M}$. In section 5, we derive estimates for certain second order derivatives of the u-foliation on $\mathcal{M}$. In section 6, we derive estimates for first order derivatives with respect to ω of the u-foliation on $\mathcal{M}$. In section 7, we derive estimates for second order derivatives with respect to ω of the u-foliation on $\mathcal{M}$. In section 8, we investigate the dependence in ω of certain norms associated to the u-foliation on $\mathcal{M}$. Finally, additional estimates are derived in section 9.

Acknowledgments. The author wishes to express his deepest gratitude to Sergiu Klainerman and Igor Rodnianski for stimulating discussions and constant encouragements during the long years where this work has matured. He also would like to stress that the basic strategy of the construction of the parametrix and how it fits into the whole proof of the bounded L^2 curvature conjecture has been done in collaboration with them. The author is supported by the ERC grant ERC-2016-CoG 725589 EPGR.

[10]Indeed, the geodesic foliation leads to a loss of derivatives in the direction transverse to the the level hypersurfaces of u. This loss of derivatives would prevent us to establish Step **C4**.

2 Main results

2.1 Maximal foliation on $\mathcal{M}$

We foliate the space-time $\mathcal{M}$ by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t. Denoting by T the unit, future oriented, normal to Σ_t and k the second fundamental form

$$k_{ij} = - < \mathbf{D}_i T, \partial_j > \tag{2.1}$$

we find,

$$k_{ij} = -\frac{1}{2}\mathcal{L}_T \mathbf{g}_{ij}$$

with $\mathcal{L}_X$ denoting the Lie derivative with respect to the vectorfield X. Let $\mathrm{Tr}(k) = g^{ij}k_{ij}$ where g is the induced metric on Σ_t and Tr is the trace. In order to be consistent with the statement of Theorem 1.1, we impose a maximal foliation

$$\mathrm{Tr}(k) = 0. \tag{2.2}$$

We also define the lapse n as

$$n^{-1} = T(t). \tag{2.3}$$

We have:

$$\mathbf{D}_T T = n^{-1}\nabla n, \tag{2.4}$$

where ∇ denotes the gradient with respect to the induced metric on Σ_t. To check (2.4) observe that $\partial_t = nT$ and therefore, for an arbitrary vectorfield X tangent to Σ_t, we easily calculate, $< \mathbf{D}_T T, X >= n^{-2}X^i < \mathbf{D}_{\partial_t}\partial_t, \partial_i >= -n^{-2}X^i < \partial_t, \mathbf{D}_{\partial_t}\partial_i >= -n^{-2}X^i < \partial_t, \mathbf{D}_{\partial_i}\partial_t >= -n^{-2}X^i\frac{1}{2}\partial_i < \partial_t, \partial_t >= n^{-2}X^i\frac{1}{2}\partial_i(n^2) = n^{-1}X(n)$.

Finally, the lapse n satisfies the following elliptic equation on Σ_t (see [4] p. 13):

$$\Delta n = |k|^2 n, \tag{2.5}$$

where one uses (2.1), (2.4), Einstein vacuum equations (1.1) and the fact that the foliation generated by t on $\mathcal{M}$ is maximal (2.2).

2.2 Geometry of the foliation generated by u on $\mathcal{M}$

Remember that u is a solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ on $\mathcal{M}$ depending on a extra parameter $\omega \in \mathbb{S}^2$. The level hypersufaces $u(t, x, \omega) = u$ of the optical function u are denoted by $\mathcal{H}_u$. Let L' denote the space-time gradient of u, i.e.:

$$L' = -\mathbf{g}^{\alpha\beta}\partial_\beta u \partial_\alpha. \tag{2.6}$$

Using the fact that u satisfies the eikonal equation, we obtain:

$$\mathbf{D}_{L'} L' = 0, \tag{2.7}$$

which implies that L' is the geodesic null generator of $\mathcal{H}_u$.

We have:

$$T(u) = \pm|\nabla u|$$

where $|\nabla u|^2 = \sum_{i=1}^3 |e_i(u)|^2$ relative to an orthonormal frame e_i on Σ_t. Since the sign of $T(u)$ is irrelevant, we choose by convention:

$$T(u) = |\nabla u|. \tag{2.8}$$

We denote by $P_{t,u}$ the surfaces of intersection between Σ_t and $\mathcal{H}_u$. They play a fundamental role in our discussion.

Remark 2.1 *Let us mention that the $\mathcal{H}_u$ are topologically 3 dimensional hyperplanes and the $P_{t,u}$ are topologically 2-planes. Note that this differs from the corresponding objects in [4] (which are respectively topologically cones and 2-spheres).*

Definition 2.2 (*Canonical null pair*) *We define the vector fields L and $\underline{L}$ by*

$$L = bL' = T + N, \qquad \underline{L} = 2T - L = T - N \tag{2.9}$$

where L' is the space-time gradient of u (2.6), b is the lapse of the null foliation (or shortly null lapse)

$$b^{-1} = - < L', T >= T(u), \tag{2.10}$$

and N is a unit normal, along Σ_t, to the surfaces $P_{t,u}$. Since u satisfies the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ on $\mathcal{M}$, this yields $L'(u) = 0$ and thus $L(u) = 0$. In view of the definition of L and (2.8), we obtain:

$$N = -\frac{\nabla u}{|\nabla u|}. \tag{2.11}$$

Remark 2.3 *u is prescribed on Σ_0 as in step* **C1**. *For any $(0, x)$ on Σ_0, L is defined as $L = T + N$ where T is the unit normal to Σ_0 at $(0, x)$ and $N = -\nabla u/|\nabla u|$ at $(0, x)$, and b is defined as $b^{-1} = |\nabla u|$. Let $\kappa_x(t)$ denote the null geodesic parametrized by t and such that $\kappa_x(0) = (0, x)$ and $\kappa_x'(0) = nL$. Then, we claim that*

$$\kappa_x'(t) = n(\kappa_x(t))L_{\kappa_x(t)} \text{ for all } t. \tag{2.12}$$

Indeed, recall that $L' = b^{-1}L$ is the geodesic null generator of $\mathcal{H}_u$ (see (2.7)). Thus, if $s \to \nu(s)$ denotes a null geodesic such that $\dot{\nu} = b^{-1}L$, then we infer in view of (2.3) and the fact that $b^{-1}L(s) = 1$[11]

$$\kappa_x'(t) = \dot{\nu}\frac{ds}{dt} = b^{-1}L\frac{b^{-1}L(s)}{b^{-1}L(t)} = \frac{1}{L(t)}L = \frac{1}{T(t)}L = nL$$

which is precisely (2.12).

[11]Note that we have $b^{-1}L(s) = 1$ since s is an affine parameter.

Definition 2.4 *A null frame e_1, e_2, e_3, e_4 at a point $p \in P_{t,u}$ consists, in addition to the null pair $e_3 = \underline{L}, e_4 = L$, of arbitrary orthonormal vectors e_1, e_2 tangent to $P_{t,u}$.*

Definition 2.5 (*Ricci coefficients*) *Let e_1, e_2, e_3, e_4 be a null frame on $P_{t,u}$ as above. The following tensors on $P_{t,u}$*

$$\begin{aligned} &\chi_{AB} =< \mathbf{D}_A e_4, e_B >, \quad && \underline{\chi}_{AB} =< \mathbf{D}_A e_3, e_B >, \\ &\zeta_A = \frac{1}{2} < \mathbf{D}_3 e_4, e_A >, \quad && \underline{\zeta}_A = \frac{1}{2} < \mathbf{D}_4 e_3, e_A >, \\ &\underline{\xi}_A = \frac{1}{2} < \mathbf{D}_3 e_3, e_A > . \end{aligned} \tag{2.13}$$

are called the Ricci coefficients associated to our canonical null pair.

We decompose χ and $\underline{\chi}$ into their trace and traceless components.

$$tr\chi = \mathbf{g}^{AB}\chi_{AB}, \qquad tr\underline{\chi} = \mathbf{g}^{AB}\underline{\chi}_{AB}, \tag{2.14}$$

$$\widehat{\chi}_{AB} = \chi_{AB} - \frac{1}{2} tr\chi \mathbf{g}_{AB}, \quad \underline{\widehat{\chi}}_{AB} = \underline{\chi}_{AB} - \frac{1}{2} tr\underline{\chi}\mathbf{g}_{AB}, \tag{2.15}$$

Definition 2.6 *The null components of the curvature tensor $\mathbf{R}$ of the space-time metric $\mathbf{g}$ are given by:*

$$\alpha_{AB} = \mathbf{R}(L, e_A, L, e_B), \qquad \beta_A = \frac{1}{2}\mathbf{R}(e_A, L, \underline{L}, L), \tag{2.16}$$

$$\rho = \frac{1}{4}\mathbf{R}(\underline{L}, L, \underline{L}, L), \qquad \sigma = \frac{1}{4}{}^\star\mathbf{R}(\underline{L}, L, \underline{L}, L) \tag{2.17}$$

$$\underline{\beta}_A = \frac{1}{2}\mathbf{R}(e_A, \underline{L}, \underline{L}, L), \qquad \underline{\alpha}_{AB} = \mathbf{R}(\underline{L}, e_A, \underline{L}, e_B) \tag{2.18}$$

where ${}^\star\mathbf{R}$ denotes the Hodge dual of $\mathbf{R}$. The null decomposition of ${}^\star\mathbf{R}$ can be related to that of $\mathbf{R}$ according to the formulas, see [4]:

$$\alpha({}^\star\mathbf{R}) = -{}^\star\alpha(\mathbf{R}), \quad \beta({}^\star\mathbf{R}) = -{}^\star\beta(\mathbf{R}), \quad \rho({}^\star\mathbf{R}) = \sigma(\mathbf{R}) \tag{2.19}$$

$$\sigma({}^\star\mathbf{R}) = -\rho(\mathbf{R}), \quad \underline{\beta}({}^\star\mathbf{R}) = {}^\star\underline{\beta}(\mathbf{R}), \quad \underline{\alpha}({}^\star\mathbf{R}) = {}^\star\underline{\alpha}(\mathbf{R}) \tag{2.20}$$

Observe that all tensors defined above are $P_{t,u}$-tangent.

Definition 2.7 *We decompose the symmetric traceless 2 tensor k into the scalar δ, the $P_{t,u}$-tangent 1-form ϵ, and the $P_{t,u}$-tangent symmetric 2-tensor η as follows:*

$$\begin{cases} k_{NN} = \delta \\ k_{AN} = \epsilon_A \\ k_{AB} = \eta_{AB}. \end{cases} \tag{2.21}$$

Note that $Tr(k) = tr(\eta) + \delta$ which together with the maximal foliation assumption (2.2) yields:

$$tr(\eta) = -\delta. \tag{2.22}$$

Remark 2.8 *Most notations in this paper such as*

$$T, k, \chi, \underline{\chi}, \zeta, \underline{\zeta}, \underline{\xi}, \delta, \epsilon, \eta, \alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}, \dots$$

are taken from [4]. Notable exceptions are n and b which are taken from [16].

The following *Ricci equations* can be easily derived from the properties of T (2.1) (2.4), the fact that L' is geodesic (2.7), and the definition (2.13) of the Ricci coefficients (see [4] p. 171):

$$\begin{aligned}
&\mathbf{D}_A e_4 = \chi_{AB} e_B - \epsilon_A e_4, && \mathbf{D}_A e_3 = \underline{\chi}_{AB} e_B + \epsilon_A e_3,\\
&\mathbf{D}_4 e_4 = -\overline{\delta} e_4, && \mathbf{D}_4 e_3 = 2\underline{\zeta}_A e_A + \overline{\delta} e_3,\\
&\mathbf{D}_3 e_4 = 2\zeta_A e_A + (\delta + n^{-1}N(n)) e_4, && \mathbf{D}_3 e_3 = 2\underline{\xi}_A e_A - (\delta + n^{-1}N(n)) e_3,\\
&\mathbf{D}_4 e_A = \nabla\!\!\!/_4 e_A + \underline{\zeta}_A e_4, && \mathbf{D}_3 e_A = \nabla\!\!\!/_3 e_A + \zeta_A e_3 + \underline{\xi}_A e_4,\\
&\mathbf{D}_B e_A = \nabla\!\!\!/_B e_A + \frac{1}{2}\chi_{AB}\, e_3 + \frac{1}{2}\underline{\chi}_{AB}\, e_4
\end{aligned} \tag{2.23}$$

where, $\nabla\!\!\!/_3$, $\nabla\!\!\!/_4$ denote the projection on $P_{t,u}$ of $\mathbf{D}_3$ and $\mathbf{D}_4$, $\nabla\!\!\!/$ denotes the induced covariant derivative on $P_{t,u}$ and $\overline{\delta}, \overline{\epsilon}$ are defined by:

$$\overline{\delta} = \delta - n^{-1}N(n),\ \overline{\epsilon}_A = \epsilon_A - n^{-1}\nabla\!\!\!/_A n. \tag{2.24}$$

Also,

$$\begin{aligned}
&\underline{\chi}_{AB} = -\chi_{AB} - 2k_{AB},\\
&\underline{\zeta}_A = -\overline{\epsilon}_A,\\
&\underline{\xi}_A = \epsilon_A + n^{-1}\nabla\!\!\!/_A n - \zeta_A.
\end{aligned} \tag{2.25}$$

Remark 2.9 *We also have the identity (see [4] p. 171):*

$$\zeta_A = b^{-1}\nabla\!\!\!/_A b + \epsilon_A. \tag{2.26}$$

Indeed, recall from the definition of b (2.10) that $b^{-1}\nabla\!\!\!/ b = -b\nabla\!\!\!/ T(u)$, which together with the fact that $e_A(u) = 0$ implies:

$$b^{-1}\nabla\!\!\!/_A b = -b\nabla\!\!\!/_A T(u) = -b[e_A, T](u) = -b(\mathbf{D}_{e_A} T - \mathbf{D}_T e_A)(u).$$

Now, using the Ricci equations (2.23) for $\mathbf{D}_{e_A} T$ and $\mathbf{D}_T e_A$ and the fact that $L(u) = e_A(u) = 0$ and $T(u) = b^{-1}$ yields (2.26).

2.3 Null structure equations

Below we write down our main structure equations.

Proposition 2.10 *The components $tr\chi$, $\widehat{\chi}$, ζ and the lapse b verify the following equations*[12]*:*

$$L(b) = -b\,\overline{\delta}, \tag{2.27}$$

$$L(tr\chi) + \frac{1}{2}(tr\chi)^2 = -|\widehat{\chi}|^2 - \overline{\delta} tr\chi, \tag{2.28}$$

$$\nabla\!\!\!/_4 \widehat{\chi} + tr\chi\widehat{\chi} = -\overline{\delta}\widehat{\chi} - \alpha, \tag{2.29}$$

$$\nabla\!\!\!/_4 \zeta_A + \frac{1}{2}(tr\chi)\zeta_A = -(\overline{\epsilon}_B + \zeta_B)\widehat{\chi}_{AB} - \frac{1}{2} tr\chi \overline{\epsilon}_A - \beta_A, . \tag{2.30}$$

Remark 2.11 *Equation* (2.28) *is known as the Raychaudhuri equation in the relativity literature, see e.g. [8].*

Proof The proof is derived from the formulas (2.23) above (see [4] chapter 9 p.266). We briefly sketch the proof for convenience. We start with (2.27). Using the fact that L' is geodesic (2.7) and the fact that $L = bL'$ by (2.9), we obtain:

$$\mathbf{D}_L L = b^{-1} L(b) L$$

which together with the Ricci equations (2.23) for $\mathbf{D}_L L$ yields (2.27).

To obtain (2.28) and (2.29), we compute:

$$\begin{aligned}
\nabla\!\!\!/_L \chi_{AB} &= L(\chi_{AB}) - \chi(\nabla\!\!\!/_L e_A, e_B) - \chi(e_A, \nabla\!\!\!/_L e_B) \\
&= \mathbf{g}(\mathbf{D}_L \mathbf{D}_{e_A} L, e_B) - \chi(\nabla\!\!\!/_L e_A, e_B) + \mathbf{g}(\mathbf{D}_{e_A} L, \mathbf{D}_L e_B) - \chi(e_A, \nabla\!\!\!/_L e_B) \\
&= \mathbf{g}(\mathbf{D}_{e_A} \mathbf{D}_L L, e_B) + \mathbf{g}(\mathbf{D}_{[L,e_A]} L, e_B) - \chi(\nabla\!\!\!/_L e_A, e_B) + \mathbf{R}_{BLLA} \\
&\quad + \mathbf{g}(\mathbf{D}_{e_A} L, \mathbf{D}_L e_B - \nabla\!\!\!/_L e_B) \\
&= \mathbf{g}(\mathbf{D}_{e_A} \mathbf{D}_L L, e_B) + \mathbf{g}(\mathbf{D}_{\mathbf{D}_L e_A - \nabla\!\!\!/_L e_A - \mathbf{D}_{e_A} L} L, e_B) - \alpha_{AB} \\
&\quad + \mathbf{g}(\mathbf{D}_{e_A} L, \mathbf{D}_L e_B - \nabla\!\!\!/_L e_B)
\end{aligned}$$

which together with the Ricci equations (2.23) yields:

$$\nabla\!\!\!/_L \chi_{AB} = -\chi_{AC}\chi_{CB} - \overline{\delta}\chi_{AB} - \alpha_{AB}. \tag{2.31}$$

Decomposing (2.31) into its trace and traceless part yields respectively (2.28) and (2.29).

Finally, we derive (2.30). We compute:

$$\begin{aligned}
\nabla\!\!\!/_L \zeta_A &= L(\zeta_A) - \zeta(\nabla\!\!\!/_L e_A) \\
&= \frac{1}{2}\mathbf{g}(\mathbf{D}_L \mathbf{D}_{\underline{L}} L, e_A) + \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} L, \mathbf{D}_L e_A) - \zeta(\nabla\!\!\!/_L e_A) \\
&= \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} \mathbf{D}_L L, e_A) + \frac{1}{2}\mathbf{g}(\mathbf{D}_{[L,\underline{L}]} L, e_A) + \frac{1}{2}\mathbf{R}_{L\underline{L}AL} + \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} L, \mathbf{D}_L e_A - \nabla\!\!\!/_L e_A) \\
&= \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} \mathbf{D}_L L, e_A) + \frac{1}{2}\mathbf{g}(\mathbf{D}_{\mathbf{D}_L \underline{L} - \mathbf{D}_{\underline{L}} L} L, e_A) - \beta_A + \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} L, \mathbf{D}_L e_A - \nabla\!\!\!/_L e_A),
\end{aligned}$$

[12] which can be interpreted as transport equations along the null geodesics generated by L. Indeed observe that if a $P_{t,u}$ tangent tensorfield Π satisfies the homogeneous equation $\nabla\!\!\!/_4 \Pi = 0$ then Π is parallel transported along null geodesics.

which together with the Ricci equations (2.23) yields (2.30). ■

To obtain estimates for χ, we may use the transport equations (2.28) (2.29). However, this does not allow us to get enough regularity. Instead, we follow [15] [14] and consider (2.28) for $\mathrm{tr}\chi$ together with an elliptic system of Hodge type for $\widehat{\chi}$.

Proposition 2.12 *The expression $(d\!\!\!/iv\widehat{\chi})_A = \nabla\!\!\!/^B\widehat{\chi}_{AB}$ verifies the following equation:*

$$(d\!\!\!/iv\widehat{\chi})_A + \widehat{\chi}_{AB}\epsilon_B = \frac{1}{2}(\nabla\!\!\!/_A tr\chi + \epsilon_A tr\chi) - \beta_A. \tag{2.32}$$

Proof The proof is derived from the formulas (2.23) (see [4] chapter 7). We briefly sketch the proof for convenience. We compute:

$$\begin{aligned}
\nabla\!\!\!/_C\chi_{AB} &= e_C(\chi_{AB}) - \chi(\nabla\!\!\!/_{e_C}e_A, e_B) - \chi(e_A, \nabla\!\!\!/_{e_C}e_B)\\
&= \mathbf{g}(\mathbf{D}_{e_C}\mathbf{D}_{e_A}L, e_B) - \chi(\nabla\!\!\!/_{e_C}e_A, e_B) + \mathbf{g}(\mathbf{D}_{e_A}L, \mathbf{D}_{e_C}e_B) - \chi(e_A, \nabla\!\!\!/_{e_C}e_B)\\
&= \mathbf{g}(\mathbf{D}_{e_A}\mathbf{D}_{e_C}L, e_B) + \mathbf{g}(\mathbf{D}_{[e_C,e_A]}L, e_B) - \chi(\nabla\!\!\!/_{e_C}e_A, e_B) + \mathbf{R}_{CBLA}\\
&\quad +\mathbf{g}(\mathbf{D}_{e_A}L, \mathbf{D}_{e_C}e_B - \nabla\!\!\!/_{e_C}e_B)\\
&= \nabla\!\!\!/_A\chi_{CB} - \mathbf{g}(\mathbf{D}_{e_C}L, \mathbf{D}_{e_A}e_B - \nabla\!\!\!/_{e_A}e_B) + \mathbf{g}(\mathbf{D}_{\mathbf{D}_{e_C}e_A - \nabla\!\!\!/_{e_C}e_A - \mathbf{D}_{e_A}e_C + \nabla\!\!\!/_{e_A}e_C}L, e_B)\\
&\quad +\mathbf{R}_{CABL} + \mathbf{g}(\mathbf{D}_{e_A}L, \mathbf{D}_L e_B - \nabla\!\!\!/_L e_B)
\end{aligned}$$

which together with the Ricci equations (2.23) yields:

$$\nabla\!\!\!/_C\chi_{AB} = \nabla\!\!\!/_B\chi_{AC} - \chi_{AB}\epsilon_C + \mathbf{R}_{CBLA} + \chi_{AC}\epsilon_B.$$

Contracting in the previous equality yields (2.32). ■

Finally, we consider the control of ζ and $\underline{L}\mathrm{tr}\chi$. To this end, we follow again [15] [14]: we derive an elliptic system of Hodge type for ζ and a transport equation for $\underline{L}(\mathrm{tr}\chi)$.

Proposition 2.13 *We have:*

$$\underline{L}(tr\chi) + \frac{1}{2}tr\underline{\chi}tr\chi = 2d\!\!\!/iv\zeta + (\delta + n^{-1}N(n))tr\chi - \widehat{\chi}\cdot\underline{\widehat{\chi}} + 2\zeta\cdot\zeta + 2\rho, \tag{2.33}$$

$$\nabla\!\!\!/_3\widehat{\chi} + \frac{1}{2}tr\underline{\chi}\widehat{\chi} = \nabla\!\!\!/\widehat{\otimes}\zeta + (\delta + n^{-1}N(n))\widehat{\chi} - \frac{1}{2}tr\chi\underline{\widehat{\chi}} + \zeta\widehat{\otimes}\zeta, \tag{2.34}$$

where for F, G $P_{t,u}$-tangent 1 forms, we denote by $\nabla\!\!\!/\widehat{\otimes}F$ the traceless part of the symmetrized covariant derivative of F, i.e. $\nabla\!\!\!/\widehat{\otimes}F_{AB} = \nabla\!\!\!/_A F_B + \nabla\!\!\!/_B F_A - d\!\!\!/iv F\delta_{AB}$ and by $F\widehat{\otimes}G$ the traceless symmetric 2-tensor $F\widehat{\otimes}G_{AB} = F_A G_B + F_B G_A - 2F_C G_C\delta_{AB}$.

Also, let

$$\mu = \underline{L}(tr\chi) - \big(\delta + n^{-1}N(n)\big)tr\chi. \tag{2.35}$$

Then, the expressions $d\!\!\!/iv\zeta = \nabla\!\!\!/^B\zeta_B$ and $cu\!\!\!/rl\zeta = \in^{AB}\nabla\!\!\!/_A\zeta_B$ verify the following equations:

$$d\!\!\!/iv\zeta = \frac{1}{2}\left(\mu + \frac{1}{2}tr\chi tr\underline{\chi} + \widehat{\chi}\cdot\underline{\widehat{\chi}} - 2|\zeta|^2\right) - \rho, \tag{2.36}$$

$$cu\!\!\!/rl\zeta = -\frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}} + \sigma, \tag{2.37}$$

where for F, G symmetric traceless $P_{t,u}$-tangent 2-tensors, we denote by $F \wedge G$ the tensor $F \wedge G_{AB} = \in_{AB} F_{AC} G_{BC}$, with $\in_{AB}$ denoting the components of the volume element of $P_{t,u}$, i.e. $\in_{11} = \in_{22} = 0$ and $\in_{12} = -\in_{21} = 1$. Furthermore, we have the Gauss equation,

$$K = \frac{1}{2}\widehat{\chi} \cdot \underline{\widehat{\chi}} - \frac{1}{4} tr\chi tr\underline{\chi} - \rho. \tag{2.38}$$

Finally, we have

$$\begin{aligned} L(\mu) + tr\chi\mu &= -2(\zeta - \underline{\zeta}) \cdot \nabla\!\!\!/ tr\chi - 2\widehat{\chi} \cdot \big(\nabla\!\!\!/\widehat{\otimes}\zeta + \zeta\widehat{\otimes}\zeta - \delta\widehat{\chi}\big) \\ &\quad -2tr\chi\Big(\rho + \zeta \cdot (2\epsilon - n^{-1}\nabla\!\!\!/ n) - n^{-1}\nabla\!\!\!/ n \cdot \epsilon + |\epsilon|^2 + |\widehat{\chi}|^2 + \widehat{\eta} \cdot \widehat{\chi}\Big). \end{aligned} \tag{2.39}$$

Proof To obtain (2.33), (2.34), (2.36) and (2.37), we compute:

$$\begin{aligned} \nabla\!\!\!/_{\underline{L}}\chi_{AB} &= \underline{L}(\chi_{AB}) - \chi(\nabla\!\!\!/_{\underline{L}} e_A, e_B) - \chi(e_A, \nabla\!\!\!/_{\underline{L}} e_B) \\ &= \mathbf{g}(\mathbf{D}_{\underline{L}}\mathbf{D}_{e_A} L, e_B) - \chi(\nabla\!\!\!/_{\underline{L}} e_A, e_B) + \mathbf{g}(\mathbf{D}_{e_A} L, \mathbf{D}_{\underline{L}} e_B) - \chi(e_A, \nabla\!\!\!/_{\underline{L}} e_B) \\ &= \mathbf{g}(\mathbf{D}_{e_A}\mathbf{D}_{\underline{L}} L, e_B) + \mathbf{g}(\mathbf{D}_{[\underline{L}, e_A]} L, e_B) - \chi(\nabla\!\!\!/_{\underline{L}} e_A, e_B) + \mathbf{R}_{BL\underline{L}A} \\ &\quad + \mathbf{g}(\mathbf{D}_{e_A} L, \mathbf{D}_{\underline{L}} e_B - \nabla\!\!\!/_{\underline{L}} e_B) \\ &= 2\nabla\!\!\!/_A \zeta_B - \mathbf{g}(\mathbf{D}_{\underline{L}} L, \mathbf{D}_{e_A} e_B - \nabla\!\!\!/_{e_A} e_B) + \mathbf{g}(\mathbf{D}_{\mathbf{D}_{\underline{L}} e_A - \nabla\!\!\!/_{\underline{L}} e_A - \mathbf{D}_{e_A}\underline{L}} L, e_B) \\ &\quad + \rho\delta_{AB} - \sigma \in_{AB} + \mathbf{g}(\mathbf{D}_{e_A} L, \mathbf{D}_{\underline{L}} e_B - \nabla\!\!\!/_{\underline{L}} e_B) \end{aligned}$$

which together with the Ricci equations (2.23) yields:

$$\nabla\!\!\!/_{\underline{L}}\chi_{AB} = 2\nabla\!\!\!/_A \zeta_B + \chi_{AB}(\delta + n^{-1}N(n)) + 2\zeta_A\zeta_B - \underline{\chi}_{AC}\chi_{CB} + \rho\delta_{AB} - \sigma \in_{AB}. \tag{2.40}$$

Taking the symmetric part of (2.40), and decomposing into its trace and traceless part yields respectively (2.33) and (2.34). (2.36) follows from (2.33). Finally, taking the antisymmetric part of (2.40) yields (2.37).

We now focus on obtaining (2.39). Differentiating the Raychaudhuri equation (2.28) with respect to $\underline{L}$ yields:

$$\begin{aligned} L(\underline{L}(\mathrm{tr}\chi)) &= [L, \underline{L}]\mathrm{tr}\chi + \underline{L}(L(\mathrm{tr}\chi)) \\ &= \overline{\delta}\underline{L}(\mathrm{tr}\chi) - (\delta + n^{-1}N(n))L(\mathrm{tr}\chi) - 2(\zeta - \underline{\zeta}) \cdot \nabla\!\!\!/\mathrm{tr}\chi - \underline{L}(\mathrm{tr}\chi)\mathrm{tr}\chi - 2\nabla\!\!\!/_{\underline{L}}(\widehat{\chi}) \cdot \widehat{\chi} \\ &\quad -\underline{L}(\overline{\delta})\mathrm{tr}\chi - \overline{\delta}\underline{L}(\mathrm{tr}\chi) \\ &= -(\delta + n^{-1}N(n))L(\mathrm{tr}\chi) - 2(\zeta - \underline{\zeta}) \cdot \nabla\!\!\!/\mathrm{tr}\chi - \underline{L}(\mathrm{tr}\chi)\mathrm{tr}\chi \\ &\quad -2\widehat{\chi} \cdot \left(-\frac{1}{2}\mathrm{tr}\underline{\chi}\widehat{\chi} + \nabla\!\!\!/\widehat{\otimes}\zeta + (\delta + n^{-1}N(n))\widehat{\chi} - \frac{1}{2}\mathrm{tr}\chi\underline{\widehat{\chi}} + \zeta\widehat{\otimes}\zeta\right) - \underline{L}(\overline{\delta})\mathrm{tr}\chi, \end{aligned}$$

where we used (2.34) in the last equality. We infer in view of the definition (2.35) of μ

$$\begin{aligned} L(\mu) + \mathrm{tr}\chi\mu &= L(\underline{L}(\mathrm{tr}\chi)) + \mathrm{tr}\chi\underline{L}(\mathrm{tr}\chi) - L(\delta + n^{1-}N(n))\mathrm{tr}\chi - (\delta + n^{-1}N(n))L(\mathrm{tr}\chi) \\ &\quad -\mathrm{tr}\chi^2(\delta + n^{-1}N(n)) \\ &= -2(\delta + n^{-1}N(n))L(\mathrm{tr}\chi) - 2(\zeta - \underline{\zeta}) \cdot \nabla\!\!\!/\mathrm{tr}\chi \\ &\quad -2\widehat{\chi} \cdot \left(-\frac{1}{2}\mathrm{tr}\underline{\chi}\widehat{\chi} + \nabla\!\!\!/\widehat{\otimes}\zeta + (\delta + n^{-1}N(n))\widehat{\chi} - \frac{1}{2}\mathrm{tr}\chi\underline{\widehat{\chi}} + \zeta\widehat{\otimes}\zeta\right) \\ &\quad -\underline{L}(\overline{\delta})\mathrm{tr}\chi - L(\delta + n^{1-}N(n))\mathrm{tr}\chi - \mathrm{tr}\chi^2(\delta + n^{-1}N(n)). \end{aligned} \tag{2.41}$$

In view of the right-hand side of (2.41), we need to compute $\underline{L}(\overline{\delta})$. We first compute $T(\delta)$. We have:

$$\begin{aligned} T(\delta) &= -\mathbf{g}(\mathbf{D}_T\mathbf{D}_N T, N) - \mathbf{g}(\mathbf{D}_N T, \mathbf{D}_T N) \\ &= -\mathbf{g}(\mathbf{D}_N\mathbf{D}_T T, N) - \mathbf{g}(\mathbf{D}_{[T,N]}T, N) + \mathbf{R}_{NTNT} - \mathbf{g}(\mathbf{D}_N T, \mathbf{D}_T N) \\ &= -N(\mathbf{g}(\mathbf{D}_T T, N)) + \mathbf{g}(\mathbf{D}_T T, \mathbf{D}_N N) - \mathbf{g}(\mathbf{D}_{\mathbf{D}_T N - \mathbf{D}_N T}T, N) + \rho - \mathbf{g}(\mathbf{D}_N T, \mathbf{D}_T N). \end{aligned}$$

Recall from (2.1) and (2.4) that

$$\mathbf{D}_N T = -\epsilon_A e_A - \delta N \text{ and } \mathbf{D}_T T = n^{-1}\nabla n,$$

which yields

$$\begin{aligned} T(\delta) &= -N(n^{-1}N(n)) + \mathbf{g}(n^{-1}\nabla n, \mathbf{D}_N N) - \mathbf{g}(\mathbf{D}_{\mathbf{D}_T N + \epsilon_A e_A + \delta N}T, N) + \rho \\ &\quad -\mathbf{g}(-\epsilon_A e_A - \delta N, \mathbf{D}_T N) \\ &= -n^{-1}N(N(n)) + (n^{-1}N(n))^2 + n^{-1}\nabla\!\!\!/_A n\mathbf{g}(e_A, \mathbf{D}_N N) \\ &\quad -\mathbf{g}(\mathbf{D}_{-\mathbf{g}(\mathbf{D}_T N, T)T + \mathbf{g}(\mathbf{D}_T N, e_A)e_A}T, N) - \epsilon_A\mathbf{g}(\mathbf{D}_A T, N) - \delta\mathbf{g}(\mathbf{D}_N T, N) + \rho \\ &\quad +\epsilon_A\mathbf{g}(e_A, \mathbf{D}_T N) \\ &= -n^{-1}N(N(n)) + (n^{-1}N(n))^2 + n^{-1}\nabla\!\!\!/_A n\mathbf{g}(e_A, \mathbf{D}_N N) \\ &\quad +\mathbf{g}(\mathbf{D}_T N, T)\mathbf{g}(\mathbf{D}_T T, N) - \mathbf{g}(\mathbf{D}_T N, e_A)\mathbf{g}(\mathbf{D}_A T, N) + |\epsilon|^2 + \delta^2 + \rho \\ &\quad +\epsilon_A\mathbf{g}(e_A, \mathbf{D}_T N) \\ &= -n^{-1}N(N(n)) + n^{-1}\nabla\!\!\!/_A n\mathbf{g}(e_A, \mathbf{D}_N N) + |\epsilon|^2 + \delta^2 + \rho + 2\epsilon_A\mathbf{g}(e_A, \mathbf{D}_T N). \end{aligned}$$

In view of the Ricci equations (2.23) and the identities (2.9) and (2.25), we have

$$\mathbf{g}(\mathbf{D}_T N, e_A) = \frac{1}{2}(-\underline{\zeta}_A + \zeta_A - \underline{\xi}_A) = \zeta_A - n^{-1}\nabla\!\!\!/_A n$$

and

$$\mathbf{g}(\mathbf{D}_N N, e_A) = \frac{1}{2}(-\underline{\zeta}_A - \zeta_A + \underline{\xi}_A) = \epsilon_A - \zeta_A$$

from which we infer

$$T(\delta) = -n^{-1}N(N(n)) + \rho + \zeta\cdot(2\epsilon - n^{-1}\nabla\!\!\!/ n) - n^{-1}\nabla\!\!\!/ n\cdot\epsilon + |\epsilon|^2 + \delta^2. \tag{2.42}$$

Now, since $L = T + N$, $\underline{L} = T - N$ and $\overline{\delta} = \delta - n^{-1}N(n)$, we have:

$$T(\delta) = \frac{1}{2}\underline{L}(\overline{\delta}) + \frac{1}{2}L(\delta + n^{-1}N(n)) - n^{-1}N(N(n)) + (n^{-1}N(n))^2,$$

which together with (2.42) yields:

$$\begin{aligned} \underline{L}(\overline{\delta}) &= 2T(\delta) - L(\delta + n^{-1}N(n)) + 2n^{-1}N(N(n)) - 2(n^{-1}N(n))^2 \qquad (2.43) \\ &= -L(\delta + n^{-1}N(n)) + 2\rho + 2\zeta\cdot(2\epsilon - n^{-1}\nabla\!\!\!/ n) - 2n^{-1}\nabla\!\!\!/ n\cdot\epsilon + 2|\epsilon|^2 + 2\delta^2 \\ &\quad -2(n^{-1}N(n))^2. \end{aligned}$$

Plugging (2.43) in (2.41), we obtain

$$\begin{aligned} L(\mu)+\mathrm{tr}\chi\mu &= -2(\delta+n^{-1}N(n))L(\mathrm{tr}\chi)-2(\zeta-\underline{\zeta})\cdot\nabla\!\!\!/\mathrm{tr}\chi \\ &\quad -2\widehat{\chi}\cdot\Big(-\frac{1}{2}\mathrm{tr}\underline{\chi}\widehat{\chi}+\nabla\!\!\!/\widehat{\otimes}\zeta+(\delta+n^{-1}N(n))\widehat{\chi}-\frac{1}{2}\mathrm{tr}\chi\underline{\widehat{\chi}}+\zeta\widehat{\otimes}\zeta\Big) \\ &\quad -\mathrm{tr}\chi\Big(2\rho+2\zeta\cdot(2\epsilon-2n^{-1}\nabla\!\!\!/ n)-2n^{-1}\nabla\!\!\!/ n\cdot\epsilon+2|\epsilon|^2+2\delta^2 \\ &\quad -2(n^{-1}N(n))^2\Big)-\mathrm{tr}\chi^2(\delta+n^{-1}N(n)). \end{aligned}$$

In view of the Raychaudhuri equation (2.28) and the first identity in (2.25), we infer

$$\begin{aligned} L(\mu)+\mathrm{tr}\chi\mu &= -2(\zeta-\underline{\zeta})\cdot\nabla\!\!\!/\mathrm{tr}\chi-2\widehat{\chi}\cdot\big(\nabla\!\!\!/\widehat{\otimes}\zeta+\zeta\widehat{\otimes}\zeta+\mathrm{tr}\chi\widehat{\chi}-\delta\widehat{\chi}+\mathrm{tr}\chi\widehat{\eta}\big) \\ &\quad -\mathrm{tr}\chi\Big(2\rho+2\zeta\cdot(2\epsilon-n^{-1}\nabla\!\!\!/ n)-2n^{-1}\nabla\!\!\!/ n\cdot\epsilon+2|\epsilon|^2+2\delta^2 \\ &\quad -2(n^{-1}N(n))^2-2\overline{\delta}(\delta+n^{-1}N(n))\Big) \end{aligned}$$

from which we derive the desired transport equation (2.39). ■

2.4 Commutation formulas

We have the following four useful commutation formulas (see [4] p. 159):

Lemma 2.14 *Let $\Pi_{\underline{A}}$ be an m-covariant tensor tangent to the surfaces $P_{t,u}$. Then,*

$$\begin{aligned} \nabla\!\!\!/_B\nabla\!\!\!/_4\Pi_{\underline{A}}-\nabla\!\!\!/_4\nabla\!\!\!/_B\Pi_{\underline{A}} &= \chi_{BC}\nabla\!\!\!/_C\Pi_{\underline{A}}-n^{-1}\nabla\!\!\!/_B n\nabla\!\!\!/_4\Pi_{\underline{A}} \\ &+ \sum_i(\chi_{A_iB}\overline{\epsilon}_C-\chi_{BC}\overline{\epsilon}_{A_i}-\in_{A_iC}{}^*\beta_B)\Pi_{A_1..\check{C}..A_m}, \end{aligned} \tag{2.44}$$

$$\begin{aligned} \nabla\!\!\!/_B\nabla\!\!\!/_3\Pi_{\underline{A}}-\nabla\!\!\!/_3\nabla\!\!\!/_B\Pi_{\underline{A}} &= \underline{\chi}_{BC}\nabla\!\!\!/_C\Pi_{\underline{A}}-\underline{\xi}_B\nabla\!\!\!/_4\Pi_{\underline{A}}-b^{-1}\nabla\!\!\!/_B b\nabla\!\!\!/_3\Pi_{\underline{A}} \\ &+ \sum_i(-\chi_{A_iB}\underline{\xi}_C+\chi_{BC}\underline{\xi}_{A_i}-\underline{\chi}_{A_iB}\zeta_C+\underline{\chi}_{BC}\zeta_{A_i} \\ &\quad +\in_{A_iC}{}^*\underline{\beta}_B)\Pi_{A_1..\check{C}..A_m}, \end{aligned} \tag{2.45}$$

$$\begin{aligned} \nabla\!\!\!/_3\nabla\!\!\!/_4\Pi_{\underline{A}}-\nabla\!\!\!/_4\nabla\!\!\!/_3\Pi_{\underline{A}} &= -\overline{\delta}\nabla\!\!\!/_3\Pi_{\underline{A}}+(\delta+n^{-1}N(n))\nabla\!\!\!/_4\Pi_{\underline{A}}+2(\zeta_B-\underline{\zeta}_B)\nabla\!\!\!/_B\Pi_{\underline{A}} \\ &+ 2\sum_i(\underline{\zeta}_{A_i}\zeta_C-\underline{\zeta}_C\zeta_{A_i}+\in_{A_iC}\sigma)\Pi_{A_1..\check{C}..A_m}. \end{aligned} \tag{2.46}$$

Finally, (2.44), (2.45) *together with the fact that $N=\frac{1}{2}(L-\underline{L})$ yield:*

$$\begin{aligned} \nabla\!\!\!/_B\nabla\!\!\!/_N\Pi_{\underline{A}}-\nabla\!\!\!/_N\nabla\!\!\!/_B\Pi_{\underline{A}} &= (\chi_{BC}+k_{BC})\nabla\!\!\!/_C\Pi_{\underline{A}}-b^{-1}\nabla\!\!\!/_B b\nabla\!\!\!/_N\Pi_{\underline{A}} \\ &+ \frac{1}{2}\sum_i(\chi_{A_iB}(\overline{\epsilon}_C+\underline{\xi}_C)-\chi_{BC}(\overline{\epsilon}_{A_i}+\underline{\xi}_{A_i})+\underline{\chi}_{A_iB}\zeta_C-\underline{\chi}_{BC}\zeta_{A_i} \\ &\quad -\in_{A_iC}{}^*(\beta_B+\underline{\beta}_B))\Pi_{A_1..\check{C}..A_m}. \end{aligned} \tag{2.47}$$

For some applications we have in mind, we would like to get rid of the term containing a $\nabla\!\!\!/_4$ derivative in the RHS of (2.44). This is achieved by considering the commutator $[\nabla\!\!\!/, \nabla\!\!\!/_{nL}]$ instead of $[\nabla\!\!\!/, \nabla\!\!\!/_4]$:

$$\begin{aligned}\nabla\!\!\!/_B\nabla\!\!\!/_{nL}\Pi_{\underline{A}} - \nabla\!\!\!/_{nL}\nabla\!\!\!/_B\Pi_{\underline{A}} &= n\chi_{BC}\nabla\!\!\!/_C\Pi_{\underline{A}} \\ &+ \sum_i (n\chi_{A_iB}\overline{\epsilon}_C - n\chi_{BC}\overline{\epsilon}_{A_i} - \in_{A_iC} n^*\beta_B)\Pi_{A_1..\check{C}..A_m}.\end{aligned} \tag{2.48}$$

(2.48) yields for any scalar function f:

$$[nL, \Delta\!\!\!/]f = -2n\chi\nabla\!\!\!/^2 f + n(2\widehat{\chi}_{AB}\overline{\epsilon}_B - \overline{\epsilon}_A \mathrm{tr}\chi - n^{-1}\nabla\!\!\!/_A n\mathrm{tr}\chi + \nabla\!\!\!/\mathrm{tr}\chi)\nabla\!\!\!/_A f. \tag{2.49}$$

Also, we would like to get rid of the term containing a $\nabla\!\!\!/_N$ derivative in the RHS of (2.47). This is achieved by considering the commutator $[\nabla\!\!\!/, \nabla\!\!\!/_{bN}]$ instead of $[\nabla\!\!\!/, \nabla\!\!\!/_N]$:

$$\begin{aligned}\nabla\!\!\!/_B\nabla\!\!\!/_{bN}\Pi_{\underline{A}} - \nabla\!\!\!/_{bN}\nabla\!\!\!/_B\Pi_{\underline{A}} &= b(\chi_{BC} + k_{BC})\nabla\!\!\!/_C\Pi_{\underline{A}} \\ &+ \frac{b}{2}\sum_i (\chi_{A_iB}(\overline{\epsilon}_C + \underline{\xi}_C) - \chi_{BC}(\overline{\epsilon}_{A_iN} + \underline{\xi}_{A_i}) + \underline{\chi}_{A_iB}\zeta_C - \underline{\chi}_{BC}\zeta_{A_i} \\ &- \in_{A_iC} {}^*(\beta_B + \underline{\beta}_B))\Pi_{A_1..\check{C}..A_m}.\end{aligned} \tag{2.50}$$

2.5 Bianchi identities

In view of the formulas on p. 161 of [4], the Bianchi equations for $\alpha, \beta, \rho, \sigma, \underline{\beta}$ are:

$$\nabla\!\!\!/_L\beta + 2\mathrm{tr}\chi\beta = \mathrm{div}\!\!\!/\alpha - \overline{\delta}\beta + (2\epsilon - \overline{\epsilon})\cdot\alpha, \tag{2.51}$$

$$\begin{aligned}\nabla\!\!\!/_{\underline{L}}\beta + \mathrm{tr}\underline{\chi}\beta &= \nabla\!\!\!/\rho + (\nabla\!\!\!/\sigma)^* + 2\widehat{\chi}\cdot\underline{\beta} + (\delta + n^{-1}N(n))\beta + \underline{\xi}\cdot\alpha \\ &+ 3(\zeta\rho + {}^*\zeta\sigma),\end{aligned} \tag{2.52}$$

$$L(\rho) + \frac{3}{2}\mathrm{tr}\chi\rho = \mathrm{div}\!\!\!/\beta - \frac{1}{2}\widehat{\chi}\cdot\alpha + (\epsilon - 2\overline{\epsilon})\cdot\beta, \tag{2.53}$$

$$\underline{L}(\rho) + \frac{3}{2}\mathrm{tr}\underline{\chi}\rho = -\mathrm{div}\!\!\!/\underline{\beta} - \frac{1}{2}\widehat{\chi}\cdot\underline{\alpha} + 2\underline{\xi}\cdot\beta + (\epsilon - 2\zeta)\cdot\underline{\beta}, \tag{2.54}$$

$$L(\sigma) + \frac{3}{2}\mathrm{tr}\chi\sigma = -\mathrm{curl}\!\!\!/\beta + \frac{1}{2}\widehat{\chi}\,{}^*\alpha + (-\epsilon + 2\overline{\epsilon})\,{}^*\beta, \tag{2.55}$$

$$\underline{L}(\sigma) + \frac{3}{2}\mathrm{tr}\underline{\chi}\sigma = -\mathrm{curl}\!\!\!/\underline{\beta} - \frac{1}{2}\widehat{\chi}\,{}^*\underline{\alpha} - 2\underline{\xi}\,{}^*\beta + (\epsilon - 2\zeta)\,{}^*\underline{\beta}, \tag{2.56}$$

$$\nabla\!\!\!/_L\underline{\beta} + \mathrm{tr}\chi\underline{\beta} = -\nabla\!\!\!/\rho + (\nabla\!\!\!/\sigma)^* + 2\underline{\widehat{\chi}}\cdot\beta + \overline{\delta}\underline{\beta} - 3(\underline{\zeta}\rho - {}^*\underline{\zeta}\sigma). \tag{2.57}$$

2.6 Assumptions on R and $u_{|_{\Sigma_0}}$

2.6.1 Assumptions on R

We introduce the L^2 curvature flux $\mathcal{R}$ relative to the time foliation:

$$\mathcal{R} = \left(\|\alpha\|^2_{L^2(\mathcal{H}_u)} + \|\beta\|^2_{L^2(\mathcal{H}_u)} + \|\rho\|^2_{L^2(\mathcal{H}_u)} + \|\sigma\|^2_{L^2(\mathcal{H}_u)} + \|\underline{\beta}\|^2_{L^2(\mathcal{H}_u)}\right)^{\frac{1}{2}}. \tag{2.58}$$

In view of the statement of Theorem 1.1, the goal of this paper is to control the geometry of the null hypersurfaces $\mathcal{H}_u$ of u up to time $t = 1$ when only assuming smallness on

$\|\mathbf{R}\|_{L^\infty_t L^2(\Sigma_t)}$ and $\mathcal{R}$. In the rest of the paper, we still denote by $\mathcal{H}_u$ the portion of the hypersurface of u between $t = 0$ and $t = 1$, and we assume for some small $\varepsilon > 0$:

$$\|\mathbf{R}\|_{L^\infty_t L^2(\Sigma_t)} \leq \varepsilon \text{ and } \sup_{\omega,u} \mathcal{R} \leq \varepsilon, \tag{2.59}$$

where the supremum is taken over all possible values $u \in \mathbb{R}$ of $u(t, x, \omega)$ and over all possible ω in $\mathbb{S}^2$, with u solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ on $\mathcal{M}$, and depending on a extra parameter $\omega \in \mathbb{S}^2$. Note that (2.59) corresponds to a bootstrap assumption[13] in the proof of Theorem 1.1 in [17] under which steps **C3** and **C4** must be achieved[14]. We refer to section 5.3 in [17] for the bootstrap assumption corresponding to (2.59) in the proof of the bounded L^2 curvature conjecture.

Remark 2.15 *Note that in* (2.59)*, all components of* $\mathbf{R}$ *are controlled in* $L^\infty_t L^2(\Sigma_t)$*, while all components but* $\underline{\alpha}$ *are controlled in* $L^\infty_u L^2(\mathcal{H}_u)$*. Thus, it will be crucial to avoid* $\underline{\alpha}$ *in our estimates in order to obtain suitable control on* $\mathcal{H}_u$*. This will be possible due to the specific form of the null structure equations of the u-foliation on* $\mathcal{M}$ *(see section 2.3)*[15]*.*

Remark 2.16 *As a byproduct of the reduction to small initial data outlined in Remark 1.4 and performed in section 2.3 of [17], we may choose* (Σ_0, g, k) *to be smooth, small and asymptotically flat outside a compact set* U *of* Σ_0 *of diameter of order 1 (see section 2.3 in [17] for details). In turn, using the finite speed of propagation, we may assume that* $(\mathcal{M}, \mathbf{g})$ *to be smooth, small and asymptotically flat outside of compact set* $\widetilde{U}$ *of* $\mathcal{M} \cap \{0 \leq t \leq 1\}$ *of diameter of order 1. This allows us to avoid issues about decay at infinity, and to solely concentrate on establishing regularity of the u-foliation on the compact set* $\widetilde{U}$ *of* $\mathcal{M} \cap \{0 \leq t \leq 1\}$*.*

2.6.2 Assumptions on $u_{|\Sigma_0}$

Recall that u is a solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ on $\mathcal{M}$ depending on a extra parameter $\omega \in \mathbb{S}^2$. Now, for u to be uniquely defined, we need to prescribe it on Σ_0 (i.e. at $t = 0$). This issue has been settled in Step **C1** (see [21]). In that step, the choice of $u(0, x, \omega)$ is such that $u(0, x, \omega)$ has enough regularity[16] to achieve step **C2**. At the same time, it is also such that u is regular enough for $t > 0$ to achieve step **C3**. More precisely, the regularity of u for $t > 0$ will involve transport equations - see for instance

[13] There should be a large enough universal bootstrap constant in front of ε in the right-hand side of (2.59) - see (5.3) and (5.4) in [17] - which we omit for the simplicity of the exposition. Note also that (5.4) in [17] holds for any suitable null hypersurface, and in particular for any $\mathcal{H}_u$ and for any $\omega \in \mathbb{S}^2$ which is is consistent with taking the supremum in (u, ω) in the second assumption of (2.59).

[14] Recall that step **C3** corresponds to the control of the u-foliation on $\mathcal{M}$, while step **C4** corresponds to the control of the error term (1.5).

[15] The only exception is the transport equation (5.105) satisfied by $\underline{LL}(b)$ which contains an $\underline{\alpha}$ term, and leads to the weak estimate (2.75).

[16] Notice that in order to study the regularity of $u(0, x, \omega)$ in [21], we do not differentiate $u(0, x, \omega)$ directly. Instead, we proceed as in this paper, i.e. we consider in [21] the regularity of the Ricci coefficients associated to a frame adapted to the $u(0, ., \omega)$-foliation of Σ_0 for $\omega \in \mathbb{S}^2$.

Proposition 2.10 - and will therefore require the same regularity at $t = 0$. We denote this regularity at $t = 0$ by the quantities $\mathcal{I}_0$ and $\mathcal{I}_{0,j}$, $j \in \mathbb{N}$, which are defined by

$$\begin{aligned}\mathcal{I}_0 &= \|b(0,.)-1\|_{L^\infty(\Sigma_0)} + \|b(0,.)-1\|_{L^\infty_u L^2(P_{0,u})} + \|\nabla b(0,.)\|_{L^\infty_u L^2(P_{0,u})} + \|\nabla b(0,.)\|_{L^\infty_u L^4(P_{0,u})} \\ &+\|\mathrm{tr}\chi(0,.)\|_{L^\infty(\Sigma_0)} + \|\mathrm{tr}\chi(0,.)\|_{L^\infty_u L^2(P_{0,u})} + \|\nabla \mathrm{tr}\chi(0,.)\|_{L^\infty_u L^2(P_{0,u})} \\ &+\sup_u \sum_j \Big(\|P_j \nabla\!\!\!/ \mathrm{tr}\chi(0,.)\|_{L^2(P_{0,u})} + \|P_j \nabla\!\!\!/ b(0,.)\|_{L^2(P_{0,u})} + \|P_j \mu(0,.)\|_{L^2(P_{0,u})} \Big) \\ &+\|\nabla \partial_\omega N(0,.)\|_{L^\infty_u L^2(P_{0,u})} + \|\partial_\omega b(0,.)\|_{L^\infty(\Sigma_0)} + \|\nabla\!\!\!/ \partial_\omega b(0,.)\|_{L^\infty_u L^2(P_{0,u})} \\ &+\|\partial_\omega \chi(0,.)\|_{L^\infty_u L^2(P_{0,u})} + \|\partial_\omega \zeta(0,.)\|_{L^\infty_u L^2(P_{0,u})} + \|\partial^2_\omega b(0,.)\|_{L^\infty_u L^2(P_{0,u})}, \end{aligned} \tag{2.60}$$

and

$$\begin{aligned}\mathcal{I}_{0,j} &= \|P_j(NN(\mathrm{tr}\chi))(0,.)\|_{L^2(\Sigma_0)} + \|P_j(NN(b))(0,.)\|_{L^2(\Sigma_0)} \\ &+\|P_j(\nabla\!\!\!/_N \Pi(\partial_\omega \chi))(0,.)\|_{L^2(\Sigma_0)} + \|P_j(N(\partial_\omega b))(0,.)\|_{L^2(\Sigma_0)} \\ &+\|P_j(\nabla\!\!\!/_N(\Pi(\partial^2_\omega N)))(0,.)\|_{L^2(\Sigma_0)} + \|P_j(\Pi(\partial^2_\omega \zeta))(0,.)\|_{L^2(\Sigma_0)}, \end{aligned} \tag{2.61}$$

where P_j denotes the geometric Littlewood-Paley projections P_j which have been constructed in [10] using the heat flow on the surfaces $P_{0,u}$ (see section 3.2), and where the projection Π of vectorfields X on $\mathcal{M}$ onto vectorfields tangent to $P_{t,u}$ is defined by

$$\Pi X = X - \mathbf{g}(X, N)N + \mathbf{g}(X, T)T$$

and extended to covariant tensors by duality. This regularity $\mathcal{I}_0$ and $\mathcal{I}_{0,j}$ required for $u(0, x, \omega)$ is consistent with the estimates derived in step **C1**, where the following estimate for the initial data quantities $\mathcal{I}_0$ and $\mathcal{I}_{0,j}$ has been derived under the curvature bound assumption (2.59) (see [21]):

$$\mathcal{I}_0 \lesssim \varepsilon, \tag{2.62}$$

and

$$\mathcal{I}_{0,j} \lesssim \varepsilon 2^{\frac{j}{2}}, \ \forall j \geq 0 \tag{2.63}$$

as well as the additional bounds

$$\|\partial_\omega N(0,.)\|_{L^\infty(\Sigma_0)} + \|\partial^2_\omega N(0,.)\|_{L^\infty_u L^2(P_{0,u})} \lesssim 1. \tag{2.64}$$

From now on, we assume that u is the solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ on $\mathcal{M}$ which is prescribed on Σ_0 as in step **C1**, and such that it satisfies on Σ_0 the smallness assumption (2.62) and (2.63) as well as the estimate (2.64).

2.7 Main results

We define some norms on $\mathcal{H}_u$. For any $1 \leq p \leq +\infty$ and for any $P_{t,u}$-tangent tensor F on $\mathcal{H}_u$, we have:

$$\|F\|_{L^p(\mathcal{H}_u)} = \left(\int_0^1 dt \int_{P_{t,u}} |F|^p d\mu_{t,u} \right)^{\frac{1}{p}},$$

where $d\mu_{t,u}$ denotes the area element of $P_{t,u}$ and where $|F|^2 = F^{AB}F_{AB}$ for a $P_{t,u}$-tangent tensor F. We also introduce the following norms:

$$\mathcal{N}_1(F) = \|F\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/ F\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/_L F\|_{L^2(\mathcal{H}_u)},$$

$$\mathcal{N}_2(F) = \mathcal{N}_1(F) + \|\nabla\!\!\!/^2 F\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/\nabla\!\!\!/_L F\|_{L^2(\mathcal{H}_u)}.$$

Let x' a coordinate system on $P_{0,u}$. By transporting this coordinate system along the null geodesics generated by L, we obtain a coordinate system (t, x') of $\mathcal{H}_u$. We define the following norms:

$$\|F\|_{L^\infty_{x'}L^2_t} = \sup_{x'\in P_{0,u}} \left(\int_0^1 |F(t,x')|^2 dt\right)^{\frac{1}{2}},$$

$$\|F\|_{L^2_{x'}L^\infty_t} = \left\| \sup_{0\leq t\leq 1} |F(t,x'))| \right\|_{L^2(P_{0,u})}.$$

Remark 2.17 *In the rest of the paper, all inequalities hold for any $\omega \in \mathbb{S}^2$ with the constant in the right-hand side being independent of ω. Thus, one may take the supremum in ω everywhere. To ease the notations, we do not explicitly write down this supremum.*

Remark 2.18 *Let a function f depending on $u \in \mathbb{R}$. In the rest of the paper, all estimates on $\mathcal{H}_u$ will be either of the following types*

$$|f(u)| \lesssim \varepsilon, \tag{2.65}$$

or

$$|f(u)| \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u), \tag{2.66}$$

where γ is a function of $L^2(\mathbb{R})$ with compact support satisfying $\|\gamma\|_{L^2(\mathbb{R})} \leq 1$. For instance, the inequalities (2.67)-(2.72)*,* (2.76)*,* (2.77)*, and* (2.80)-(2.86) *below are of the first type, while the inequalities* (2.73)-(2.75)*,* (2.78) *and* (2.88) *below are of the second type. All inequalities of the first type hold for any u with the constant in the right-hand side being independent of u. Thus, one may take the supremum in u in these inequalities. To ease the notations, we do not explicitly write down the supremum in u for all estimates of the type* (2.65)*.*

Remark 2.19 *The contribution $2^{\frac{j}{2}}\gamma(u)$ to* (2.66) *will always come from the initial data term of a transport equation estimate which is controlled using* (2.63)*. In the particular case of the estimate* (2.75) *below, it will also come from the presence of an term involving $\underline{\alpha}$ in the transport equation satisfied by $\underline{LL}(b)$ (see* (5.105)*).*

Remark 2.20 *In the right-hand side of* (2.66)*, u is used as a coordinate. Note that the precise asymptotic of u at infinity is irrelevant since γ is assumed to be compactly supported. In turn, the fact that γ can be chosen so follows immediately from Remark 2.16.*

The existence of a solution u to the eikonal equation initialized on a timelike hypersurface is classical, see for instance the beginning of Chapter 9 of [4]. The following theorem investigates the regularity of u with respect to (t, x):

Theorem 2.21 *Assume that u is the solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ on $\mathcal{M}$ such that u is prescribed on Σ_0 as in section 2.6.2 where it satisfies in particular* (2.62)*. Assume also that the estimate* (2.59) *is satisfied. Then, null geodesics generating $\mathcal{H}_u$ do not have conjugate points (i.e. there are no caustics) and distinct null geodesics do not intersect. Furthermore, the following estimates are satisfied:*

$$\|n-1\|_{L^\infty(\mathcal{H}_u)} + \|\nabla n\|_{L^\infty(\mathcal{H}_u)} + \|\nabla n\|_{L^\infty_{t}L^2_{x'}} + \|\nabla^2 n\|_{L^\infty_{t}L^2_{x'}} + \|\nabla T(n)\|_{L^\infty_{t}L^2_{x'}} \lesssim \varepsilon, \quad (2.67)$$

$$\mathcal{N}_1(k) + \|\nabla\!\!\!/_{\underline{L}}\epsilon\|_{L^2(\mathcal{H}_u)} + \|\underline{L}(\delta)\|_{L^2(\mathcal{H}_u)} + \|\overline{\epsilon}\|_{L^\infty_{x'}L^2_t} + \|\overline{\delta}\|_{L^\infty_{x'}L^2_t} \lesssim \varepsilon, \quad (2.68)$$

$$\|b-1\|_{L^\infty(\mathcal{H}_u)} + \mathcal{N}_2(b-1) + \|\underline{L}(b)\|_{L^2_{x'}L^\infty_t} + \|\underline{L}(b)\|_{L^\infty_t L^4_{x'}} \lesssim \varepsilon, \quad (2.69)$$

$$\|tr\chi\|_{L^\infty(\mathcal{H}_u)} + \|tr\chi\|_{L^2_{x'}L^\infty_t} + \|\nabla\!\!\!/ tr\chi\|_{L^2_{x'}L^\infty_t} + \|\underline{L}tr\chi\|_{L^2_{x'}L^\infty_t} \lesssim \varepsilon, \quad (2.70)$$

$$\|\widehat{\chi}\|_{L^\infty_{x'}L^2_t} + \mathcal{N}_1(\widehat{\chi}) + \|\nabla\!\!\!/_{\underline{L}}\widehat{\chi}\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon, \quad (2.71)$$

$$\|\zeta\|_{L^2_{x'}L^\infty_t} + \mathcal{N}_1(\zeta) \lesssim \varepsilon. \quad (2.72)$$

Remark 2.22 *Recall that we work in a time foliation under the assumption* (2.59)*. Under the same assumption, in the case of a geodesic foliation, the estimates* (2.70) (2.71) (2.72) *have been obtained in [14] on a truncated null cone (see also [28] for the case of a non truncated null cone). These estimates have also been obtained in the case of a time foliation in [29], albeit under an additional assumption of a priori control on k and $\nabla(\log(n))$. Finally, recall from Remark 2.1 that the null hypersurfaces $\mathcal{H}_u$ are topologically 3 dimensional hyperplanes. This allows us to avoid issues in [28] concerning the vertex of a null cone.*

We introduce the family of intrinsic Littlewood-Paley projections P_j which have been constructed in [10] using the heat flow on the surfaces $P_{t,u}$ (see section 3.2). This allows us to state our second theorem which investigates the regularity of $\underline{L}\underline{L}\mathrm{tr}\chi$, $\nabla\!\!\!/_{\underline{L}}\zeta$ and $\underline{L}\underline{L}b$.

Theorem 2.23 *Assume that u is the solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ on $\mathcal{M}$ such that u is prescribed on Σ_0 as in section 2.6.2 where it satisfies in particular* (2.62) *and* (2.63)*. Assume also that the estimate* (2.59) *is satisfied. Then, there exists a function γ in $L^2(\mathbb{R})$ with compact support satisfying $\|\gamma\|_{L^2(\mathbb{R})} \leq 1$, such that for all $j \geq 0$, we have:*

$$\|P_j\underline{L}\underline{L}tr\chi\|_{L^2(\mathcal{H}_u)} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u), \quad (2.73)$$

$$\|P_j\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon + 2^{-\frac{j}{2}}\varepsilon\gamma(u), \quad (2.74)$$

and

$$\|P_j\underline{L}\underline{L}b\|_{L^\infty_t L^2_{x'}} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u). \quad (2.75)$$

The following theorem investigates the regularity with respect to the parameter $\omega \in \mathbb{S}^2$.

Theorem 2.24 *Assume that u is the solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ on $\mathcal{M}$ such that u is prescribed on Σ_0 as in section 2.6.2 where it satisfies in particular* (2.62)*,* (2.63) *and* (2.64)*. Assume also that the estimate* (2.59) *is satisfied. Then, we have the following estimates:*

$$\|\partial_\omega N\|_{L^\infty(\mathcal{H}_u)} \lesssim 1, \quad (2.76)$$

$$\|\mathbf{D}\partial_\omega N\|_{L^2_{x'}L^\infty_t} + \|\partial_\omega b\|_{L^\infty(\mathcal{H}_u)} + \|\nabla\!\!\!/\partial_\omega b\|_{L^2_{x'}L^\infty_t} + \|\partial_\omega \chi\|_{L^2_{x'}L^\infty_t} + \|\partial_\omega \zeta\|_{L^2_{x'}L^\infty_t} \lesssim \varepsilon, \quad (2.77)$$

and

$$\|P_j \nabla\!\!\!/_{\underline{L}} \Pi(\partial_\omega \chi)\|_{L^p_t L^2_{x'}} \lesssim 2^j \varepsilon + 2^{\frac{j}{2}} \varepsilon \gamma(u), \quad (2.78)$$

where p is any real number such that $2 \leq p < +\infty$, and where γ is a function of $L^2(\mathbb{R})$ with compact support satisfying $\|\gamma\|_{L^2(\mathbb{R})} \leq 1$.

Also, there exists χ_1 and χ_2 two symmetric traceless $P_{t,u}$-tangent 2-tensors such that we have the following decomposition for $\widehat{\chi}$:

$$\widehat{\chi} = \chi_1 + \chi_2, \quad (2.79)$$

where χ_1 and χ_2 satisfy:

$$\|\partial_\omega \chi_1\|_{L^\infty_t L^2_{x'}} + \mathcal{N}_1(\chi_1) + \mathcal{N}_1(\chi_2) + \|\nabla\!\!\!/_{\underline{L}} \chi_2\|_{L^2(\mathcal{H}_u)} + \|\chi_2\|_{L^\infty_{x'} L^2_t} + \|\partial_\omega \chi_2\|_{L^\infty_t L^2_{x'}} \lesssim \varepsilon \quad (2.80)$$

and for any $2 \leq p < +\infty$, we have:

$$\|\nabla\!\!\!/ \chi_1\|_{L^\infty_t L^2_{x'}} + \|\chi_1\|_{L^p_t L^\infty_{x'}} + \|\partial_\omega \chi_2\|_{L^p_t L^{4-}_{x'}} + \|\partial_\omega \chi_2\|_{L^{6-}(\mathcal{H}_u)} + \|\nabla\!\!\!/ \partial_\omega \chi_2\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon. \quad (2.81)$$

Furthermore, for any $2 \leq q < 4$, we have:

$$\|\nabla\!\!\!/_L \chi_1\|_{L^\infty_t L^2_{x'} + L^2_t L^q_{x'}} + \|\nabla\!\!\!/_{\underline{L}} \chi_1\|_{L^\infty_t L^2_{x'} + L^2_t L^q_{x'}} \lesssim \varepsilon. \quad (2.82)$$

Finally, let ω and ω' in $\mathbb{S}^2$. Then, there holds the following lower bound

$$|N(\cdot, \omega) - N(\cdot, \omega')| \gtrsim |\omega - \omega'|. \quad (2.83)$$

Remark 2.25 *Notice from* (2.80) *that χ_1 and χ_2 have at least the same regularity as $\widehat{\chi}$. Now, the point of the decomposition* (2.79) *is that both χ_1 and χ_2 have better regularity properties than $\widehat{\chi}$. Indeed, in view of* (2.81)*, χ_1 has better regularity with respect to (t, x) while χ_2 has better regularity with respect to ω.*

Remark 2.26 *Let ω and ω' in $\mathbb{S}^2$. The estimate* (2.76) *for N yields the following upper bound for $N(\cdot, \omega) - N(\cdot, \omega')$:*

$$|N(\cdot, \omega) - N(\cdot, \omega')| \lesssim |\omega - \omega'|.$$

Note that (2.83) *establishes the corresponding lower bound.*

Finally, the following theorem contains estimates for second order derivatives with respect to ω.

Theorem 2.27 *Assume that u is the solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ on $\mathcal{M}$ such that u is prescribed on Σ_0 as in section 2.6.2 where it satisfies in particular* (2.62)*,* (2.63) *and* (2.64)*. Assume also that the estimate* (2.59) *is satisfied. Then, we have the following estimates:*

$$\|\partial^2_\omega N\|_{L^2_{x'}L^\infty_t} \lesssim 1, \quad (2.84)$$

$$\|\nabla\!\!\!/_L \partial_\omega^2 N\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{2.85}$$

$$\|\partial_\omega^2 b\|_{L^\infty_{\underline{t}} L^2_{x'}} \lesssim \varepsilon, \tag{2.86}$$

$$\|P_j \Pi(\partial_\omega^2 \chi)\|_{L^\infty_{\underline{t}} L^2_{x'}} \lesssim 2^j \varepsilon, \tag{2.87}$$

and

$$\|P_j \nabla\!\!\!/_{\underline{L}} \Pi(\partial_\omega^2 N)\|_{L^p_{\underline{t}} L^2_{x'}} + \|P_j \Pi(\partial_\omega^2 \zeta)\|_{L^p_{\underline{t}} L^2_{x'}} \lesssim 2^j \varepsilon + 2^{\frac{j}{2}} \varepsilon \gamma(u), \tag{2.88}$$

where p is any real number such that $2 \leq p < +\infty$, and where γ is a function of $L^2(\mathbb{R})$ with compact support satisfying $\|\gamma\|_{L^2(\mathbb{R})} \leq 1$.

Remark 2.28 *Our assumption on curvature* (2.59) *is critical with respect to the control of the Eikonal equation as can be seen throughout the paper where numerous log-losses are barely overcome. In order to prove Theorem 2.21, Theorem 2.23, Theorem 2.24, and Theorem 2.27 we will rely in particular on the null transport equations and the elliptic systems of Hodge type on $P_{t,u}$ of section 2.3, the geometric Littlewood-Paley theory of [10], sharp trace theorems, and an extensive use of the crucial structure of the Bianchi identities* (2.51)-(2.57).

Remark 2.29 *The regularity with respect to (t, x) for u is clearly limited as a consequence of the fact that we only assume L^2 bounds on $\mathbf{R}$. On the other hand, $\mathbf{R}$ is independent of the parameter ω, and one might infer that u is smooth with respect to ω. Surprisingly, this is not at all the case. Indeed, we are even not able to go beyond estimates for the second order derivatives with respect to ω which are given in Theorem 2.27. This is due to the fact that we rely in a fundamental way on the null transport equations of Proposition 2.10. Now, the commutator between L and ∂_ω gives rise to a tangential derivative with respect to $P_{t,u}$ (see* (6.5)*) for which we have less control. This leads to a loss of one derivative for each derivative taken with respect to ω for all quantities estimated through transport equations. This is best seen by comparing the estimate* (2.70) (2.71) *for χ, the estimate* (2.77) *for $\partial_\omega \chi$ and the estimate* (2.87) *for $\partial_\omega^2 \chi$.*

2.8 Dependance of the norm $L^\infty_u L^2(\mathcal{H}_u)$ on $\omega \in \mathbb{S}^2$

Let ω and ω' in $\mathbb{S}^2$ such that

$$|\omega - \omega'| \lesssim 2^{-\frac{j}{2}}.$$

Let $u = u(\cdot, \omega)$ and $u' = u(\cdot, \omega')$. In this section, we compare the norm in $L^\infty_u L^2(\mathcal{H}_u)$ with the norm in $L^\infty_{u'} L^2(\mathcal{H}_{u'})$ for various scalars and tensors, relying on the estimates of the previous section. Let us first stress the difficulty by considering the decomposition for $\mathrm{tr}\chi$ in Proposition 2.32 below. A naive approach consists in writing the following decomposition

$$\mathrm{tr}\chi(t, x, \omega) = \mathrm{tr}\chi(t, x, \omega') + (\mathrm{tr}\chi(t, x, \omega) - \mathrm{tr}\chi(t, x, \omega')) = f_1^j + f_2^j.$$

f_1^j does not depend on ω and satisfies, in view of the estimate (2.70)

$$\|f_1^j\|_{L^\infty} \lesssim \|\mathrm{tr}\chi(\cdot, \omega')\|_{L^\infty} \lesssim \varepsilon.$$

Also, we have (see Remark 2.30 below)

$$f_2^j = (\omega - \omega') \int_0^1 \partial_\omega \mathrm{tr}\chi(t, x, \omega_\sigma) d\sigma, \tag{2.89}$$

which together with the fact that $|\omega - \omega'| \lesssim 2^{-\frac{j}{2}}$ yields

$$\|f_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}} \left\| \int_0^1 \partial_\omega \mathrm{tr}\chi(t, x, \omega_\sigma) d\sigma \right\|_{L_u^\infty L^2(\mathcal{H}_u)}.$$

Unfortunately, we can not obtain the desired estimate for f_2^j since we have $\partial_\omega \mathrm{tr}\chi(\cdot, \omega_\sigma) \in L_{u_\sigma}^\infty L^2(\mathcal{H}_{u_\sigma})$, and $L_u^\infty L^2(\mathcal{H}_u)$ and $L_{u_\sigma}^\infty L^2(\mathcal{H}_{u_\sigma})$ are not directly comparable, where $u_\sigma = u(\cdot, \omega_\sigma)$. Nevertheless, relying on the geometric Littlewood-Paley projections of [10], on well-suited coordinate systems, and on various commutator estimates, we are able to improve on this naive approach in order to obtain the decompositions below.

Remark 2.30 *The identity* (2.89) *relies on a notation that will be used throughout section 8. Let ω and ω' in $\mathbb{S}^2$ and f a function on $\mathbb{S}^2$. Also, let γ the length minimizing geodesic of $\mathbb{S}^2$ between ω and ω' parametrized by an affine parameter σ such that $\gamma(0) = \omega$ and $\gamma(1) = \omega'$. Then, applying the fundamental theorem of calculus to $f \circ \gamma$, we infer*

$$f(\omega') = f(\omega) + \int_0^1 \dot{\gamma}(\sigma) \partial_\omega f(\omega_\sigma) d\sigma$$

where ω_σ denotes the point $\gamma(\sigma) \in \mathbb{S}^2$ on the curve γ. From now on, we rewrite for convenience, by a slight abuse of notation, this identity as

$$f(\omega') = f(\omega) + (\omega - \omega') \int_0^1 \partial_\omega f(\omega_\sigma) d\sigma, \tag{2.90}$$

from which (2.89) *is a particular case. Note that* (2.90) *will always be used when ω and ω' are such that $|\omega - \omega'| \lesssim 2^{-\frac{j}{2}}$, and hence close to each other.*

Proposition 2.31 *Let ω and ω' in $\mathbb{S}^2$ such that $|\omega - \omega'| \lesssim 2^{-\frac{j}{2}}$. Let $N = N(\cdot, \omega)$ and $N' = N(\cdot, \omega')$. For any $j \geq 0$, we have the following decomposition for $N - N'$:*

$$N - N' = (F_1^j + F_2^j)(\omega - \omega')$$

where F_1^j only depends on ω' and satisfies:

$$\|F_1^j\|_{L^\infty} \lesssim 1,$$

and where F_2^j satisfies:

$$\|F_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}}.$$

Proposition 2.32 *Let ω and ω' in $\mathbb{S}^2$ such that $|\omega - \omega'| \lesssim 2^{-\frac{j}{2}}$. For any $j \geq 0$, we have the following decomposition for $tr\chi(\cdot, \omega)$:*

$$tr\chi(\cdot, \omega) = f_1^j + f_2^j$$

where f_1^j does not depend on ω and satisfies:

$$\|f_1^j\|_{L^\infty} \lesssim \varepsilon,$$

and where f_2^j satisfies:

$$\|f_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}}\varepsilon.$$

Proposition 2.33 *Let ω and ω' in $\mathbb{S}^2$. Let $p \in \mathbb{Z}$. For any $j \geq 0$, we have the following estimate for $b^p(\cdot, \omega) - b^p(, \omega')$:*

$$\|b^p(\cdot, \omega) - b^p(\cdot, \omega')\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim |\omega - \omega'|\varepsilon.$$

Lemma 2.34 *Let ω and ω' in $\mathbb{S}^2$. Let χ_2 the symmetric traceless $P_{t,u}$-tangent 2-tensor introduced in* (2.79) *which satisfies the estimates* (2.80) (2.81)*. For any $j \geq 0$, we have the following estimate for $\chi_2(\cdot, \omega) - \chi_2(\cdot, \omega')$:*

$$\|\chi_2(\cdot, \omega) - \chi_2(\cdot, \omega')\|_{L_u^\infty L^{4-}(\mathcal{H}_u)} \lesssim |\omega - \omega'|\varepsilon.$$

Proposition 2.35 *Let ω and ω' in $\mathbb{S}^2$ such that $|\omega - \omega'| \lesssim 2^{-\frac{j}{2}}$. For any $j \geq 0$, we have the following decomposition for $\chi(\cdot, \omega)$ and $\widehat{\chi}(\cdot, \omega)$:*

$$\chi(\cdot, \omega),\, \widehat{\chi}(\cdot, \omega) = \chi_2(\cdot, \omega') + F_1^j + F_2^j$$

where F_1^j does not depend on ω and satisfies for any $2 \leq p < +\infty$:

$$\|F_1^j\|_{L_{u'}^\infty L_t^p L^\infty(P_{t,u'})} \lesssim \varepsilon,$$

and where F_2^j satisfies:

$$\|F_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}}\varepsilon.$$

Proposition 2.36 *Let ω and ω' in $\mathbb{S}^2$ such that $|\omega - \omega'| \lesssim 2^{-\frac{j}{2}}$. For any $j \geq 0$, we have the following decomposition for $\chi(\cdot, \omega)$ and $\widehat{\chi}(\cdot, \omega)$:*

$$\chi(\cdot, \omega),\, \widehat{\chi}(\cdot, \omega) = F_1^j + F_2^j$$

where F_1^j does not depend on ω and satisfies:

$$\|F_1^j\|_{L_{u'}^\infty L^\infty(P_{t,u'})L_t^2} \lesssim \varepsilon,$$

and where F_2^j satisfies:

$$\|F_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}}\varepsilon.$$

Proposition 2.37 *Let ω and ω' in $\mathbb{S}^2$ such that $|\omega - \omega'| \lesssim 2^{-\frac{j}{2}}$. For any $j \geq 0$, we have the following decomposition for $\widehat{\chi}(\cdot, \omega)^2$:*

$$\widehat{\chi}(\cdot,\omega)^2 = \chi_2(\cdot,\omega')^2 + \chi_2(\cdot,\omega')F_1^j + \chi_2(\cdot,\omega')F_2^j + F_3^j + F_4^j + F_5^j$$

where F_1^j and F_3^j do not depend on ω and satisfy:

$$\|F_1^j\|_{L^\infty_{u'}L^2_t L^\infty(P_{t,u'})} + \|F_3^j\|_{L^\infty_{u'}L^2_t L^\infty(P_{t,u'})} \lesssim \varepsilon,$$

where F_2^j and F_4^j satisfy:

$$\|F_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|F_4^j\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}}\varepsilon,$$

and where F_5^j satisfies

$$\|F_5^j\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{-j}.$$

Proposition 2.38 *Let ω and ω' in $\mathbb{S}^2$ such that $|\omega - \omega'| \lesssim 2^{-\frac{j}{2}}$. For any $j \geq 0$, we have the following decomposition for $\widehat{\chi}(\cdot, \omega)^3$:*

$$\begin{aligned}\widehat{\chi}(\cdot,\omega)^3 &= \chi_2(\cdot,\omega')^3 + \chi_2(\cdot,\omega')^2 F_1^j + \chi_2(\cdot,\omega')^2 F_2^j + \chi_2(\cdot,\omega')F_3^j + \chi_2(\cdot,\omega')F_4^j \\ &\quad + \chi_2(\cdot,\omega')F_5^j + F_6^j + F_7^j + F_8^j + F_9^j\end{aligned}$$

where F_1^j, F_3^j and F_6^j do not depend on ω and satisfy:

$$\|F_1^j\|_{L^\infty_{u'}L^2_t L^\infty(P_{t,u'})} + \|F_3^j\|_{L^\infty_{u'}L^2_t L^\infty(P_{t,u'})} + \|F_6^j\|_{L^\infty_{u'}L^2_t L^\infty(P_{t,u'})} \lesssim \varepsilon,$$

where F_2^j, F_4^j and F_7^j satisfy:

$$\|F_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|F_4^j\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|F_7^j\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}}\varepsilon,$$

where F_5^j and F_8^j satisfy

$$\|F_5^j\|_{L^2(\mathcal{M})} + \|F_8^j\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{-j}.$$

and where F_9^j satisfies

$$\|F_9^j\|_{L^{2-}(\mathcal{M})} \lesssim \varepsilon 2^{-\frac{3j}{2}}.$$

Proposition 2.39 *Let ω and ω' in $\mathbb{S}^2$ such that $|\omega - \omega'| \lesssim 2^{-\frac{j}{2}}$. For any $j \geq 0$, we have the following decomposition for $\zeta(\cdot, \omega)$ and $\nabla\!\!\!/ b(\cdot, \omega)$:*

$$\zeta(\cdot,\omega),\, \nabla\!\!\!/ b(\cdot,\omega) = F_1^j + F_2^j$$

where F_1^j does not depend on ω and satisfies for any $2 \leq p < +\infty$:

$$\|F_1^j\|_{L^\infty_{u'}L^2_t L^p(P_{t,u'})} \lesssim \varepsilon,$$

and where F_2^j satisfies:

$$\|F_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{4}}\varepsilon.$$

Proposition 2.40 *Let ω and ω' in $\mathbb{S}^2$ such that $|\omega - \omega'| \lesssim 2^{-\frac{j}{2}}$. For any $j \geq 0$, we have the following decomposition for $b(\cdot, \omega) - b(\cdot, \omega')$:*

$$b(\cdot, \omega) - b(\cdot, \omega') = (f_1^j + f_2^j)(\omega - \omega')$$

where f_1^j does not depend on ω and satisfies:

$$\|f_1^j\|_{L^\infty} \lesssim \varepsilon,$$

and where f_2^j satisfies:

$$\|f_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{4}}\varepsilon.$$

2.9 Additional estimates for trχ

In this section, we state estimates for trχ involving the geometric Littlewood-Paley projections P_j on $P_{t,u}$ constructed in [10], that are not direct consequences of the estimate (2.70) for trχ and basic properties of P_j.

Proposition 2.41 *trχ satisfies the following estimates*

$$\|P_j tr\chi\|_{L^2_{x'}L^\infty_t} \lesssim 2^{-j}\varepsilon, \tag{2.91}$$

and

$$\|P_j(nL tr\chi)\|_{L^2_{x'}L^1_t} \lesssim 2^{-j}\varepsilon. \tag{2.92}$$

Proposition 2.42 *trχ satisfies the following estimates*

$$\|\nabla\!\!\!/ P_{\leq j} tr\chi\|_{L^2_{x'}L^\infty_t} \lesssim \varepsilon, \tag{2.93}$$

and

$$\|\nabla\!\!\!/ P_{\leq j}(nL tr\chi)\|_{L^2_{x'}L^1_t} \lesssim \varepsilon. \tag{2.94}$$

2.10 Organization of the paper

The rest of the paper is as follows.

- In section 3,
  - we derive several embeddings on $P_{t,u}$, $\mathcal{H}_u$ and Σ_t which are compatible with the regularity stated in Theorem 2.21,
  - we also discuss the Littlewood-Paley projections of [10] as well as several elliptic systems of Hodge type on $P_{t,u}$.
- In section 4, we prove Theorem 2.21. More precisely
  - The fact that the null geodesics generating $\mathcal{H}_u$ do not have conjugate points and distinct null geodesics do not intersect is proved in section 4.1.
  - (2.67) is proved in section 4.4.

- (2.68) is proved in section 4.5 except for the $L^\infty_{x'}L^2_t$ estimates which are proved in section 4.7.
- (2.69) is proved in section 4.8.
- (2.70) is proved in section 4.9 except for the bound in L^∞ which is proved in section 4.6.2.
- (2.71) and (2.72) are proved in section 4.9 except for the bounds in $L^\infty_{x'}L^2_t$ which is proved in section 4.6.2.

- In section 5, we prove Theorem 2.23. More precisely
  - (2.74) is proved in section 5.3.
  - (2.73) is proved in section 5.4.
  - (2.75) is proved in section 5.5.

- In section 6, we prove Theorem 2.24. More precisely
  - (2.76) and (2.77) are proved in section 6.2.3.
  - (2.78) is proved in section 6.3.
  - (2.79)-(2.82) are proved in section 6.4.
  - (2.83) is proved in section 6.6.

- In section 7 we prove Theorem 2.27. More precisely
  - (2.84) and (2.85) are proved in section 7.2.1.
  - (2.86) is proved in section 7.2.2.
  - (2.87) is proved in section 7.2.3.
  - (2.88) is proved in sections 7.2.4 and 7.2.5.

- In section 8, we derive the various decompositions of section 2.8. More precisely
  - Proposition 2.31 is proved in section 8.2.1.
  - Proposition 2.32 is proved in section 8.2.2.
  - Proposition 2.33 is proved in section 8.2.3.
  - Lemma 2.34 is proved in section 8.4.1
  - Proposition 2.35 is proved in section 8.4.2
  - Proposition 2.36 is proved in section 8.4.3
  - Proposition 2.37 is proved in section 8.4.6
  - Proposition 2.38 is proved in section 8.4.8
  - Corollary 2.39 is proved in section 8.6.1
  - Corollary 2.40 is proved in section 8.6.2

- In section 9, we prove Proposition 2.41 in section 9.3.1 and Proposition 2.42 in section 9.3.2.

- Finally, technical results of sections 4-9 are proved in Appendices A to E.

3 Calculus inequalities on $P_{t,u}$, $\mathcal{H}_u$ and Σ_t

In this section, we first recall some calculus inequalities from [10] on the 2-surfaces $P_{t,u}$. We then discuss the Littlewood-Paley projections of [10] as well as several elliptic systems of Hodge type on $P_{t,u}$. We establish calculus inequalities on $\mathcal{H}_u$. Finally, we establish calculus inequalities on Σ_t, and we construct geometric Littlewood-Paley projections on Σ_t in the spirit of [10].

3.1 Calculus inequalities on $P_{t,u}$

We denote by γ the metric induced by $\mathbf{g}$ on $P_{t,u}$. A coordinate chart $U \subset P_{t,u}$ with coordinates x^1, x^2 is admissible if, relative to these coordinates, there exists a constant $c > 0$ such that,

$$c^{-1}|\xi|^2 \leq \gamma_{AB}(p)\xi^A\xi^B \leq c|\xi|^2, \qquad \text{uniformly for all } \; p \in U. \tag{3.1}$$

We assume that $P_{t,u}$ can be covered by a finite number of admissible coordinate charts, i.e., charts satisfying the conditions (3.1). Furthermore, we assume that the constant c in (3.1) and the number of charts is independent of t and u.

Remark 3.1 *The existence of a covering of $P_{t,u}$ by coordinate charts satisfying* (3.1) *with a constant $c > 0$ and the number of charts independent of t and u will be shown in section 4.2.1.*

Under these assumptions, the following calculus inequality has been proved in [10]:

Proposition 3.2 *Let f be a real scalar function. Then,*

$$\|f\|_{L^2(P_{t,u})} \lesssim \|\nabla\!\!\!/ f\|_{L^1(P_{t,u})} + \|f\|_{L^1(P_{t,u})}. \tag{3.2}$$

As a corollary of the estimate (3.2), the following Gagliardo-Nirenberg inequality is derived in [10]:

Corollary 3.3 *Given an arbitrary tensorfield F on $P_{t,u}$ and any $2 \leq p < \infty$, we have:*

$$\|F\|_{L^p(P_{t,u})} \lesssim \|\nabla\!\!\!/ F\|^{1-\frac{2}{p}}_{L^2(P_{t,u})}\|F\|^{\frac{2}{p}}_{L^2(P_{t,u})} + \|F\|_{L^2(P_{t,u})}. \tag{3.3}$$

As a corollary of (3.2), it also classical to derive the following inequality (for a proof, see for example [7] page 157):

Corollary 3.4 *For any tensorfield F on $P_{t,u}$ and any $p > 2$:*

$$\|F\|_{L^\infty(P_{t,u})} \lesssim \|\nabla\!\!\!/ F\|_{L^p(P_{t,u})} + \|F\|_{L^p(P_{t,u})}. \tag{3.4}$$

We recall the Bochner identity on $P_{t,u}$ (which has dimension 2). This allows us to control the L^2 norm of the second derivatives of a tensorfield in terms of the L^2 norm of the Laplacian and geometric quantities associated with $P_{t,u}$ (see for example [10] for a proof).

Proposition 3.5 *Let K denote the Gauss curvature of $P_{t,u}$. Then*
i) *For a scalar function f:*

$$\int_{P_{t,u}} |\nabla\!\!\!/^2 f|^2 d\mu_{t,u} = \int_{P_{t,u}} |\Delta\!\!\!/ f|^2 d\mu_{t,u} - \int_{P_{t,u}} K|\nabla\!\!\!/ f|^2 d\mu_{t,u}. \tag{3.5}$$

ii) *For a vectorfield F:*

$$\begin{aligned}\int_{P_{t,u}} |\nabla\!\!\!/^2 F|^2 d\mu_{t,u} &= \int_{P_{t,u}} |\Delta\!\!\!/ F|^2 d\mu_{t,u} - \int_{P_{t,u}} K(2\,|\nabla\!\!\!/ F|^2 - |d\!\!\!/iv F|^2 - |cu\!\!\!/rl F|^2) d\mu_{t,u} \\ &\quad + \int_{P_{t,u}} K^2 |F|^2 d\mu_{t,u},\end{aligned} \tag{3.6}$$

where $d\!\!\!/iv F = \gamma^{AB}\nabla\!\!\!/_B F_A$, $cu\!\!\!/rl F = d\!\!\!/iv(^*F) = \in^{AB}\nabla\!\!\!/_A F_B$.

Using (3.3) and (3.6), the following Bochner inequality is derived in [10] for a tensor F. For all $2 \le p < +\infty$, we have:

$$\begin{aligned}\|\nabla\!\!\!/^2 F\|_{L^2(P_{t,u})} &\lesssim \|\Delta\!\!\!/ F\|_{L^2(P_{t,u})} + (\|K\|_{L^2(P_{t,u})} + \|K\|^{\frac{1}{2}}_{L^2(P_{t,u})})\|\nabla\!\!\!/ F\|_{L^2(P_{t,u})} \\ &\quad + \|K\|^{\frac{p}{p-1}}_{L^2(P_{t,u})}(\|\nabla\!\!\!/ F\|^{\frac{p-2}{p-1}}_{L^2(P_{t,u})}\|F\|^{\frac{1}{p-1}}_{L^2(P_{t,u})} + \|F\|_{L^2(P_{t,u})}).\end{aligned} \tag{3.7}$$

3.2 Geometric Littlewood Paley theory on $P_{t,u}$

We recall the properties of the heat equation for arbitrary tensorfields F on $P_{t,u}$.

$$\partial_\tau U(\tau)F - \Delta\!\!\!/ U(\tau)F = 0,\ U(0)F = F.$$

The following L^2 estimates for the operator $U(\tau)$ are proved in [10].

Proposition 3.6 *We have the following estimates for the operator $U(\tau)$:*

$$\|U(\tau)F\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/ U(\tau')F\|^2_{L^2(P_{t,u})} d\tau' \lesssim \|F\|^2_{L^2(P_{t,u})}, \tag{3.8}$$

$$\|\nabla\!\!\!/ U(\tau)F\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\Delta\!\!\!/ U(\tau')F\|^2_{L^2(P_{t,u})} d\tau' \lesssim \|\nabla\!\!\!/ F\|^2_{L^2(P_{t,u})}, \tag{3.9}$$

$$\tau\|\nabla\!\!\!/ U(\tau)F\|^2_{L^2(P_{t,u})} + \int_0^\tau \tau'\|\Delta\!\!\!/ U(\tau')F\|^2_{L^2(P_{t,u})} d\tau' \lesssim \|F\|^2_{L^2(P_{t,u})}. \tag{3.10}$$

We also introduce the nonhomogeneous heat equation:

$$\partial_\tau V(\tau) - \Delta\!\!\!/ V(\tau) = F(\tau),\ V(0) = 0,$$

for which we easily derive the following estimates:

Proposition 3.7 *We have the following estimates for the operator $V(\tau)$:*

$$\|\nabla\!\!\!/ V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\Delta\!\!\!/ V(\tau')\|^2_{L^2(P_{t,u})} d\tau' \lesssim \int_0^\tau \|F(\tau')\|^2_{L^2(P_{t,u})} d\tau', \tag{3.11}$$

$$\|V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(P_{t,u})} d\tau' \lesssim \int_0^\tau \int_{P_{t,u}} V(\tau') F(\tau') d\mu_{t,u} d\tau'. \tag{3.12}$$

To build a Littlewood-Paley theory on $P_{t,u}$, we will need the following simple lemma (see for example Lemma 5.4 in [10] for a proof).

Lemma 3.8 *There exists a smooth and compactly supported in $(0,1)$ function m such that*[17]

$$\int_0^{+\infty} m(\tau) d\tau = 0 \tag{3.13}$$

and

$$\sum_{k\in\mathbb{Z}} 2^{2k} m(2^{2k}\tau) = \delta(\tau). \tag{3.14}$$

We now recall the definition of the geometric Littlewood-Paley projections P_j constructed in [10]:

Definition 3.9 *Consider the function m constructed in Lemma 3.8. We set, $m_j(\tau) = 2^{2j} m(2^{2j}\tau)$ and define the geometric Littlewood -Paley (LP) projections P_j, for arbitrary tensorfields F on $P_{t,u}$ to be*

$$P_j F = \int_0^\infty m_j(\tau) U(\tau) F d\tau. \tag{3.15}$$

Given an interval $I \subset \mathbb{Z}$ we define

$$P_I = \sum_{j\in I} P_j F.$$

In particular we shall use the notation $P_{<k}, P_{\leq k}, P_{>k}, P_{\geq k}$.

Observe that P_j are selfadjoint, i.e., $P_j = P_j^*$, in the sense,

$$< P_j F, G > = < F, P_j G >,$$

where, for any given m-tensors F, G

$$< F, G > = \int_{P_{t,u}} \gamma^{i_1 j_1} \dots \gamma^{i_m j_m} F_{i_1 \dots i_m} G_{j_1 \dots j_m} d\mu_u$$

[17] In [10], the function m may verify the vanishing of higher order moments in addition to (3.13). Now, (3.13) suffices for the basic properties of LP projections of [10] used in this paper and recalled in this section.

denotes the usual L^2 scalar product. Also, note from (3.14) that the LP-projections associated to m verify:

$$\sum_{j\in\mathbb{Z}} P_j = I. \tag{3.16}$$

The following properties of the LP-projections P_j have been proved in [10]:

Theorem 3.10 *The LP-projections P_j verify the following properties:*
i) L^p-boundedness For any $1 \le p \le \infty$, and any interval $I \subset \mathbb{Z}$,

$$\|P_I F\|_{L^p(P_{t,u})} \lesssim \|F\|_{L^p(P_{t,u})} \tag{3.17}$$

ii) Bessel inequality

$$\sum_j \|P_j F\|^2_{L^2(P_{t,u})} \lesssim \|F\|^2_{L^2(P_{t,u})}$$

iii) Finite band property For any $1 \le p \le \infty$.

$$\begin{array}{lll} \|\not\Delta P_j F\|_{L^p(P_{t,u})} & \lesssim & 2^{2j}\|F\|_{L^p(P_{t,u})} \\ \|P_j F\|_{L^p(P_{t,u})} & \lesssim & 2^{-2j}\|\not\Delta F\|_{L^p(P_{t,u})}. \end{array} \tag{3.18}$$

In addition, the L^2 estimates

$$\begin{array}{lll} \|\not\nabla P_j F\|_{L^2(P_{t,u})} & \lesssim & 2^{j}\|F\|_{L^2(P_{t,u})} \\ \|P_j F\|_{L^2(P_{t,u})} & \lesssim & 2^{-j}\|\not\nabla F\|_{L^2(P_{t,u})} \end{array} \tag{3.19}$$

hold together with the dual estimate

$$\|P_j \not\nabla F\|_{L^2(P_{t,u})} \lesssim 2^j \|F\|_{L^2(P_{t,u})}$$

iv) Weak Bernstein inequality For any $2 \le p < \infty$

$$\begin{aligned} &\|P_j F\|_{L^p(P_{t,u})} \lesssim (2^{(1-\frac{2}{p})j} + 1)\|F\|_{L^2(P_{t,u})}, \\ &\|P_{<0} F\|_{L^p(P_{t,u})} \lesssim \|F\|_{L^2(P_{t,u})} \end{aligned}$$

together with the dual estimates

$$\begin{aligned} &\|P_j F\|_{L^2(P_{t,u})} \lesssim (2^{(1-\frac{2}{p})j} + 1)\|F\|_{L^{p'}(P_{t,u})}, \\ &\|P_{<0} F\|_{L^2(P_{t,u})} \lesssim \|F\|_{L^{p'}(P_{t,u})} \end{aligned}$$

In addition, in [10], the following estimate is derived

$$\sum_j 2^{2j}\|P_j F\|^2_{L^2(P_{t,u})} \lesssim \|\not\nabla F\|^2_{L^2(P_{t,u})}. \tag{3.20}$$

We further recall the definition of the negative fractional powers of $\Lambda^2 = I - \not\Delta$ on any smooth tensorfield F on $P_{t,u}$ used in [10].

$$\Lambda^\alpha F = \frac{1}{\Gamma(-\alpha/2)} \int_0^\infty \tau^{-\frac{\alpha}{2}-1} e^{-\tau} U(\tau) F d\tau \tag{3.21}$$

where α is an arbitrary complex number with $\Re(\alpha) < 0$ and Γ denotes the Gamma function. We extend the definition of fractional powers of Λ to the range of α with $\Re(\alpha) > 0$, on smooth tensorfields F, by defining first

$$\Lambda^{\alpha} F = \Lambda^{\alpha-2} \cdot (I - \not{\Delta}) F$$

for $0 < \Re(\alpha) \leq 2$ and then, in general, for $0 < \Re(\alpha) \leq 2n$, with an arbitrary positive integer n, according to the formula

$$\Lambda^{\alpha} F = \Lambda^{\alpha-2n} \cdot (I - \not{\Delta})^n F.$$

With this definition, Λ^{α} is symmetric and verifies the group property $\Lambda^{\alpha}\Lambda^{\beta} = \Lambda^{\alpha+\beta}$. We also have by standard complex interpolation the following inequality:

$$\|\Lambda^{\mu\alpha+(1-\mu)\beta} F\|_{L^2(P_{t,u})} \lesssim \|\Lambda^{\alpha} F\|^{\mu}_{L^2(P_{t,u})} \|\Lambda^{\beta} F\|^{1-\mu}_{L^2(P_{t,u})}. \tag{3.22}$$

We now investigate the boundedness of Λ^{-a} on $L^p(P_{t,u})$ spaces for $0 \leq a \leq 1$. For any tensor F on $P_{t,u}$ and any $a \in \mathbb{R}$, integrating by parts and using the definition of Λ, we get:

$$\begin{aligned} \|\Lambda^a F\|^2_{L^2(P_{t,u})} + \|\not{\nabla}\Lambda^a F\|^2_{L^2(P_{t,u})} &= \int_{P_{t,u}} \Lambda^a F \cdot \Lambda^a F d\mu_{t,u} + \int_{P_{t,u}} \not{\nabla}\Lambda^a F \cdot \not{\nabla}\Lambda^a F d\mu_{t,u} \\ &= \int_{P_{t,u}} (1 - \not{\Delta})\Lambda^a F \cdot \Lambda^a F d\mu_{t,u} = \int_{P_{t,u}} \Lambda^2\Lambda^a F \cdot \Lambda^a F d\mu_{t,u} \\ &= \|\Lambda^{a+1} F\|^2_{L^2(P_{t,u})}. \end{aligned} \tag{3.23}$$

Taking $a = -1$ in (3.23), we obtain:

$$\|\not{\nabla}\Lambda^{-1} F\|_{L^2(P_{t,u})} \lesssim \|F\|_{L^2(P_{t,u})}. \tag{3.24}$$

Below, we deduce several estimates from (3.24). Taking the adjoint of (3.24), we obtain for any tensor F:

$$\|\Lambda^{-1}\not{\nabla} F\|_{L^2(P_{t,u})} \lesssim \|F\|_{L^2(P_{t,u})}. \tag{3.25}$$

Also, (3.3) and (3.24) imply for any tensor F on $P_{t,u}$:

$$\|\Lambda^{-1} F\|_{L^p(P_{t,u})} \lesssim \|F\|_{L^2(P_{t,u})} \text{ for all } 2 \leq p < +\infty. \tag{3.26}$$

Taking the adjoint of (3.26) yields:

$$\|\Lambda^{-1} F\|_{L^2(P_{t,u})} \lesssim \|F\|_{L^p(P_{t,u})} \text{ for all } 1 < p \leq 2. \tag{3.27}$$

Interpolating between the identity and Λ^{-1}, we deduce form (3.27):

$$\|\Lambda^{-a} F\|_{L^2(P_{t,u})} \lesssim \|F\|_{L^p(P_{t,u})} \text{ for all } 0 < a < 1, \ \frac{2}{1+a} < p \leq 2. \tag{3.28}$$

The proposition below completes the estimates for the heat flow recalled at the beginning of the section:

Proposition 3.11 *Let $a \in \mathbb{R}$ and $d > 0$. We have the following estimates for the homogeneous heat equation satisfied by $U(\tau)F$ and the nonhomogeneous heat equation satisfied by $V(\tau)$:*

$$\tau\|U(\tau)F\|^2_{L^2(P_{t,u})} + \int_0^\tau \tau' \|\nabla\!\!\!/ U(\tau')F\|^2_{L^2(P_{t,u})} d\tau' \lesssim (1+\tau)\|\Lambda^{-1}F\|^2_{L^2(P_{t,u})}, \tag{3.29}$$

$$\|\Lambda^a V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/ \Lambda^a V(\tau')\|^2_{L^2(P_{t,u})} d\tau' \lesssim \int_0^\tau \int_{P_{t,u}} \Lambda^{2a} V(\tau') F(\tau') d\mu_u d\tau', \tag{3.30}$$

$$\tau^{2d}\|V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \tau'^{2d} \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(P_{t,u})} d\tau' \lesssim \int_0^\tau \int_{P_{t,u}} \tau'^{2d} V(\tau') F(\tau') d\mu_u d\tau' + \int_0^\tau \tau'^{2d-1} \|V(\tau')\|^2_{L^2(P_{t,u})} d\tau'. \tag{3.31}$$

Proof We have

$$\frac{d}{dt}(\tau\|U(\tau)F\|^2_{L^2(P_{t,u})}) = 2\tau \int_{P_{t,u}} U(\tau)F \Delta\!\!\!/ U(\tau)F + \|U(\tau)F\|^2_{L^2(P_{t,u})}$$

from which we infer

$$\tau\|U(\tau)F\|^2_{L^2(P_{t,u})} + \int_0^\tau \tau' \|\nabla\!\!\!/ U(\tau')F\|^2_{L^2(P_{t,u})} d\tau' \lesssim \int_0^\tau \|U(\tau')F\|^2_{L^2(P_{t,u})}.$$

To estimate the RHS, we use (3.8) with F replaced by $\Lambda^{-1}F$ which yields

$$\|\Lambda^{-1}U(\tau)F\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/ \Lambda^{-1} U(\tau')F\|^2_{L^2(P_{t,u})} d\tau' \lesssim \|\Lambda^{-1}F\|^2_{L^2(P_{t,u})}$$

where we used $[\Lambda^{-1}, U(\tau)]F = 0$. Together with (3.23) with $a = -1$, we infer

$$\int_0^\tau \|U(\tau')F\|^2_{L^2(P_{t,u})} d\tau' \lesssim (1+\tau)\|\Lambda^{-1}F\|^2_{L^2(P_{t,u})}$$

and hence, in view of the second estimate of the proof

$$\tau\|U(\tau)F\|^2_{L^2(P_{t,u})} + \int_0^\tau \tau' \|\nabla\!\!\!/ U(\tau')F\|^2_{L^2(P_{t,u})} d\tau' \lesssim (1+\tau)\|\Lambda^{-1}F\|^2_{L^2(P_{t,u})}$$

which is (3.29).

The proof of (3.30) follows from the nonhomogenous heat equation for V, the fact that $V(0) = 0$, and the fact that $[\Lambda^a, \Delta\!\!\!/] = 0$. The proof of (3.31) is straightforward and left to the reader. This concludes the proof of the lemma. ∎

Finally, we conclude this section by recalling the sharp Bernstein inequality for scalars obtained in [10]. It is derived under the additional assumption that the Christoffel symbols Γ^A_{BC} of the coordinate system (3.1) on $P_{t,u}$ verify:

$$\sum_{A,B,C} \int_U |\Gamma^A_{BC}|^2 dx^1 dx^2 \leq c^{-1}, \tag{3.32}$$

with a constant $c > 0$ independent of u and where U is a coordinate chart.

Remark 3.12 *The existence of a covering of $P_{t,u}$ by coordinate charts satisfying* (3.1) *and* (3.32) *with a constant $c > 0$ and the number of charts independent of u will be shown in section 4.2.1.*

Let $0 \leq \gamma < 1$, and let K_γ be defined by:

$$K_\gamma := \|\Lambda^{-\gamma} K\|_{L^2(P_{t,u})}. \tag{3.33}$$

Then, we have the following sharp Bernstein inequality for any scalar function f on $P_{t,u}$, $0 \leq \gamma < 1$, any $j \geq 0$, and an arbitrary $2 \leq p < \infty$ (see Theorem 10.1 in [10]):

$$\|P_j f\|_{L^\infty(P_{t,u})} \lesssim 2^j \big(1 + 2^{-\frac{j}{p}} \big(K_\gamma^{\frac{1}{p(1-\gamma)}} + K_\gamma^{\frac{1}{2p}}\big)\big) \|f\|_{L^2(P_{t,u})} \tag{3.34}$$

$$\|P_{<0} f\|_{L^\infty(P_{t,u})} \lesssim \big(1 + K_\gamma^{\frac{2}{p(1-\gamma)}} + K_\gamma^{\frac{1}{2p}}\big) \|f\|_{L^2(P_{t,u})}. \tag{3.35}$$

Also, the Bochner identity (3.5) together with the properties of Λ implies the following inequality (see the equation (69) on page 158 of [10]):

$$\int_{P_{t,u}} |\nabla\!\!\!/^2 f|^2 \lesssim \int_{P_{t,u}} |\Delta\!\!\!/ f|^2 + \big(K_\gamma^{\frac{2}{1-\gamma}} + K_\gamma\big) \int_{P_{t,u}} |\nabla\!\!\!/ f|^2. \tag{3.36}$$

Thus, we need to bound K_γ in order to be able to use (3.34), (3.35), and (3.36). For $0 < a < 1$, we will use the fact that for any tensor F on $P_{t,u}$:

$$\|\Lambda^{-a} F\|^2_{L^2(P_{t,u})} \lesssim \|P_{<0} F\|^2_{L^2(P_{t,u})} + \sum_{j=0}^{+\infty} 2^{-2aj} \|P_j F\|^2_{L^2(P_{t,u})} \tag{3.37}$$

which follows from the methods in [10] (Corollary 7.12 page 140 of [10] holds for the case $a < 0$ and can be immediately extended to the easier case $0 < a < 1$).

Finally, we also recall from [10] (see estimate (10) in that paper) the following estimate for the L^∞ norm of any tensor F on $P_{t,u}$:

$$\|F\|_{L^\infty} \lesssim \|\nabla\!\!\!/^2 F\|^{\frac{1}{p}}_{L^2(P_{t,u})} (\|\nabla\!\!\!/ F\|^{\frac{p-2}{p}}_{L^2(P_{t,u})} \|F\|^{\frac{1}{p}}_{L^2(P_{t,u})} + \|F\|^{\frac{p-1}{p}}_{L^2(P_{t,u})}) + \|\nabla\!\!\!/ F\|_{L^2(P_{t,u})} \tag{3.38}$$

which is valid for any $2 \leq p < +\infty$. This estimate requires the assumption (3.32).

3.3 Hodge systems

We consider the following Hodge operators acting on 2 surface $P_{t,u}$:

1. The operator $\mathcal{D}_1$ takes any 1-form F into the pairs of functions $(\mathrm{div}\!\!\!/ F\,,\, \mathrm{curl}\!\!\!/ F)$.
2. The operator $\mathcal{D}_2$ takes any 2-covariant $P_{t,u}$ tangent symmetric, traceless tensor F into the $P_{t,u}$ tangent one form $\mathrm{div}\!\!\!/ F$.
3. The operator ${}^*\mathcal{D}_1$ takes the pair of scalar functions (ρ, σ) into the $P_{t,u}$-tangent 1-form $-\nabla\!\!\!/ \rho + (\nabla\!\!\!/ \sigma)^\star$.

4. The operator ${}^*\mathcal{D}_2$ takes 1-forms F on $P_{t,u}$ into the 2-covariant, symmetric, traceless tensors $-\frac{1}{2}\widehat{\mathcal{L}_F\gamma}$ with $\widehat{\mathcal{L}_F\gamma}$ the traceless part of the Lie derivative of the metric γ relative to F, i.e.

$$\widehat{(\mathcal{L}_F\gamma)}_{AB} = \nabla\!\!\!/_B F_A + \nabla\!\!\!/_A F_B - (\mathrm{div}\!\!\!/ F)\gamma_{AB}.$$

Observe that ${}^*\mathcal{D}_1$, resp. ${}^*\mathcal{D}_2$ are the L^2 adjoints of $\mathcal{D}_1$, respectively $\mathcal{D}_2$.

We record the following simple identities (see [4] page 38)

$$\begin{aligned} {}^*\mathcal{D}_1 \cdot \mathcal{D}_1 &= -\Delta\!\!\!/ + K, \qquad \mathcal{D}_1 \cdot {}^*\mathcal{D}_1 = -\Delta\!\!\!/, & (3.39)\\ {}^*\mathcal{D}_2 \cdot \mathcal{D}_2 &= -\frac{1}{2}\Delta\!\!\!/ + K, \qquad \mathcal{D}_2 \cdot {}^*\mathcal{D}_2 = -\frac{1}{2}(\Delta\!\!\!/ + K). & (3.40)\end{aligned}$$

Using integration by parts, this immediately yields the following identities for Hodge systems:

Proposition 3.13 *Let $(P_{t,u}, \gamma)$ be a two dimensional manifold with Gauss curvature K.*

i.) *The following identity holds for 1-form F on $P_{t,u}$:*

$$\int_{P_{t,u}} \left(|\nabla\!\!\!/ F|^2 + K|F|^2\right) = \int_{P_{t,u}} \left(|\mathrm{div}\!\!\!/ F|^2 + |\mathrm{curl}\!\!\!/ F|^2\right) = \int_{P_{t,u}} |\mathcal{D}_1 F|^2 \tag{3.41}$$

ii.) *The following identity holds for symmetric, traceless, 2-tensor F on $P_{t,u}$:*

$$\int_{P_{t,u}} \left(|\nabla\!\!\!/ F|^2 + 2K|F|^2\right) = 2\int_{P_{t,u}} |\mathrm{div}\!\!\!/ F|^2 = 2\int_{P_{t,u}} |\mathcal{D}_2 F|^2 \tag{3.42}$$

iii.) *The following identity holds for pairs of scalar functions (ρ, σ) on $P_{t,u}$:*

$$\int_{P_{t,u}} \left(|\nabla\!\!\!/ \rho|^2 + |\nabla\!\!\!/ \sigma|^2\right) = \int_{P_{t,u}} |-\nabla\!\!\!/ \rho + (\nabla\!\!\!/ \sigma)^\star|^2 = \int_{P_{t,u}} |{}^*\mathcal{D}_1(\rho, \sigma)|^2 \tag{3.43}$$

iv.) *The following identity holds for 1-forms F on $P_{t,u}$,*

$$\int_{P_{t,u}} \left(|\nabla\!\!\!/ F|^2 - K|F|^2\right) = 2\int_{P_{t,u}} |{}^*\mathcal{D}_2 F|^2 \tag{3.44}$$

We recall the following estimate from [10]. Let $0 \leq \gamma < 1$ and let F a $P_{t,u}$-tangent tensor. Then, we have

$$\int_{P_{t,u}} K|F|^2 \lesssim \|\Lambda^{-\gamma} K\|_{L^\infty_t L^2_{x'}} \|\nabla\!\!\!/ F\|^{1+\gamma}_{L^2(P_{t,u})} \|F\|^{1-\gamma}_{L^2(P_{t,u})}.$$

We need to refine this estimate outside of a compact set of $P_{t,u}$. Recall from Remark 2.16 that there is a compact set $\widetilde{U}$ of $\mathcal{M} \cap \{0 \leq t \leq 1\}$ of diameter of order 1, such that $(\mathcal{M}, \mathbf{g})$ is smooth, small and asymptotically flat outside of $\widetilde{U}$. Let ψ a smooth cut-off function

which is one on $\widetilde{U}$ and compactly supported in $\mathcal{M} \cap \{0 \leq t \leq 1\}$, and let $\widetilde{V}$ a compact set containing the support of ψ is its interior. Then, since $\widetilde{V}$ is compact, relying on the coordinate charts on $P_{t,u}$ satisfying (3.1), we easily obtain for any scalar f on $P_{t,u}$

$$\|f\|_{L^2(P_{t,u}\cap\widetilde{V})} \lesssim \|\nabla\!\!\!/ f\|_{L^2(P_{t,u})}.$$

Choosing $f = |F|$, we deduce for any tensor F

$$\|F\|_{L^2(P_{t,u}\cap\widetilde{V})} \lesssim \|\nabla\!\!\!/ F\|_{L^2(P_{t,u})}.$$

We infer from the previous estimates

$$\begin{aligned}\int_{P_{t,u}} K|\psi F|^2 &\lesssim \|\Lambda^{-\gamma}K\|_{L^\infty_t L^2_{x'}}\|\nabla\!\!\!/(\psi F)\|^{1+\gamma}_{L^2(P_{t,u})}\|\psi F\|^{1-\gamma}_{L^2(P_{t,u})}\\ &\lesssim \|\Lambda^{-\gamma}K\|_{L^\infty_t L^2_{x'}}(\|\nabla\!\!\!/ F\|_{L^2(P_{t,u})} + \|F\|_{L^2(P_{t,u}\cap\widetilde{V})})^2\\ &\lesssim \|\Lambda^{-\gamma}K\|_{L^\infty_t L^2_{x'}}\|\nabla\!\!\!/ F\|^2_{L^2(P_{t,u})}.\end{aligned}$$

Since $\psi = 1$ on $\widetilde{U}$, we deduce

$$\int_{P_{t,u}} K|F|^2 \lesssim \|\Lambda^{-\gamma}K\|_{L^\infty_t L^2_{x'}}\|\nabla\!\!\!/ F\|^2_{L^2(P_{t,u})} + \int_{P_{t,u}\setminus\widetilde{U}} K|F|^2$$

Together with Proposition 3.13, we immediately obtain the following corollary.

Corollary 3.14 *Assume that $\|\Lambda^{-\gamma}K\|_{L^\infty_t L^2_{x'}} \lesssim \varepsilon$ for some $0 \leq \gamma < 1$. The following estimates hold on an arbitrary 2-surface $P_{t,u}$:*

i.) *Let a $P_{t,u}$-tangent 1-form H, and let the pair of scalars $F = (\rho, \sigma)$ such that $\mathrm{div}\!\!\!/ H = \rho$, $\mathrm{curl}\!\!\!/ H = \sigma$. Then, we formally write $H = \mathcal{D}_1^{-1}F$, and we have the following estimate*

$$\|\nabla\!\!\!/ \cdot \mathcal{D}_1^{-1}F\|_{L^2(P_{t,u})} \lesssim \|F\|_{L^2(P_{t,u})} + \sqrt{\int_{P_{t,u}\setminus\widetilde{U}} K|\mathcal{D}_1^{-1}F|^2}. \tag{3.45}$$

ii.) *Let a $P_{t,u}$-tangent symmetric, traceless, 2-tensorfields H, and let the $P_{t,u}$ tangent 1-form F such that $\mathrm{div}\!\!\!/ H = F$. Then, we formally write $H = \mathcal{D}_2^{-1}F$, and we have the following estimate*

$$\|\nabla\!\!\!/ \cdot \mathcal{D}_2^{-1}F\|_{L^2(P_{t,u})} \lesssim \|F\|_{L^2(P_{t,u})} + \sqrt{\int_{P_{t,u}\setminus\widetilde{U}} K|\mathcal{D}_2^{-1}F|^2}. \tag{3.46}$$

iii.) *Let (ρ, σ) a pair of scalars on $P_{t,u}$, and let the $P_{t,u}$-tangent L^2 1-form F such that $-\nabla\!\!\!/\rho + (\nabla\!\!\!/\sigma)^\star = F$. Then, we formally write $(\rho, \sigma) = {}^*\mathcal{D}_1^{-1}F$, and we have the following estimate*

$$\|\nabla\!\!\!/ \cdot {}^*\mathcal{D}_1^{-1}F\|_{L^2(P_{t,u})} \lesssim \|F\|_{L^2(P_{t,u})} + \sqrt{\int_{P_{t,u}\setminus\widetilde{U}} K|{}^*\mathcal{D}_1^{-1}F|^2}. \tag{3.47}$$

iv.) *Let a $P_{t,u}$ tangent 1-form H, and let F the $P_{t,u}$-tangent symmetric, traceless, 2-tensor such that ${}^*\mathcal{D}_2 H = F$. Then, we formally write $H = {}^*\mathcal{D}_2^{-1} F$, and we have the following estimate*

$$\|\nabla\!\!\!/ \cdot {}^*\mathcal{D}_2^{-1} F\|_{L^2(P_{t,u})} \lesssim \|F\|_{L^2(P_{t,u})} + \sqrt{\int_{P_{t,u}\setminus\widetilde{U}} K |{}^*\mathcal{D}_2^{-1} F|^2}. \tag{3.48}$$

In view of (3.45), (3.46), (3.47) and (3.48), we have schematically

$$\|\nabla\!\!\!/ \cdot \mathcal{D}^{-1} F\|_{L^2(P_{t,u})} + \|\mathcal{D}^{-1} F\|_{L^2(P_{t,u})} \lesssim \|F\|_{L^2(P_{t,u})} + \sqrt{\int_{P_{t,u}\setminus\widetilde{U}} K |\mathcal{D}^{-1} F|^2}, \tag{3.49}$$

where $\mathcal{D} = \mathcal{D}_1$, $\mathcal{D}_2$, ${}^*\mathcal{D}_1$ or ${}^*\mathcal{D}_2$. Now, due to the fact that $(\mathcal{M}, \mathbf{g})$ is smooth, small and asymptotically flat outside of $\widetilde{U}$ as recalled above, all scalars and tensors estimated in this paper are sufficiently smooth and decaying outside of $\widetilde{U}$ so that the last term in the right-hand side of (3.49) will always be harmless. Indeed, for that term, we just need

$$K|\mathcal{D}^{-1} F|^2 \lesssim \frac{\varepsilon^2}{1+|x|^{2+}} \text{ in } \mathcal{M}\setminus\widetilde{U}$$

while due to the asymptotic flatness assumption, we have $K = O(\varepsilon|x|^{-3})$ and $\mathcal{D}^{-1}F = O(\varepsilon|x|^{-1})$ as $|x| \to +\infty$ for all choices of F satisfying a Hodge type system considered in this paper, so there is in fact $|x|^{-3}$ room for that term. For the simplicity of the exposition, by a slight abuse of notation, we will replace (3.49) with the following estimate in the rest of the paper

$$\|\nabla\!\!\!/ \cdot \mathcal{D}^{-1} F\|_{L^2(P_{t,u})} + \|\mathcal{D}^{-1} F\|_{L^2(P_{t,u})} \lesssim \|F\|_{L^2(P_{t,u})}, \tag{3.50}$$

where $\mathcal{D} = \mathcal{D}_1$, $\mathcal{D}_2$, ${}^*\mathcal{D}_1$ or ${}^*\mathcal{D}_2$.

Remark 3.15 *The estimate* (3.50) *together with the Gagliardo-Nirenberg inequality* (3.3) *yields for any $2 \leq p < +\infty$:*

$$\|\mathcal{D}^{-1} F\|_{L^p(P_{t,u})} \lesssim \|F\|_{L^2(P_{t,u})}$$

where F is a $P_{t,u}$-tangent tensor and $\mathcal{D}^{-1}$ denotes one of the operators $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$, ${}^*\mathcal{D}_1^{-1}$, ${}^*\mathcal{D}^{-1}$. We also obtain the dual inequality for any $1 < p \leq 2$:

$$\|\mathcal{D}^{-1} F\|_{L^2(P_{t,u})} \lesssim \|F\|_{L^p(P_{t,u})}.$$

Lemma 3.16 *For all $2 \leq p < +\infty$, and $j \geq 0$ we have:*

$$\|\mathcal{D}^{-1} P_j F\|_{L^p(P_{t,u})} \lesssim 2^{-\frac{2j}{p}} \|F\|_{L^2(P_{t,u})}, \tag{3.51}$$

where F is a $P_{t,u}$-tangent tensor and $\mathcal{D}^{-1}$ denotes one of the operators $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$, ${}^*\mathcal{D}_1^{-1}$, ${}^*\mathcal{D}^{-1}$.

Proof First, note that the finite band property implies

$$\begin{aligned}\|P_j\mathcal{D}^{-1}F\|_{L^2(P_{t,u})} &\lesssim 2^{-j}\|\nabla\!\!\!/\mathcal{D}^{-1}F\|_{L^2(P_{t,u})}\\ &\lesssim 2^{-j}\|F\|_{L^2(P_{t,u})}\end{aligned}$$

where we have used in particular (3.50) in the last inequality. Since $\mathcal{D}^{-1}$ denotes one of the operators $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$, ${}^*\mathcal{D}_1^{-1}$, ${}^*\mathcal{D}^{-1}$, we infer that the dual of $\mathcal{D}^{-1}$ is also of the type $\mathcal{D}^{-1}$. Thus, we deduce by duality

$$\|\mathcal{D}^{-1}P_jF\|_{L^2(P_{t,u})} \lesssim 2^{-j}\|F\|_{L^2(P_{t,u})}$$

Together with the Gagliardo-Nirenberg inequality (3.3), we infer:

$$\begin{aligned}\|\mathcal{D}^{-1}P_jF\|_{L^p(P_{t,u})} &\lesssim \|\mathcal{D}^{-1}P_jF\|_{L^2(P_{t,u})}^{\frac{2}{p}}\|\nabla\!\!\!/\mathcal{D}^{-1}P_jF\|_{L^2(P_{t,u})}^{1-\frac{2}{p}} + \|\mathcal{D}^{-1}P_jF\|_{L^2(P_{t,u})}\\ &\lesssim \|\mathcal{D}^{-1}P_jF\|_{L^2(P_{t,u})}^{\frac{2}{p}}\|F\|_{L^2(P_{t,u})}^{1-\frac{2}{p}} + \|\mathcal{D}^{-1}P_jF\|_{L^2(P_{t,u})}\\ &\lesssim 2^{-\frac{2j}{p}}\|\mathcal{D}^{-1}P_jF\|_{L^2(P_{t,u})}\end{aligned}$$

which concludes the proof of the lemma. ∎

The following lemma generalizes Remark 3.15.

Lemma 3.17 *For all $1 < p \leq 2 \leq q < +\infty$ such that $\frac{1}{p} < \frac{1}{q} + \frac{1}{2}$, we have:*

$$\|\mathcal{D}^{-1}F\|_{L^q(P_{t,u})} \lesssim \|F\|_{L^p(P_{t,u})},$$

where F is a $P_{t,u}$-tangent tensor and $\mathcal{D}^{-1}$ denotes one of the operators $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$, ${}^*\mathcal{D}_1^{-1}$, ${}^*\mathcal{D}^{-1}$.

Proof Let F, p, q as in the statement of Lemma 3.17. We decompose $\|\mathcal{D}^{-1}F\|_{L^q(P_{t,u})}$ using the property (3.16) of the geometric Littlewood-Paley projections:

$$\|\mathcal{D}^{-1}F\|_{L^q(P_{t,u})} \lesssim \|P_{<0}\mathcal{D}^{-1}F\|_{L^q(P_{t,u})} + \sum_{l\geq 0}\|P_l\mathcal{D}^{-1}F\|_{L^q(P_{t,u})}. \tag{3.52}$$

We focus on the second term in the right-hand side of (3.52), the other being easier to handle. Since $2 \leq q < +\infty$, we may use the weak Bernstein inequality for P_l:

$$\begin{aligned}\|P_l\mathcal{D}^{-1}F\|_{L^q(P_{t,u})} &\lesssim 2^{l(1-\frac{2}{q})}\|P_l\mathcal{D}^{-1}F\|_{L^2(P_{t,u})} \qquad (3.53)\\ &\lesssim 2^{l(1-\frac{2}{q})}\|P_l\mathcal{D}^{-1}\|_{\mathcal{L}(L^p(P_{t,u}),L^2(P_{t,u}))}\|F\|_{L^p(P_{t,u})}\\ &\lesssim 2^{l(1-\frac{2}{q})}\|\mathcal{D}^{-1}P_l\|_{\mathcal{L}(L^2(P_{t,u}),L^{p'}(P_{t,u}))}\|F\|_{L^p(P_{t,u})}\end{aligned}$$

where p' is the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

Next, note that $2 \leq p' < +\infty$ and hence from (3.51) we have

$$\|\mathcal{D}^{-1}P_l\|_{\mathcal{L}(L^2(P_{t,u}),L^{p'}(P_{t,u}))} \lesssim 2^{-\frac{2l}{p'}}$$

which together with (3.52) and (3.53) implies:

$$\begin{aligned}\|\mathcal{D}^{-1}F\|_{L^q(P_{t,u})} &\lesssim \left(1+\sum_{l\geq 0}2^{l(1-\frac{2}{q}-\frac{2}{p'})}\right)\|F\|_{L^p(P_{t,u})}\\ &\lesssim \left(1+\sum_{l\geq 0}2^{l(-1-\frac{2}{q}+\frac{2}{p})}\right)\|F\|_{L^p(P_{t,u})}\\ &\lesssim \|F\|_{L^p(P_{t,u})},\end{aligned}$$

where we used the fact that $\frac{1}{p}<\frac{1}{q}+\frac{1}{2}$ in the last inequality. This concludes the proof of Lemma 3.17. ■

We end this section with an algebraic expression for the commutators between L and $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$, ${}^*\mathcal{D}_1^{-1}$.

Lemma 3.18 *Let $\mathcal{D}^{-1}$ be any of the operators $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$, ${}^*\mathcal{D}_1^{-1}$. Then,*

$$[L,\mathcal{D}^{-1}]=\mathcal{D}^{-1}[\mathcal{D},L]\mathcal{D}^{-1} \tag{3.54}$$

3.4 Calculus inequalities on $\mathcal{H}_u$

For all integrable function f on $\mathcal{H}_u$, the coarea formula implies:

$$\int_{\mathcal{H}} f d\mathcal{H}=\int_0^1\int_{P_{t,u}} f b d\mu_{t,u}dt. \tag{3.55}$$

It is also well-known that for a scalar function f:

$$\frac{d}{dt}\left(\int_{P_{t,u}} f d\mu_{t,u}\right)=\int_{P_{t,u}}\left(nL(f)+n\mathrm{tr}\chi f\right)d\mu_{t,u}. \tag{3.56}$$

We have the classical Sobolev inequality on $\mathcal{H}$:

Lemma 3.19 *For any tensor F on $\mathcal{H}$, we have:*

$$\|F\|_{L^6(\mathcal{H}_u)}\lesssim \mathcal{N}_1(F), \tag{3.57}$$

and

$$\|F\|_{L^\infty_t L^4_{x'}}\lesssim \mathcal{N}_1(F). \tag{3.58}$$

Proof Using (3.2), we have:

$$\begin{aligned}\|F(t,.)\|^6_{L^6(P_{t,u})} &= \||F(t,.)|^3\|^2_{L^2(P_{t,u})}\\ &\lesssim \|\nabla\!\!\!/F(t,.)\cdot F(t,.)|F(t,.)|\|^2_{L^1(P_{t,u})}+\||F(t,.)|^3\|^2_{L^1(P_{t,u})}\\ &\lesssim \|\nabla\!\!\!/F(t,.)\|^2_{L^2(P_{t,u})}\|F(t,.)\|^4_{L^4(P_{t,u})}+\|F(t,.)\|^2_{L^2(P_{t,u})}\|F(t,.)\|^4_{L^4(P_{t,u})},\end{aligned}$$

which yields:

$$\|F\|_{L^6(\mathcal{H}_u)} \lesssim (\|\nabla\!\!\!/ F\|_{L^2(\mathcal{H}_u)} + \|F\|_{L^2(\mathcal{H}_u)})^{\frac{1}{3}} \|F\|^{\frac{2}{3}}_{L^\infty_t L^4_{x'}}.$$

and hence

$$\|F\|_{L^6(\mathcal{H}_u)} \lesssim \mathcal{N}_1(F)^{\frac{1}{3}} \|F\|^{\frac{2}{3}}_{L^\infty_t L^4_{x'}}. \tag{3.59}$$

Using (3.56) and (3.57), we have:

$$\begin{aligned}
\|F(t,.)\|^4_{L^4(P_{t,u})} &= \|F(0,.)\|^4_{L^4(P_{0,u})} \\
&\quad +4\int_0^t \int_{P_{\tau,u}} n\mathbf{D}_L F(\tau,x')\cdot F(\tau,x')|F(\tau,x')|^2 d\tau d\mu_{\tau,u} \\
&\quad +\int_0^t \int_{P_{\tau,u}} n\mathrm{tr}\chi |F(\tau,x')|^4 d\tau d\mu_{\tau,u} \\
&\lesssim \|F(0,.)\|^4_{L^4(P_{0,u})} + \|\mathbf{D}_L F\|_{L^2(\mathcal{H}_u)} \|F\|^3_{L^6(\mathcal{H}_u)} \\
&\quad +\|n\|_{L^\infty(\mathcal{H}_u)} \|\mathrm{tr}\chi\|_{L^\infty(\mathcal{H}_u)} \|F\|^4_{L^4(\mathcal{H}_u)} \\
&\lesssim \|F(0,.)\|^4_{L^4(P_{0,u})} + \mathcal{N}_1(F)^4 + \|F\|^4_{L^6(\mathcal{H}_u)}.
\end{aligned} \tag{3.60}$$

Replacing F with $\varphi(t)F$ where φ is a smooth function such that $\varphi(0)=1$ and $\varphi(1)=0$, and proceeding as in (3.60), we obtain:

$$\begin{aligned}
\|F(0,.)\|^4_{L^4(P_{0,u})} &= -4\int_0^1 \int_{P_{\tau,u}} n\varphi(\tau)^4 \mathbf{D}_L F(\tau,x')\cdot F(\tau,x')|F(\tau,x')|^2 d\tau d\mu_{\tau,u} \\
&\quad -4\int_0^1 \int_{P_{\tau,u}} \varphi'(\tau)\varphi(\tau)^3 |F(\tau,x')|^4 n d\tau d\mu_{\tau,u} \\
&\quad -\int_0^1 \int_{P_{\tau,u}} n\mathrm{tr}\chi \varphi(\tau)^4 |F(\tau,x')|^4 d\tau d\mu_{\tau,u} \\
&\lesssim \mathcal{N}_1(F)^4 + \|F\|^4_{L^6(\mathcal{H}_u)},
\end{aligned}$$

which together with (3.60) yields:

$$\|F(t,.)\|_{L^4(P_{t,u})} \lesssim \mathcal{N}_1(F) + \|F\|_{L^6(\mathcal{H}_u)}.$$

Taking the supremum in t yields

$$\|F\|_{L^\infty_t L^4_{x'}} \lesssim \mathcal{N}_1(F) + \|F\|_{L^6(\mathcal{H}_u)}. \tag{3.61}$$

Finally, (3.59) and (3.61) imply (3.57) and (3.58). This concludes the proof. ■

Lemma 3.20 *For any tensor F:*

$$\|F\|_{L^\infty_t L^2_{x'}} \lesssim \|\mathbf{D}_L F\|^{\frac{1}{2}}_{L^2(\mathcal{H}_u)} \|F\|^{\frac{1}{2}}_{L^2(\mathcal{H}_u)} + \|F\|_{L^2(\mathcal{H}_u)}. \tag{3.62}$$

Furthermore, if $F(0,.)$ belongs to $L^2(P_{0,u})$, we have:

$$\|F\|_{L^\infty_t L^2_{x'}} \lesssim \|F(0,.)\|_{L^2(P_{0,u})} + \|\mathbf{D}_L F\|_{L^2(\mathcal{H}_u)}. \tag{3.63}$$

Proof Using (3.56), we have:

$$\begin{aligned}
\|F(t,.)\|^2_{L^2(P_{t,u})} &= \|F(0,.)\|^2_{L^2(P_{0,u})} \\
&\quad +2\int_0^t\int_{P_{\tau,u}} n\mathbf{D}_L F(\tau,x')\cdot F(\tau,x')d\tau d\mu_{\tau,u} \\
&\quad +\int_0^t\int_{P_{\tau,u}} n\mathrm{tr}\chi|F(\tau,x')|^2 d\tau d\mu_{\tau,u} \qquad (3.64)\\
&\lesssim \|F(0,.)\|^2_{L^2(P_{0,u})} + \|\mathbf{D}_L F\|_{L^2(\mathcal{H}_u)}\|F\|_{L^2(\mathcal{H}_u)} \\
&\quad +\|n\|_{L^\infty(\mathcal{H}_u)}\|\mathrm{tr}\chi\|_{L^\infty(\mathcal{H}_u)}\|F\|^2_{L^2(\mathcal{H}_u)} \\
&\lesssim \|F(0,.)\|^2_{L^2(P_{0,u})} + \|\mathbf{D}_L F\|_{L^2(\mathcal{H}_u)}\|F\|_{L^2(\mathcal{H}_u)} + \|F\|^2_{L^2(\mathcal{H}_u)}.
\end{aligned}$$

Replacing F with $\varphi(t)F$ where φ is a smooth function such that $\varphi(0)=1$ and $\varphi(1)=0$, and proceeding as in (3.64), we obtain:

$$\begin{aligned}
\|F(0,.)\|^2_{L^2(P_{0,u})} &= -2\int_0^1\int_{P_{\tau,u}} \varphi(\tau)^2 n\mathbf{D}_L F(\tau,x')\cdot F(\tau,x')d\tau d\mu_{\tau,u} \\
&\quad -2\int_0^1\int_{P_{\tau,u}} \varphi(\tau)'\varphi(\tau)|F(\tau,x')|^2 d\tau d\mu_{\tau,u} \\
&\quad -\int_0^1\int_{P_{\tau,u}} n\mathrm{tr}\chi\varphi(\tau)^2|F(\tau,x')|^2 d\tau d\mu_{\tau,u} \\
&\lesssim \|\mathbf{D}_L F\|_{L^2(\mathcal{H}_u)}\|F\|_{L^2(\mathcal{H}_u)} + \|F\|^2_{L^2(\mathcal{H}_u)},
\end{aligned}$$

which together with (3.64) yields:

$$\|F(t,.)\|^2_{L^2(P_{t,u})} \lesssim \|\mathbf{D}_L F\|_{L^2(\mathcal{H}_u)}\|F\|_{L^2(\mathcal{H}_u)} + \|F\|^2_{L^2(\mathcal{H}_u)}. \qquad (3.65)$$

Taking the supremum in t yields (3.62).

To obtain (3.63), we combine (3.64) with Gronwall's lemma. This concludes the proof. ∎

The following lemma will be useful to estimate the various transport equations arising in the null structure equations. Its proof is immediate.

Lemma 3.21 *Let W and F two $P_{t,u}$-tangent tensors such that $\nabla\!\!\!/_L W = F$. Then, for any $p \geq 1$, we have:*

$$\|W\|_{L^p_{x'}L^\infty_t} \lesssim \|W(0)\|_{L^p(P_{0,u})} + \|F\|_{L^p_{x'}L^1_t}. \qquad (3.66)$$

Finally, let us recall the Minkowski inequality which will be often useful to estimate the right-hand side of (3.66):

$$\|F\|_{L^p_{x'}L^q_t} \lesssim \|F\|_{L^q_t L^p_{x'}}, \quad \text{provided } 1 \leq q \leq p \leq +\infty. \qquad (3.67)$$

3.5 Calculus inequalities on Σ_t

Recall that g is the metric induced by $\mathbf{g}$ on Σ_t. A coordinate chart $U \subset \Sigma_t$ with coordinates $x = (x_1, x_2, x_3)$ is admissible if, relative to these coordinates, there exists a constant $c > 0$ such that,

$$c^{-1}|\xi|^2 \leq g_{ij}(p)\xi^i\xi^j \leq c|\xi|^2, \qquad \text{uniformly for all } \ p \in U. \tag{3.68}$$

We assume that Σ_t can be covered by a global admissible coordinates system, i.e., a chart satisfying the conditions (3.68) with $U = \Sigma_t$. Furthermore, we assume that the constant c in (3.68) is independent of t.

Remark 3.22 *The existence of a global coordinate system Σ_t satisfying* (3.68) *with a constant $c > 0$ independent of t will be shown in section 4.2.2.*

Lemma 3.23 *Let f a real scalar function on Σ_t. Then:*

$$\|f\|_{L^{\frac{3}{2}}(\Sigma_t)} \lesssim \|\nabla f\|_{L^1(\Sigma_t)}. \tag{3.69}$$

Proof We may assume that f has compact support in Σ_t. In the global coordinate system $x = (x_1, x_2, x_3)$ on Σ_t satisfying (3.68), we have:

$$\begin{aligned}
|f(x_1,x_2,x_3)|^{\frac{3}{2}} &= \left|\int_{-\infty}^{x_1}\partial_1 f(y,x_2,x_3)dy \int_{-\infty}^{x_2}\partial_2 f(x_1,y,x_3)dy \int_{-\infty}^{x_3}\partial_3 f(x_1,x_2,y)dy\right|^{\frac{1}{2}} \\
&\lesssim \left(\int_{\mathbb{R}}|\partial_1 f(y,x_2,x_3)|dy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}|\partial_2 f(x_1,y,x_3)|dy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}|\partial_3 f(x_1,x_2,y)|dy\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^3}|f(x_1,x_2,x_3)|^{\frac{3}{2}}dx_1dx_2dx_3 \\
\lesssim\ &\left(\int_{\mathbb{R}^3}|\partial_1 f(x_1,x_2,x_3)|dx_1dx_2dx_3\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}|\partial_2 f(x_1,x_2,x_3)|dx_1dx_2dx_3\right)^{\frac{1}{2}} \\
&\left(\int_{\mathbb{R}^3}|\partial_3 f(x_1,x_2,x_3)|dx_1dx_2dx_3\right)^{\frac{1}{2}} \\
\lesssim\ &\left(\int_{\mathbb{R}^3}|\nabla f(x_1,x_2,x_3)|dx_1dx_2dx_3\right)^{\frac{3}{2}}.
\end{aligned}$$

Now in view of the bootstrap assumption (4.1) (4.4), and the coordinates system properties (4.19) and (4.20), we have $\frac{1}{5} \leq \sqrt{|g_t|} \leq 5$ which together with the previous estimate yields:

$$\left(\int_{\mathbb{R}^3}|f(x)|^{\frac{3}{2}}\sqrt{|g_t|}dx_1dx_2dx_3\right)^{\frac{2}{3}} \lesssim \int_{\mathbb{R}^3}|\nabla f(x)|\sqrt{|g_t|}dx_1dx_2dx_3$$

as desired. ∎

As a corollary of the estimate (3.69), we may derive the following Sobolev embeddings.

Corollary 3.24 *Given an arbitrary tensorfield F on Σ_t, we have:*

$$\|F\|_{L^3(\Sigma_t)} \lesssim \|\nabla F\|_{L^{\frac{3}{2}}(\Sigma_t)} \tag{3.70}$$

and

$$\|F\|_{L^6(\Sigma_t)} \lesssim \|\nabla F\|_{L^2(\Sigma_t)}. \tag{3.71}$$

Proof We use (3.69) with $f = |F|^2$:

$$\|F\|^2_{L^3(\Sigma_t)} = \||F|^2\|_{L^{\frac{3}{2}}(\Sigma_t)} \lesssim \|F\nabla F\|_{L^1(\Sigma_t)} \lesssim \|\nabla F\|_{L^{\frac{3}{2}}(\Sigma_t)}\|F\|_{L^3(\Sigma_t)}$$

which yields (3.70). To obtain (3.71), we use (3.69) with $f = |F|^4$:

$$\|F\|^4_{L^6(\Sigma_t)} = \||F|^4\|_{L^{\frac{3}{2}}(\Sigma_t)} \lesssim \||F|^2F\nabla F\|_{L^1(\Sigma_t)} \lesssim \|\nabla F\|_{L^2(\Sigma_t)}\|F\|^3_{L^6(\Sigma_t)}$$

which yields (3.71). ∎

As a corollary of (3.69), it is classical to derive the following inequality (for a proof, see for example [7] page 157):

Corollary 3.25

$$\|F\|_{L^\infty(\Sigma_t)} \lesssim \|\nabla F\|_{L^p(\Sigma_t)} + \|F\|_{L^p(\Sigma_t)}, \tag{3.72}$$

where p is any real number $p > 3$.

As a corollary of (3.71) and (3.72), we immediately obtain:

$$\|F\|_{L^\infty(\Sigma_t)} \lesssim \|\nabla^2 F\|_{L^2(\Sigma_t)} + \|F\|_{L^2(\Sigma_t)}. \tag{3.73}$$

Lemma 3.26 *For any tensor F on $\mathcal{M}$:*

$$\|F\|_{L^\infty_u L^2_{x'}} \lesssim \|\nabla F\|_{L^\infty_t L^2(\Sigma_t)} + \|F\|_{L^\infty_t L^2(\Sigma_t)}. \tag{3.74}$$

and

$$\|F\|_{L^\infty_u L^2_{x'}} \lesssim \|\nabla F\|_{L^\infty_t L^{\frac{3}{2}}(\Sigma_t)} + \|F\|_{L^\infty_t L^2(\Sigma_t)}. \tag{3.75}$$

Proof We first recall the analogous formula to (3.55) (3.56). For all integrable functions on Σ_t, the coarea formula implies:

$$\int_{\Sigma_t} f d\Sigma_t = \int_u \int_{P_{t,u}} f b d\mu_{t,u} du \tag{3.76}$$

Also, we have for all integrable scalar functions f:

$$\frac{d}{du}\left(\int_{P_{t,u}} f d\mu_{t,u}\right) = \int_{P_{t,u}} b(\nabla_N f + \mathrm{tr}\theta f) d\mu_{t,u} \tag{3.77}$$

where θ is the second fundamental form of $P_{t,u}$ in Σ_t, i.e. $\theta_{ij} = \nabla_i N_j$. Note that from the definition of k, χ and θ, we have:

$$\chi_{AB} = < \mathbf{D}_A L, e_B > = < \nabla_A T, e_B > + < \nabla_A N, e_B > = -k_{AB} + \theta_{AB}. \tag{3.78}$$

The proof of (3.74) is easier, so we focus on (3.75). Using (3.76)-(3.78), we obtain:

$$\begin{aligned}\|F\|^2_{L^\infty_u L^2_{x'}} &\lesssim \int_u \int_{P_{t,u}} (2F\cdot\nabla_N F + \mathrm{tr}\theta|F|^2) d\mu_{t,u}dx' \\ &\lesssim \|\nabla_N F\|_{L^\infty_t L^{\frac{3}{2}}(\Sigma_t)}\|F\|_{L^\infty_t L^3(\Sigma_t)} + \|\mathrm{tr}\chi\|_{L^\infty}\|F\|^2_{L^\infty_t L^2(\Sigma_t)} \\ &\quad + \|\mathrm{tr}k\|_{L^\infty_t L^6(\Sigma_t)}\|F\|_{L^\infty_t L^3(\Sigma_t)}\|F\|_{L^\infty_t L^2(\Sigma_t)} \\ &\lesssim \|\nabla F\|^2_{L^\infty_t L^{\frac{3}{2}}(\Sigma_t)} + \|F\|^2_{L^\infty_t L^2(\Sigma_t)} \end{aligned} \tag{3.79}$$

where we have used the bootstrap assumptions (4.3) (4.4) and the Gagliardo-Nirenberg inequality (3.70) in the last inequality. We thus have obtained (3.75). ∎

We have the following corollary of the estimate (3.75):

Corollary 3.27 *For any tensor F on $\mathcal{M}$, we have*

$$\|F\|_{L^\infty_t L^4_{x'}} \lesssim \|\nabla F\|_{L^\infty_t L^2(\Sigma_t)} + \|F\|_{L^\infty_t L^2(\Sigma_t)}. \tag{3.80}$$

Proof Using (3.75) with F replaced by $|F|^2$, we obtain

$$\begin{aligned}\|F\|^2_{L^\infty_u L^4_{x'}} &\lesssim \|F\cdot\nabla F\|_{L^\infty_t L^{\frac{3}{2}}(\Sigma_t)} + \|F\|^2_{L^\infty_t L^4(\Sigma_t)} \\ &\lesssim \|F\|_{L^\infty_t L^6(\Sigma_t)}\|\nabla F\|_{L^\infty_t L^2(\Sigma_t)} + \|F\|^{\frac{3}{2}}_{L^\infty_t L^6(\Sigma_t)}\|F\|^{\frac{1}{2}}_{L^\infty_t L^2(\Sigma_t)} \\ &\lesssim \|\nabla F\|^2_{L^\infty_t L^2(\Sigma_t)} + \|F\|^2_{L^\infty_t L^2(\Sigma_t)}, \end{aligned}$$

where we used in the last inequality the Sobolev embedding (3.71). This concludes the proof of the corollary. ∎

Proposition 3.28 *For any tensor F on Σ_t, we have the following inequality:*

$$\|\nabla^2 F\|_{L^2(\Sigma_t)} \lesssim \|\Delta F\|_{L^2(\Sigma_t)} + \|F\|_{L^2(\Sigma_t)}. \tag{3.81}$$

Proof We recall the Bochner identity on the 3 dimensional manifold Σ_t for a tensor F (see for example page 133 in [10]):

$$\begin{aligned}\int_{\Sigma_t}|\nabla^2 F|^2 d\Sigma_t &= \int_{\Sigma_t}|\Delta F|^2 d\Sigma_t - \int_{\Sigma_t}(R_t)_{ij}\nabla_i F_l \nabla_j F_l d\Sigma_t \\ &\quad + \int_{\Sigma_t}(R_t)_{ijlm}\nabla_m F_j \nabla_l F_i d\Sigma_t - \frac{1}{2}\int_{\Sigma_t}(R_t)_{mijl}(R_t)_{nijl}F_m F_n d\Sigma_t \end{aligned} \tag{3.82}$$

where R_t is the curvature tensor of the induced metric on Σ_t. The bound (4.26) on R_t together with the Sobolev inequality (3.71) and (3.82) implies:

$$\begin{aligned}\int_{\Sigma_t}|\nabla^2 F|^2 d\Sigma_t &\lesssim \|\Delta f\|^2_{L^2(\Sigma_t)} + \|R_t\|_{L^2(\Sigma_t)}\|\nabla F\|^2_{L^4(\Sigma_t)} + \|R_t\|^2_{L^2(\Sigma_t)}\|F\|^2_{L^\infty(\Sigma_t)} \\ &\lesssim \|\Delta F\|^2_{L^2(\Sigma_t)} + \varepsilon\|\nabla F\|^{\frac{1}{2}}_{L^2(\Sigma_t)}\|\nabla F\|^{\frac{3}{2}}_{L^6(\Sigma_t)} + \varepsilon^2(\|\nabla^2 F\|^2_{L^2(\Sigma_t)} + \|F\|^2_{L^2(\Sigma_t)}) \\ &\lesssim \|\Delta F\|^2_{L^2(\Sigma_t)} + \|\nabla F\|^{\frac{1}{2}}_{L^2(\Sigma_t)}\|\nabla^2 F\|^{\frac{3}{2}}_{L^2(\Sigma_t)} + \|F\|^2_{L^2(\Sigma_t)} \end{aligned}$$

which yields (3.81). ∎

Proposition 3.29 *For any tensor F on Σ_t, we have the following inequality:*

$$\|F\|_{L^\infty(\Sigma_t)} \lesssim \|F\|_{L^2(\Sigma_t)} + \|\nabla F\|_{L^2(\Sigma_t)} + \|\nabla\!\!\!/\nabla F\|_{L^2(\Sigma_t)}. \tag{3.83}$$

Proof Using (3.4) with $p = 4$, we have:

$$\|F\|_{L^\infty(\Sigma_t)} \lesssim \|\nabla\!\!\!/ F\|_{L^\infty_u L^4_{x'}} + \|F\|_{L^\infty_u L^4_{x'}}. \tag{3.84}$$

Pick any real number u_0. Now, using the coarea formula (3.76) and (3.77), as well as the Sobolev embedding (3.71), we have:

$$\begin{aligned}
& \|\nabla\!\!\!/ F(u,.)\|^4_{L^\infty_u L^4_{x'}} + \|F(u,.)\|^4_{L^\infty_u L^4_{x'}} \qquad (3.85)\\
\lesssim\ & \|\nabla\!\!\!/ F(u_0,.)\|^4_{L^4_{x'}} + \|F(u_0,.)\|^4_{L^4_{x'}} + \int_{\Sigma_t} \nabla_N \nabla\!\!\!/ F \cdot \nabla\!\!\!/ F |\nabla\!\!\!/ F|^2\\
& + \int_{\Sigma_t} |\nabla\!\!\!/ F|^4 \mathrm{tr}\theta + \int_{\Sigma_t} \nabla_N F \cdot F |F|^2 + \int_{\Sigma_t} |F|^4 \mathrm{tr}\theta\\
\lesssim\ & \|\nabla\!\!\!/ F(u_0,.)\|^4_{L^\infty_u L^4_{x'}} + \|F(u_0,.)\|^4_{L^\infty_u L^4_{x'}} + \|\nabla\!\!\!/_N \nabla\!\!\!/ F\|_{L^2(\Sigma_t)} \|\nabla\!\!\!/ F\|^3_{L^6(\Sigma_t)}\\
& + \|\nabla\!\!\!/_N F\|_{L^2(\Sigma_t)} \|F\|^3_{L^6(\Sigma_t)} + (\|\nabla\!\!\!/ F\|^4_{L^6(\Sigma_t)} + \|F\|^4_{L^6(\Sigma_t)}) \|\mathrm{tr}\theta\|_{L^3(\Sigma_t)}\\
\lesssim\ & \|\nabla\!\!\!/ F(u_0,.)\|^4_{L^\infty_u L^4_{x'}} + \|F(u_0,.)\|^4_{L^\infty_u L^4_{x'}} + \|[\nabla\!\!\!/_N, \nabla\!\!\!/] F\|^4_{L^2(\Sigma_t)} + \|\nabla\!\!\!/\nabla F\|^4_{L^2(\Sigma_t)}\\
& + \|\nabla F\|^2_{L^2(\Sigma_t)},
\end{aligned}$$

where we used in the last inequality the estimates (2.68)-(2.70) for b and $\mathrm{tr}\theta$.

In view of (3.84) and (3.85), we need to estimate $\|[\nabla_N, \nabla\!\!\!/] F\|_{L^2(\Sigma_t)}$. Using the commutator formula (2.47), we have:

$$\begin{aligned}
\|[\nabla_N, \nabla\!\!\!/] F\|_{L^2(\Sigma_t)} \lesssim\ & \|\theta\|_{L^\infty_u L^4_{x'}} \|\nabla\!\!\!/ F\|_{L^2_u L^4(P_{t,u})} + \|b^{-1} \nabla\!\!\!/ b\|_{L^\infty_u L^4_{x'}} \|\nabla\!\!\!/_N F\|_{L^2_u L^4(P_{t,u})}\\
+\ & (\|\widehat{\chi}\|_{L^\infty_u L^4_{x'}} (\|\overline{\epsilon}\|_{L^\infty_u L^4_{x'}} + \|\underline{\xi}\|_{L^\infty_u L^4_{x'}}) + \|\underline{\chi}\|_{L^\infty_u L^4_{x'}} \|\zeta\|_{L^\infty_u L^4_{x'}})) \|F\|_{L^2_u L^\infty_{x'}}\\
& + (\|\beta\|_{L^2(\Sigma_t)} + \|\underline{\beta}\|_{L^2(\Sigma_t)}) \|F\|_{L^\infty(\Sigma_t)}\\
& + \|\mathrm{tr}\chi\|_{L^\infty(\Sigma_t)} (\|\overline{\epsilon}\|_{L^\infty_u L^4_{x'}} + \|\underline{\xi}\|_{L^\infty_u L^4_{x'}}) \|F\|_{L^2_u L^4_{x'}}\\
\lesssim\ & D\varepsilon \|\nabla\!\!\!/^2 F\|_{L^2(\Sigma_t)} + D\varepsilon \|\nabla\!\!\!/ \nabla_N F\|_{L^2(\Sigma_t)} + D\varepsilon \|F\|_{L^\infty(\Sigma_t)}\\
& + D\varepsilon \|F\|_{L^2 L^\infty_{x'}} + D\varepsilon \|F\|_{L^2 L^4_{x'}}\\
\lesssim\ & D\varepsilon \|F\|_{L^\infty(\Sigma_t)} + \|\nabla\!\!\!/\nabla F\|_{L^2(\Sigma_t)} + \|\nabla F\|_{L^2(\Sigma_t)} + \|F\|_{L^2(\Sigma_t)}, \qquad (3.86)
\end{aligned}$$

where we have used the curvature bound (2.59) for β and $\underline{\beta}$, the bootstrap assumptions (4.1)-(4.6) for $b, \theta, \chi, \underline{\chi}, \zeta$ and $\underline{\xi}$, the estimate (3.3) to bound the $L^2_u L^4_{x'}$ norm of F, the estimate (3.38) to bound the $L^2_u L^\infty_{x'}$ norm of F, and the estimate:

$$\sup_t \|H\|_{L^\infty_u L^4_{x'}} = \sup_u \|H\|_{L^\infty_t L^4_{x'}} \lesssim \mathcal{N}_1(H),$$

which is valid for any tensor H and follows from (3.61).

Now, in view of (3.84)-(3.86), we have for any real number u_0:

$$\begin{aligned}
& \|F\|_{L^\infty(\Sigma_t)} + \|\nabla\!\!\!/ F\|_{L^\infty_u L^4_{x'}} + \|F\|_{L^\infty_u L^4_{x'}}\\
\lesssim\ & \|\nabla\!\!\!/ F(u_0,.)\|^4_{L^\infty_u L^4_{x'}} + \|F(u_0,.)\|^4_{L^\infty_u L^4_{x'}} + \|\nabla\!\!\!/\nabla F\|_{L^2(\Sigma_t)} + \|\nabla F\|_{L^2(\Sigma_t)}\\
& + \|F\|_{L^2(\Sigma_t)} + D\varepsilon \|F\|_{L^\infty(\Sigma_t)},
\end{aligned}$$

which yields:

$$\begin{aligned} & \|F\|_{L^\infty(\Sigma_t)} + \|\not\nabla F\|_{L^\infty_u L^4_{x'}} + \|F\|_{L^\infty_u L^4_{x'}} \qquad (3.87) \\ \lesssim\ & \|\not\nabla F(u_0,.)\|^4_{L^\infty_u L^4_{x'}} + \|F(u_0,.)\|^4_{L^\infty_u L^4_{x'}} + \|\not\nabla\nabla F\|_{L^2(\Sigma_t)} + \|\nabla F\|_{L^2(\Sigma_t)} + \|F\|_{L^2(\Sigma_t)}. \end{aligned}$$

Let φ a smooth compactly supported scalar function on Σ_t. Applying (3.87) respectively to φF with u_0 outside of the support of φ, and then to F with u_0 inside the support of φ finally yields (3.83). This concludes the proof of the proposition. ■

For the following proposition, we assume that for each $\delta > 0$, there exists a constant $C(\delta) > 0$ and a finite covering of Σ_t by charts U with coordinates systems relative to which we have

$$(1+\delta)^{-1}|\xi|^2 \leq g_{ij}(p)\xi^i\xi^j \leq (1+\delta)|\xi|^2,\ p \in U \qquad (3.88)$$

and

$$\int_U |\partial^2 g_{ij}|^2\sqrt{|g|}dx \leq C(\delta). \qquad (3.89)$$

Remark 3.30 *The existence of a finite covering of Σ_t by coordinates systems relative to which we have* (3.88) (3.89) *with $C(\delta)$ and the number of charts being independent of t will be shown in section 4.2.3.*

Proposition 3.31 *Assume that for each $\delta > 0$, there is a finite covering of Σ_t by coordinates systems relative to which we have* (3.88) (3.89). *For an arbitrary tensorfield F on Σ_t, we have the following inequality:*

$$\|\nabla^2 F\|_{L^{\frac{3}{2}}(\Sigma_t)} \lesssim \|\Delta F\|_{L^{\frac{3}{2}}(\Sigma_t)} + \|\nabla F\|_{L^2(\Sigma_t)}. \qquad (3.90)$$

Proof (3.90) may be reduced by partition of unity to the case where F has compact support in a coordinate chart U. Let $x = (x_1, x_2, x_3)$ a coordinate system on U satisfying (3.88) (3.89). We have:

$$\begin{aligned} & \|\nabla^2 F - \partial^2 F\|_{L^{\frac{3}{2}}(U)} \\ \lesssim\ & \|(g_{ij}-\delta_{ij})\partial^2 F\|_{L^{\frac{3}{2}}(U)} + \|\partial g \partial F\|_{L^{\frac{3}{2}}(U)} + \|\partial^2 g F\|_{L^{\frac{3}{2}}(U)} \\ \lesssim\ & \|g_{ij}-\delta_{ij}\|_{L^\infty(U)}\|\partial^2 F\|_{L^{\frac{3}{2}}(U)} + \|\partial g\|_{L^6(U)}\|\nabla F\|_{L^2(U)} \\ & + \|\partial^2 g\|_{L^2(U)}(1+\|\partial g\|_{L^3(U)})\|F\|_{L^6(U)} \\ \lesssim\ & \delta\|\partial^2 F\|_{L^{\frac{3}{2}}(U)} + C(\delta)\|\nabla F\|_{L^2(U)}, \end{aligned}$$

where we have used the Sobolev embedding (3.71) in the last inequality. Thus, we now fix $\delta > 0$ small enough such that for a constant $C > 0$, we have:

$$\|\nabla^2 F - \partial^2 F\|_{L^{\frac{3}{2}}(U)} \leq \frac{1}{2}\|\partial^2 F\|_{L^{\frac{3}{2}}(U)} + C\|\nabla F\|_{L^2(U)}. \qquad (3.91)$$

Note that $C = C(\delta) > 0$ is now a fixed number. Similarly, we also have:

$$\left\|\Delta F - \sum_{j=1}^3 \partial_j^2 F\right\|_{L^{\frac{3}{2}}(U)} \leq \frac{1}{2}\left\|\sum_{j=1}^3 \partial_j^2 F\right\|_{L^{\frac{3}{2}}(U)} + C\|\nabla F\|_{L^2(U)}, \qquad (3.92)$$

where $\sum_{j=1}^3 \partial_j^2$ is the usual Laplacian in $\mathbb{R}^3$. Now, from usual Calderon-Zygmund theory, we have:

$$\|\partial^2 F\|_{L^{\frac{3}{2}}(U)} \lesssim \left\| \sum_{j=1}^3 \partial_j^2 F \right\|_{L^{\frac{3}{2}}(U)}$$

which together with (3.91) and (3.92) yields (3.90). ∎

Finally, we have the following useful commutation formula for any scalar function f on $\mathcal{M}$:

$$[\Delta, \mathbf{D}_T] f = -2k\nabla^2 f + 2n^{-1}\nabla n \nabla T(f) + n^{-1}\Delta n T(f) - \nabla k \nabla f - 2n^{-1} k \nabla n \nabla f \quad (3.93)$$

where we used the fact that we are on a maximal foliation (see (2.2)), so that the term $\mathrm{Tr}(k)\Delta f$ vanishes. We also used the fact that the Einstein equations (1.1) are satisfied, so that the term of type $\mathbf{R}\nabla f$ vanishes as well. We also provide commutation formulas with tensors. Let $\Pi_{\underline{A}}$ be an m-covariant tensor tangent to Σ_t. Then, we have:

$$\begin{aligned} \nabla_j \mathbf{D}_T \Pi_{\underline{A}} - \mathbf{D}_T \nabla_j \Pi_{\underline{A}} &= k_{jl}\nabla_l \Pi_{\underline{A}} + n^{-1}\nabla_j n \mathbf{D}_T \Pi_{\underline{A}} + \sum_i (n^{-1} k_{A_i j} \nabla_l n \quad (3.94) \\ &\quad - n^{-1} k_{jl} \nabla_{A_i} n + \mathbf{R}_{TA_i}(g_t)^l_j - \mathbf{R}_{Tl}(g_t)^{A_i}_j) \Pi_{A_1 .. \check{l} .. A_m}. \end{aligned}$$

For some applications we have in mind, we would like to get rid of the term containing a $\mathbf{D}_T$ derivative in the right-hand side of (3.94). This is achieved by considering the commutator $[\nabla, \mathbf{D}_{nT}]$ instead of $[\nabla, \mathbf{D}_T]$:

$$\begin{aligned} \nabla_j \mathbf{D}_{nT} \Pi_{\underline{A}} - \mathbf{D}_{nT} \nabla_j \Pi_{\underline{A}} &= n k_{jl} \nabla_l \Pi_{\underline{A}} \quad (3.95) \\ &+ \sum_i (k_{A_i j} \nabla_l n - k_{jl} \nabla_{A_i} n + n \mathbf{R}_{TA_i}(g_t)^l_j - n \mathbf{R}_{Tl}(g_t)^{A_i}_j) \Pi_{A_1 .. \check{l} .. A_m}. \end{aligned}$$

3.6 Geometric Littlewood-Paley theory on Σ_t

3.6.1 The Gagliardo-Nirenberg inequality on Σ_t

We first consider the case of $L^p(\Sigma_t)$ with $2 \leq p \leq 6$. Using the Sobolev inequality (3.71) and interpolation implies for any tensor F on Σ_t

$$\|F\|_{L^p(\Sigma_t)} \lesssim \|\nabla F\|^{3(\frac{1}{2}-\frac{1}{p})}_{L^2(\Sigma_t)} \|F\|^{-\frac{1}{2}+\frac{3}{p}}_{L^2(\Sigma_t)} \ \forall 2 \leq p \leq 6. \quad (3.96)$$

Next, we derive the following analog of Lemma 3.23

Lemma 3.32 *Assume that for $\delta = \frac{1}{2}$, there is a finite covering of Σ_t by coordinates systems relative to which we have* (3.88) (3.89)*. Let f a real scalar function on Σ_t. Then:*

$$\|f\|_{L^\infty(\Sigma_t)} \lesssim \|\nabla^2 f\|^{\frac{1}{2}}_{L^2(\Sigma_t)} \|\nabla f\|^{\frac{1}{2}}_{L^2(\Sigma_t)} + \|\nabla f\|_{L^2(\Sigma_t)}. \quad (3.97)$$

Proof The statement may be reduced to the case where f has compact support in an admissible local chart U of Σ_t satisfying (3.88) (3.89) with $\delta = \frac{1}{2}$. Let $x = (x_1, x_2, x_3)$ denote the corresponding coordinate system. We start by proving the following estimate on $\mathbb{R}^3$

$$\|f^2\|_{L^\infty(\mathbb{R}^3)} \lesssim \|\partial f\|_{L^2(\mathbb{R}^3)} \|\partial^2 f\|_{L^2(\mathbb{R}^3)}. \tag{3.98}$$

To this end, we introduce a standard Littlewood-Paley decomposition on $\mathbb{R}^3$. Let φ a positive function in $C_0^\infty(\mathbb{R}^3)$ equal to 1 for $|\xi| \leq 1/2$ and to 0 for $|\xi| \geq 1$. For all integer p we define the Littlewood-Paley projection Δ_p by $\widehat{\Delta_p f}(\xi) = \psi(2^{-p}\xi)\hat{f}(\xi)$ where $\psi(\xi) = \varphi(\xi/2) - \varphi(\xi)$. We also define Δ_{-1} by $\widehat{\Delta_{-1} f}(\xi) = \varphi(\xi)\hat{f}(\xi)$. The Littlewood-Paley decomposition of f is:

$$f = \sum_{p \geq -1} \Delta_p(f).$$

Using the Littlewood-Paley decomposition for f^2, we have:

$$\begin{aligned} \|f^2\|_{L^\infty(\mathbb{R}^3)} &\lesssim \sum_{j \geq -1} \|\Delta_j(f^2)\|_{L^\infty(\mathbb{R}^3)} \\ &\lesssim \sum_{j \geq -1} 2^{\frac{3j}{2}} \|\Delta_j(f^2)\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \sum_{j,l,m \geq -1} 2^{\frac{3j}{2}} \|\Delta_j(\Delta_l f \Delta_m f)\|_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{3.99}$$

The expression being symmetric in (l, m), we may assume $m \leq l$. We consider the two cases $l \leq j$ and $j < l$ separately. If $j < l$, we use the boundedness of Δ_j on $L^2(\mathbb{R}^3)$ and the Bernstein inequality for Δ_m to obtain

$$\|\Delta_j(\Delta_l f \Delta_m f)\|_{L^2(\mathbb{R}^3)} \lesssim 2^{\frac{3m}{2}} \|\Delta_l f\|_{L^2(\mathbb{R}^3)} \|\Delta_m f\|_{L^2(\mathbb{R}^3)}. \tag{3.100}$$

If $l \leq j$, we use the finite band property for Δ_j, Δ_l and Δ_m, and the Bernstein inequality for Δ_m to obtain

$$\begin{aligned} \|\Delta_j(\Delta_l f \Delta_m f)\|_{L^2(\mathbb{R}^3)} &\lesssim 2^{-2j} \|\Delta(\Delta_l f \Delta_m f)\|_{L^2(\mathbb{R}^3)} \\ &\lesssim 2^{-2j} \|\Delta(\Delta_l f)\Delta_m f\|_{L^2(\mathbb{R}^3)} + 2^{-2j} \|\Delta_l f \Delta(\Delta_m f)\|_{L^2(\mathbb{R}^3)} \\ &\quad + 2^{-2j} \|\nabla(\Delta_l f)\nabla(\Delta_m f)\|_{L^2(\mathbb{R}^3)} \\ &\lesssim 2^{-2j}(2^{2l+\frac{3m}{2}} + 2^{l+\frac{5m}{2}} + 2^{\frac{7m}{2}}) \|\Delta_l f\|_{L^2(\mathbb{R}^3)} \|\Delta_m f\|_{L^2(\mathbb{R}^3)} \\ &\lesssim 2^{-2j+2l+\frac{3m}{2}} \|\Delta_l f\|_{L^2(\mathbb{R}^3)} \|\Delta_m f\|_{L^2(\mathbb{R}^3)}, \end{aligned} \tag{3.101}$$

where we used the fact that $m \leq l$ in the last inequality. Now, (3.100) and (3.101) imply

$$2^{\frac{3j}{2}} \|\Delta_j(\Delta_l f \Delta_m f)\|_{L^2(\mathbb{R}^3)} \lesssim 2^{-\frac{|j-l|}{4} - \frac{|j-m|}{4}} (2^{2l} \|\Delta_l f\|_{L^2(\mathbb{R}^3)})(2^m \|\Delta_m f\|_{L^2(\mathbb{R}^3)}).$$

Together with (3.99), we infer

$$\|f^2\|_{L^\infty(\mathbb{R}^3)} \lesssim \left(\sum_{l \geq -1} (2^{2l} \|\Delta_l f\|_{L^2(\mathbb{R}^3)})^2 \right) \left(\sum_{m \geq -1} (2^m \|\Delta_m f\|_{L^2(\mathbb{R}^3)})^2 \right) \lesssim \|\partial f\|_{L^2(\mathbb{R}^3)} \|\partial^2 f\|_{L^2(\mathbb{R}^3)},$$

which is (3.98). Now in view of the assumptions (3.88) (3.89) with $\delta = \frac{1}{2}$, we have $\frac{1}{8} \leq \sqrt{|g_t|} \leq 8$ and the estimate $\|\Gamma\|_{L^3(U)} \lesssim 1$ where Γ is the corresponding Christoffel symbol, which together with (3.98) yields:

$$\begin{aligned}
&\|f\|_{L^\infty(\mathbb{R}^3)} \\
\lesssim\ & \left(\int_{\mathbb{R}^3} (|\nabla^2 f(x)| + |\Gamma(x)||\nabla f(x)|)^2 \sqrt{|g_t|} dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\nabla f(x)|^2 \sqrt{|g_t|} dx\right)^{\frac{1}{2}} \\
\lesssim\ & \left(\left(\int_{\mathbb{R}^3} |\nabla^2 f(x)|^2 \sqrt{|g_t|} dx\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^3} |\nabla f(x)|^6 \sqrt{|g_t|} dx\right)^{\frac{1}{6}}\right) \left(\int_{\mathbb{R}^3} |\nabla f(x)|^2 \sqrt{|g_t|} dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Coming back to Σ_t, we obtain

$$\|f\|_{L^\infty(\Sigma_t)} \lesssim (\|\nabla^2 f\|^{\frac{1}{2}}_{L^2(\Sigma_t)} + \|\nabla f\|^{\frac{1}{2}}_{L^6(\Sigma_t)}) \|\nabla f\|^{\frac{1}{2}}_{L^2(\Sigma_t)} + \|\nabla f\|_{L^2(\Sigma_t)},$$

which together with the Sobolev embedding (3.71) yields (3.97). ∎

Let F a tensor on Σ_t. Then (3.97) with the choice $f = |F|^2$ yields

$$\begin{aligned}
\|F\|^2_{L^\infty(\Sigma_t)} &\lesssim \|F \cdot \nabla^2 F + |\nabla F|^2\|^{\frac{1}{2}}_{L^2(\Sigma_t)} \|F \cdot \nabla F\|^{\frac{1}{2}}_{L^2(\Sigma_t)} + \|F \cdot \nabla F\|_{L^2(\Sigma_t)} \\
&\lesssim (\|F\|_{L^\infty(\Sigma_t)} \|\nabla^2 F\|_{L^2(\Sigma_t)} + \|\nabla F\|^2_{L^4(\Sigma_t)})^{\frac{1}{2}} (\|F\|_{L^\infty(\Sigma_t)} \|\nabla F\|_{L^2(\Sigma_t)})^{\frac{1}{2}} \\
&\quad + \|F\|_{L^\infty(\Sigma_t)} \|\nabla F\|_{L^2(\Sigma_t)}.
\end{aligned}$$

Using the Gagliardo-Nirenberg inequality (3.96) to evaluate $\|\nabla F\|_{L^4(\Sigma_t)}$, we deduce

$$\begin{aligned}
\|F\|^2_{L^\infty(\Sigma_t)} &\lesssim \|F\|_{L^\infty(\Sigma_t)} \|\nabla^2 F\|^{\frac{1}{2}}_{L^2(\Sigma_t)} \|\nabla F\|^{\frac{1}{2}}_{L^2(\Sigma_t)} + \|\nabla^2 F\|^{\frac{3}{4}}_{L^2(\Sigma_t)} \|\nabla F\|^{\frac{3}{4}}_{L^2(\Sigma_t)} \|F\|^{\frac{1}{2}}_{L^\infty(\Sigma_t)} \\
&\quad + \|F\|_{L^\infty(\Sigma_t)} \|\nabla F\|_{L^2(\Sigma_t)}.
\end{aligned}$$

Thus, we finally obtain for any tensor F on Σ_t

$$\|F\|_{L^\infty(\Sigma_t)} \lesssim \|\nabla^2 F\|^{\frac{1}{2}}_{L^2(\Sigma_t)} \|\nabla F\|^{\frac{1}{2}}_{L^2(\Sigma_t)} + \|\nabla F\|_{L^2(\Sigma_t)}.$$

Interpolating with the Sobolev embedding (3.71) on Σ_t, we finally obtain the following Gagliardo-Nirenberg inequality on Σ_t

$$\|F\|_{L^p(\Sigma_t)} \lesssim \|\nabla^2 F\|^{\frac{1}{2}-\frac{3}{p}}_{L^2(\Sigma_t)} \|\nabla F\|^{\frac{1}{2}+\frac{3}{p}}_{L^2(\Sigma_t)} \ \forall 6 \leq p \leq +\infty. \tag{3.102}$$

3.6.2 Heat equation on Σ_t

In this section we study the properties of the heat equation for arbitrary tensorfields F on Σ_t.

$$\partial_\tau \mathcal{U}(\tau) F - \Delta \mathcal{U}(\tau) F = 0,\ \mathcal{U}(0) F = F.$$

Observe that the operators $\mathcal{U}(\tau)$ are selfadjoint and form a semigroup for $\tau > 0$. In other words for all, real valued, smooth tensorfields F, G,

$$\int_{\Sigma_t} \mathcal{U}(\tau) F \cdot G = \int_{\Sigma_t} F \cdot \mathcal{U}(\tau) G, \qquad \mathcal{U}(\tau_1)\mathcal{U}(\tau_2) = \mathcal{U}(\tau_1 + \tau_2). \tag{3.103}$$

We have the following $L^2(\Sigma_t)$ estimates for the operator $\mathcal{U}(\tau)$.

$$\|\mathcal{U}(\tau)F\|_{L^2(\Sigma_t)} \leq \|F\|_{L^2(\Sigma_t)}, \tag{3.104}$$
$$\|\nabla\mathcal{U}(\tau)F\|_{L^2(\Sigma_t)} \leq \|\nabla F\|_{L^2(\Sigma_t)}. \tag{3.105}$$

They are obtained after multiplication of the Heat equation satisfied by $\mathcal{U}(\tau)F$ respectively with $\mathcal{U}(\tau)F$ and $\Delta\mathcal{U}(\tau)F$, and then integration over Σ_t.

In the next proposition we establish a simple $L^p(\Sigma_t)$ estimate for $\mathcal{U}(\tau)$.

Proposition 3.33 *For every $2 \leq p \leq \infty$, we have*

$$\|\mathcal{U}(\tau)F\|_{L^p(\Sigma_t)} \leq \|F\|_{L^p(\Sigma_t)}.$$

Proof : The proof is identical to the one in [10] on compact 2-surfaces. We reproduce it here for the convenience of the reader. We shall first prove the Lemma for scalar functions f. We multiply the equation $\partial_\tau\mathcal{U}(\tau)f - \Delta\mathcal{U}(\tau)f = 0$ by $\big(\mathcal{U}(\tau)f\big)^{2p-1}$ and integrate by parts. We get,

$$\frac{1}{2p}\frac{d}{d\tau}\|\mathcal{U}(\tau)f\|^{2p}_{L^{2p}(\Sigma_t)} + (2p-1)\int|\nabla\mathcal{U}(\tau)f|^2|\mathcal{U}(\tau)f|^{2p-2} = 0.$$

Therefore,

$$\|\mathcal{U}(\tau)f\|_{L^{2p}(\Sigma_t)} \leq \|f\|_{L^{2p}(\Sigma_t)}.$$

The case when F is a tensorfield can be treated in the same manner with multiplier $\big(|\mathcal{U}(\tau)F|^2\big)^{p-1}\mathcal{U}(\tau)F$. ∎

3.6.3 Invariant Littlewood-Paley theory on Σ_t

In this section we shall develop an invariant, fully tensorial, Littlewood-Paley theory on Σ_t. We follow the analog construction in [10] for two dimensional compact manifolds. Now, the results essentially rely on the properties of the heat flow discussed in the previous section. Since these properties are true for manifolds of arbitrary dimensions, both compact and noncompact, the results in [10] extend in a straightforward fashion. Thus, we recall below the main objects introduced in [10], and we refer to [10] for the proofs.

Definition 3.34 *Consider the function m constructed in Lemma 3.8. We set, $m_k(\tau) = 2^{2k}m(2^{2k}\tau)$ and define the geometric Littlewood -Paley (LP) projections Q_k, associated to the LP- representative function $m \in \mathfrak{M}$, for arbitrary tensorfields F on Σ_t to be*

$$Q_kF = \int_0^\infty m_k(\tau)\mathcal{U}(\tau)Fd\tau \tag{3.106}$$

Given an interval $I \subset \mathbb{Z}$ we define

$$Q_I = \sum_{k\in I} Q_kF.$$

In particular we shall use the notation $Q_{<k}, Q_{\leq k}, Q_{>k}, Q_{\geq k}$.

Observe that Q_k are selfadjoint, i.e., $Q_k = Q_k^*$, in the sense,

$$< Q_k F, G >=< F, Q_k G >,$$

where, for any given m-tensors F, G

$$< F, G >= \int_{\Sigma_t} g^{i_1 j_1} \dots g^{i_m j_m} F_{i_1 \dots i_m} G_{j_1 \dots j_m} d\mathrm{vol}_g$$

denotes the usual $L^2(\Sigma_t)$ scalar product. Also, note from (3.14) that the LP-projections associated to m

$$\sum_k Q_k = I. \tag{3.107}$$

The following theorem summarizes the main properties of the Littlewood-Paley decompositions Q_k.

Theorem 3.35 *The LP-projections Q_k associated verify the following properties:*

i) $L^p(\Sigma_t)$-boundedness For any $1 \le p \le \infty$, and any interval $I \subset \mathbb{Z}$,

$$\|Q_I F\|_{L^p(\Sigma_t)} \lesssim \|F\|_{L^p(\Sigma_t)} \tag{3.108}$$

ii) Bessel inequality

$$\sum_k \|Q_k F\|^2_{L^2(\Sigma_t)} \lesssim \|F\|^2_{L^2(\Sigma_t)}$$

iii) Finite band property For any $1 \le p \le \infty$.

$$\begin{aligned} \|\Delta Q_k F\|_{L^p(\Sigma_t)} &\lesssim 2^{2k} \|F\|_{L^p(\Sigma_t)} \\ \|Q_k F\|_{L^p(\Sigma_t)} &\lesssim 2^{-2k} \|\Delta F\|_{L^p(\Sigma_t)} \end{aligned}$$

Moreover given $m \in \mathfrak{M}$ we can find $\bar{m} \in \mathfrak{M}$ such that $\Delta Q_k = 2^{2k} \bar{P}_k$ with $\bar{P}_k$ the LP projections associated to $\bar{m}$.

In addition, the $L^2(\Sigma_t)$ estimates

$$\begin{aligned} \|\nabla Q_k F\|_{L^2(\Sigma_t)} &\lesssim 2^k \|F\|_{L^2(\Sigma_t)} \\ \|Q_k F\|_{L^2(\Sigma_t)} &\lesssim 2^{-k} \|\nabla F\|_{L^2(\Sigma_t)} \end{aligned}$$

hold together with the dual estimate

$$\|Q_k \nabla F\|_{L^2(\Sigma_t)} \lesssim 2^k \|F\|_{L^2(\Sigma_t)}$$

iv) Bernstein inequality For any $2 \le p \le +\infty$

$$\begin{aligned} &\|Q_k F\|_{L^p(\Sigma_t)} \lesssim (2^{3(\frac{1}{2}-\frac{1}{p})k} + 1) \|F\|_{L^2(\Sigma_t)}, \\ &\|Q_{<0} F\|_{L^p(\Sigma_t)} \lesssim \|F\|_{L^2(\Sigma_t)} \end{aligned}$$

together with the dual estimates

$$\begin{aligned} &\|Q_k F\|_{L^2(\Sigma_t)} \lesssim (2^{3(\frac{1}{2}-\frac{1}{p})k} + 1) \|F\|_{L^{p'}(\Sigma_t)}, \\ &\|Q_{<0} F\|_{L^2(\Sigma_t)} \lesssim \|F\|_{L^{p'}(\Sigma_t)} \end{aligned}$$

Proof We refer to [10] for the proof of i)-iii). Next, we turn to the proof of iv). In the case $2 \leq p \leq 6$, it is an easy consequence of the Gagliardo-Nirenberg inequality (3.96):

$$\begin{aligned} \|Q_k F\|_{L^p(\Sigma_t)} &\lesssim \|\nabla Q_k F\|_{L^2(\Sigma_t)}^{3(\frac{1}{2}-\frac{1}{p})} \|Q_k F\|_{L^2(\Sigma_t)}^{-\frac{1}{2}+\frac{3}{p}} \\ &\lesssim 2^{3(\frac{1}{2}-\frac{1}{p})k} \|F\|_{L^2(\Sigma_t)}, \end{aligned}$$

where we used the finite band property and the boundedness on $L^2(\Sigma_t)$ for Q_k. Next, we consider the case $6 < p \leq +\infty$. Using the Gagliardo-Nirenberg inequality (3.102), we have

$$\begin{aligned} \|Q_k F\|_{L^p(\Sigma_t)} &\lesssim \|\nabla^2 Q_k F\|_{L^2(\Sigma_t)}^{\frac{1}{2}-\frac{3}{p}} \|\nabla Q_k F\|_{L^2(\Sigma_t)}^{\frac{1}{2}+\frac{3}{p}} \\ &\lesssim (\|\Delta Q_k F\|_{L^2(\Sigma_t)} + \|Q_k F\|_{L^2(\Sigma_t)})^{\frac{1}{2}-\frac{3}{p}} 2^{k(\frac{1}{2}+\frac{3}{p})} \|F\|_{L^2(\Sigma_t)}^{\frac{1}{2}+\frac{3}{p}} \\ &\lesssim (2^{3(\frac{1}{2}-\frac{1}{p})k} + 1) \|F\|_{L^2(\Sigma_t)} \end{aligned}$$

where we used the Bochner inequality (3.81), and the finite band property and the boundedness on $L^2(\Sigma_t)$ for Q_k. This concludes the proof of vi), and of the theorem. ■

3.6.4 Besov spaces on Σ_t

Using the Littlewood-Paley projections of the previous section, we introduce Besov spaces on Σ_t.

Definition 3.36 *Let $a \geq 0$. We define the Besov norms*

$$\|F\|_{\widehat{\mathcal{B}}^a} = \sum_{j \geq 0} 2^{aj} \|Q_j F\|_{L^2(\Sigma_t)} + \|F\|_{L^2(\Sigma_t)},$$

where F is an arbitrary tensor on Σ_t.

In view of the definition of $\widehat{\mathcal{B}}^{\frac{3}{2}}$ and the Bernstein inequality for Q_j, we immediately obtain the following embedding

$$\|F\|_{L^\infty(\Sigma_t)} \lesssim \|F\|_{\widehat{\mathcal{B}}^{\frac{3}{2}}(\Sigma_t)} \tag{3.109}$$

where F is an arbitrary tensor on Σ_t.

Next, we consider the action of ∇ on $\widehat{\mathcal{B}}^{\frac{5}{2}}$.

Lemma 3.37 *Let f a scalar function on Σ_t. Then, we have the following estimate*

$$\|\nabla f\|_{\widehat{\mathcal{B}}^{\frac{3}{2}}} \lesssim \|f\|_{\widehat{\mathcal{B}}^{\frac{5}{2}}}. \tag{3.110}$$

Proof We have

$$\|Q_j \nabla f\|_{L^2(\Sigma_t)} \lesssim \|Q_j \nabla Q_{<0} f\|_{L^2(\Sigma_t)} + \sum_{l \geq 0} \|Q_j \nabla Q_l f\|_{L^2(\Sigma_t)}. \tag{3.111}$$

Next, we estimate the right-hand side of (3.111). We start with the case $j < l$. Using the finite band property for Q_j, we have

$$\|Q_j \nabla Q_l f\|_{L^2(\Sigma_t)} \lesssim 2^j \|Q_l f\|_{L^2(\Sigma_t)}. \tag{3.112}$$

Next, we consider the case $l \leq j$. Using the finite band property for Q_j, we have

$$\|Q_j \nabla Q_l f\|_{L^2(\Sigma_t)} \lesssim 2^{-2j} \|\Delta \nabla Q_l f\|_{L^2(\Sigma_t)}. \tag{3.113}$$

Furthermore, we have $[\Delta, \nabla] h = R_t \nabla h$ for any scalar h on Σ_t, where R_t is the curvature tensor of the induced metric on Σ_t. Thus, we obtain

$$\begin{aligned} \|\Delta \nabla Q_l f\|_{L^2(\Sigma_t)} &\lesssim \|\nabla \Delta Q_l f\|_{L^2(\Sigma_t)} + \|[\Delta, \nabla] Q_l f\|_{L^2(\Sigma_t)} \\ &\lesssim \|\nabla \Delta Q_l f\|_{L^2(\Sigma_t)} + \|R_t \nabla Q_l f\|_{L^2(\Sigma_t)} \\ &\lesssim \|\nabla \Delta Q_l f\|_{L^2(\Sigma_t)} + \|R_t\|_{L^2(\Sigma_t)} \|\nabla Q_l f\|_{L^\infty(\Sigma_t)} \\ &\lesssim 2^{3l} \|Q_l f\|_{L^2(\Sigma_t)} + \|\nabla Q_l f\|_{L^\infty(\Sigma_t)}, \end{aligned} \tag{3.114}$$

where we used in the last inequality the finite band property for P_l and the bound (4.26) for R_t. Next, we evaluate the second term in the right-hand side of (3.112). Using the Gagliardo-Nirenberg inequality (3.102) with $p = +\infty$, we have

$$\|\nabla Q_l f\|_{L^\infty(\Sigma_t)} \lesssim \|\nabla^3 Q_l f\|^{\frac{1}{2}}_{L^\infty(\Sigma_t)} \|\nabla^2 Q_l f\|^{\frac{1}{2}}_{L^\infty(\Sigma_t)} + \|\nabla^2 Q_l f\|_{L^\infty(\Sigma_t)}.$$

Together with the Bochner inequality (3.81) on Σ_t, we obtain

$$\begin{aligned} \|\nabla Q_l f\|_{L^\infty(\Sigma_t)} &\lesssim (\|\Delta \nabla Q_l f\|_{L^2(\Sigma_t)} + \|\Delta Q_l f\|_{L^2(\Sigma_t)} + \|\nabla Q_l f\|_{L^2(\Sigma_t)})^{\frac{1}{2}} \\ &\quad \times (\|\Delta Q_l f\|_{L^2(\Sigma_t)} + \|\nabla Q_l f\|_{L^2(\Sigma_t)})^{\frac{1}{2}} + \|\Delta Q_l f\|_{L^2(\Sigma_t)} + \|\nabla Q_l f\|_{L^2(\Sigma_t)} \\ &\lesssim \|\Delta \nabla Q_l f\|^{\frac{1}{2}}_{L^2(\Sigma_t)} 2^l \|Q_l f\|^{\frac{1}{2}}_{L^2(\Sigma_t)} + 2^{2l} \|Q_l f\|_{L^2(\Sigma_t)} \end{aligned} \tag{3.115}$$

where we used the finite band property for Q_l in the last inequality. (3.114) and (3.115) imply

$$\|\Delta \nabla Q_l f\|_{L^2(\Sigma_t)} \lesssim \|\Delta \nabla Q_l f\|^{\frac{1}{2}}_{L^2(\Sigma_t)} 2^l \|Q_l f\|^{\frac{1}{2}}_{L^2(\Sigma_t)} + 2^{3l} \|Q_l f\|_{L^2(\Sigma_t)},$$

which yields

$$\|\Delta \nabla Q_l f\|_{L^2(\Sigma_t)} \lesssim 2^{3l} \|Q_l f\|_{L^2(\Sigma_t)}.$$

Together with (3.113), we obtain

$$\|Q_j \nabla Q_l f\|_{L^2(\Sigma_t)} \lesssim 2^{-2j+3l} \|Q_l f\|_{L^2(\Sigma_t)}. \tag{3.116}$$

Finally, using (3.112) for $j < l$ and (3.116) for $j \geq l$, we obtain

$$2^{\frac{3j}{2}} \|Q_j \nabla Q_l f\|_{L^2(\Sigma_t)} \lesssim 2^{-\frac{|j-l|}{2}} (2^{\frac{5l}{2}} \|Q_l f\|_{L^2(\Sigma_t)}),$$

which together with (3.111) and the definition of $\widehat{\mathcal{B}}^{\frac{3}{2}}$ implies (3.110). This concludes the proof of the lemma. ∎

We conclude this section with two estimates for the product in the Besov space $\widehat{\mathcal{B}}^{\frac{1}{2}}$.

Lemma 3.38 *We have*

$$\||F|^2\|_{\widehat{\mathcal{B}}^{\frac{1}{2}}} \lesssim (\|\nabla F\|_{L^2(\Sigma_t)} + \|F\|_{L^2(\Sigma_t)})^2 \tag{3.117}$$

for any tensor F on Σ_t.

Proof We have

$$\|Q_j(|F|^2)\|_{L^2(\Sigma_t)} \lesssim \sum_{l,m\geq 0} \|Q_j(Q_l F \cdot Q_m F)\|_{L^2(\Sigma_t)}, \tag{3.118}$$

where we dropped the terms involving $Q_{<0}$ which are easier to handle. Next, we estimate the right-hand side of (3.118). By symmetry, we may assume $m \leq l$. We start with the case $j < m$. Using the dual Bernstein inequality for Q_j, we have

$$\|Q_j(Q_l F \cdot Q_m F)\|_{L^2(\Sigma_t)} \lesssim 2^{\frac{3j}{2}} \|Q_l F \cdot Q_m F\|_{L^1(\Sigma_t)} \lesssim 2^{\frac{3j}{2}} \|Q_l F\|_{L^2(\Sigma_t)} \|Q_m F\|_{L^2(\Sigma_t)}. \tag{3.119}$$

Next, we consider the case $m \leq j < l$. Using the boundedness on $L^2(\Sigma_t)$ of Q_j and the Bernstein inequality for Q_m, we have

$$\|Q_j(Q_l F \cdot Q_m F)\|_{L^2(\Sigma_t)} \lesssim \|Q_l F\|_{L^2(\Sigma_t)} \|Q_m F\|_{L^\infty(\Sigma_t)} \lesssim 2^{\frac{3m}{2}} \|Q_l F\|_{L^2(\Sigma_t)} \|Q_m F\|_{L^2(\Sigma_t)}. \tag{3.120}$$

Finally, we consider the case $l \leq j$. Using the finite band property for Q_j, Q_l and Q_m, we have

$$\begin{aligned}
& \|Q_j(Q_l F \cdot Q_m F)\|_{L^2(\Sigma_t)} \\
\lesssim\ & 2^{-2j} \|\Delta(Q_l F \cdot Q_m F)\|_{L^2(\Sigma_t)} \\
\lesssim\ & 2^{-2j} \|\Delta(Q_l F) \cdot Q_m F\|_{L^2(\Sigma_t)} + 2^{-2j} \|\nabla(Q_l F) \cdot \nabla(Q_m F)\|_{L^2(\Sigma_t)} \\
& + 2^{-2j} \|Q_l F \cdot \Delta(Q_m F)\|_{L^2(\Sigma_t)} \\
\lesssim\ & (2^{-2j+2l} + 2^{-2j+2m}) \|Q_l F\|_{L^2(\Sigma_t)} \|Q_m F\|_{L^\infty(\Sigma_t)} + 2^{-2j} \|\nabla Q_l F\|_{L^4(\Sigma_t)} \|\nabla Q_m F\|_{L^4(\Sigma_t)}.
\end{aligned}$$

Together with the Bernstein inequality for Q_m, the Gagliardo-Nirenberg inequality (3.96), and the fact that $m \leq l$, we obtain

$$\begin{aligned}
& \|Q_j(Q_l F \cdot Q_m F)\|_{L^2(\Sigma_t)} \\
\lesssim\ & 2^{-2j+2l+\frac{3m}{2}} \|Q_l F\|_{L^2(\Sigma_t)} \|Q_m F\|_{L^2(\Sigma_t)} \\
& + 2^{-2j} \|\nabla^2 Q_l F\|^{\frac{3}{4}}_{L^2(\Sigma_t)} \|\nabla Q_l F\|^{\frac{1}{4}}_{L^2(\Sigma_t)} \|\nabla^2 Q_m F\|^{\frac{3}{4}}_{L^2(\Sigma_t)} \|\nabla Q_m F\|^{\frac{1}{4}}_{L^2(\Sigma_t)}.
\end{aligned}$$

Using the finite band property for Q_l and Q_m, the Bochner inequality (3.81) on Σ_t and the fact that $m \leq l$, we obtain

$$\|Q_j(Q_l F \cdot Q_m F)\|_{L^2(\Sigma_t)} \lesssim 2^{-2j+2l+\frac{3m}{2}} \|Q_l F\|_{L^2(\Sigma_t)} \|Q_m F\|_{L^2(\Sigma_t)}. \tag{3.121}$$

In the end, (3.119), (3.120) and (3.121) imply

$$2^{\frac{j}{2}} \|Q_j(Q_l F \cdot Q_m F)\|_{L^2(\Sigma_t)} \lesssim 2^{-\frac{|l-j|}{4} - \frac{|m-j|}{4}} (2^l \|Q_l F\|_{L^2(\Sigma_t)})(2^m \|Q_m F\|_{L^2(\Sigma_t)}),$$

which together with (3.118) and the definition of $\widehat{\mathcal{B}}^{\frac{1}{2}}$ implies (3.117). This concludes the proof of the lemma. ∎

Lemma 3.39 *We have*

$$\|fh\|_{\widehat{\mathcal{B}}^{\frac{1}{2}}} \lesssim (\|f\|_{L^\infty(\Sigma_t)} + \|\nabla h\|_{L^3(\Sigma_t)})\|h\|_{\widehat{\mathcal{B}}^{\frac{1}{2}}} \tag{3.122}$$

for any scalars f and h on Σ_t.

Proof We have

$$\|Q_j(fh)\|_{L^2(\Sigma_t)} \lesssim \sum_{l\geq 0} \|Q_j(fQ_l h)\|_{L^2(\Sigma_t)}, \tag{3.123}$$

where we dropped the term involving $Q_{<0}$ which is easier to handle. Next, we estimate the right-hand side of (3.123). We start with the case $j < l$. Using the boundedness on $L^2(\Sigma_t)$ of Q_j, we have

$$\|Q_j(fQ_l h)\|_{L^2(\Sigma_t)} \lesssim \|fQ_l h\|_{L^2(\Sigma_t)} \lesssim \|f\|_{L^\infty(\Sigma_t)}\|Q_l h\|_{L^2(\Sigma_t)}. \tag{3.124}$$

Next, we consider the case $l \leq j$. Using the finite band property for Q_j, we have

$$\begin{aligned} \|Q_j(fQ_l h)\|_{L^2(\Sigma_t)} &\lesssim 2^{-j}\|\nabla(fQ_l h)\|_{L^2(\Sigma_t)} \\ &\lesssim 2^{-j}\|\nabla f\|_{L^3(\Sigma_t)}\|Q_l h\|_{L^6(\Sigma_t)} + 2^{-j}\|f\|_{L^\infty(\Sigma_t)}\|\nabla Q_l h\|_{L^2(\Sigma_t)}. \end{aligned}$$

Using the Bernstein inequality and the finite band property for Q_l, this yields

$$\|Q_j(fQ_l h)\|_{L^2(\Sigma_t)} \lesssim 2^{-j+l}(\|\nabla f\|_{L^3(\Sigma_t)} + \|f\|_{L^\infty(\Sigma_t)})\|Q_l h\|_{L^2(\Sigma_t)}. \tag{3.125}$$

Finally, (3.124) and (3.125) imply

$$2^{\frac{j}{2}}\|Q_j(fQ_l h)\|_{L^2(\Sigma_t)} \lesssim 2^{-\frac{|l-j|}{2}}(2^{\frac{l}{2}}\|Q_l h\|_{L^2(\Sigma_t)}),$$

which together with (3.123) and the definition of $\widehat{\mathcal{B}}^{\frac{1}{2}}$ implies (3.122). This concludes the proof of the lemma. ∎

4 Regularity with respect to (t, x)

This section is devoted to the proof of Theorem 2.21. We assume the following bootstrap assumptions:

$$\|n-1\|_{L^\infty(\mathcal{H}_u)} + \|b-1\|_{L^\infty(\mathcal{H}_u)} \leq \frac{1}{10}, \tag{4.1}$$

$$\begin{aligned} &\|\nabla n\|_{L^\infty_t L^2_{x'}} + \|\nabla^2 n\|_{L^\infty_t L^2_{x'}} + \|\nabla^2 n\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla T(n)\|_{L^\infty_t L^2_{x'}} \\ &\qquad + \|\nabla T(n)\|_{L^\infty_t L^2(\Sigma_t)} + \mathcal{N}_2(b-1) + \|\underline{L}(b)\|_{L^2_{x'}L^\infty_t} \leq D\varepsilon, \end{aligned} \tag{4.2}$$

$$\mathcal{N}_1(k) + \|\nabla\!\!\!/_{\underline{L}}\epsilon\|_{L^2(\mathcal{H}_u)} + \|\mathbf{D}_{\underline{L}}\delta\|_{L^2(\mathcal{H}_u)} + \|\overline{\epsilon}\|_{L^\infty_{x'}L^2_t} + \|\overline{\delta}\|_{L^\infty_{x'}L^2_t} \leq D\varepsilon, \tag{4.3}$$

$$\|\mathrm{tr}\chi\|_{L^\infty(\mathcal{H}_u)} + \|\mathrm{tr}\chi\|_{L^2_{x'}L^\infty_t} + \|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2_{x'}L^\infty_t} + \|\underline{L}\mathrm{tr}\chi\|_{L^2_{x'}L^\infty_t} \leq D\varepsilon, \tag{4.4}$$

$$\|\widehat{\chi}\|_{L^\infty_{x'}L^2_t} + \mathcal{N}_1(\widehat{\chi}) + \|\nabla\!\!\!/_{\underline{L}}\widehat{\chi}\|_{L^2(\mathcal{H}_u)} \leq D\varepsilon, \tag{4.5}$$

$$\|\zeta\|_{L^2_{x'}L^\infty_t} + \mathcal{N}_1(\zeta) \leq D\varepsilon, \tag{4.6}$$

where $D > 0$ is a large enough constant. We will improve on these estimates. To this end, we show in section 4.1 that the null hypersurfaces $\mathcal{H}_u$ are well-behaved for $0 \leq t \leq 1$, in the sense that there are neither caustics nor intersection of distinct null geodesics generating $\mathcal{H}_u$. In section 4.2, we construct coordinate systems on $P_{t,u}$ and Σ_t needed for the validity of the estimates derived in section 3. In section 4.3, we derive an estimate for the Gauss curvature K necessary to obtain a useful strong Bernstein inequality, as well as a useful Bochner inequality on $P_{t,u}$. In sections 4.4 and 4.5, we improve on the bootstrap bounds for n and k in (4.1)-(4.3), with the exception of the trace estimates for $\overline{\epsilon}$ and $\overline{\delta}$ in (4.3). In section 4.6, we show how to infer estimates in the time foliation from corresponding estimates in the geodesic foliation obtained in [14]. This allows us to improve on the L^∞ bound for $\mathrm{tr}\chi$ and the trace bounds on $\widehat{\chi}$ and ζ in the bootstrap bounds (4.4) and (4.6). In section 4.7, we improve on the bootstrap bound (4.3) for the trace estimates of $\overline{\delta}$ and $\overline{\epsilon}$. In section 4.8, we improve on the bootstrap bounds for b in (4.1) (4.2), and we also derive an estimate for b in $L^\infty_t L^4_{x'}$. Finally, we improve on the remaining bootstrap bounds in (4.4)-(4.6) in section 4.9.

Remark 4.1 *This section concerns the regularity of the foliation generated by u on $\mathcal{M}$ with respect to (t, x). Thus, the dependance in the angle $\omega \in \mathbb{S}^2$ plays no role in this section.*

4.1 Lower bound on the injectivity radius on $\mathcal{H}_u$

The control we obtain on the geometric quantities associated to our foliation is only valid as long as no caustic form and null geodesics do not intersect on $\mathcal{H}_u$. The goal of this section is to prove the absence of caustic and that null geodesics do not intersect at least until $t = 1$, i.e. the null radius of injectivity of $\mathcal{H}_u$ is at least 1. In addition to the bound (2.59) on the curvature tensor $\mathbf{R}$ of $\mathbf{g}$, we make the following regularity assumption on $\mathbf{g}$. There exists a coordinate chart on $\mathcal{M}$ such that

$$\|\mathbf{g}\|_{C^2(\mathcal{M})} \leq M, \tag{4.7}$$

where M is a very large constant.

Remark 4.2 *The assumption* (4.7) *is only used to prove the absence of caustic and that null geodesics do not intersect at least until $t = 1$, which is a qualitative property. On the other hand, we only rely on the bound* (2.59) *on $\mathbf{R}$ to prove the various quantitative bounds of Theorems 2.21, 2.23, 2.24 and 2.27.*

For $(0, x)$ in Σ_0, recall the definition in Remark 2.3 of the null geodesic $\kappa_x(t)$. For all $0 \leq t \leq 1$, let $\Phi_t : \Sigma_0 \to \Sigma_t$ defined by $\Phi_t(0, x) = \kappa_x(t)$. We have $\Phi_0(0, x) = (0, x)$ on Σ_0 which together with (4.7) and the global inversion theorem shows that Φ_t is a C^1 global diffeomorphism from Σ_0 to Σ_t for $0 \leq t \leq \frac{1}{100M}$. We define $t_0 \geq 0$ as the supremum of $0 \leq t \leq 1$ such that Φ_t is bijective from Σ_0 to Σ_t. Our goal is to show that we have in fact $t_0 = 1$. We will first show the absence of caustic which is a consequence of the fact that Φ_t is locally injective. We will then show that Σ_t is covered by the u-foliation which is

equivalent to the surjectivity of Φ_t. Finally, we will show the nonintersection of distinct null geodesics which is equivalent to the global injectivity of Φ_t.

Remark 4.3 *As long as $0 \leq t < t_0$, there are no caustics and no distinct null geodesic intersections. Thus, we may assume that the u-foliation exists and satisfies the bounds* (4.1)-(4.6) *given by the bootstrap assumptions. Furthermore, we may assume the identity* (2.12) *for the null geodesics $\kappa_x(t)$.*

4.1.1 Absence of caustic

The absence of caustic is equivalent to the absence of conjugate points and is a consequence of the fact that Φ_t is locally injective. Since Φ_t preserves the u-foliation, it is enough to show that Φ_t is locally injective as a map from $P_{0,u}$ to $P_{t,u}$. We will actually show that Φ_t as a map from $P_{0,u}$ to $P_{t,u}$ is a local C^1 diffeomorphism.

Let $(0, x)$ a point in $P_{0,u}$. From (2.12), we have $\kappa_x'(t) = nL_{\kappa_x(t)}$ for all $0 \leq t < t_0$. Since $\Phi_t(0, x) = \kappa_x(t)$, we obtain the following differential equation for the Jacobian matrix $D\Phi_t$ of Φ_t:

$$\frac{d}{dt}\left(D\Phi_t\right) = n\chi(D\Phi_t, e_B)e_B$$

which together with the fact that χ is symmetric yields:

$$\frac{d}{dt}\left(\det\left(D\Phi_t(D\Phi_t)^T\right)\right) = 2n\mathrm{tr}\chi \det\left(D\Phi_t(D\Phi_t)^T\right)$$

and after integration in time:

$$\det\left(D\Phi_t(D\Phi_t)^T\right) = \exp\left(2\int_0^t n\mathrm{tr}\chi d\tau\right) \sim 1 \tag{4.8}$$

where we used the bootstrap assumption (4.4). In particular, the local inversion Theorem together with (4.8) and (4.7) yields the fact that Φ_t as a map from $P_{0,u}$ to $P_{t,u}$ is a local C^1 diffeomorphism. In particular, no caustic form for all $0 \leq t < t_0$.

4.1.2 Covering of Σ_t by the u-foliation

We will prove that for all $0 \leq t < t_0$, Σ_t is covered by the u-foliation, i.e.:

$$\Sigma_t = \cup_u P_{t,u}$$

which is equivalent to the surjectivity of Φ_t as a map from Σ_0 to Σ_t.

Let $A = \{t \,/\, \Sigma_t = \cup_u P_{t,u}\}$. We start by showing that A is closed in $0 \leq t < t_0$. Consider a sequence of times $t_p \to \underline{t}$ such that t_p belongs to A for all p. Let $(\underline{t}, x)$ an arbitrary point in $\Sigma_{\underline{t}}$. There exists a sequence (t_p, x_p) in Σ_{t_p} such that (t_p, x_p) converges to $(\underline{t}, x)$. Since (t_p, x_p) is in Σ_{t_p} and t_p belongs to A, (t_p, x_p) belong to $\cup\Sigma_{t_p}$ and therefore there is $(0, x_p^0)$ in Σ_0 such that $(t_p, x_p) = \kappa_{x_p^0}(t_p)$. Now, the bound (4.7) together with the fact that (t_p, x_p) is a bounded sequence implies that $(0, x_p^0)$ is a bounded sequence in Σ_0.

Thus, up to a subsequence, it converges to $(0, x_0)$ in Σ_0. Finally, using again the bound (4.7) together with the fact that t_p converges to $\underline{t}$ and $(0, x_p^0)$ converges to $(0, x_0)$ implies that $\kappa_{x_p^0}(t_p)$ converges to $\kappa_{x_0}(\underline{t})$. Thus, $(\underline{t}, x) = \kappa_{x_0}(\underline{t})$ which shows that $(\underline{t}, x)$ belongs to $\cup_u P_{t,u}$. Therefore, $\underline{t}$ belongs to A which implies that A is closed.

Let us now prove that A is open in $0 \leq t < t_0$. Let $\underline{t} \in A$ and consider times t satisfying $|t - \underline{t}| < \frac{1}{100M}$ where M is the constant appearing in (4.7). Let (t, x_0) an arbitrary point in Σ_t. We may assume $t > \underline{t}$ since the case $t < \underline{t}$ is treated in the exact same way. Let C^- denote the backward null cone with vertex (t, x_0) (we would consider the forward null cone in the case $t < \underline{t}$). Let S^- denote the intersection of the backward null cone C^- with $\Sigma_{\underline{t}}$. Then, the assumption $|t - \underline{t}| < \frac{1}{100M}$ together with the bound (4.7) implies that S^- is a C^1 compact orientable surface in $\Sigma_{\underline{t}}$. In particular, since any compact set of $\Sigma_{\underline{t}}$ is included in $\{-B < u < B\}$ for a large enough constant B, the set $\{u \,/\, P_{t,u} \cap S^- \neq \emptyset\}$ is a bounded subset of $\mathbb{R}$. Using the fact that S^- is compact, $P_{t,u}$ is closed in $\Sigma_{\underline{t}}$, and $\underline{t} \in A$, we obtain the existence of u_0 such that $P_{t,u_0} \cap S^- \neq \emptyset$ and:

$$u_0 = \min\{u \,/\, P_{t,u} \cap S^- \neq \emptyset\}.$$

Let $(\underline{t}, x_1)$ a point in $P_{t,u_0} \cap S^-$. Then, by definition of u_0 we have $P_{t,u} \cap S^- = \emptyset$ for all $u < u_0$ which implies that $N = -N_{S^-}$ at $(\underline{t}, x_1)$ where $N = \nabla u / |\nabla u|$ is the normal to $P_{t,u}$ and N_{S^-} is the outward normal to S^-. In turn, this implies that L coincides with the null generator of the backward null cone C^- at $(\underline{t}, x_1)$. From (2.3), let $(0, x_2)$ on Σ_0 such that $b^{-1}L = \kappa'_{x_2}(\underline{t})$. Since $\kappa'_{x_2}(\underline{t})$ coincides with the null generator of the backward null cone C^- at $(\underline{t}, x_1)$, we obtain $\kappa_{x_2}(t) = (t, x_0)$. Therefore, (t, x_0) belongs to P_{t,u_1} where $u_1 = u(0, x_2)$. This implies that $\Sigma_t = \cup_u P_{t,u}$ for all $|t - \underline{t}| < \frac{1}{100M}$ so that A is open.

Finally, A is closed and open in $0 \leq t < t_0$. Furthermore, $\Sigma_0 = \cup_u P_{0,u}$ from the construction of u on Σ_0 in [21]. Therefore, $A = \{0 \leq t < t_0\}$, i.e. $\Sigma_t = \cup_u P_{t,u}$ for all $0 \leq t < t_0$.

4.1.3 Nonintersection of distinct null geodesics

We would like to show that $t_0 \geq 1$. Assume by contradiction that $0 < t_0 < 1$.

Let us first show that there exist two distinct null geodesics intersecting at $t = t_0$. Assume by contradiction that there exists $\delta > 0$ such that no distinct null geodesics intersect on $0 \leq t < t_0 + \delta$. Then, u exists on $0 \leq t < t_0 + \delta$ unless a caustic forms at a time $0 < t_1 < t_0 + \delta$. Assume that t_1 is the first such time. Then, u exists on $0 \leq t < t_1$ and b and $\mathrm{tr}\chi$ satisfy the bootstrap assumptions (4.1) (4.4) on $0 \leq t < t_1$ so that (4.8) holds on $0 \leq t < t_1$. Now, since Φ_t is C^1 from the assumption (4.7), this implies that:

$$\det\left(D\Phi_t (D\Phi_t)^T\right) \sim 1,\ 0 \leq t < t_1 + \delta_1$$

for some $\delta_1 > 0$. In turn, this yields the absence of caustic for $0 \leq t < t_1 + \delta_1$ contradicting the definition of t_1. In particular, we obtain the absence of caustic for $0 \leq t < t_0 + \delta$, the existence of u on the same time interval, and in turn $\Sigma_t = \cup_u P_{t,u}$ from section 4.1.2.

Finally, on $0 \leq t < t_0 + \delta$, no distinct null geodesic intersect and $\Sigma_t = \cup_u P_{t,u}$ so that Φ_t which is both injective and surjective. This contradicts the definition of t_0. We conclude that there exist two distinct null geodesics that intersect at t_0.

From the previous paragraph, u exists on the time interval $0 \leq t < t_0$ where it satisfies $\Sigma_t = \cup_u P_{t,u}$ and the bootstrap assumptions (4.1)-(4.6). Furthermore, two distinct null geodesics intersect at t_0. Let $(0, x_1) \neq (0, x_2)$ two points in Σ_0 such that $\kappa_{x_1}(t_0) = \kappa_{x_2}(t_0) = (t_0, x_0)$. In view of (4.7), there exists a coordinate chart $U \subset \mathcal{M}$ which is a neighborhood of (t_0, x_0) such that relative to this coordinate system, we have:

$$\|\mathbf{g}_{\alpha\beta}\|_{C^2(U)} \lesssim M, \, \forall \alpha, \beta = 0, \ldots, 3. \tag{4.9}$$

Now, from the Ricci equations (2.23) we have:

$$\begin{aligned} & \|\mathbf{D}_L L\|_{L^\infty_u L^6(\mathcal{H}_u)} + \|\mathbf{D}_{\underline{L}} L\|_{L^\infty_u L^6(\mathcal{H}_u)} + \|\nabla\!\!\!/ L\|_{L^\infty_u L^6(\mathcal{H}_u)} \\ \lesssim \; & \|\chi\|_{L^\infty_u L^6(\mathcal{H}_u)} + \|\zeta\|_{L^\infty_u L^6(\mathcal{H}_u)} + \|k\|_{L^\infty_u L^6(\mathcal{H}_u)} + \|\nabla n\|_{L^\infty_u L^6(\mathcal{H}_u)}, \end{aligned} \tag{4.10}$$

which together with the Sobolev embedding (3.57) and the bootstrap assumptions (4.1)-(4.6) yields:

$$\|\mathbf{D} L\|_{L^\infty_u L^6(\mathcal{H}_u)} \lesssim 1. \tag{4.11}$$

From the bootstrap assumption (4.2) and the Sobolev embedding (3.57), we have:

$$\|T(n)\|_{L^\infty_t L^6(\Sigma_t)} + \|\nabla n\|_{L^\infty_t L^6(\Sigma_t)} \lesssim 1. \tag{4.12}$$

(4.11) and (4.12) yield:

$$\|\mathbf{D}(nL)\|_{L^6(\mathcal{M} \cap \{0 \leq t < t_0\})} \lesssim 1. \tag{4.13}$$

In particular, the same bound holds in $L^6(\mathcal{M} \cap \{0 \leq t < t_0\})$ which together with (4.9) implies in the coordinate chart U:

$$\|\partial(nL)\|_{L^6(U \cap \{0 \leq t < t_0\})} \lesssim M.$$

Together with the usual Sobolev embedding in dimension 4, this yields, in the coordinate chart U:

$$\|nL\|_{C^{\frac{1}{12}}(U \cap \{0 \leq t < t_0\})} \lesssim M. \tag{4.14}$$

Now, using the fact that $\kappa_{x_1}(t_0) = \kappa_{x_2}(t_0)$, and the fact that $\kappa_x(t)$ is continuous in t from (4.7), we have

$$\lim_{t \to t_{0-}} \mathrm{dist}(\kappa_{x_1}(t), \kappa_{x_2}(t)) = 0$$

where dist denotes the geodesic distance in Σ_t. Together with (4.14), this implies that the distance between $nL_{\kappa_{x_1}(t)}$ and $nL_{\kappa_{x_2}(t)}$ as vectors of $\mathbb{R}^4$ in the coordinate chart U converges to 0 as $t \to t_{0-}$. As $nL_{\kappa_{x_j}(t)} = \kappa'_{x_j}(t)$ for $0 \leq t < t_0$ and $j = 1, 2$ by (2.12), and since $\kappa'_x(t)$ is continuous in t from (4.7), we conclude that $\kappa'_{x_1}(t_0) = \kappa'_{x_2}(t_0)$. From the classical uniqueness result for ODEs, we deduce that $\kappa_{x_1}(t) = \kappa_{x_2}(t)$ for all t. In particular, taking $t = 0$, we obtain $(0, x_1) = (0, x_2)$ which yields a contradiction.

Finally, we have proved that $t_0 \geq 1$. In particular, we have:

On $0 \leq t \leq 1$, there are no caustics and no intersection of distinct null geodesics. In particular, u exists on $0 \leq t \leq 1$ and the bootstrap assumption (4.1)-(4.6) hold. Furthermore, $\Sigma_t = \cup_u P_{t,u}$ for all $0 \leq t \leq 1$. (4.15)

4.2 Coordinate systems on Σ_t and $P_{t,u}$

4.2.1 A global coordinate system on $P_{t,u}$

Lemma 4.4 *There exists a global coordinate system x' on $P_{t,u}$ satisfying:*

$$(1 - O(\varepsilon))|\xi|^2 \leq \gamma_{AB}(p)\xi^A\xi^B \leq (1 + O(\varepsilon))|\xi|^2, \qquad \text{uniformly for all } p \in P_{t,u}, \quad (4.16)$$

and the Christoffel symbols Γ^A_{BC} of the coordinate system verify:

$$\sum_{A,B,C} \int_{P_{t,u}} |\Gamma^A_{BC}|^2 dx^1 dx^2 \lesssim \varepsilon. \quad (4.17)$$

Remark 4.5 *Lemma 4.4 provides the existence of a global coordinate system on $P_{t,u}$ satisfying assumptions* (3.1) *and* (3.32)*. Thus, we may use the results of sections 3.1 and 3.2 in the rest of the paper.*

Proof : In **step B1**, we have constructed a global coordinate system $x' = (x^1, x^2)$ on $P_{0,u}$ (see [21]). By transporting this coordinate system along the null geodesics generated by L, we obtain a coordinate system x' of $P_{t,u}$, and a coordinate system (t, x') of $\mathcal{H}$. Let γ_t denote the restriction of $\mathbf{g}$ to $P_{t,u}$. We claim that relative to the coordinates (t, x') on $\mathcal{H}$, the metric γ_t verifies:

$$\frac{d}{dt}\gamma_{AB} = 2n\chi_{AB}. \quad (4.18)$$

Indeed relative to the coordinates t, x' on $\mathcal{H}$ we have $nL = \frac{\partial}{\partial t}$ and since $[\frac{\partial}{\partial t}, \frac{\partial}{\partial x^A}] = 0$ we infer from $\nabla_{nL}\gamma = 0$, and $\gamma_{AB} = \gamma(\frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B})$,

$$\begin{aligned} 0 &= (\nabla_{nL}\gamma)(\frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B}) = \frac{d}{dt}\gamma_{AB} - n\gamma(\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B}) - n\gamma(\frac{\partial}{\partial x^A}, \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial x^B}) \\ &= \frac{d}{dt}\gamma_{AB} - n\gamma(\nabla_{\frac{\partial}{\partial x^A}}L, \frac{\partial}{\partial x^B}) - n\gamma(\frac{\partial}{\partial x^A}, \nabla_{\frac{\partial}{\partial x^B}}L) \\ &= \frac{d}{dt}\gamma_{AB} - 2n\chi_{AB} \end{aligned}$$

as desired.

Now, using the bootstrap assumptions (4.1) and (4.4) (4.5), we have $|n - 1| \leq \frac{1}{2}$ and $\|\chi\|_{L^\infty_{x'}L^2_t} \leq D\varepsilon$. Together with (4.18) and the fact that (4.16) is satisfied on $P_{0,u}$, this yields (4.16).

Differentiating (4.18) and using the fact that derivatives $\frac{\partial}{\partial x^A}$ commute with $\frac{d}{dt}$, we obtain:

$$\begin{aligned}\frac{d}{dt}\partial_C\gamma_{AB} &= 2\nabla_C(n)\chi_{AB} + 2n\partial_C\chi_{AB}\\ &= 2\nabla_C(n)\chi_{AB} + 2n\nabla_C\chi_{AB} + (\partial\gamma)\cdot\chi\end{aligned}$$

with $(\partial\gamma)\cdot\chi$ denoting sum of terms involving only products between derivatives of the metric coefficients and components of χ. Therefore, using the bootstrap assumptions (4.1) and (4.4) (4.5), we obtain:

$$\begin{aligned}\|\partial\gamma\|_{L^\infty_t L^2_{x'}} &\lesssim \|\nabla\!\!\!/ n\|_{L^4(\mathcal{H}_u)}\|\chi\|_{L^4(\mathcal{H}_u)} + \|n\|_{L^\infty(\mathcal{H}_u)}\|\nabla\!\!\!/\chi\|_{L^2(\mathcal{H}_u)} + \|\chi\|_{L^\infty_{x'}L^2_t}\|\partial\gamma\|_{L^2(\mathcal{H}_u)}\\ &\lesssim \varepsilon + \varepsilon\|\partial\gamma\|_{L^\infty_t L^2_{x'}},\end{aligned}$$

which yields (4.17). This concludes the proof of lemma 4.4. ■

Remark 4.6 *Denoting $|\gamma| = \det(\gamma_{AB})$, we obtain from* (4.18)*:*

$$\frac{d}{dt}\log|\gamma| = \gamma^{AB}\frac{d}{dt}\gamma_{AB} = 2ntr\chi$$

or,

$$\frac{d}{dt}\sqrt{|\gamma|} = tr\chi\sqrt{|\gamma|}. \tag{4.19}$$

Now, relative to the coordinates t, x^1, x^2, $\int_{P_{t,u}} f d\mu_{t,u} = \int\int f\sqrt{|\gamma|}\,dx^1dx^2$, therefore,

$$\frac{d}{dt}\int_{P_{t,u}} f d\mu_{t,u} = \int\int \frac{d}{dt}(f\sqrt{|\gamma|})\,dx^1dx^2 = \int_{P_{t,u}} \big(\frac{d}{dt}f + ntr\chi f\big)d\mu_{t,u}$$

which proves (3.56)*.*

Remark 4.7 *Since the global coordinate system x' on $P_{t,u}$ is obtained by transporting the coordinate system on $P_{0,u}$ along the null geodesics generated by L, it requires in particular that null geodesics generating $\mathcal{H}$ have no conjugate points, and that two distinct null geodesics do not intersect. This fact has been proved in section 4.1 (see* (4.15)*).*

4.2.2 A global coordinate system on Σ_t

Recall that we have constructed a global coordinate system on $P_{t,u}$ in section 4.2.1. Let us denote x' such a coordinate system. We obtain a global coordinate system on Σ_t as follows.

- First, recall from (4.15) that $\Sigma_t = \cup P_{t,u}$ so that u is defined on Σ_t.
- To any $p \in \Sigma_t$, we associate the coordinates $(u(p), x'(p))$ where $u(p)$ is the value of the optical function u at p, and $x'(p)$ are the coordinate of p in the coordinate system of $P_{t,u}$ constructed in section 4.2.1.

- Recall that the coordinate system of $P_{t,u}$ constructed in section 4.2.1 is global, i.e. $p \to x'(p)$ from $P_{t,u}$ to $\mathbb{R}^2$ is a bijection. Hence, $p \to (u(p), x'(p))$ from Σ_t to $\mathbb{R}^3$ is a bijection so that (u, x') provides a global coordinate system on Σ_t.

- Finally, by construction, and in view of the control the coordinates system of $P_{t,u}$ in section 4.2.1, the map $p \to (u(p), x'(p))$ is a diffeomorphism provided the lapse of the u-foliation of Σ_t does not degenerate. This is indeed the case since the lapse of the u-foliation of Σ_t is given by $|\nabla u|^{-1} = b$, with b under control in view of (4.1).

In this coordinate system, the metric g_t on Σ_t (i.e. the restriction of $\mathbf{g}$ on Σ_t) takes the following form:

$$g_t = \begin{pmatrix} b^{-2} & 0 \\ 0 & \gamma \end{pmatrix}, \tag{4.20}$$

where γ is the induced metric on $P_{t,u}$. Together with the estimate (4.1) for b and (4.16) for γ, we obtain

$$\left(\frac{10}{11} + O(\varepsilon)\right) |\xi|^2 \leq (g_t)_{ij}(p)\xi^i\xi^j \leq \left(\frac{11}{10} + O(\varepsilon)\right) |\xi|^2,$$

and thus, for $\varepsilon > 0$ small enough, we deduce

$$\frac{5}{6}|\xi|^2 \leq (g_t)_{ij}(p)\xi^i\xi^j \leq \frac{6}{5}|\xi|^2. \tag{4.21}$$

This coordinate system allows us in particular to get a lower bound on the volume radius of the Riemannian manifold Σ_t. We recall below the definition of the volume radius on a general Riemannian manifold M.

Definition 4.8 *Let $B_r(p)$ denote the geodesic ball of center p and radius r. The volume radius $r_{vol}(p, r)$ at a point $p \in M$ and scales $\leq r$ is defined by*

$$r_{vol}(p, r) = \inf_{r' \leq r} \frac{|B_{r'}(p)|}{r'^3},$$

with $|B_r|$ the volume of B_r relative to the metric on M. The volume radius $r_{vol}(M, r)$ of M on scales $\leq r$ is the infimum of $r_{vol}(p, r)$ over all points $p \in M$.

Denote by $B^c_r(p)$ the euclidean ball of center p and radius r in the coordinate system (4.21) of Σ_t. Then, clearly $B^c_{\frac{5r}{6}}(p) \subset B_r(p)$. Thus, we obtain a lower bound for any $p \in \Sigma_t$:

$$|B_r(p)| \geq \left|B^c_{\frac{5r}{6}}(p)\right| = \int_{B^c_{\frac{5r}{6}}(p)} \sqrt{|g_t|} du dx' \geq \frac{5}{6}\left|B^c_{\frac{5r}{6}}(p)\right| \geq \frac{4\pi}{3}\left(\frac{5}{6}\right)^4 r^3,$$

which yields the following lower bound on the volume radius of Σ_t at scales ≤ 1:

$$r_{vol}(\Sigma_t, 1) \geq \frac{4\pi}{3}\left(\frac{5}{6}\right)^4. \tag{4.22}$$

4.2.3 Harmonic coordinates on Σ_t

We will need a second coordinate system on Σ_t since the metric coefficients in the coordinates system of section 4.2.2 are not regular enough for some of the applications we have in mind. Indeed, we only control some Christoffel symbols in this coordinate system (see for example (4.17)), but no second order derivative of the metric coefficients. The second coordinate system we have in mind are the harmonic coordinates. To obtain an appropriate covering of Σ_t by harmonic coordinates, we rely on the following general result based on Cheeger-Gromov convergence of Riemannian manifolds, see [1] or Theorem 5.4 in [18].

Theorem 4.9 *Given $c_1 > 0, c_2 > 0, c_3 > 0$, there exists $r_0 > 0$ such that any 3-dimensional, complete, Riemannian manifold (M, g) with $\|R\|_{L^2(M)} \leq c_1$ and volume radius at scales ≤ 1 bounded from below by c_2, i.e. $r_{vol}(M, 1) \geq c_2$, verifies the following property:*

Every geodesic ball $B_r(p)$ with $p \in M$ and $r \leq r_0$ admits a system of harmonic coordinates $x = (x_1, x_2, x_3)$ relative to which we have

$$(1 + c_3)^{-1}\delta_{ij} \leq g_{ij} \leq (1 + c_3)\delta_{ij}, \tag{4.23}$$

and

$$r \int_{B_r(p)} |\partial^2 g_{ij}|^2 \sqrt{|g|} dx \leq c_3. \tag{4.24}$$

To apply Theorem 4.9, we need to bound the curvature tensor R_t on Σ_t in $L^\infty_t L^2(\Sigma_t)$. Since Σ_t has dimension 3, it is enough to bound its Ricci tensor. Now, we have the following formula relating the Ricci tensor on Σ_t to the curvature tensor $\mathbf{R}$ on $\mathcal{M}$ and k:

$$(R_t)_{ij} = k_{il}k^l_j + \mathbf{R}_{iTjT}$$

which yields:

$$\|R_t\|_{L^\infty_t L^2(\Sigma_t)} \leq \|\mathbf{R}\|_{L^\infty_t L^2(\Sigma_t)} + \mathcal{N}_1(k)^2. \tag{4.25}$$

The curvature bound (2.59), the bootstrap assumption (4.3) and (4.25) imply:

$$\|R_t\|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon. \tag{4.26}$$

Let $\delta > 0$. (4.26) together with the volume lower bound (4.22) and Theorem (4.9) yields the existence of $r_0(\delta) > 0$ and a finite covering of Σ_t by geodesic balls of radius $r_0(\delta)$ such that each geodesic ball in the covering admits a system of harmonic coordinates $x = (x_1, x_2, x_3)$ relative to which we have

$$(1 + \delta)^{-1}\delta_{ij} \leq g_{ij} \leq (1 + \delta)\delta_{ij}, \tag{4.27}$$

and

$$r_0(\delta) \int_{B_{r_0}(p)} |\partial^2 g_{ij}|^2 \sqrt{|g|} dx \leq \delta. \tag{4.28}$$

Remark 4.10 *Σ_t is asymptotically flat and therefore admits a harmonic coordinates system in a neighborhood of infinity. Therefore, the problem of covering Σ_t with harmonic coordinates charts is reduced to a compact region which explains why we may chose finitely many harmonic coordinates charts covering Σ_t and satisfying* (4.27) (4.28).

4.3 Bound on the Gauss curvature K

The following proposition will be crucial to obtain useful strong Bernstein and Bochner inequalities.

Proposition 4.11 *Let K the gauss curvature on $P_{t,u}$. Then, K satisfies the following bounds:*

$$\|K\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon \tag{4.29}$$

and

$$\|\Lambda^{-\frac{1}{2}} K\|_{L^\infty_t L^2_{x'}} \lesssim \varepsilon. \tag{4.30}$$

The proof of Proposition 4.11 is postponed to section A.1. The following consequence of Proposition 4.11 will be useful in the sequel. Proposition 4.11 and (3.37) with the choice $a = 1/2$ imply:

$$\|K_{\frac{1}{2}}\|_{L^\infty(0,1)} = \|\Lambda^{-\frac{1}{2}} K\|_{L^\infty_t L^2_{x'}} \lesssim \varepsilon, \tag{4.31}$$

where $K_{1/2}$ has been defined in (3.33). Together with (3.34) and (3.35) with the choice $\gamma = 1/2$, we obtain for any scalar function f on $P_{t,u}$ and any $j \geq 0$ the following sharp Bernstein inequality:

$$\begin{aligned} \|P_j f\|_{L^\infty(P_{t,u})} &\lesssim 2^j \|f\|_{L^2(P_{t,u})}, & (4.32)\\ \|P_{<0} f\|_{L^\infty(P_{t,u})} &\lesssim \|f\|_{L^2(P_{t,u})}. & (4.33) \end{aligned}$$

Also, (4.31) and (3.36) with the choice $\gamma = 1/2$ imply the following Bochner inequality:

$$\int_{P_{t,u}} |\nabla\!\!\!/^2 f|^2 \lesssim \int_{P_{t,u}} |\Delta\!\!\!/ f|^2 + \varepsilon \int_{P_{t,u}} |\nabla\!\!\!/ f|^2. \tag{4.34}$$

Finally, using the Gagliardo-Nirenberg inequality (3.3) and (4.34), we obtain for any $2 \leq p < +\infty$, any $j \geq 0$, and any scalar function f:

$$\begin{aligned} \|\nabla\!\!\!/ P_j f\|_{L^p(P_{t,u})} &\lesssim \|\nabla\!\!\!/^2 P_j f\|^{1-\frac{2}{p}}_{L^2(P_{t,u})} \|\nabla\!\!\!/ P_j f\|^{\frac{2}{p}}_{L^2(P_{t,u})} + \|\nabla\!\!\!/ P_j f\|_{L^2(P_{t,u})} & (4.35)\\ &\lesssim (\|\Delta\!\!\!/ P_j f\|_{L^2(P_{t,u})} + \|\nabla\!\!\!/ P_j f\|_{L^2(P_{t,u})})^{1-\frac{2}{p}} 2^{\frac{2j}{p}} \|f\|^{\frac{2}{p}}_{L^2(P_{t,u})} + 2^j \|f\|_{L^2(P_{t,u})} \\ &\lesssim 2^{2(1-\frac{1}{p})j} \|f\|_{L^2(P_{t,u})}. \end{aligned}$$

Taking the dual of (4.35), we obtain for any $1 < p \leq 2$, any $j \geq 0$, and any $P_{t,u}$-tangent 1-form F:

$$\|P_j \mathrm{div}\!\!\!/(F)\|_{L^p(P_{t,u})} \lesssim 2^{\frac{2j}{p}} \|f\|_{L^2(P_{t,u})}. \tag{4.36}$$

Remark 4.12 (4.32) *and* (4.33) *only hold for scalar functions f on $P_{t,u}$. For tensors F on $P_{t,u}$, and for arbitrary $2 \leq p < +\infty$, we have the following sharp Bernstein inequality (see [10] for the proof):*

$$\begin{aligned} \|P_j F\|_{L^\infty(P_{t,u})} &\lesssim 2^j (1 + 2^{-\frac{j}{p}} \|K\|^{\frac{1}{p}}_{L^2(P_{t,u})} + 2^{-\frac{j}{p-1}} \|K\|^{\frac{1}{p-1}}_{L^2(P_{t,u})}) \|F\|_{L^2(P_{t,u})}, & (4.37)\\ \|P_{<0} F\|_{L^\infty(P_{t,u})} &\lesssim (1 + \|K\|^{\frac{1}{p}}_{L^2(P_{t,u})} + \|K\|^{\frac{1}{p-1}}_{L^2(P_{t,u})}) \|F\|_{L^2(P_{t,u})}. & (4.38) \end{aligned}$$

4.4 Estimates for the lapse n

The goal of this section is to improve the estimate for n in the bootstrap assumptions (4.1) (4.2).

4.4.1 Estimates for k on Σ_t

We recall the definition of E and H used in the standard electric-magnetic decomposition of the tensor $\mathbf{R}$ (see [4] chapter 7). We have:

$$E_{\alpha\beta} = \mathbf{R}_{\mu\alpha\nu\beta}T^\mu T^\nu,\ H_{\alpha\beta} = {}^*\mathbf{R}_{\mu\alpha\nu\beta}T^\mu T^\nu. \tag{4.39}$$

Then, k satisfies the following symmetric Hodge system on Σ_t:

$$\begin{cases} \mathrm{curl} k_{ij} = H_{ij}, \\ \nabla^j k_{ij} = 0, \\ \mathrm{Tr} k = 0, \end{cases} \tag{4.40}$$

where $\mathrm{curl} k_{ij} = \frac{1}{2}(\in_i^{lm} \nabla_l k_{mj} + \in_j^{lm} \nabla_l k_{mi})$. The solution k of the symmetric Hodge system (4.40) in 3 dimensions satisfies the following estimate (see [4] chapter 4):

$$\int_{\Sigma_t} \left(|\nabla k|^2 + 3(R_t)_{jl}k^{ij}k_i^l - \frac{1}{2}R_t|k|^2 \right) d\Sigma_t = \int_{\Sigma_t} |H|^2 d\Sigma_t. \tag{4.41}$$

The bound (2.59) on $\mathbf{R}$, the bound (4.25) on R_t, the definition of H (4.39) and (4.40) yield:

$$\|\nabla k\|^2_{L^2(\Sigma_t)} \lesssim \varepsilon \|k\|^2_{L^6(\Sigma_t)} + \varepsilon^2 \tag{4.42}$$

which together with the Sobolev embedding (3.71) implies:

$$\|\nabla k\|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon. \tag{4.43}$$

Remark 4.13 *To obtain* (4.42) *from* (4.41)*, we rely on* $L^2(\Sigma_t)$ *bounds for* R_t *and* $\mathbf{R}$*. This is enough on compacts, but not at infinity. Fortunately,* Σ_t *is asymptotically flat so that* R_t *and* $\mathbf{R}$ *decay at least like* r^{-3} *at infinity which is fast enough to obtain* (4.42)*. Furthermore, the fact that* Σ_t *is asymptotically flat also implies that* k *decays at least like* r^{-2} *at infinity which together with the Sobolev embedding* (3.71) *and the estimate* (4.43) *yields:*

$$\|k\|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon. \tag{4.44}$$

4.4.2 Improvement of the bootstrap assumptions on n

We first improve the L^∞ bound for $n-1$. Using the Sobolev embedding (3.71), (3.72) and the consequence of the Bochner identity (3.81), we have:

$$\begin{aligned} \|n-1\|_{L^\infty(\mathcal{M})} &\lesssim \|\nabla n\|_{L^\infty_t L^6(\Sigma_t)} + \|n-1\|_{L^\infty_t L^6(\Sigma_t)} \\ &\lesssim \|\nabla^2 n\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla n\|_{L^\infty_t L^2(\Sigma_t)} \\ &\lesssim \|\Delta n\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla n\|_{L^\infty_t L^2(\Sigma_t)}. \end{aligned}$$

Together with the equation of the lapse (2.5) on Σ_t, the bootstrap assumption (4.1) , the Sobolev embedding (3.71), and the estimates (4.43) (4.44), we obtain:

$$\begin{aligned} \|n-1\|_{L^\infty(\mathcal{M})} &\lesssim \||n|k|^2\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla n\|_{L^\infty_t L^2(\Sigma_t)} \\ &\lesssim \|n\|_{L^\infty(\mathcal{M})}\|k\|^2_{L^\infty_t L^4(\Sigma_t)} + \|\nabla n\|_{L^\infty_t L^2(\Sigma_t)} \\ &\lesssim \varepsilon^2 + \|\nabla n\|_{L^\infty_t L^2(\Sigma_t)} \end{aligned} \tag{4.45}$$

Multiplying the equation for the lapse (2.5) by $n-1$ on Σ_t, integrating by parts yields:

$$\begin{aligned} \|\nabla n\|^2_{L^2(\Sigma_t)} &= \int_{\Sigma_t} |k|^2 n(n-1) d\Sigma_t \lesssim \|k\|^2_{L^2(\Sigma_t)}\|n\|_{L^\infty(\Sigma_t)}\|n-1\|_{L^\infty(\Sigma_t)} \\ &\lesssim D^2\varepsilon^2\|n-1\|_{L^\infty(\Sigma_t)} \end{aligned}$$

where we used the bootstrap assumption (4.1) and (4.3). Together with (4.45), this yields:

$$\|n-1\|_{L^\infty(\mathcal{M})} + \|\nabla n\|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon. \tag{4.46}$$

Furthermore, the equation for the lapse (2.5), the Bochner identity (3.81), together with the estimates (4.43) (4.44) and (4.46) yields:

$$\begin{aligned} \|\nabla^2 n\|_{L^\infty_t L^2(\Sigma_t)} &\lesssim \|\Delta n\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla n\|_{L^\infty_t L^2(\Sigma_t)} \\ &\lesssim \||n|k|^2\|_{L^\infty_t L^2(\Sigma_t)} + \varepsilon \\ &\lesssim \|n\|_{L^\infty(\mathcal{M})}\|k\|^2_{L^\infty_t L^4(\Sigma_t)} + \varepsilon \\ &\lesssim \varepsilon. \end{aligned} \tag{4.47}$$

Using (3.74), (4.46) and (4.47), we also obtain:

$$\|\nabla n\|_{L^\infty_t L^2_{x'}} \lesssim \varepsilon. \tag{4.48}$$

We differentiate the equation of the lapse (2.5) with respect to ∇. We obtain:

$$\Delta(\nabla n) = \nabla(n|k|^2) + [\Delta, \nabla]n = |k|^2\nabla n + 2nk\nabla k + R_t\nabla n, \tag{4.49}$$

which together with the bound (4.26) on R_t, the Sobolev embedding (3.71), and the estimates (4.43), (4.44), (4.46) and (4.47), yields:

$$\begin{aligned} &\|\Delta(\nabla n)\|_{L^\infty_t L^{\frac{3}{2}}(\Sigma_t)} \\ \lesssim\ & \|k\|^2_{L^\infty_t L^3(\Sigma_t)}\|\nabla n\|_{L^\infty_t L^2(\Sigma_t)} + \|n\|_{L^\infty(\mathcal{M})}\|k\|_{L^\infty_t L^6(\Sigma_t)}\|\nabla k\|_{L^\infty_t L^2(\Sigma_t)} \\ &+\|R_t\|_{L^\infty_t L^2(\Sigma_t)}\|\nabla n\|_{L^\infty_t L^6(\Sigma_t)} \\ \lesssim\ & \varepsilon. \end{aligned} \tag{4.50}$$

(3.90), (4.47) and (4.50) imply:

$$\|\nabla^3 n\|_{L^\infty_t L^{\frac{3}{2}}(\Sigma_t)} \lesssim \|\Delta(\nabla n)\|_{L^\infty_t L^{\frac{3}{2}}(\Sigma_t)} + \|\nabla^2 n\|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon. \tag{4.51}$$

We now differentiate the equation of the lapse (2.5) with respect to $\mathbf{D}_T$. Using the commutation formula (3.93), we obtain:

$$\begin{aligned}\Delta(T(n)) &= T(\Delta n) + [\Delta, \mathbf{D}_T]n \\ &= |k|^2 T(n) + nk\mathbf{D}_T k - 2k\nabla^2 n + 2n^{-1}\nabla n \nabla T(n) + n^{-1}\Delta n T(n) \\ &\quad -\nabla k \nabla n - 2n^{-1}k\nabla n \nabla n.\end{aligned} \tag{4.52}$$

We need an estimate for $\mathbf{D}_T k$. We have the following identity (see [4] chapter 11):

$$\mathbf{D}_T k_{ij} = -n\nabla^2 n_{ij} + 2n^{-1}\nabla_i n \nabla_j n + (R_t)_{ij}$$

which together with the bound (4.26), (4.46) and (4.47) yields:

$$\|\mathbf{D}_T k\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \|n\nabla^2 n\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla n\|^2_{L_t^\infty L^4(\Sigma_t)} + \|R_t\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \varepsilon. \tag{4.53}$$

We multiply (4.52) by $T(n)$ and integrate by parts, which yields:

$$\begin{aligned}&\|\nabla(T(n))\|^2_{L_t^\infty L^2(\Sigma_t)} \\ \lesssim\ & \Big(\|k\|^2_{L_t^\infty L^3(\Sigma_t)} \|T(n)\|_{L_t^\infty L^6(\Sigma_t)} + \|n\|_{L^\infty} \|k\|_{L_t^\infty L^3(\Sigma_t)} \|\mathbf{D}_T k\|_{L_t^\infty L^2(\Sigma_t)} \\ &+ \|k\|_{L_t^\infty L^3(\Sigma_t)} \|\nabla^2 n\|_{L_t^\infty L^2(\Sigma_t)} + \|n^{-1}\nabla n\|_{L_t^\infty L^3(\Sigma_t)} \|\nabla T(n)\|_{L_t^\infty L^2(\Sigma_t)} \\ &+ \|n^{-1}\Delta n\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \|T(n)\|_{L_t^\infty L^6(\Sigma_t)} + \|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} \|\nabla n\|_{L_t^\infty L^3(\Sigma_t)} \\ &+ \|n^{-1}k\|_{L_t^\infty L^2(\Sigma_t)} \|\nabla n\|^2_{L_t^\infty L^6(\Sigma_t)} \Big) \|T(n)\|_{L_t^\infty L^6(\Sigma_t)} \\ \lesssim\ & \Big(\varepsilon + \varepsilon \|T(n)\|_{L_t^\infty L^6(\Sigma_t)} + \varepsilon \|\nabla T(n)\|_{L_t^\infty L^2(\Sigma_t)} \Big) \|T(n)\|_{L_t^\infty L^6(\Sigma_t)}\end{aligned} \tag{4.54}$$

where we used (4.43), (4.44), (4.46), (4.47) and (4.53) in the last inequality. (4.54) and the Sobolev embedding (3.71) imply:

$$\|\nabla(T(n))\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \varepsilon. \tag{4.55}$$

We now estimate (4.52) in $L_t^\infty L^{\frac{3}{2}}(\Sigma_t)$:

$$\begin{aligned}&\|\Delta(T(n))\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \\ \lesssim\ & \|k\|^2_{L_t^\infty L^4(\Sigma_t)} \|T(n)\|_{L_t^\infty L^6(\Sigma_t)} + \|nk\|_{L_t^\infty L^6(\Sigma_t)} \|\mathbf{D}_T k\|_{L_t^\infty L^2(\Sigma_t)} \\ &+ \|k\|_{L_t^\infty L^6(\Sigma_t)} \|\nabla^2 n\|_{L_t^\infty L^2(\Sigma_t)} + \|n^{-1}\nabla n\|_{L_t^\infty L^6(\Sigma_t)} \|\nabla T(n)\|_{L_t^\infty L^2(\Sigma_t)} \\ &+ \|n^{-1}\Delta n\|_{L_t^\infty L^2(\Sigma_t)} \|T(n)\|_{L_t^\infty L^6(\Sigma_t)} + \|\nabla k\|_{L_t^\infty L^6(\Sigma_t)} \|\nabla n\|_{L_t^\infty L^2(\Sigma_t)} \\ &+ \|n^{-1}k\|_{L_t^\infty L^3(\Sigma_t)} \|\nabla n\|^2_{L_t^\infty L^6(\Sigma_t)} \\ \lesssim\ & \varepsilon\end{aligned} \tag{4.56}$$

where we used (4.43), (4.44), (4.46), (4.47), (4.53) and (4.55) in the last inequality. (3.90), (4.55) and (4.56) imply:

$$\|\nabla^2 T(n)\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \lesssim \|\Delta(T(n))\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} + \|\nabla T(n)\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \varepsilon. \tag{4.57}$$

Finally, (3.75), (4.47), (4.51), (4.55) and (4.57) imply:

$$\|\nabla^2 n\|_{L^\infty_t L^2_{x'}} + \|\nabla T(n)\|_{L^\infty_t L^2_{x'}} \lesssim \varepsilon. \tag{4.58}$$

Note that (4.46), (4.47), (4.48), (4.55) and (4.58) improve the estimates for n in the bootstrap assumptions (4.1) (4.2).

4.4.3 An $L^\infty(\mathcal{M})$ estimate for ∇n

In view of the embedding (3.109), we have

$$\|\nabla n\|_{L^\infty(\Sigma_t)} \lesssim \|\nabla n\|_{\widehat{\mathcal{B}}^{\frac{3}{2}}}.$$

Together with the estimate (3.110), this yields

$$\|\nabla n\|_{L^\infty(\Sigma_t)} \lesssim \|n-1\|_{\widehat{\mathcal{B}}^{\frac{5}{2}}}. \tag{4.59}$$

Now, in view of the definition of the Besov spaces $\widehat{\mathcal{B}}^a$ and the finite band property for Q_j, we have

$$\|n-1\|_{\widehat{\mathcal{B}}^{\frac{5}{2}}} \lesssim \|\Delta n\|_{\widehat{\mathcal{B}}^{\frac{1}{2}}}.$$

Injecting the equation for the lapse (2.5) in the right-hand side, we obtain

$$\|n-1\|_{\widehat{\mathcal{B}}^{\frac{5}{2}}} \lesssim \|n|k|^2\|_{\widehat{\mathcal{B}}^{\frac{1}{2}}}.$$

Using the product estimates (3.117) and (3.122), we deduce

$$\begin{aligned}\|n-1\|_{\widehat{\mathcal{B}}^{\frac{5}{2}}} &\lesssim (\|n\|_{L^\infty(\mathcal{M})} + \|\nabla n\|_{L^\infty_t L^3(\Sigma_t)})\||k|^2\|_{\widehat{\mathcal{B}}^{\frac{1}{2}}} \\ &\lesssim (\|n\|_{L^\infty(\mathcal{M})} + \|\nabla n\|_{L^\infty_t L^3(\Sigma_t)})(\|\nabla k\|_{L^2(\Sigma_t)} + \|k\|_{L^2(\Sigma_t)})^2.\end{aligned}$$

Together with the estimates (4.43) and (4.44) for k, the estimates (4.46) and (4.47) for n, and the Sobolev embedding (3.71), we deduce

$$\|n-1\|_{\widehat{\mathcal{B}}^{\frac{5}{2}}} \lesssim \varepsilon.$$

In view of (4.59), we finally obtain

$$\|\nabla n\|_{L^\infty(\Sigma_t)} \lesssim \varepsilon.$$

4.5 Estimates for k on $\mathcal{H}_u$

The goal of this section is to improve the estimate for $\mathcal{N}_1(k)$, $\|\nabla\!\!\!/_{\underline{L}}\epsilon\|_{L^2(\mathcal{H}_u)}$ and $\|\nabla\!\!\!/_{\underline{L}}\delta\|_{L^2(\mathcal{H}_u)}$ given by the bootstrap assumption (4.3). The improvement of $\|\overline{\epsilon}\|_{L^\infty_{x'}L^2_t}$ and $\|\overline{\delta}\|_{L^\infty_{x'}L^2_t}$ is postponed to section 4.7. Note that the bootstrap assumption (4.3) yields:

$$\mathcal{N}_1(\delta) + \mathcal{N}_1(\epsilon) + \mathcal{N}_1(\eta) \leq D\varepsilon. \tag{4.60}$$

4.5.1 A Hodge type system on $\mathcal{H}_u$

The first step is to derive a Hodge type system analog to (4.40) on $\mathcal{H}$. We first recall the formula p. 106/107 in [4] relating the derivatives of k to the derivatives of η, ϵ, δ:

$$\begin{aligned}
\nabla_N k_{NN} &= \nabla_N \delta + 2b^{-1} \nabla\!\!\!/ b \cdot \epsilon \\
\nabla_B k_{NA} &= \nabla\!\!\!/_B \epsilon_A + \frac{3}{4}\delta \mathrm{tr}\theta \gamma_{AB} - \frac{1}{2}\mathrm{tr}\theta \widehat{\eta}_{AB} - \widehat{\eta}_{AC}\widehat{\theta}_{CB} + \frac{3}{2}\delta\widehat{\theta}_{AB} \\
\nabla_C k_{AB} &= \nabla\!\!\!/_C \eta_{AB} + \theta_{AC}\epsilon_B + \theta_{BC}\epsilon_A \\
\nabla_A k_{NN} &= \nabla_A \delta - 2\theta_{AB}\epsilon_B \\
\nabla_N k_{AN} &= \nabla\!\!\!/_N \epsilon_A - \frac{3}{2}\delta b^{-1}\nabla\!\!\!/_A b + b^{-1}\nabla\!\!\!/_B b\widehat{\eta}_{AB} \\
\nabla_N k_{AB} &= \nabla\!\!\!/_N \eta_{AB} - b^{-1}\nabla\!\!\!/_A b\epsilon_B - b^{-1}\nabla\!\!\!/_B b\epsilon_A
\end{aligned}$$

where θ is the second fundamental form of $P_{t,u}$ in Σ_t. Since $L = T + N$, θ is connected to the second fundamental form k of Σ_t and the null second fundamental form χ of $P_{t,u}$ through the formula:

$$\theta_{AB} = \chi_{AB} + \eta_{AB}. \tag{4.61}$$

Together with the Hodge system (4.40), we obtain:

$$\begin{aligned}
\mathrm{di}\!\!\!/\eta_A + \nabla\!\!\!/_N \epsilon_A &= -\theta_{AB}\epsilon_B - \mathrm{tr}\theta\epsilon_A + \tfrac{3}{2}\delta b^{-1}\nabla\!\!\!/_A b - b^{-1}\nabla_B b\widehat{\eta}_{AB} \\
\mathrm{di}\!\!\!/\epsilon + \nabla_N \delta &= -\tfrac{3}{2}\delta\mathrm{tr}\theta + \widehat{\eta}\widehat{\theta} - 2b^{-1}\nabla\!\!\!/_A b\epsilon_A \\
\nabla\!\!\!/_C \eta_{AB} - \nabla\!\!\!/_B \eta_{AC} &= \mathbf{R}_{ATBC} - \theta_{AC}\epsilon_B + \theta_{AB}\epsilon_C \\
\nabla\!\!\!/_N \eta_{AB} - \nabla\!\!\!/_B \epsilon_A &= \mathbf{R}_{ATBN} + b^{-1}\nabla\!\!\!/_A b\epsilon_B + b^{-1}\nabla\!\!\!/_B b\epsilon_A + \tfrac{3}{4}\delta\mathrm{tr}\theta\gamma_{AB} \\
&\quad -\tfrac{1}{2}\mathrm{tr}\theta\widehat{\eta}_{AB} - \widehat{\eta}_{AC}\widehat{\theta}_{CB} + \tfrac{3}{2}\delta\widehat{\theta}_{AB} \\
\nabla\!\!\!/_N \epsilon_A - \nabla_A \delta &= \mathbf{R}_{NTAN} + \tfrac{3}{2}\delta b^{-1}\nabla\!\!\!/_A b - b^{-1}\nabla\!\!\!/_B b\widehat{\eta}_{AB} - 2\theta_{AB}\epsilon_B
\end{aligned} \tag{4.62}$$

In order to obtain a Hodge system on $\mathcal{H}$, we need to replace the derivatives in the N-direction with derivatives in the L-direction in (4.62). We use the following formula for $\mathbf{D}_T\delta, \nabla\!\!\!/_T\varepsilon, \nabla\!\!\!/_T\eta$ (see [4] p. 337):

$$\begin{aligned}
\mathbf{D}_T\delta &= -n^{-1}\nabla_N^2 n + \rho + \delta^2 - \zeta\underline{\zeta} + \zeta\epsilon - \underline{\zeta}\epsilon \\
\nabla\!\!\!/_T\epsilon &= -n^{-1}\nabla\!\!\!/ N(n) + \tfrac{1}{2}(\underline{\beta} + \beta) + b^{-1}\nabla\!\!\!/ b n^{-1}N(n) - \tfrac{3}{2}(\zeta - n^{-1}\nabla\!\!\!/ n)\delta \\
&\quad +(\zeta - n^{-1}\nabla\!\!\!/ n + \epsilon)\widehat{\eta} + \tfrac{1}{2}\delta\epsilon \\
\nabla\!\!\!/_T\eta &= -n^{-1}\nabla\!\!\!/^2 n + \tfrac{1}{4}(\underline{\alpha} + \alpha) - \delta\eta + \epsilon\epsilon - (\zeta - n^{-1}\nabla\!\!\!/ n)\epsilon
\end{aligned} \tag{4.63}$$

Now, (4.62) and (4.63) yield:

$$\begin{aligned}
\mathrm{di}\!\!\!/\eta_A + \nabla\!\!\!/_L \epsilon_A &= \tfrac{1}{2}(\underline{\beta} + \beta) - n^{-1}\nabla\!\!\!/_A N(n) + b^{-1}\nabla\!\!\!/_A b n^{-1}N(n) \\
&\quad -\tfrac{3}{2}(\zeta_A - n^{-1}\nabla\!\!\!/_A n)\delta - \theta_{AB}\epsilon_B - \mathrm{tr}\theta\epsilon_A + \tfrac{3}{2}\delta b^{-1}\nabla\!\!\!/_A b - b^{-1}\nabla_B b\widehat{\eta}_{AB} \\
\mathrm{di}\!\!\!/\epsilon + \mathbf{D}_L\delta &= \rho - n^{-1}\nabla_N^2 n + \delta^2 - \zeta\underline{\zeta} + \zeta\epsilon - \underline{\zeta}\epsilon - \tfrac{3}{2}\delta\mathrm{tr}\theta + \widehat{\eta}\widehat{\theta} - 2b^{-1}\nabla\!\!\!/_A b\epsilon_A \\
\nabla\!\!\!/_C \eta_{AB} - \nabla\!\!\!/_B \eta_{AC} &= -\in_{BC} {}^*\beta_A + \in_{BC} {}^*\underline{\beta}_A - \theta_{AC}\epsilon_B + \theta_{AB}\epsilon_C \\
\nabla\!\!\!/_L \eta_{AB} - \nabla\!\!\!/_B \epsilon_A &= \tfrac{1}{2}\alpha_{AB} - n^{-1}\nabla\!\!\!/^2_{AB} n - \tfrac{1}{2}\delta\eta_{AB} + 2\epsilon_A\epsilon_B - (\zeta_A - n^{-1}\nabla\!\!\!/_A n)\epsilon_B \\
&\quad -(\zeta_B - n^{-1}\nabla\!\!\!/_B n)\epsilon_A + b^{-1}\nabla\!\!\!/_A b\epsilon_B + b^{-1}\nabla\!\!\!/_B b\epsilon_A + \tfrac{3}{4}\delta\mathrm{tr}\theta\gamma_{AB} \\
&\quad -\tfrac{1}{2}\mathrm{tr}\theta\widehat{\eta}_{AB} - \widehat{\eta}_{AC}\widehat{\theta}_{CB} + \tfrac{3}{2}\delta\widehat{\theta}_{AB} \\
\nabla\!\!\!/_L \epsilon_A - \nabla_A \delta &= \beta_A - n^{-1}\nabla\!\!\!/_A N(n) + b^{-1}\nabla\!\!\!/_A b n^{-1}N(n) - \tfrac{3}{2}(\zeta_A - n^{-1}\nabla\!\!\!/_A n - b^{-1}\nabla\!\!\!/_A b)\delta \\
&\quad -b^{-1}\nabla\!\!\!/_B b\widehat{\eta}_{AB} - 2\theta_{AB}\epsilon_B
\end{aligned} \tag{4.64}$$

Using the curvature bound (2.59), the bootstrap assumptions (4.1)-(4.6), the bound (4.60) on η, ϵ, δ together with (4.64), we obtain:

$$\|\mathrm{d}\!\!\!/\mathrm{iv}\eta_A + \nabla\!\!\!/_L \epsilon_A\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\mathrm{d}\!\!\!/\mathrm{iv}\epsilon + \mathbf{D}_L\delta\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \epsilon + D^2\varepsilon^2 \lesssim \varepsilon \tag{4.65}$$

and

$$\begin{aligned} & \|\nabla\!\!\!/_C \eta_{AB} - \nabla\!\!\!/_B \eta_{AC}\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/_L \eta_{AB} - \nabla\!\!\!/_B \epsilon_A\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/_L \epsilon_A - \nabla_A \delta\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ \lesssim\ & \epsilon + D^2\varepsilon^2 \lesssim \varepsilon. \end{aligned} \tag{4.66}$$

4.5.2 Estimates for η, ϵ, δ

We start with the following identity:

$$\begin{aligned} & \int_{\mathcal{H}} |\nabla\!\!\!/_C \eta_{AB} - \nabla\!\!\!/_B \eta_{AC}|^2 + 2|\nabla\!\!\!/_L \eta_{AB} - \nabla\!\!\!/_B \epsilon_A|^2 + 2|\nabla\!\!\!/_L \epsilon_A - \nabla_A \delta|^2 \\ =\ & 2\int_{\mathcal{H}} |\nabla\!\!\!/\eta|^2 + |\nabla\!\!\!/_L \eta|^2 + |\nabla\!\!\!/\epsilon|^2 + |\mathrm{d}\!\!\!/\mathrm{iv}\epsilon|^2 + 2|\nabla\!\!\!/_L \epsilon| + |\nabla\!\!\!/\delta|^2 + |\mathbf{D}_L\delta|^2 \\ & -2\int_{\mathcal{H}} \nabla\!\!\!/_C \eta_{AB} \nabla\!\!\!/_B \eta_{AC} - 4\int_{\mathcal{H}} \nabla\!\!\!/_L \eta_{AB} \nabla\!\!\!/_B \epsilon_A - 4\int_{\mathcal{H}} \nabla\!\!\!/_L \epsilon_A \nabla_A \delta \\ & -2\int_{\mathcal{H}} |\mathrm{d}\!\!\!/\mathrm{iv}\epsilon|^2 - 2\int_{\mathcal{H}} |\nabla\!\!\!/_L \epsilon|^2 - 2\int_{\mathcal{H}} |\mathbf{D}_L\delta|^2. \end{aligned} \tag{4.67}$$

We compute the last terms in the right hand side of (4.67) using integration by parts and the coarea formula (3.55) on $\mathcal{H}$:

$$\begin{aligned} & -2\int_{\mathcal{H}} \nabla\!\!\!/_C \eta_{AB} \nabla\!\!\!/_B \eta_{AC} - 4\int_{\mathcal{H}} \nabla\!\!\!/_L \eta_{AB} \nabla\!\!\!/_B \epsilon_A - 4\int_{\mathcal{H}} \nabla\!\!\!/_L \epsilon_A \nabla_A \delta \\ & -2\int_{\mathcal{H}} |\mathrm{d}\!\!\!/\mathrm{iv}\epsilon|^2 - 2\int_{\mathcal{H}} |\nabla\!\!\!/_L \epsilon|^2 - 2\int_{\mathcal{H}} |\mathbf{D}_L\delta|^2 \\ =\ & 2\int_{\mathcal{H}} \eta^{AB}(\nabla\!\!\!/_B \mathrm{d}\!\!\!/\mathrm{iv}\eta_A + [\nabla\!\!\!/_C, \nabla\!\!\!/_B]\eta_{AC}) + 4\int_{\mathcal{H}} \eta_{AB}(\nabla\!\!\!/_B \nabla\!\!\!/_L \epsilon_A + [\nabla\!\!\!/_L, \nabla\!\!\!/_B]\epsilon_A) \\ & -4\int_{\mathcal{H}} (-n^{-1}L(n) - \overline{\delta} + \mathrm{tr}\chi)\eta_{AB}\nabla\!\!\!/_B \epsilon_A - 4\int_{P_{t,u}} \eta_{AB}\nabla\!\!\!/_B \epsilon_A \\ & +4\int_{\mathcal{H}} \delta(\nabla\!\!\!/_L \mathrm{d}\!\!\!/\mathrm{iv}\epsilon + [\nabla\!\!\!/_A, \nabla\!\!\!/_L]\epsilon_A) - 2\int_{\mathcal{H}} |\mathrm{d}\!\!\!/\mathrm{iv}\epsilon|^2 - 2\int_{\mathcal{H}} |\nabla\!\!\!/_L \epsilon|^2 - 2\int_{\mathcal{H}} |\mathbf{D}_L\delta|^2 \\ =\ & -2\int_{\mathcal{H}} |\mathrm{d}\!\!\!/\mathrm{iv}\eta + \nabla\!\!\!/_L \epsilon|^2 - 2\int_{\mathcal{H}} |\mathrm{d}\!\!\!/\mathrm{iv}\epsilon + \mathbf{D}_L\delta|^2 + 2\int_{\mathcal{H}} \eta_{AB}\mathbf{R}_{AB}{}^{CD}\eta_{CD} \\ & +4\int_{\mathcal{H}} \eta_{AB}(\chi\nabla\!\!\!/\eta - n^{-1}\nabla\!\!\!/ n \nabla\!\!\!/_L \eta + (\chi\overline{\epsilon} + {}^*\beta)\eta) + 4\int_{\mathcal{H}} \delta(\chi\nabla\!\!\!/\epsilon - n^{-1}\nabla\!\!\!/ n \nabla\!\!\!/_L \epsilon + (\chi\overline{\epsilon} + {}^*\beta)\epsilon) \\ & -4\int_{\mathcal{H}} (-n^{-1}L(n) - \overline{\delta} + \mathrm{tr}\chi)\eta_{AB}\nabla\!\!\!/_B \epsilon_A - 4\int_{P_{t,u}} \eta_{AB}\nabla\!\!\!/_B \epsilon_A \\ & -4\int_{\mathcal{H}} (-n^{-1}L(n) - \overline{\delta} + \mathrm{tr}\chi)\delta\mathrm{d}\!\!\!/\mathrm{iv}\epsilon + 4\int_{P_{t,u}} \delta\mathrm{d}\!\!\!/\mathrm{iv}\epsilon \end{aligned}$$

Together with the curvature bound (2.59) and the bootstrap assumptions (4.1)-(4.6), we obtain:

$$\begin{aligned}
&-2\int_{\mathcal{H}} \nabla\!\!\!/_C \eta_{AB} \nabla\!\!\!/_B \eta_{AC} - 4\int_{\mathcal{H}} \nabla\!\!\!/_L \eta_{AB} \nabla\!\!\!/_B \epsilon_A - 4\int_{\mathcal{H}} \nabla\!\!\!/_L \epsilon_A \nabla_A \delta \\
&-2\int_{\mathcal{H}} |\mathrm{di}\!\!\!/\mathrm{v}\epsilon|^2 - 2\int_{\mathcal{H}} |\nabla\!\!\!/_L \epsilon|^2 - 2\int_{\mathcal{H}} |\mathbf{D}_L \delta|^2 \\
= \;& -2\int_{\mathcal{H}} |\mathrm{di}\!\!\!/\mathrm{v}\eta + \nabla\!\!\!/_L \epsilon|^2 - 2\int_{\mathcal{H}} |\mathrm{di}\!\!\!/\mathrm{v}\epsilon + \mathbf{D}_L \delta|^2 - 4\int_{P_{t,u}} \eta_{AB} \nabla\!\!\!/_B \epsilon_A + 4\int_{P_{t,u}} \delta \mathrm{di}\!\!\!/\mathrm{v}\epsilon + O(D^3\varepsilon^3).
\end{aligned}$$

The bounds (4.65) (4.66) together with (4.67) and (4.68) yield:

$$\int_{\mathcal{H}} |\nabla\!\!\!/\eta|^2 + |\nabla\!\!\!/\epsilon|^2 + |\nabla\!\!\!/_L \epsilon|^2 + |\nabla\!\!\!/\delta|^2 + |\mathbf{D}_L \delta|^2 \lesssim \left|\int_{P_{t,u}} \eta_{AB} \nabla\!\!\!/_B \epsilon_A\right| + \left|\int_{P_{t,u}} \delta \mathrm{di}\!\!\!/\mathrm{v}\epsilon\right| + \varepsilon^2. \quad (4.68)$$

We now state a lemma which will allow us to control the integrals over $P_{t,u}$ in (4.68).

Lemma 4.14 *Let F and G tensors on Σ_t such that $F \cdot \nabla\!\!\!/ G$ is a scalar. Then, we have:*

$$\left|\int_{P_{t,u}} F \cdot \nabla\!\!\!/ G\right| \lesssim \|F\|_{H^1(\Sigma_t)} \|G\|_{H^1(\Sigma_t)}. \quad (4.69)$$

The proof of Lemma 4.14 is postponed to section A.2. We now use Lemma 4.14 to obtain estimates for η, ϵ, δ. The bounds (4.43) (4.44) for k on Σ_t together with (4.69) yield the following estimate:

$$\left|\int_{P_{t,u}} \eta_{AB} \nabla\!\!\!/_B \epsilon_A\right| + \left|\int_{P_{t,u}} \delta \mathrm{di}\!\!\!/\mathrm{v}\epsilon\right| \lesssim \|k\|^2_{H^1(\Sigma_t)} \lesssim \varepsilon^2.$$

Together with (4.68), this implies:

$$\int_{\mathcal{H}} |\nabla\!\!\!/\eta|^2 + |\nabla\!\!\!/\epsilon|^2 + |\nabla\!\!\!/_L \epsilon|^2 + |\nabla\!\!\!/\delta|^2 + |\mathbf{D}_L \delta|^2 \lesssim \varepsilon^2. \quad (4.70)$$

Using also (3.63), we finally obtain:

$$\mathcal{N}_1(\eta) + \mathcal{N}_1(\epsilon) + \mathcal{N}_1(\delta) \lesssim \varepsilon. \quad (4.71)$$

Now, in view of (4.63), we have:

$$\|\mathbf{D}_T \delta\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/_T \epsilon\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon + D^2\varepsilon^2 \lesssim \varepsilon \quad (4.72)$$

where we have used the curvature bound (2.59) and the bootstrap assumptions (4.1)-(4.6). (4.71) and (4.72) yield:

$$\|\mathbf{D}_{\underline{L}} \delta\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/_{\underline{L}} \epsilon\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon. \quad (4.73)$$

(4.71) and (4.73) improve the estimate for $\mathcal{N}_1(k)$, $\|\nabla\!\!\!/_{\underline{L}} \epsilon\|_{L^\infty_u L^2(\mathcal{H}_u)}$ and $\|\mathbf{D}_{\underline{L}} \delta\|_{L^\infty_u L^2(\mathcal{H}_u)}$ given by the bootstrap assumption (4.3).

4.6 Time foliation versus geodesic foliation

While we work with a time foliation, we recall that the estimates corresponding to the bootstrap assumptions on χ and ζ have already been proved in the context of a geodesic foliation in the sequence of papers [14] [10] [11]. One may reprove these estimates by adapting the proofs to the context of a time foliation. However, this would be rather lengthy and we suggest here a more elegant solution which consists in translating certain estimates from the geodesic foliation to the time foliation, and in obtaining directly the rest of the estimates. More precisely, we wish to obtain the L^∞ bound for $\mathrm{tr}\chi$, and the trace bounds for $\widehat{\chi}$ and ζ by exploiting the corresponding estimates in the geodesic foliation. We will also obtain the trace bounds for δ and ϵ by reducing to estimates in the geodesic foliation in section 4.7. Finally, these trace bounds and the null structure equations will allow us to get all the remaining estimates in sections 4.8 and 4.9. We start by recalling some of the results obtained in the context of the geodesic foliation in the sequence of papers [14] [10] [11].

4.6.1 The case of the geodesic foliation

Remember that u is a solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ on $\mathcal{M}$. The level hypersufaces $u(t,x,\omega) = u$ of the optical function u are denoted by $\mathcal{H}_u$. $L' = -\mathbf{g}^{\alpha\beta}\partial_\beta u\partial_\alpha$ is the geodesic null generator of $\mathcal{H}_u$. In particular, we have:

$$\mathbf{D}_{L'}L' = 0,\ \mathbf{g}(L',L') = 0.$$

Let s denote its affine parameter, i.e. $L'(s) = 1$. We denote by $P'_{s,u}$ the level surfaces of s in $\mathcal{H}_u$.

Definition 4.15 *A null frame e'_1, e'_2, e'_3, e'_4 at a point $p \in P'_{s,u}$ consists, in addition to $e'_4 = L'$, of arbitrary orthonormal vectors e'_1, e'_2 tangent to $P'_{s,u}$ and the unique vectorfield $e'_3 = \underline{L}'$ satisfying the relations:*

$$\mathbf{g}(e'_3,e'_4) = -2,\ \mathbf{g}(e'_3,e'_3) = 0,\ \mathbf{g}(e'_3,e'_1) = 0,\ \mathbf{g}(e'_3,e'_2) = 0.$$

Definition 4.16 (*Ricci coefficients in the geodesic foliation*) *Let e'_1, e'_2, e'_3, e'_4 be a null frame on $P'_{s,u}$ as above. The following tensors on $P'_{s,u}$*

$$\begin{array}{ll} \chi'_{AB} =< \mathbf{D}_{e'_A}e'_4, e'_B >, & \underline{\chi}'_{AB} =< \mathbf{D}_{e'_A}e'_3, e'_B >, \\ \zeta'_A = \frac{1}{2} < \mathbf{D}_{e'_A}e'_4, e'_3 > & \end{array} \tag{4.74}$$

are called the Ricci coefficients associated to the geodesic foliation.

We decompose χ' and $\underline{\chi}'$ into their trace and traceless components.

$$tr\chi' = \mathbf{g}^{AB}\chi'_{AB}, \qquad tr\underline{\chi}' = \mathbf{g}^{AB}\underline{\chi}'_{AB}, \tag{4.75}$$

$$\widehat{\chi}'_{AB} = \chi'_{AB} - \frac{1}{2}tr\chi'\mathbf{g}_{AB}, \quad \widehat{\underline{\chi}}'_{AB} = \underline{\chi}'_{AB} - \frac{1}{2}tr\underline{\chi}'\mathbf{g}_{AB}. \tag{4.76}$$

Definition 4.17 *The null components of the curvature tensor* $\mathbf{R}$ *of the space-time metric* $\mathbf{g}$ *in the geodesic foliation are given by:*

$$\begin{aligned}
\alpha'_{AB} &= \mathbf{R}(L', e'_A, L', e'_B)\,, \qquad \beta'_A = \frac{1}{2}\mathbf{R}(e'_A, L', \underline{L}', L'), && (4.77)\\
\rho' &= \frac{1}{4}\mathbf{R}(\underline{L}', L', \underline{L}', L')\,, \qquad \sigma' = \frac{1}{4}{}^\star\mathbf{R}(\underline{L}', L', \underline{L}', L') && (4.78)\\
\underline{\beta}'_A &= \frac{1}{2}\mathbf{R}(e'_A, \underline{L}', \underline{L}', L')\,, \qquad \underline{\alpha}'_{AB} = \mathbf{R}(\underline{L}', e'_A, \underline{L}', e'_B) && (4.79)
\end{aligned}$$

where ${}^\star\mathbf{R}$ *denotes the Hodge dual of* $\mathbf{R}$*.*

The following *Ricci equations* can be easily derived (see [14]):

$$\begin{aligned}
&\mathbf{D}_{e'_A} e'_4 = \chi'_{AB} e'_B - \zeta'_A e'_4, \quad \mathbf{D}_{e'_A} e'_3 = \underline{\chi}'_{AB} e'_B + \zeta'_A e'_3,\\
&\mathbf{D}_{e'_4} e'_4 = 0, \qquad\qquad\quad \mathbf{D}_{e'_4} e'_3 = -2\zeta'_A e'_A, \qquad\qquad (4.80)\\
&\mathbf{D}_{e'_4} e'_A = \nabla\!\!\!/_{e'_4} e'_A - \zeta'_A e'_4, \quad \mathbf{D}_{e'_B} e'_A = \nabla\!\!\!/'_{e'_B} e'_A + \frac{1}{2}\chi'_{AB}\, e'_3 + \frac{1}{2}\underline{\chi}'_{AB}\, e'_4
\end{aligned}$$

where, $\nabla\!\!\!/_{e'_3}$, $\nabla\!\!\!/_{e'_4}$ denote the projection on $P'_{s,u}$ of $\mathbf{D}_{e'_3}$ and $\mathbf{D}_{e'_4}$, and $\nabla\!\!\!/'$ denotes the induced covariant derivative on $P'_{s,u}$.

We now recall some of the main estimates obtained in the sequence of papers [14] [10] [11]. Under the following assumption on the initial data

$$\begin{aligned}
\|\mathrm{tr}\chi'(0,.)\|_{L^\infty(P_{0,u})} + \sum_j \Big(&\|P_j \nabla\!\!\!/' \mathrm{tr}\chi'(0,.)\|_{L^2(P_{0,u})} + \|P_j \mu'(0,.)\|_{L^2(P_{0,u})}\\
&+ \|P_j \mathrm{tr}\underline{\chi}'(0,.)\|_{L^2(P_{0,u})} + \|P_j \widehat{\underline{\chi}}'(0,.)\|_{L^2(P_{0,u})}\Big) \;\leq\; \varepsilon, \qquad (4.81)
\end{aligned}$$

we have

$$\|\mathrm{tr}\chi'\|_{L^\infty(\mathcal{H}_u)} + \|\widehat{\chi}'\|_{L^2_{x'}L^\infty_s} + \|\zeta'\|_{L^2_{x'}L^\infty_s} \lesssim \varepsilon \qquad (4.82)$$

and

$$\|\underline{\chi}'\|_{L^2_{x'}L^\infty_s} + \mathcal{N}'_1(\chi') + \mathcal{N}'_1(\zeta') \lesssim \varepsilon, \qquad (4.83)$$

where the norm $\mathcal{N}'_1$ is given by

$$\mathcal{N}'_1(F) = \|F\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/' F\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/_{L'} F\|_{L^2(\mathcal{H}_u)}.$$

Remark 4.18 *Note that* (4.81) *is implied by our initial data assumption* $\mathcal{I}_0 \leq \varepsilon$*, where* $\mathcal{I}_0$ *is defined* (2.60)*. Indeed, at* $t = 0$*, we have*

$$L' = b^{-1}L, \quad \underline{L}' = b\underline{L}, \quad e'_A = e_A, A = 1, 2$$

and we easily check[18]

$$tr\chi' = b^{-1}tr\chi, \quad \widehat{\chi}' = b^{-1}\widehat{\chi}, \quad \zeta' = \zeta, \quad \mu' = \mu, \quad tr\underline{\chi}' = btr\underline{\chi}, \quad \widehat{\underline{\chi}}' = b\widehat{\underline{\chi}}.$$

[18]This corresponds to some of the computations of the next section in the trivial case where $F' = 0$.

In view of the definition (2.60) *of* $\mathcal{I}_0$*, and together with the analog at* $s = 0$ *of the estimates* (4.88) *and* (4.90) *below, we immediately obtain*

$$\|tr\chi'(0,.)\|_{L^\infty(P_{0,u})} + \sum_j \Big(\|P_j \nabla\!\!\!/' tr\chi'(0,.)\|_{L^2(P_{0,u})} + \|P_j \mu'(0,.)\|_{L^2(P_{0,u})} \\ + \|P_j tr\underline{\chi}'(0,.)\|_{L^2(P_{0,u})} + \|P_j \widehat{\underline{\chi}}'(0,.)\|_{L^2(P_{0,u})} \Big) \leq \mathcal{I}_0,$$

as desired.

Remark 4.19 *Note that the norm* $L^\infty(\mathcal{H}_u)$ *does not depend on the particular foliation. Now, this is also the case for the trace norm* $L^\infty_{x'} L^2_s$*. Indeed, recall the definition of the null geodesic* κ_x *in Remark 2.3. Then, we have:*

$$\|F\|^2_{L^\infty_{x'} L^2_t} = \sup_{(0,x)\in\Sigma_0} \int_0^1 |F(\kappa_x(t))|^2 dt = \sup_{(0,x)\in\Sigma_0} \int_0^1 |F(\kappa_x(s))|^2 n^{-1} b^{-1} ds \sim \|F\|^2_{L^\infty_{x'} L^2_s}$$

where we used the fact that $\frac{dt}{ds} = n^{-1}b^{-1}$ *and the fact that* $nb \sim 1$ *by the bootstrap assumption* (4.1)*.*

In the next section, we will obtain the estimates corresponding to (4.82) in the time foliation. For now, we conclude this section by recalling the definition and properties of the Besov spaces constructed in the sequence of papers [14] [10] [11]. For $P'_{s,u}$-tangent tensors F on $\mathcal{H}_u$, $0 \leq a \leq 1$, we introduce the Besov norms:

$$\|F\|_{\mathcal{B}'^a} = \sum_{j\geq 0} 2^{ja} \sup_{0\leq s\leq 1} \|P'_j F\|_{L^2(P'_{s,u})} + \sup_{0\leq s\leq 1} \|P'_{<0} F\|_{L^2(P'_{s,u})}, \tag{4.84}$$

$$\|F\|_{\mathcal{P}'^a} = \sum_{j\geq 0} 2^{ja} \|P'_j F\|_{L^2(\mathcal{H}_u)} + \|P'_{<0} F\|_{L^2(\mathcal{H}_u)} \tag{4.85}$$

where P'_j are the geometric Littlewood-Paley projections on the 2-surfaces $P'_{s,u}$. Using the definition of these Besov spaces, we recall another estimate obtained in the sequence of papers [14] [10] [11]. We have:

$$\|\underline{\chi}'\|_{\mathcal{B}'^0} \lesssim \varepsilon. \tag{4.86}$$

We now recall some properties of these Besov spaces obtained in the sequence of papers [14] [10] [11]. We have for scalar functions on $\mathcal{H}_u$ (see [14] section 5):

$$\|f\|_{L^\infty(\mathcal{H}_u)} \lesssim \|f\|_{\mathcal{B}'^1} \lesssim \|f\|_{L^\infty_s L^2_{x'}} + \|\nabla\!\!\!/' f\|_{\mathcal{B}'^0}. \tag{4.87}$$

Furthermore, for any $P'_{s,u}$-tangent tensors F, G on $\mathcal{H}_u$, we have:

$$\|F \cdot G\|_{\mathcal{B}'^0} \lesssim \big(\|\nabla\!\!\!/' F\|_{L^\infty_s L^2_x} + \|F\|_{L^\infty} \big) \|G\|_{\mathcal{B}'^0} \tag{4.88}$$

and

$$\|F \cdot G\|_{\mathcal{P}'^0} \lesssim \big(\|\nabla\!\!\!/' F\|_{L^\infty_s L^2_x} + \|F\|_{L^\infty} \big) \|G\|_{\mathcal{P}'^0}. \tag{4.89}$$

To bound Besov norms, we sometime use the following non sharp embedding estimate. For any $0 \leq a < \frac{1}{2}$, we have:

$$\|F\|_{\mathcal{B}'^a} \lesssim \mathcal{N}'_1(F). \tag{4.90}$$

We also have the following non sharp product estimate:

$$\|FG\|_{\mathcal{P}'^a} \lesssim \mathcal{N}_1'(F)\mathcal{N}_1'(G). \tag{4.91}$$

The following proposition is the key tool used in [14] to control the transport equations appearing in the null structure.

Proposition 4.20 *Assume that the scalar function U satisfies $U(0) = 0$ and the following transport equation along $\mathcal{H}_u$:*

$$\frac{d}{ds}U + atr\chi U = F_1 \cdot \nabla\!\!\!/_{L'} P + F_2 \cdot W,$$

where a is some positive number. Then,

$$\|U\|_{\mathcal{B}'^0} \lesssim \big(\mathcal{N}_1'(F_1) + \|F_1\|_{L^\infty_{x'}L^2_s}\big) \cdot \mathcal{N}_1'(P)\big(\mathcal{N}_1'(F_2) + \|F_2\|_{L^\infty_{x'}L^2_s}\big) \cdot \|W\|_{\mathcal{P}'^0}. \tag{4.92}$$

Finally, using the previous proposition, we may prove the following version of the sharp classical trace theorem which is a slight generalization of Corollary 5.10 in [14].

Corollary 4.21 *Assume F is an $P'_{s,u}$-tangent tensor which admits a decomposition of the form, $\nabla\!\!\!/' F = A\nabla\!\!\!/_{L'} P + E$. Then,*

$$\|F\|_{L^\infty_{x'}L^2_s} \lesssim \mathcal{N}_1'(F) + \mathcal{N}_1'(P)(\|A\|_{L^\infty} + \|\nabla\!\!\!/' A\|_{L^2_{x'}L^\infty_s} + \|\nabla\!\!\!/_{L'} A\|_{L^2_{x'}L^\infty_s}) + \|E\|_{\mathcal{P}'^0}. \tag{4.93}$$

Proof The scalar function $f(t) = \int_0^t |F|^2$ verifies the transport equation,

$$Lf = |F|^2, \qquad f(0) = 0.$$

Recall the following commutator formula in the geodesic foliation:

$$[\nabla\!\!\!/_{L'}, \nabla\!\!\!/'_A]f = -\chi'_{AB}\nabla\!\!\!/'_B f.$$

Differentiating and applying the commutator formula, we derive,

$$\begin{aligned}\nabla\!\!\!/_{L'}(\nabla\!\!\!/' f) &= 2F \cdot \nabla\!\!\!/' F - \chi' \cdot (\nabla\!\!\!/' f)\\ &= F \cdot A\nabla\!\!\!/_{L'} P + F \cdot E - \chi' \cdot (\nabla\!\!\!/' f)\end{aligned}$$

Applying (4.92), we deduce:

$$\begin{aligned}\|\nabla\!\!\!/' f\|_{\mathcal{B}^0} \lesssim\ & \big(\mathcal{N}_1'(FA) + \|FA\|_{L^\infty_{x'}L^2_s}\big)\mathcal{N}_1'(P)\big(\mathcal{N}_1'(F) + \|F\|_{L^\infty_{x'}L^2_s}\big)\|E\|_{\mathcal{P}'^0}\\ &+\big(\mathcal{N}_1'(\chi') + \|\chi'\|_{L^\infty_{x'}L^2_s}\big)\|\nabla\!\!\!/' f\|_{\mathcal{P}'^0},\end{aligned}$$

which together with the estimates (4.82) and (4.83) for χ' and the fact that ε is small yields:

$$\|\nabla\!\!\!/' f\|_{\mathcal{B}'^0} \lesssim \big(\mathcal{N}_1'(FA) + \|FA\|_{L^\infty_{x'}L^2_s}\big)\mathcal{N}_1'(P) + \big(\mathcal{N}_1'(F) + \|F\|_{L^\infty_{x'}L^2_s}\big)\|E\|_{\mathcal{P}'^0}. \tag{4.94}$$

We have:

$$\begin{aligned}\mathcal{N}_1'(FA)+\|FA\|_{L^\infty_{x'}L^2_s} &\lesssim \|A\nabla\!\!\!/' F\|_{L^2(\mathcal{H}_u)}+\|A\nabla\!\!\!/_{L'}F\|_{L^2(\mathcal{H}_u)}+\|F\nabla\!\!\!/' A\|_{L^2(\mathcal{H}_u)}\\ &\quad+\|F\nabla\!\!\!/_{L'}A\|_{L^2(\mathcal{H}_u)}+\|AF\|_{L^2(\mathcal{H}_u)}+\|F\|_{L^\infty_{x'}L^2_s}\|A\|_{L^\infty}\\ &\lesssim (\|F\|_{L^\infty_{x'}L^2_s}+\mathcal{N}_1'(F))(\|A\|_{L^\infty}+\|\nabla\!\!\!/' A\|_{L^2_{x'}L^\infty_s}+\|\nabla\!\!\!/_{L'}A\|_{L^2_{x'}L^\infty_s}),\end{aligned}$$

which together with (4.94) yields:

$$\|\nabla\!\!\!/' f\|_{\mathcal{B}'^0}\lesssim \big(\mathcal{N}_1'(F)+\|F\|_{L^\infty_{x'}L^2_s}\big)\big((\|A\|_{L^\infty}+\|\nabla\!\!\!/' A\|_{L^2_{x'}L^\infty_s}+\|\nabla\!\!\!/_{L'}A\|_{L^2_{x'}L^\infty_s})\mathcal{N}_1'(P)+\|E\|_{\mathcal{P}'^0}\big).$$

Now, in view of estimate (4.87), we infer that,

$$\begin{aligned}\|f\|_{L^\infty} &\lesssim \|f\|_{L^\infty_s L^2_{x'}}+\|\nabla\!\!\!/' f\|_{\mathcal{B}'^0}\\ &\lesssim \big(\mathcal{N}_1'(F)+\|F\|_{L^\infty_{x'}L^2_s}\big)\big((\|A\|_{L^\infty}+\|\nabla\!\!\!/' A\|_{L^2_{x'}L^\infty_s}+\|\nabla\!\!\!/_{L'}A\|_{L^2_{x'}L^\infty_s})\mathcal{N}_1'(P)\\ &\quad+\|E\|_{\mathcal{P}'^0}\big)+\|F\|^2_{L^2_sL^4_{x'}}.\end{aligned}$$

Thus, recalling the definition of $f=\int_0^t|F|^2$, and the estimate $\|F\|_{L^2_sL^4_{x'}}\lesssim\mathcal{N}_1'(F)$, we obtain:

$$\begin{aligned}\|F\|^2_{L^\infty_{x'}L^2_s} &\lesssim \big(\mathcal{N}_1'(F)+\|F\|_{L^\infty_{x'}L^2_s}\big)\big((\|A\|_{L^\infty}+\|\nabla\!\!\!/' A\|_{L^2_{x'}L^\infty_s}+\|\nabla\!\!\!/_{L'}A\|_{L^2_{x'}L^\infty_s})\mathcal{N}_1'(P)\\ &\quad+\|E\|_{\mathcal{P}'^0}\big)+\mathcal{N}_1'(F)^2\end{aligned}$$

which yields the desired estimate (4.93). ∎

4.6.2 Estimates in the time foliation

In this section, we obtain the L^∞ bound from $\mathrm{tr}\chi$, and the trace bounds for $\widehat{\chi}$ and ζ by exploiting the corresponding estimates in the geodesic foliation (4.82). We start by establishing the relation between the Ricci coefficients in the time and geodesic foliation. Recall first from the definition of L and L' (2.9) that $L=bL'$. Since (e_1,e_2) and (e_1',e_2') are both orthonormal vectors in the tangent space of $\mathcal{H}_u$ which are both orthogonal to L, we may chose these vectors such that there is a tensor F' on $P'_{s,u}$ satisfying:

$$e_A=e_A'+F_A'L',\ A=1,2.$$

Also, writing $\underline{L}$ in the frame $e_1',e_2',L',\underline{L}'$, and using the fact that $g(L,\underline{L})=-2$, $g(\underline{L},\underline{L})=0$ and $g(\underline{L},e_A)=0$, $A=1,2$, we obtain:

$$\underline{L}=b^{-1}\underline{L}'+2b^{-1}F_A'e_A'+b^{-1}|F'|^2L'.$$

Finally, we have established the following relations:

$$\begin{aligned}&L=bL',\\ &e_A=e_A'+F_A'L',\ A=1,2,\\ &\underline{L}=b^{-1}\underline{L}'+2b^{-1}F_A'e_A'+b^{-1}|F'|^2L'.\end{aligned}\tag{4.95}$$

We now use the definition (2.13) and (4.74) of the Ricci coefficients respectively in the time and geodesic foliation. We first establish the relation between χ and χ'. Using the definition (2.13) of χ and (4.74) of χ', we have:

$$\chi_{AB} = \mathbf{g}(\mathbf{D}_{e_A} L, e_B) = \mathbf{g}(\mathbf{D}_{e'_A + F'_A L'}(bL'), e'_B + F'_B L') = b\chi'_{AB}$$

where we used the Ricci equations (4.80) and the identities $\mathbf{g}(L', L') = \mathbf{g}(L', e'_A) = 0$, $A = 1, 2$. In particular, we obtain:

$$\chi = b\chi',\ \mathrm{tr}\chi = b\mathrm{tr}\chi',\ \widehat{\chi} = b\widehat{\chi'}. \tag{4.96}$$

(4.96) together with the bootstrap assumption (4.1) and the estimate (4.82) yields:

$$\begin{aligned} &\|\mathrm{tr}\chi\|_{L^\infty(\mathcal{H}_u)} \leq \|b\|_{L^\infty(\mathcal{H}_u)} \|\mathrm{tr}\chi'\|_{L^\infty(\mathcal{H}_u)} \lesssim \varepsilon, \\ &\|\widehat{\chi}\|_{L^\infty_{x'} L^2_t} \leq \|b\|_{L^\infty(\mathcal{H}_u)} \|\widehat{\chi'}\|_{L^\infty_{x'} L^2_s} \lesssim \varepsilon, \end{aligned} \tag{4.97}$$

where we have used the fact that the trace norms $L^\infty_{x'} L^2_t$ and $L^\infty_{x'} L^2_s$ are equivalent by Remark 4.19. Note that (4.97) is an improvement of the corresponding estimates in the bootstrap assumptions (4.4) (4.5).

Next, we establish the relation between $\underline{\chi}$ and $\underline{\chi}'$. Using the definition (2.13) of $\underline{\chi}$ and (4.74) of $\underline{\chi}'$, we have:

$$\begin{aligned} \underline{\chi}_{AB} &= \mathbf{g}(\mathbf{D}_{e_A} \underline{L}, e_B) = \mathbf{g}(\mathbf{D}_{e'_A + F'_A L'}(b^{-1}\underline{L}' + 2b^{-1}F'_C e'_C + b^{-1}|F'|^2 L'), e'_B + F'_B L') \\ &= b^{-1}\big(\mathbf{g}(\mathbf{D}_{e'_A + F'_A L'}\underline{L}', e'_B + F'_B L') + 2\mathbf{g}(\mathbf{D}_{e'_A + F'_A L'}F', e'_B + F'_B L') \\ &\quad + |F'|^2 \mathbf{g}(\mathbf{D}_{e'_A + F'_A L'} L', e'_B + F'_B L')\big) \\ &= b^{-1}\big(\underline{\chi}'_{AB} - 2F'_B\zeta'_A - 2F'_A\zeta'_B + 2\mathbf{g}(\mathbf{D}_{e'_A}F', e'_B) - 2F'_B\chi'_{AC}F'_C + 2F'_A\mathbf{g}(\mathbf{D}_{L'}F', e'_B) \\ &\quad + |F'|^2\chi'_{AB}\big) \end{aligned} \tag{4.98}$$

where we used the Ricci equations (4.80).

We establish the relation between ζ and ζ'. Using the definition (2.13) of ζ and (4.74) of ζ', we have:

$$\begin{aligned} \zeta_A &= \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} L, e_A) = \frac{1}{2}\mathbf{g}(\mathbf{D}_{b^{-1}\underline{L}' + 2b^{-1}F'_C e'_C + b^{-1}|F'|^2 L'}(bL'), e'_A + F'_A L') \\ &= \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}'} L', e'_A) + \chi'_{AC}F'_C. \end{aligned} \tag{4.99}$$

Now, we have:

$$\begin{aligned} \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}'} L', e'_A) &= -\frac{1}{2}\mathbf{g}(L', \mathbf{D}_{\underline{L}'} e'_A) \\ &= -\frac{1}{2}\mathbf{g}(L', [\underline{L}', e'_A]) - \frac{1}{2}\mathbf{g}(L', \mathbf{D}_{e'_A}\underline{L}') \\ &= \zeta'_A - \frac{1}{2}\mathbf{g}(L', [\underline{L}', e'_A]), \end{aligned} \tag{4.100}$$

where we used the definition of ζ' (4.74) in the last equality. The last term in (4.100) is given by:

$$-\frac{1}{2}\mathbf{g}(L', [\underline{L}', e'_A]) = -\frac{1}{2}[\underline{L}', e'_A](u) = -\frac{1}{2}\underline{L}'(e'_A(u)) + \frac{1}{2}e'_A(\underline{L}'(u)) = 0 \tag{4.101}$$

where we used the fact that $e'_A(u) = 0$ and $\underline{L}'(u) = -2$. Finally, (4.99)-(4.101) yield:

$$\zeta_A = \zeta'_A + \chi'_{AC} F'_C,$$

which together with the estimate (4.82) and Remark 4.19 implies:

$$\|\zeta\|_{L^\infty_{x'} L^2_t} \lesssim \|\zeta'\|_{L^\infty_{x'} L^2_s} + \|\chi'\|_{L^\infty_{x'} L^2_s} \|F'\|_{L^\infty} \lesssim \varepsilon + \varepsilon \|F'\|_{L^\infty}. \tag{4.102}$$

In view of (4.102), we need to estimate $\|F'\|_{L^\infty}$. We make the bootstrap assumption:

$$\|F'\|_{L^\infty} \leq D^2 \varepsilon \tag{4.103}$$

where D is the large constant appearing in the bootstrap assumptions (4.1)-(4.6). Our goal is to improve on the constant in the right-hand side of (4.103). We first estimate $\mathbf{D}_{L'} F$. In view of the Ricci equations (2.23), we have:

$$\begin{aligned} \bar{\epsilon}_A &= -\frac{1}{2} \mathbf{g}(\mathbf{D}_L \underline{L}, e_A) = -\frac{1}{2} \mathbf{g}(\mathbf{D}_{bL'}(b^{-1} \underline{L}' + 2b^{-1} F'_C e'_C + b^{-1} |F'|^2 L'), e'_A + F'_A L') \\ &= -\frac{1}{2} \mathbf{g}(\mathbf{D}_{L'} \underline{L}', e'_A) - \mathbf{g}(\mathbf{D}_{L'}(F), e'_A) \\ &= \zeta'_A - \mathbf{g}(\mathbf{D}_{L'}(F), e'_A), \end{aligned} \tag{4.104}$$

where we used the Ricci equations (4.80) to obtain the last equality. (4.104) implies:

$$\|\nabla\!\!\!/_{L'} F'\|_{L^\infty_{x'} L^2_s} + \mathcal{N}'_1(\nabla\!\!\!/_{L'} F') \lesssim \|\zeta'\|_{L^\infty_{x'} L^2_s} + \|\bar{\epsilon}\|_{L^\infty_{x'} L^2_s} + \mathcal{N}'_1(\zeta') + \mathcal{N}'_1(\bar{\epsilon}). \tag{4.105}$$

Now, in view of the definition of $\mathcal{N}_1$ and $\mathcal{N}'_1$, and the relation (4.95) between $\nabla\!\!\!/$ and $\nabla\!\!\!/'$, we have for any tensor G:

$$\mathcal{N}_1(G) \lesssim (1 + \|F'\|_{L^\infty}) \mathcal{N}'_1(G) \lesssim \mathcal{N}'_1(G) \text{ and } \mathcal{N}'_1(G) \lesssim (1 + \|F'\|_{L^\infty}) \mathcal{N}_1(G) \lesssim \mathcal{N}_1(G) \tag{4.106}$$

where we used the bootstrap assumption (4.103). Remark 4.19, the estimates (4.82), (4.83), (4.105), (4.106) and the bootstrap assumptions (4.2) (4.3) imply:

$$\|\nabla\!\!\!/_{L'} F'\|_{L^\infty_{x'} L^2_s} + \mathcal{N}'_1(\nabla\!\!\!/_{L'} F') \lesssim D\varepsilon. \tag{4.107}$$

We now estimate $\nabla\!\!\!/' F'$. In view of (4.98), we have:

$$\begin{aligned} \mathbf{g}(\mathbf{D}_{e'_A} F', e'_B) &= \frac{1}{2} b \underline{\chi}_{AB} - \frac{1}{2} \underline{\chi}'_{AB} + F'_B \zeta'_A + F'_A \zeta'_B + F'_B \chi'_{AC} F'_C - F'_A \mathbf{g}(\mathbf{D}_{L'} F', e'_B) \\ &\quad - \frac{1}{2} |F'|^2 \chi'_{AB} \end{aligned} \tag{4.108}$$

which yields:

$$\begin{aligned} \|\nabla\!\!\!/' F'\|_{\mathcal{B}'^0} &\lesssim \|b\underline{\chi}\|_{\mathcal{B}'^0} + \|\underline{\chi}'\|_{\mathcal{B}'^0} + \|F'\zeta'\|_{\mathcal{B}'^0} + \|F'^2 \chi'\|_{\mathcal{B}'^0} + \|F' \nabla\!\!\!/_{L'} F'\|_{\mathcal{B}'^0} \qquad (4.109) \\ &\lesssim \mathcal{N}'_1(b\underline{\chi}) + \|\underline{\chi}'\|_{\mathcal{B}'^0} + (\|\nabla\!\!\!/' F'\|_{L^\infty_s L^2_{x'}} + \|F'\|_{L^\infty}) \|\zeta'\|_{\mathcal{B}'^0} \\ &\quad + (\|F' \nabla\!\!\!/' F'\|_{L^\infty_s L^2_{x'}} + \|F'\|^2_{L^\infty}) \|\chi'\|_{\mathcal{B}'^0} + (\|\nabla\!\!\!/' F'\|_{L^\infty_s L^2_{x'}} + \|F'\|_{L^\infty}) \|\nabla\!\!\!/_{L'} F'\|_{\mathcal{B}'^0} \\ &\lesssim \mathcal{N}_1(b\underline{\chi}) + \|\underline{\chi}'\|_{\mathcal{B}'^0} + (\|\nabla\!\!\!/' F'\|_{L^\infty_s L^2_{x'}} + \|F'\|_{L^\infty}) \mathcal{N}'_1(\zeta') \\ &\quad + \|F'\|_{L^\infty} (\|\nabla\!\!\!/' F'\|_{L^\infty_s L^2_{x'}} + \|F'\|_{L^\infty}) \mathcal{N}'_1(\chi') \\ &\quad + (\|\nabla\!\!\!/' F'\|_{L^\infty_s L^2_{x'}} + \|F'\|_{L^\infty}) \mathcal{N}'_1(\nabla\!\!\!/_{L'} F') \end{aligned}$$

where we used several times the inequalities (4.88) and (4.90) as well as (4.106). The estimates (4.83) and (4.86) together with the bootstrap assumptions (4.1)-(4.5), (4.107) and (4.109) yield:

$$\|\nabla\!\!\!/' F'\|_{\mathcal{B}'^0} \lesssim D\varepsilon + D\varepsilon(1+\|F'\|_{L^\infty})(\|\nabla\!\!\!/' F'\|_{\mathcal{B}'^0} + \|F'\|_{L^\infty}). \tag{4.110}$$

Finally, the bootstrap assumption (4.103) together with (4.110) and the fact that ε is small yields:

$$\|\nabla\!\!\!/' F'\|_{\mathcal{B}'^0} \lesssim D\varepsilon$$

which together with (4.87) implies:

$$\|F'\|_{L^\infty} \lesssim D\varepsilon. \tag{4.111}$$

(4.111) is an improvement of the bootstrap assumption (4.103) which shows that F' is indeed in L^∞ and satisfies the bound (4.111). In particular, (4.102) and (4.111) imply:

$$\|\zeta\|_{L^\infty_{x'}L^2_t} \lesssim \varepsilon. \tag{4.112}$$

Note that (4.97) and (4.112) are improvements of the corresponding estimates in the bootstrap assumptions (4.4)-(4.6).

Remark 4.22 *We also have an estimate for $\nabla\!\!\!/' F'$ in $L^2_{x'}L^\infty_s$. Indeed,* (4.108) *yields:*

$$\begin{aligned}
\|\nabla\!\!\!/' F'\|_{L^2_{x'}L^\infty_s} &\lesssim \|b\underline{\chi}\|_{L^2_{x'}L^\infty_s} + \|\underline{\chi}'\|_{L^2_{x'}L^\infty_s} + \|F'\zeta'\|_{L^2_{x'}L^\infty_s} + \|F'^2\chi'\|_{L^2_{x'}L^\infty_s} + \|F'\nabla\!\!\!/_{L'}F'\|_{L^2_{x'}L^\infty_s}\\
&\lesssim \|b\|_{L^\infty}\mathcal{N}_1'(\underline{\chi}) + \|\underline{\chi}'\|_{L^2_{x'}L^\infty_s} + \|F'\|_{L^\infty}\|\zeta'\|_{L^2_{x'}L^\infty_s}\\
&\quad + \|F'\|^2_{L^\infty}\|\chi'\|_{L^2_{x'}L^\infty_s} + \|F'\|_{L^\infty}\|\nabla\!\!\!/_{L'}F'\|_{L^2_{x'}L^\infty_s}\\
&\lesssim \|b\|_{L^\infty}\mathcal{N}_1(\underline{\chi}) + \varepsilon + D\varepsilon\mathcal{N}_1'(\zeta') + D^2\varepsilon^2\mathcal{N}_1'(\chi') + D\varepsilon\mathcal{N}_1'(\nabla\!\!\!/_{L'}F')\\
&\lesssim D\varepsilon
\end{aligned} \tag{4.113}$$

where we used (4.106), (4.107), (4.111), *the bootstrap assumptions* (4.1)-(4.6) *on b and $\underline{\chi}$, and the estimates* (4.83).

Remark 4.23 (4.108) *implies the following estimate for $\nabla\!\!\!/' F' + \frac{1}{2}\underline{\chi}'$:*

$$\begin{aligned}
\mathcal{N}_1'\left(\nabla\!\!\!/' F' + \frac{1}{2}\underline{\chi}'\right) &\lesssim \mathcal{N}_1'(b\underline{\chi}) + \mathcal{N}_1'(F'\zeta') + \mathcal{N}_1'(F'^2\chi') + \mathcal{N}_1'(F'\nabla\!\!\!/_{L'}F')\\
&\lesssim \mathcal{N}_1(b\underline{\chi}) + (\|F'\|_{L^\infty} + \|\nabla\!\!\!/' F'\|_{L^2_{x'}L^\infty_s})(\mathcal{N}_1'(\zeta') + \|\zeta'\|_{L^\infty_{x'}L^2_s}\\
&\quad + \|F'\|_{L^\infty}(\mathcal{N}_1'(\chi') + \|\chi'\|_{L^\infty_{x'}L^2_s}) + \mathcal{N}_1'(\nabla\!\!\!/_{L'}F') + \|\nabla\!\!\!/_{L'}F'\|_{L^\infty_{x'}L^2_s})\\
&\lesssim D\varepsilon,
\end{aligned} \tag{4.114}$$

where we used (4.106), *the estimates* (4.82) *and* (4.83), *the estimates* (4.107) (4.111) *and* (4.113) *for F', and the bootstrap assumptions* (4.1)-(4.6) *for b and $\underline{\chi}$.*

4.7 Trace norm bounds for $\overline{\delta}$ and $\overline{\epsilon}$

The goal of this section is to improve the estimate for $\|\overline{\delta}\|_{L^\infty_{x'}L^2_t}$ and $\|\overline{\epsilon}\|_{L^\infty_{x'}L^2_t}$ given by the bootstrap assumption (4.3). Let us first define k_{LL} and k_{LA}:

$$k_{LL} = -\mathbf{g}(\mathbf{D}_L T, L),\ k_{LA} = -\mathbf{g}(\mathbf{D}_L T, e_A),\ A = 1, 2. \tag{4.115}$$

Then, using the definition of $\overline{k}$ (2.24) and the computation of $\mathbf{D}_T T$ (2.4), we have:

$$\begin{aligned} \overline{\delta} &= \delta - n^{-1}N(n) = -\mathbf{g}(\mathbf{D}_N T, N) - \mathbf{g}(\mathbf{D}_T T, N) = -\mathbf{g}(\mathbf{D}_L T, N) = -\mathbf{g}(\mathbf{D}_L T, L) \\ &= k_{LL}. \end{aligned} \tag{4.116}$$

and

$$\begin{aligned} \overline{\epsilon}_A &= \epsilon_A - n^{-1}\nabla\!\!\!/_A n = -\mathbf{g}(\mathbf{D}_N T, e_A) - \mathbf{g}(\mathbf{D}_T T, e_A) = -\mathbf{g}(\mathbf{D}_L T, e_A) \\ &= k_{LA}. \end{aligned} \tag{4.117}$$

We also define $k_{L'L'}$ and $k_{L'A}$:

$$k_{L'L'} = -\mathbf{g}(\mathbf{D}_{L'} T, L'),\ k_{L'A} = -\mathbf{g}(\mathbf{D}_{L'} T, e'_A),\ A = 1, 2. \tag{4.118}$$

Then, the relations (4.95) between L, e_1, e_2 and L', e'_1, e'_2 together with the definitions (4.115) and (4.118) yield:

$$k_{LL} = b^2 k_{L'L'} \text{ and } k_{LA} = b k_{L'A} + b F'_A k_{L'L'}. \tag{4.119}$$

Thus, (4.116), (4.117) and (4.119) imply:

$$\begin{aligned} \|\overline{\delta}\|_{L^\infty_{x'}L^2_t} &\lesssim \|b k_{L'L'}\|_{L^\infty_{x'}L^2_s} \lesssim \|k_{L'L'}\|_{L^\infty_{x'}L^2_s} \\ \|\overline{\epsilon}\|_{L^\infty_{x'}L^2_t} &\lesssim \|b k_{L'A}\|_{L^\infty_{x'}L^2_s} + \|b F'_A k_{L'L'}\|_{L^\infty_{x'}L^2_s} \lesssim \|k_{L'L'}\|_{L^\infty_{x'}L^2_s} + \|k_{L'A}\|_{L^\infty_{x'}L^2_s} \end{aligned} \tag{4.120}$$

where we used the bootstrap assumption (4.1), the L^∞ bound for F' (4.111) and Remark 4.19.

In view of (4.120), it is enough to bound the trace norms $\|k_{L'L'}\|_{L^\infty_{x'}L^2_s}$ and $\|k_{L'A}\|_{L^\infty_{x'}L^2_s}$. First, note that the bootstrap assumptions (4.1) (4.3) together with the L^∞ bound for F' (4.111) and the identity (4.119) yield:

$$\|k_{L'L'}\|_{L^\infty_{x'}L^2_s} + \|k_{L'A}\|_{L^\infty_{x'}L^2_s} \lesssim D\varepsilon. \tag{4.121}$$

Our goal in this section is to improve the constant D in the right-hand side of (4.121). We will rely on the trace estimate (4.93). The improved estimates for n (4.48) (4.58) and the improved estimate for δ, ϵ (4.71) imply:

$$\mathcal{N}_1(\overline{\delta}) + \mathcal{N}_1(\overline{\epsilon}) \lesssim \varepsilon. \tag{4.122}$$

(4.116), (4.119), (4.122), (4.106) and the bootstrap assumptions (4.1) (4.2) for b yield:

$$\mathcal{N}'_1(k_{L'L'}) \lesssim \mathcal{N}_1(b^{-2}\overline{\delta}) \lesssim \varepsilon. \tag{4.123}$$

(4.121) and (4.123) yield:

$$\begin{aligned}\mathcal{N}_1'(F'k_{L'L'}) &\lesssim \|F'\|_{L^\infty}\mathcal{N}_1'(k_{L'L'}) + (\|\nabla\!\!\!/' F'\|_{L^2_{x'}L^\infty_s} + \|\nabla\!\!\!/_{L'}F'\|_{L^2_{x'}L^\infty_s})\|k_{L'L'}\|_{L^\infty_{x'}L^2_s}\\ &\lesssim D\varepsilon^2 + D^2\varepsilon^2 + D\varepsilon\mathcal{N}_1'(\nabla\!\!\!/_{L'}F')\\ &\lesssim \varepsilon \end{aligned} \tag{4.124}$$

where we used (4.107), (4.111) and (4.113). Finally, (4.117), (4.119), (4.122), (4.124), (4.106) and the bootstrap assumptions (4.1) (4.2) for b yield:

$$\mathcal{N}_1'(k_{L'A}) \lesssim \mathcal{N}_1(b^{-1}\overline{\epsilon}) + \mathcal{N}_1'(F'_A k_{L'L'}) \lesssim \varepsilon. \tag{4.125}$$

In order to apply the trace estimate (4.93), we need to show that $\nabla\!\!\!/' k_{L'L'}$ and $\nabla\!\!\!/' k_{L'A}$ admit a decomposition of the form, $A\nabla\!\!\!/_{L'}P + E$. We start with $k_{L'L'}$. We have:

$$\begin{aligned}\nabla\!\!\!/'_{e'_A}k_{L'L'} &= -\mathbf{D}_{e'_A}\mathbf{g}(\mathbf{D}_{L'}T, L') = -\mathbf{g}(\mathbf{D}_{e'_A}\mathbf{D}_{L'}T, L') - \mathbf{g}(\mathbf{D}_{L'}T, D_{e'_A}L')\\ &= -\mathbf{g}(\mathbf{D}_{L'}\mathbf{D}_{e'_A}T, L') - \mathbf{R}_{e'_A L'TL'} - \mathbf{g}(\mathbf{D}_{[e'_A,L']}T, L')\\ &\quad -b^{-1}\mathbf{g}(-\overline{\delta}N + \underline{\zeta}_C e_C, \chi'_{AB}e'_B - \zeta'_A L')\\ &= -\nabla\!\!\!/_{L'}[\mathbf{g}(\mathbf{D}_\cdot T, L')]_A - b^{-1}F'_B\alpha'_{AB} - \frac{b^{-1}}{2}\beta'_A\\ &\quad +b^{-1}\chi'_{AB}(k_{BN} - \underline{\zeta}_B) - b^{-2}\overline{\delta}(2\chi'_{AB}F'_B + \zeta'_A). \end{aligned} \tag{4.126}$$

Relying on the Bianchi identities, the following decomposition for α', β' were obtained in [14]:

$$\alpha' = \nabla\!\!\!/_{L'}(P_1) + E_1,\ \beta' = \nabla\!\!\!/_{L'}(P_2) + E_2, \tag{4.127}$$

where $P_1 = \mathcal{D}'^{-1}_2\beta'$, $P_2 = \mathcal{D}'^{-1}_1(\rho', -\sigma')$, and

$$\mathcal{N}_1'(P_1) + \mathcal{N}_1'(P_2) + \|E_1\|_{\mathcal{P}'^0} + \|E_2\|_{\mathcal{P}'^0} \lesssim \varepsilon. \tag{4.128}$$

We define the tensors A_1, A_2, P_3, E_3 as:

$$\begin{aligned}&P_{3A} = -\mathbf{g}(\mathbf{D}_{e'_A}T, L'),\ A_1 = -b^{-1}F',\ A_2 = -\tfrac{b^{-1}}{2},\\ &E_3 = b^{-1}\chi'_{AB}(k_{BN} - \underline{\zeta}_B) - b^{-2}\overline{\delta}(2\chi'_{AB}F'_B + \zeta'_A),\end{aligned} \tag{4.129}$$

which together with (4.126) and (4.127) yields:

$$\nabla\!\!\!/' k_{L'L'} = A_1\nabla\!\!\!/_{L'}P_1 + A_2\nabla\!\!\!/_{L'}P_2 + \nabla\!\!\!/_{L'}P_3 + E_1 + E_2 + E_3. \tag{4.130}$$

Now, we have:

$$P_{3A} = -\mathbf{g}(\mathbf{D}_{e'_A}T, L') = b^{-1}k_{AN} - F'_A b^{-2}\overline{\delta}$$

which yields:

$$\begin{aligned}\mathcal{N}_1'(P_3) &\lesssim \mathcal{N}_1(b^{-1}k_{AN})\\ &\quad +(\|F'\|_{L^\infty} + \|\nabla\!\!\!/' F\|_{L^2_{x'}L^\infty_s} + \|\nabla\!\!\!/'_{L'}F\|_{L^2_{x'}L^\infty_s})(\mathcal{N}_1(b^{-2}\overline{\delta}) + \|b^{-2}\overline{\delta}\|_{L^\infty_{x'}L^2_t})\\ &\lesssim \varepsilon + D^2\varepsilon^2\\ &\lesssim \varepsilon \end{aligned} \tag{4.131}$$

where we used the bootstrap assumptions (4.1) (4.2) for b, the bootstrap assumption (4.3) for $\overline{\delta}$, the improved estimate (4.71) for k_{AN}, the estimates (4.107) (4.111) and (4.113) for F together with Remark 4.19 and (4.106). Using the bootstrap assumptions (4.1) (4.2) for b and the estimates (4.107) (4.111) and (4.113) for F, we also have:

$$\begin{aligned}
&\|A_1\|_{L^\infty} + \|\nabla\!\!\!/' A_1\|_{L^2_{x'}L^\infty_s} + \|\nabla\!\!\!/_{L'} A_1\|_{L^2_{x'}L^\infty_s} + \|A_2\|_{L^\infty} + \|\nabla\!\!\!/' A_2\|_{L^2_{x'}L^\infty_s} + \|\nabla\!\!\!/_{L'} A_2\|_{L^2_{x'}L^\infty_s} \\
\lesssim\ & (\|b\|_{L^\infty} + \mathcal{N}_2(b-1))(1 + \|F'\|_{L^\infty} + \|\nabla\!\!\!/' F'\|_{L^2_{x'}L^\infty_s} + \|\nabla\!\!\!/_{L'} F'\|_{L^2_{x'}L^\infty_s}) \\
\lesssim\ & D\varepsilon. \qquad (4.132)
\end{aligned}$$

The functional inequalities (4.89) and (4.91) yield:

$$\begin{aligned}
\|E_3\|_{\mathcal{P}'^0} \lesssim\ & (\|b\|_{L^\infty} + \|\nabla\!\!\!/' b\|_{L^\infty_s L^2_{x'}})(1 + \|F'\|_{L^\infty} + \|\nabla\!\!\!/' F'\|_{L^\infty_s L^2_{x'}}) \\
& \times(\mathcal{N}'_1(\chi')^2 + \mathcal{N}'_1(\zeta')^2 + \mathcal{N}'_1(\overline{\delta})^2 + \mathcal{N}'_1(\epsilon)^2 + \mathcal{N}'_1(\underline{\zeta})^2) \\
\lesssim\ & D^2\varepsilon^2 \\
\lesssim\ & \varepsilon, \qquad (4.133)
\end{aligned}$$

where we used the bootstrap assumptions (4.1)-(4.6) for b, $\overline{\delta}$, k_{BN} and $\underline{\zeta}$, the estimates (4.107) (4.111) and (4.113) for F, and the estimate (4.83) for χ' and ζ'. Finally, the sharp trace estimate (4.93) together with (4.130) and the estimates (4.123) (4.128), (4.131), (4.132) and (4.133) yields:

$$\begin{aligned}
\|k_{L'L'}\|_{L^\infty_{x'}L^2_s} \lesssim\ & \mathcal{N}'_1(k_{L'L'}) + (\|A_1\|_{L^\infty} + \|\nabla\!\!\!/' A_1\|_{L^2_{x'}L^\infty_s})\mathcal{N}'_1(P_1) + (\|A_2\|_{L^\infty} \qquad (4.134) \\
& + \|\nabla\!\!\!/' A_2\|_{L^2_{x'}L^\infty_s})\mathcal{N}'_1(P_2) + \mathcal{N}'_1(P_3) + \|E_1\|_{\mathcal{P}'^0} + \|E_2\|_{\mathcal{P}'^0} + \|E_3\|_{\mathcal{P}'^0} \\
\lesssim\ & \varepsilon,
\end{aligned}$$

which is an improvement of (4.121) for $k_{L'L'}$.

Next, we show that $\nabla\!\!\!/' k_{L'A}$ admit a decomposition of the form, $A\nabla\!\!\!/_{L'}P + E$. We have:

$$\begin{aligned}
\nabla\!\!\!/'_{e'_B}[k_{L'.}]_{e'_A} =\ & -e'_B[\mathbf{g}(\mathbf{D}_{L'}T, e'_A)] + \mathbf{g}(\mathbf{D}_{L'}T, \nabla\!\!\!/_{e'_B} e'_A) \\
=\ & -\mathbf{g}(\mathbf{D}_{e'_B}\mathbf{D}_{L'}T, e'_A) - \mathbf{g}(\mathbf{D}_{L'}T, D_{e'_B}e'_A - \nabla\!\!\!/_{e'_B}e'_A) \\
=\ & -\mathbf{g}(\mathbf{D}_{L'}\mathbf{D}_{e'_B}T, e'_A) - \mathbf{R}_{e'_B L'T e'_A} - \mathbf{g}(\mathbf{D}_{[e'_B,L']}T, e'_A) \\
& -b^{-1}\mathbf{g}(-\overline{\delta}N + \underline{\zeta}_C e_C, \frac{1}{2}\chi'_{AB}\underline{L}' + \frac{1}{2}\underline{\chi}'_{AB}L') \\
=\ & -\nabla\!\!\!/_{L'}[\mathbf{g}(\mathbf{D}_{.}T, .)]_{AB} - \mathbf{R}_{e'_B L'T e'_A} - \mathbf{g}(\mathbf{D}_{\mathbf{D}_{e'_B}L'}T, e'_A) \\
& +\mathbf{g}(\mathbf{D}_{\mathbf{D}_{L'}e'_B - \nabla\!\!\!/_{L'}e'_B}T, e'_A) + \mathbf{g}(\mathbf{D}_{e'_B}T, \mathbf{D}_{L'}e'_A - \nabla\!\!\!/_{L'}e'_A) \\
& -b^{-1}\mathbf{g}(-\overline{\delta}N + \underline{\zeta}_C e_C, \frac{1}{2}\chi'_{AB}\underline{L}' + \frac{1}{2}\underline{\chi}'_{AB}L') \\
=\ & -\nabla\!\!\!/_{L'}[\mathbf{g}(\mathbf{D}_{.}T, .)]_{AB} - \frac{1}{2}b^{-1}\alpha'_{AB} \\
& +\frac{1}{2}b^{-1}(\rho'\delta_{AB} - \sigma' \in_{AB} + 2F'_C \in_{AC} {}^*\beta'_B - |F'|^2\alpha'_{AB}) \\
& +\chi'_{BC}(k_{CA} - F'_A b^{-1}k_{CN} - F'_C b^{-1}\overline{\epsilon}_A + F'_C F'_A b^{-2}\overline{\delta}) \\
& +b^{-1}\zeta'_A(k_{BN} - b^{-1}F'_B\overline{\delta}) + \frac{1}{2}\chi'_{AB}(-\overline{\delta} + b^{-2}\overline{\delta}|F'|^2 \\
& +2b^{-1}\underline{\zeta}_C F'_C) + \frac{1}{2}b^{-2}\underline{\chi}'_{AB}\overline{\delta}. \qquad (4.135)
\end{aligned}$$

Define as in [14] $\check{\rho}', \check{\sigma}'$ as:

$$\check{\rho}' = \rho' - \frac{1}{2}\widehat{\chi}' \cdot \underline{\widehat{\chi}}', \; \check{\sigma} = \sigma - \frac{1}{2}\widehat{\chi}' \wedge \underline{\widehat{\chi}}'. \tag{4.136}$$

Relying on the Bianchi identities, the following decomposition for $\check{\rho}', \check{\sigma}'$ was obtained in [14]:

$$(\check{\rho}', \check{\sigma}') = \nabla\!\!\!/_{L'}(P_4) + E_4, \tag{4.137}$$

where $P_4 = {}^*\mathcal{D}'^{-1}_1 \underline{\beta}'$, and

$$\mathcal{N}'_1(P_4) + \|E_4\|_{\mathcal{P}'^0} \lesssim \varepsilon. \tag{4.138}$$

We have

$$\mathbf{g}(\mathbf{D}_{e'_B} T, e'_A) = -k_{AB} + F'_B b^{-1} \overline{\epsilon}_A + b^{-1} F'_A k_{BN} - b^{-2} F'_A F'_B \overline{\delta}$$

which yields:

$$-\mathbf{g}(\mathbf{D}_{e'_B} T, e'_A) = \nabla\!\!\!/_{L'}(P_5) + A_6 \nabla\!\!\!/_{L'}(P_6) + E_6 \tag{4.139}$$

where P_5, A_6, P_6, E_6 are defined by:

$$\begin{array}{l} P_{5AB} = -k_{AB} + F'_B b^{-1} \overline{\epsilon}_A - b^{-2} F'_A F'_B \overline{\delta}, \\ A_{6B} = b^{-1} F'_B, \; P_{6B} = k_{BN}, \; E_{6AB} = \nabla\!\!\!/'_L [b^{-1} F']_A k_{BN}. \end{array} \tag{4.140}$$

We define the tensors A_4, A_7, A_8, E_7 as:

$$\begin{array}{l} A_4 = \frac{1}{2} b^{-1}, \; A_7 = b^{-1} F', \; A_8 = -b^{-1} |F'|^2 \\ E_{7AB} = \chi'_{BC}(k_{CA} - F'_A b^{-1} k_{CN} - F'_C b^{-1} \overline{\epsilon}_A + F'_C F'_A b^{-2} \overline{\delta}) \\ + b^{-1} \zeta'_A (k_{BN} - b^{-1} F'_B \overline{\delta}) + \frac{1}{2} \chi'_{AB} (-\overline{\delta} + b^{-2} \overline{\delta} |F'|^2 + 2 b^{-1} \underline{\zeta}_C F'_C), \end{array} \tag{4.141}$$

which together with (4.127), (4.135), (4.137) and (4.139) yields:

$$\begin{aligned} \nabla\!\!\!/' k_{L'A} \;=\; & A_7 \nabla\!\!\!/_{L'} P_2 + A_8 \nabla\!\!\!/_{L'} P_1 + A_4 \nabla\!\!\!/_{L'} P_4 + \nabla\!\!\!/_{L'} P_5 + A_6 \nabla\!\!\!/_{L'} P_6 + E_1 + E_2 + E_4 \\ & + E_6 + E_7 + \frac{1}{4} b^{-1} (\widehat{\chi}' \cdot \underline{\widehat{\chi}}' \delta_{AB} - \widehat{\chi}' \wedge \underline{\widehat{\chi}}' \in_{AB}) + \frac{1}{2} b^{-2} \underline{\chi}'_{AB} \overline{\delta} \end{aligned} \tag{4.142}$$

In view of (4.114), we define E_8 as:

$$E_8 = \frac{1}{2} b^{-1} (\widehat{\chi}' \cdot (\nabla\!\!\!/' F' + \frac{1}{2} \underline{\widehat{\chi}}') \delta_{AB} - \widehat{\chi}' \wedge (\nabla\!\!\!/' F' + \frac{1}{2} \underline{\widehat{\chi}}') \in_{AB}) + b^{-2} (\nabla\!\!\!/'_B F'_A + \frac{1}{2} \underline{\chi}'_{AB}) \overline{\delta}. \tag{4.143}$$

Note that the non sharp product estimate (4.91) together with the bootstrap assumptions (4.1)-(4.3) for b and $\overline{\delta}$, the estimate (4.83) for χ' and the estimate (4.114) for $\nabla\!\!\!/' F' + \frac{1}{2}\underline{\chi}'$ yields:

$$\|E_8\|_{\mathcal{P}'^0} \lesssim D^2 \varepsilon^2 \lesssim \varepsilon. \tag{4.144}$$

Now, we recall the following result from [14] section 7:

$$\nabla\!\!\!/' \chi' = \nabla\!\!\!/_{L'}(P_{10}) + E_{10},$$

with $\mathcal{N}'_1(P_{10}) + \|E_{10}\|_{\mathcal{P}'^0} \lesssim \varepsilon$ which together with (4.116), (4.130) and (4.143) yields:

$$\begin{aligned} & \frac{1}{4} b^{-1} (\widehat{\chi}' \cdot \underline{\widehat{\chi}}' \delta_{AB} - \widehat{\chi}' \wedge \underline{\widehat{\chi}}' \in_{AB}) + \frac{1}{2} b^{-2} \underline{\chi}'_{AB} \overline{\delta} \\ = \;& A_{11} \nabla\!\!\!/_{L'}(P_{11}) + E_{11} - \nabla\!\!\!/' (\frac{1}{2} b^{-1} (\widehat{\chi}' \cdot F' \delta_{AB} - \widehat{\chi}' \wedge F' \in_{AB}) + b^{-2} F'_A \overline{\delta}). \end{aligned} \tag{4.145}$$

Using (4.144), the fact that $\mathcal{N}_1'(P_{10}) + \|E_{10}\|_{\mathcal{P}'^0} \lesssim \varepsilon$ from [14], the estimate (4.83) for χ', the bootstrap assumptions (4.1)-(4.3) for b and $\overline{\delta}$, and the estimates (4.107) (4.111) and (4.113) for F implies that A_{11}, P_{11}, E_{11} satisfy:

$$\|A_{11}\|_{L^\infty} + \|\nabla\!\!\!/' A_{11}\|_{L^2_{x'}L^\infty_s} + \|\nabla\!\!\!/_{L'} A_{11}\|_{L^2_{x'}L^\infty_s} + \mathcal{N}_1'(P_{11}) + \|E_{11}\|_{\mathcal{P}'^0} \lesssim \varepsilon. \tag{4.146}$$

Now, (4.142) and (4.145) yield:

$$\begin{aligned} & \nabla\!\!\!/'[k_{L'A} + \frac{1}{2}b^{-1}(\widehat{\chi}' \cdot F'\delta_{AB} - \widehat{\chi}' \wedge F' \in_{AB}) + b^{-2}F'_A\overline{\delta}] \\ = \;& A_7\nabla\!\!\!/_{L'}P_2 + A_8\nabla\!\!\!/_{L'}P_1 + A_4\nabla\!\!\!/_{L'}P_4 + \nabla\!\!\!/_{L'}P_5 + A_6\nabla\!\!\!/_{L'}P_6 + A_{11}\nabla\!\!\!/_{L'}P_{11} \\ & + E_1 + E_2 + E_4 + E_6 + E_7 + E_{11}. \end{aligned} \tag{4.147}$$

(4.140), (4.141), the bootstrap assumptions (4.1) (4.2) for b, the bootstrap assumption (4.3) for $\overline{\epsilon}, \overline{\delta}$, the improved estimate (4.71) for k_{AN}, the estimates (4.107) (4.111) and (4.113) for F' together with Remark 4.19 and (4.106), the estimate (4.83) for χ' and ζ', the trace estimate (4.82), the inequality (4.89) and the non sharp product estimate (4.91) yield:

$$\begin{aligned} & \|A_j\|_{L^\infty} + \|\nabla\!\!\!/' A_j\|_{L^2_{x'}L^\infty_s} + \|\nabla\!\!\!/_{L'} A_j\|_{L^2_{x'}L^\infty_s} \lesssim D\varepsilon \text{ for } j = 4, 6, 7, 8, \\ & \mathcal{N}_1'(P_j) \lesssim \varepsilon \text{ for } j = 5, 6, \\ & \|E_j\|_{\mathcal{P}'^0} \lesssim \varepsilon \text{ for } j = 6, 7. \end{aligned} \tag{4.148}$$

Note also that (4.125), the bootstrap assumptions (4.1) (4.2) for b, the bootstrap assumption (4.3) for $\overline{\delta}$, the estimates (4.107) (4.111) and (4.113) for F' together with the inequality (4.106), the estimate (4.83) for χ' and the trace estimate (4.82) imply:

$$\mathcal{N}_1'\left(k_{L'A} + \frac{1}{2}b^{-1}(\widehat{\chi}' \cdot F'\delta_{AB} - \widehat{\chi}' \wedge F' \in_{AB}) + b^{-2}F'_A\overline{\delta}\right) \lesssim \varepsilon + D^2\varepsilon^2 \lesssim \varepsilon. \tag{4.149}$$

Finally, the sharp trace estimate (4.93) together with (4.147) and the estimates (4.128), (4.138), (4.146), (4.148) and (4.149) yields:

$$\left\|k_{L'A} + \frac{1}{2}b^{-1}(\widehat{\chi}' \cdot F'\delta_{AB} - \widehat{\chi}' \wedge F' \in_{AB}) + b^{-2}F'_A\overline{\delta}\right\|_{L^\infty_{x'}L^2_s} \lesssim \varepsilon + D^2\varepsilon^2 \lesssim \varepsilon. \tag{4.150}$$

(4.150), the bootstrap assumption (4.1) for b, the bootstrap assumption (4.3) for $\overline{\delta}$, the estimate (4.111) for F' and the trace estimate (4.82) for χ' imply:

$$\|k_{L'A}\|_{L^\infty_{x'}L^2_s} \lesssim \varepsilon, \tag{4.151}$$

which is an improvement of (4.121) for $k_{L'A}$. (4.120), (4.134) and (4.151) yield:

$$\|\overline{\delta}\|_{L^\infty_{x'}L^2_t} + \|\overline{\epsilon}\|_{L^\infty_{x'}L^2_t} \lesssim \varepsilon, \tag{4.152}$$

which improves the trace estimates for $\overline{\delta}$ and $\overline{\epsilon}$ given by the bootstrap assumption (4.3).

4.8 Estimates for b

The goal of this section is to improve the bootstrap assumption for b given by (4.1) and (4.2), and to derive an estimate for $\underline{L}(b)$ in $L^\infty_t L^4_{x'}$. Using the transport equation for b (2.27) and the estimate for transport equations (3.66), we obtain:

$$\begin{aligned} \|b-1\|_{L^\infty} &\lesssim \|b(0)-1\|_{L^\infty} + \|b\overline{\delta}\|_{L^\infty_{x'}L^1_t} \\ &\lesssim \varepsilon + (1+\|b-1\|_{L^\infty})\|\overline{\delta}\|_{L^\infty_{x'}L^2_t} \\ &\lesssim \varepsilon + D\varepsilon\|b-1\|_{L^\infty} \end{aligned} \tag{4.153}$$

where we used the bootstrap assumption (4.3) for $\overline{\delta}$ in the last inequality. (4.153) yields:

$$\|b-1\|_{L^\infty} \lesssim \varepsilon \tag{4.154}$$

which improves the estimate for b given by the bootstrap assumption (4.1). Using again the estimate for transport equations (3.66), we also have:

$$\|b-1\|_{L^\infty_t L^2_{x'}} \lesssim \|b(0)-1\|_{L^2_{x'}} + \|b\overline{\delta}\|_{L^2_{x'}L^1_t} \lesssim \varepsilon$$

which together with (2.27) and (2.26) implies:

$$\begin{aligned} \mathcal{N}_2(b-1) &\lesssim \varepsilon + \mathcal{N}_1(L(b)) + \mathcal{N}_1(\nabla\!\!\!/ b) \\ &\lesssim \varepsilon + \mathcal{N}_1(b\overline{\delta}) + \mathcal{N}_1(b\zeta) + \mathcal{N}_1(b\epsilon) \\ &\lesssim \varepsilon + (\|b\|_{L^\infty} + \mathcal{N}_2(b-1))(\mathcal{N}_1(\overline{\delta}) + \mathcal{N}_1(\zeta) + \mathcal{N}_1(\epsilon)) \\ &\lesssim \varepsilon + \varepsilon\mathcal{N}_2(b-1) \end{aligned} \tag{4.155}$$

where we used (4.154) and the improved estimates (4.71) for $\overline{\delta}$ and ϵ, and (4.180) for ζ. (4.155) yields:

$$\mathcal{N}_2(b-1) \lesssim \varepsilon. \tag{4.156}$$

We also derive an estimate for $\underline{L}(b)$. Differentiating the transport equation for b (2.27) with respect to $\underline{L}$, we obtain:

$$\begin{aligned} L(\underline{L}(b)) &= [L,\underline{L}](b) - \underline{L}(b)\overline{\delta} - b\underline{L}(\overline{\delta}) \\ &= (\delta + n^{-1}N(n))\overline{\delta}b - 2(\zeta-\underline{\zeta})\cdot\nabla\!\!\!/ b - b\underline{L}(\overline{\delta}), \end{aligned} \tag{4.157}$$

where we used the commutator formula (2.46) in the last equality. This yields:

$$\begin{aligned} &\|L(\underline{L}(b))\|_{L^2(\mathcal{H}_u)} \\ \lesssim\ & (1+\|b\|_{L^\infty})\bigg(\|\delta + n^{-1}N(n)\|_{L^4(\mathcal{H}_u)}\|\overline{\delta}\|_{L^4(\mathcal{H}_u)} + \|\zeta-\underline{\zeta}\|_{L^4(\mathcal{H}_u)}\|\nabla\!\!\!/ b\|_{L^4(\mathcal{H}_u)} \\ & + \|\underline{L}(\overline{\delta})\|_{L^2(\mathcal{H}_u)}\bigg) \\ \lesssim\ & (1+\|b\|_{L^\infty})\bigg(\mathcal{N}_1(\delta)^2 + \mathcal{N}_1(n^{-1}\nabla n)^2 + \mathcal{N}_1(\zeta)^2 + \mathcal{N}_1(\underline{\zeta})^2 + \mathcal{N}_1(\nabla\!\!\!/ b)^2 + \|\underline{L}(\overline{\delta})\|_{L^2(\mathcal{H}_u)}\bigg) \\ \lesssim\ & \varepsilon + D^2\varepsilon^2 \\ \lesssim\ & \varepsilon, \end{aligned} \tag{4.158}$$

where we used the bootstrap assumptions (4.1)-(4.6). Together with the estimate for transport equations (3.66), we obtain:

$$\|\underline{L}(b)\|_{L^2_{x'}L^\infty_t} \lesssim \|L(\underline{L}(b))\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{4.159}$$

(4.156) and (4.159) improve the estimate for b given by the bootstrap assumption (4.2).

Finally, we derive an estimate for $\underline{L}(b)$ in $L^\infty_t L^4_{x'}$. In view of (2.43), we have

$$\begin{aligned} b\underline{L}(\overline{\delta}) &= -L(b(\delta+n^{-1}N(n))) + L(b)(\delta+n^{-1}N(n)) + 2b\rho + 2b\zeta\cdot(2\epsilon - n^{-1}\nabla\!\!\!/ n) \\ &\quad -2bn^{-1}\nabla\!\!\!/ n\cdot\epsilon + 2b|\epsilon|^2 + 2b\delta^2 - 2b(n^{-1}N(n))^2, \end{aligned}$$

which together with (4.157) implies

$$L\left(\underline{L}(b) - b(\delta+n^{-1}N(n))\right) = -2b\rho + h_1, \tag{4.160}$$

where h_1 is given by

$$\begin{aligned} h_1 &= (\delta+n^{-1}N(n))\overline{\delta}b - 2(\zeta-\underline{\zeta})\cdot\nabla\!\!\!/ b - L(b)(\delta+n^{-1}N(n)) - 2b\zeta\cdot(2\epsilon - n^{-1}\nabla\!\!\!/ n) \\ &\quad +2bn^{-1}\nabla\!\!\!/ n\cdot\epsilon - 2b|\epsilon|^2 - 2b\delta^2 + 2b(n^{-1}N(n))^2. \end{aligned}$$

In view of the bootstrap assumptions (4.1)-(4.6), we have the following estimate for h_1

$$\begin{aligned} \|h_1\|_{L^1_t L^4_{x'}} &\lesssim (\mathcal{N}_1(\delta)+\mathcal{N}_1(n^{-1}\nabla n)+\mathcal{N}_2(b-1)+\mathcal{N}_1(\epsilon)+\mathcal{N}_1(\zeta)+\mathcal{N}_1(\underline{\zeta}))^2(1+\|b\|_{L^\infty}) \\ &\lesssim D^2\varepsilon^2 \\ &\lesssim \varepsilon. \end{aligned} \tag{4.161}$$

Next, we decompose the term involving ρ in the right-hand side of (4.160). In view of the Bianchi identity (2.57), we have:

$$(n\rho, n\sigma) = {}^*\mathcal{D}_1^{-1}\left(\nabla\!\!\!/_{nL}(\underline{\beta}) - \nabla\!\!\!/(n)\rho + \nabla\!\!\!/(n)\sigma + n\mathrm{tr}\chi\underline{\beta} - 2n\underline{\widehat{\chi}}\cdot\beta - n\overline{\delta}\underline{\beta} + 3n(\underline{\zeta}\rho - {}^*\underline{\zeta}\sigma)\right)$$

which yields

$$(\rho,\sigma) = L({}^*\mathcal{D}_1^{-1}(\underline{\beta})) + n^{-1}[{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}]\underline{\beta} + h_2, \tag{4.162}$$

where h_2 is given by

$$h_2 = n^{-1}{}^*\mathcal{D}_1^{-1}\left(-\nabla\!\!\!/(n)\rho + \nabla\!\!\!/(n)\sigma + n\mathrm{tr}\chi\underline{\beta} - 2n\underline{\widehat{\chi}}\cdot\beta - n\overline{\delta}\underline{\beta} + 3n(\underline{\zeta}\rho - {}^*\underline{\zeta}\sigma)\right).$$

In view of the bootstrap assumptions (4.1)-(4.6) and Lemma 3.17, we have the following estimate for h_2

$$\begin{aligned} &\|h_2\|_{L^1_t L^4_{x'}} + \|h_2\|_{L^2_t L^3_{x'}} \qquad (4.163) \\ \lesssim\ & \|n^{-1}\|_{L^\infty}\left\|-\nabla\!\!\!/(n)\rho + \nabla\!\!\!/(n)\sigma + n\mathrm{tr}\chi\underline{\beta} - 2n\underline{\widehat{\chi}}\cdot\beta - n\overline{\delta}\underline{\beta} + 3n(\underline{\zeta}\rho - {}^*\underline{\zeta}\sigma)\right\|_{L^1_t L^{\frac{3}{2}}_{x'}} \\ & + \|n^{-1}\|_{L^\infty}\left\|-\nabla\!\!\!/(n)\rho + \nabla\!\!\!/(n)\sigma + n\mathrm{tr}\chi\underline{\beta} - 2n\underline{\widehat{\chi}}\cdot\beta - n\overline{\delta}\underline{\beta} + 3n(\underline{\zeta}\rho - {}^*\underline{\zeta}\sigma)\right\|_{L^2_t L^{\frac{4}{3}}_{x'}} \\ \lesssim\ & (\mathcal{N}_1(\nabla\!\!\!/ n)+\mathcal{N}_1(\mathrm{tr}\chi)+\mathcal{N}_1(\widehat{\chi})+\mathcal{N}_1(\overline{\delta})+\mathcal{N}_1(\underline{\zeta}))\|(\beta,\rho,\sigma,\underline{\beta})\|_{L^\infty_u L^2(\mathcal{H}_u)}\|n^{-1}\|_{L^\infty}(1+\|n\|_{L^\infty}) \\ \lesssim\ & D\varepsilon^2 \\ \lesssim\ & \varepsilon, \end{aligned}$$

where we used the bootstrap assumptions (4.1)-(4.6) and the curvature bound (2.59). Next, we estimate the commutator term in the right-hand side of (4.162). This is done in the following lemma.

Lemma 4.24 *$[{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta})$ satisfies the following estimate:*

$$\|[{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta})\|_{L^2_t L^3_{x'}} + \|[{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta})\|_{L^1_t L^4_{x'}} \lesssim \varepsilon.$$

We postpone the proof of Lemma 4.24 to section A.3 and conclude the estimate for $\underline{L}(b)$ in $L^\infty_t L^4_{x'}$. In view of (4.160) and (4.162), we have

$$\begin{aligned} & L\left(\underline{L}(b) - b(\delta + n^{-1}N(n)) + 2b\pi_1({}^*\mathcal{D}_1^{-1}(\underline{\beta}))\right) \\ = \;& 2L(b)\pi_1({}^*\mathcal{D}_1^{-1}(\underline{\beta})) - 2b\pi_1([{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta})) - 2b\pi_1(h_2) + h_1, \end{aligned}$$

where π_1 denote the projection in $\mathbb{R}^2$ on the first coordinate. Together with the estimate for transport equations (3.66), we obtain

$$\begin{aligned} & \left\|\underline{L}(b) - b(\delta + n^{-1}N(n)) + 2b\pi_1({}^*\mathcal{D}_1^{-1}(\underline{\beta}))\right\|_{L^\infty_t L^4_{x'}} \\ \lesssim \;& \|L(b)\|_{L^2_t L^8_{x'}} \|{}^*\mathcal{D}_1^{-1}(\underline{\beta})\|_{L^2_t L^8_{x'}} + \|b\|_{L^\infty}(\|[{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta})\|_{L^1_t L^4_{x'}} + \|h_2\|_{L^1_t L^4_{x'}}) + \|h_1\|_{L^1_t L^4_{x'}} \\ \lesssim \;& \mathcal{N}_2(b-1)\|\nabla\!\!\!/ {}^*\mathcal{D}_1^{-1}(\underline{\beta})\|_{L^2(\mathcal{H}_u)} + \varepsilon, \end{aligned}$$

where we used in the last inequality the Gagliardo-Nirenberg inequality (3.3), the bootstrap assumption (4.1) for b, Lemma 4.24, the estimate (4.163) for h_2 and the estimate (4.161) for h_1. Together with the bootstrap assumption (4.2) for b, the estimate (3.50) for ${}^*\mathcal{D}_1^{-1}$, and the curvature bound (2.59) for $\underline{\beta}$, we deduce

$$\left\|\underline{L}(b) - b(\delta + n^{-1}N(n)) + 2b\pi_1({}^*\mathcal{D}_1^{-1}(\underline{\beta}))\right\|_{L^\infty_t L^4_{x'}} \lesssim \varepsilon. \tag{4.164}$$

This yields

$$\begin{aligned} \|\underline{L}(b)\|_{L^\infty_t L^4_{x'}} \;&\lesssim\; \left\|-b(\delta + n^{-1}N(n)) + 2b\pi_1({}^*\mathcal{D}_1^{-1}(\underline{\beta}))\right\|_{L^\infty_t L^4_{x'}} + \varepsilon \qquad (4.165) \\ &\lesssim\; \|b\|_{L^\infty}(\mathcal{N}_1(\delta) + \mathcal{N}_1(n^{-1}N(n))) + \|b\|_{L^\infty}\mathcal{N}_1({}^*\mathcal{D}_1^{-1}(\underline{\beta})) + \varepsilon \\ &\lesssim\; D\varepsilon + \mathcal{N}_1({}^*\mathcal{D}_1^{-1}(\underline{\beta})). \end{aligned}$$

Next, we estimate the right-hand side of (4.165). In view of the estimate (3.50) for ${}^*\mathcal{D}_1^{-1}$ and the curvature bound (2.59) for $\underline{\beta}$, we have

$$\|\nabla\!\!\!/ {}^*\mathcal{D}_1^{-1}(\underline{\beta})\|_{L^2(\mathcal{H}_u)} \lesssim \|\underline{\beta}\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{4.166}$$

Also, in view of (4.162), we have

$$\begin{aligned} & \|L({}^*\mathcal{D}_1^{-1}(\underline{\beta})\|_{L^2(\mathcal{H}_u)} \lesssim \|(\rho,\sigma)\|_{L^2(\mathcal{H}_u)} + \|n^{-1}\|_{L^\infty}\|[{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}]\underline{\beta}\|_{L^2(\mathcal{H}_u)} + \|h_2\|_{L^2(\mathcal{H}_u)} \\ \lesssim \;& \varepsilon, \end{aligned} \tag{4.167}$$

where we used in the last inequality the curvature bound (2.59) for (ρ,σ), the bootstrap assumption (4.1) for n, the commutator estimate of Lemma 4.24, and the estimate (4.163) for h_2. Finally, (4.166) and (4.167) imply

$$\mathcal{N}_1({}^*\mathcal{D}_1^{-1}(\underline{\beta})) \lesssim \varepsilon$$

which together with (4.165) yields the following estimate for $\underline{L}(b)$

$$\|\underline{L}(b)\|_{L^\infty_t L^4_{x'}} \lesssim D\varepsilon. \tag{4.168}$$

Remark 4.25 *The estimate* (4.168) *contains the bootstrap constant D in its right-hand side. This is not an issue since such an estimate is not part of our bootstrap assumptions* (4.1)-(4.6).

4.9 Remaining estimates for $\mathrm{tr}\chi$, $\widehat{\chi}$ and ζ

We first estimate $\mathrm{tr}\chi$. In view of the estimate for transport equations (3.66) and the Raychaudhuri equation (2.28), we have

$$\begin{aligned}\|\mathrm{tr}\chi\|_{L^2_{x'}L^\infty_t} &\lesssim \|\mathrm{tr}\chi(0)\|_{L^2(P_{0,u})} + \|L\mathrm{tr}\chi\|_{L^2_{x'}L^1_t}\\ &\lesssim \|\mathrm{tr}\chi(0)\|_{L^2(P_{0,u})} + \|\chi\|^2_{L^2_tL^4_{x'}} + \|\chi\|_{L^2_tL^4_{x'}}\|\overline{\delta}\|_{L^2_tL^4_{x'}}\\ &\lesssim \|\mathrm{tr}\chi(0)\|_{L^2(P_{0,u})} + \mathcal{N}_1(\chi)^2 + \mathcal{N}_1(k)^2,\end{aligned}$$

which together with the assumption on initial data (2.62) and the bootstrap assumptions (4.3)-(4.5) implies

$$\begin{aligned}\|\mathrm{tr}\chi\|_{L^2_{x'}L^\infty_t} &\lesssim \varepsilon + D^2\varepsilon^2\\ &\lesssim \varepsilon.\end{aligned} \tag{4.169}$$

Next, we estimate $\nabla\!\!\!/\mathrm{tr}\chi$. Differentiating the Raychaudhuri equation (2.28) and using the commutation formula (2.44), we obtain:

$$\nabla\!\!\!/_L\nabla\!\!\!/\mathrm{tr}\chi = -\left(\frac{3}{2}\mathrm{tr}\chi + \widehat{\chi} + \overline{\delta}\right)\nabla\!\!\!/\mathrm{tr}\chi - 2\widehat{\chi}\nabla\!\!\!/\widehat{\chi} + n^{-1}\nabla\!\!\!/ nL(\mathrm{tr}\chi) - \nabla\!\!\!/(\overline{\delta})\mathrm{tr}\chi, \tag{4.170}$$

which together with the bootstrap assumptions (4.1)-(4.6) and the estimate for transport equations (3.66) yields:

$$\begin{aligned}\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2_{x'}L^\infty_t} &\lesssim \left\|\frac{3}{2}\mathrm{tr}\chi + \widehat{\chi} + \overline{\delta}\right\|_{L^2_{x'}L^\infty_t}\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2(\mathcal{H}_u)} + \|\widehat{\chi}\|_{L^\infty_{x'}L^2_t}\|\nabla\!\!\!/\widehat{\chi}\|_{L^2(\mathcal{H}_u)}\\ &\quad + \|n^{-1}\nabla\!\!\!/ n\|_{L^2_tL^8_{x'}}\|L(\mathrm{tr}\chi)\|_{L^2_tL^{\frac{8}{3}}_{x'}} + \|\nabla\!\!\!/(\overline{\delta})\|_{L^2(\mathcal{H}_u)}\|\mathrm{tr}\chi\|_{L^\infty}\\ &\lesssim D^2\varepsilon^2\\ &\lesssim \varepsilon\end{aligned} \tag{4.171}$$

where we used the Raychaudhuri equation (2.28), the embeddings (3.3) and (3.58), and the bootstrap assumption to bound $L(\mathrm{tr}\chi)$:

$$\|L(\mathrm{tr}\chi)\|_{L^2_tL^{\frac{8}{3}}_{x'}} \lesssim \|\chi\|_{L^2_tL^8_{x'}}\|\chi\|_{L^\infty_tL^4_{x'}} + \|\chi\|_{L^2_tL^8_{x'}}\|\overline{\delta}\|_{L^\infty_tL^4_{x'}} \lesssim \mathcal{N}_1(\chi)^2 + \mathcal{N}_1(\overline{\delta})^2 \lesssim \varepsilon.$$

Note that (4.171) improves the estimate for $\nabla\!\!\!/\mathrm{tr}\chi$ given by the bootstrap assumption (4.4).

We now estimate $\mathcal{N}_1(\chi)$. Using the transport equation for $\widehat{\chi}$ (2.29), we obtain:

$$\|\nabla\!\!\!/_L\widehat{\chi}\|_{L^2(\mathcal{H}_u)} \lesssim \|\mathrm{tr}\chi\|_{L^\infty}\|\widehat{\chi}\|_{L^2(\mathcal{H}_u)} + \|\overline{\delta}\|_{L^4(\mathcal{H}_u)}\|\widehat{\chi}\|_{L^4(\mathcal{H}_u)} + \|\alpha\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon + D^2\varepsilon^2 \lesssim \varepsilon \tag{4.172}$$

where we have used the curvature bound (2.59) for α, and the bootstrap assumptions (4.2)-(4.5) for χ and $\overline{\delta}$. Next, using the codazzi equation (2.32) for $\widehat{\chi}$, we obtain:

$$\|\mathcal{D}_2\widehat{\chi}\|_{L^2(\mathcal{H}_u)} \lesssim \|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2(\mathcal{H}_u)} + \|\beta\|_{L^2(\mathcal{H}_u)} + \|\chi\|_{L^4(\mathcal{H}_u)}\|\epsilon\|_{L^4(\mathcal{H}_u)} \lesssim \varepsilon + D^2\varepsilon^2 \lesssim \varepsilon \tag{4.173}$$

where we have used (4.171), the curvature bound (2.59) for β, and the bootstrap assumptions (4.3)-(4.5) for χ and ϵ. The Hodge estimate (3.50) together with (4.173) yields:

$$\|\nabla\!\!\!/\widehat{\chi}\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{4.174}$$

(4.172) and (4.174) imply:

$$\mathcal{N}_1(\widehat{\chi}) \lesssim \varepsilon. \tag{4.175}$$

Note that (4.175) improves the estimate for $\mathcal{N}_1(\widehat{\chi})$ given by the bootstrap assumption (4.5).

We now estimate $\underline{L}\mathrm{tr}\chi$. Using the transport equation for μ (2.39) and the estimate for transport equations (3.66), we obtain:

$$\begin{aligned}
\|\mu\|_{L^2_{x'}L^\infty_t} \lesssim & \Big\| -\mathrm{tr}\chi\mu - 2(\zeta-\underline{\zeta})\cdot\nabla\!\!\!/\mathrm{tr}\chi - 2\widehat{\chi}\cdot\big(\nabla\!\!\!/\widehat{\otimes}\zeta + \zeta\widehat{\otimes}\zeta - \delta\widehat{\chi}\big) \\
& -2\mathrm{tr}\chi\Big(\rho + \zeta\cdot(2\epsilon - n^{-1}\nabla\!\!\!/ n) - n^{-1}\nabla\!\!\!/ n\cdot\epsilon + |\epsilon|^2 + |\widehat{\chi}|^2 + \widehat{\eta}\cdot\widehat{\chi}\Big)\Big\|_{L^2_{x'}L^1_t} \\
\lesssim & \|\mathrm{tr}\chi\|_{L^\infty}\|\mu\|_{L^2_{x'}L^\infty_t} + (\|\underline{\zeta}\|_{L^\infty_{x'}L^2_t} + \|\zeta\|_{L^\infty_{x'}L^2_t})\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2(\mathcal{H}_u)} \\
& + \|\widehat{\chi}\|_{L^\infty_{x'}L^2_t}\Big(\|\nabla\!\!\!/\zeta\|_{L^2(\mathcal{H}_u)} + \|\zeta\|^2_{L^4(\mathcal{H}_u)} + \|\delta\widehat{\chi}\|_{L^2(\mathcal{H}_u)}\Big) \\
& + \|\mathrm{tr}\chi\|_{L^\infty}\Big\|\rho + \zeta\cdot(2\epsilon - n^{-1}\nabla\!\!\!/ n) - n^{-1}\nabla\!\!\!/ n\cdot\epsilon + |\epsilon|^2 + |\widehat{\chi}|^2 + \widehat{\eta}\cdot\widehat{\chi}\Big\|_{L^2(\mathcal{H}_u)} \\
\lesssim & D\varepsilon\|\mu\|_{L^2_{x'}L^\infty_t} + D^2\varepsilon^2
\end{aligned} \tag{4.176}$$

where we used the curvature bound (2.59), the bootstrap assumptions (4.1)-(4.6) and the Sobolev inequality (3.57). (4.176) yields:

$$\|\mu\|_{L^2_{x'}L^\infty_t} \lesssim \varepsilon$$

which together with the bootstrap assumptions (4.1)-(4.6) and the definition (2.35) of μ implies:

$$\|\underline{L}\mathrm{tr}\chi\|_{L^2_{x'}L^\infty_t} \lesssim \big\|\mu + \big(\delta + n^{-1}N(n)\big)\mathrm{tr}\chi\big\|_{L^2_{x'}L^\infty_t} \lesssim \varepsilon + D^2\varepsilon^2 \lesssim \varepsilon. \tag{4.177}$$

Note that (4.177) improves the estimate for $\underline{L}\mathrm{tr}\chi$ given by the bootstrap assumption (4.4).

We now estimate $\mathcal{N}_1(\zeta)$. Using the transport equation for ζ (2.30), we obtain:

$$\|\nabla\!\!\!/_L\zeta\|_{L^2(\mathcal{H}_u)} \lesssim \|\chi\|_{L^4(\mathcal{H}_u)}(\|\zeta\|_{L^4(\mathcal{H}_u)} + \|\overline{\epsilon}\|_{L^4(\mathcal{H}_u)}) + \|\beta\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon + D^2\varepsilon^2 \lesssim \varepsilon \tag{4.178}$$

where we have used the curvature bound (2.59) for β, and the bootstrap assumptions (4.2)-(4.5) for χ and $\overline{\epsilon}$. Next, using the div-curl system of equations (2.36) (2.37) for ζ, we obtain:

$$\begin{aligned}
\|\mathcal{D}_1\zeta\|_{L^2(\mathcal{H}_u)} &\lesssim \|\mu\|_{L^2(\mathcal{H}_u)} + \|\rho\|_{L^2(\mathcal{H}_u)} + \|\sigma\|_{L^2(\mathcal{H}_u)} + \|\chi\|^2_{L^4(\mathcal{H}_u)} + \|k\|^2_{L^4(\mathcal{H}_u)} + \|\zeta\|^2_{L^4(\mathcal{H}_u)} \\
&\lesssim \varepsilon + D^2\varepsilon^2 \\
&\lesssim \varepsilon
\end{aligned} \tag{4.179}$$

where we have used (4.177), the curvature bound (2.59) for ρ and σ, and the bootstrap assumptions (4.3)-(4.6) for χ, k and ζ. The Hodge estimate (3.50) together with (4.179) yields:

$$\|\nabla\!\!\!/\zeta\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{4.180}$$

(4.178) and (4.180) imply:

$$\mathcal{N}_1(\zeta) \lesssim \varepsilon. \tag{4.181}$$

Note that (4.181) improves the estimate for $\mathcal{N}_1(\zeta)$ given by the bootstrap assumption (4.6).

We now estimate $\nabla\!\!\!/_{\underline{L}}\widehat{\chi}$. Using the null structure equation (2.34), we obtain:

$$\begin{aligned} & \|\nabla\!\!\!/_{\underline{L}}\widehat{\chi}\|_{L^2(\mathcal{H}_u)} \qquad (4.182)\\ \lesssim\ & \|\nabla\!\!\!/\zeta\|_{L^2(\mathcal{H}_u)} + \|\chi\|_{L^4(\mathcal{H}_u)}(\|\underline{\chi}\|_{L^4(\mathcal{H}_u)} + \|\delta\|_{L^4(\mathcal{H}_u)} + \|n^{-1}N(n)\|_{L^4(\mathcal{H}_u)}) + \|\zeta\|^2_{L^4(\mathcal{H}_u)}\\ \lesssim\ & \varepsilon + D^2\varepsilon^2\\ \lesssim\ & \varepsilon \end{aligned}$$

where we have used (4.180) and the bootstrap assumptions (4.1)-(4.6) for n, χ, $\underline{\chi}$, δ and ζ. Note that (4.182) improves the estimate for $\nabla\!\!\!/_{\underline{L}}\widehat{\chi}$ given by the bootstrap assumption (4.5).

Finally, (4.46), (4.48), (4.58), (4.71), (4.73), (4.97), (4.112), (4.152), (4.169), (4.171), (4.175), (4.177), (4.181), (4.182), (4.154), (4.156) and (4.159) improve the bootstrap assumptions (4.1)-(4.6). Thus, there exists a universal constant $D > 0$ such that (4.1)-(4.6) and (4.168) hold. This yields (2.67)-(2.72) which concludes the proof of Theorem 2.21.

5 Estimates for $\underline{LL}\mathrm{tr}\chi$, $\nabla\!\!\!/_{\underline{L}}(\zeta)$ and $\underline{LL}(b)$

This section is devoted to the proof of Theorem 2.23. We assume the following bootstrap assumptions. There exists a function γ in $L^2(\mathbb{R})$ with $\|\gamma\|_{L^2(\mathbb{R})} \leq 1$ such that for all $j \geq 0$, we have:

$$\|P_j\underline{LL}\mathrm{tr}\chi\|_{L^2(\mathcal{H}_u)} \lesssim 2^j D\varepsilon + 2^{\frac{j}{2}} D\varepsilon\gamma(u), \tag{5.1}$$

$$\|P_j(\nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)} \lesssim D^2\varepsilon + 2^{-\frac{j}{2}} D^2\varepsilon\gamma(u), \tag{5.2}$$

where $D > 0$ is a large enough constant. We will improve on these estimates. Using the estimates obtained in Theorem 2.21, in particular for $\mathrm{tr}\chi$ and $\widehat{\chi}$, would yield an upper bound for $\underline{LL}\mathrm{tr}\chi$ of the following type

$$\|P_j\underline{LL}\mathrm{tr}\chi\|_{L^2(\mathcal{H}_u)} \lesssim 2^{\frac{j}{2}}\varepsilon\gamma(u) + \sum_{l,q} 2^j 2^{-\frac{|q-l|}{2}}\gamma_q^{(1)}\gamma_l^{(2)}, \text{ where } \gamma_q^{(1)} \in \ell^2(\mathbb{N}) \text{ and } \gamma_l^{(2)} \in \ell^\infty(\mathbb{N}) \tag{5.3}$$

which is not summable. This forces us to rely on a Besov improvement for $\mathrm{tr}\chi$, as well as a suitable decomposition for $\nabla\!\!\!/\widehat{\chi}$ (see (5.40)). This is done in section 5.1. Then, we derive a system of equations for $\underline{LL}\mathrm{tr}\chi$ and $\nabla\!\!\!/_{\underline{L}}(\zeta)$ in section 5.2. This allows us to improve on the bootstrap assumption (5.2) in section 5.3, and (5.1) in section 5.4. Finally, the estimate for $\underline{LL}(b)$ is then derived in section 5.5.

5.1 Besov improvement for trχ in the time foliation

In this section, we first define Besov spaces, and then explain how to adapt the ideas in the sequence of papers [14] [10] [11] to obtain the Besov improvement for trχ.

5.1.1 Definition of the Besov spaces and first properties

Following [10] [14], we introduce for $0 \leq a \leq 1$ and for tensors F on $P_{t,u}$ the Besov norm:

$$\|F\|_{B^a_{2,1}(P_{t,u})} = \sum_{j\geq 0} 2^{ja} \|P_j F\|_{L^2(P_{t,u})} + \|P_{<0} F\|_{L^2(P_{t,u})}, \tag{5.4}$$

where P_j are the geometric Littlewood-Paley projections on the 2-surfaces $P_{t,u}$. Furthermore, for $P_{t,u}$-tangent tensors F on $\mathcal{H}_u$, $0 \leq a \leq 1$, we introduce the Besov norms:

$$\|F\|_{\mathcal{B}^a} = \sum_{j\geq 0} 2^{ja} \sup_{0\leq t\leq 1} \|P_j F\|_{L^2(P_{t,u})} + \sup_{0\leq t\leq 1} \|P_{<0} F\|_{L^2(P_{t,u})}, \tag{5.5}$$

$$\|F\|_{\mathcal{P}^a} = \sum_{j\geq 0} 2^{ja} \|P_j F\|_{L^2(\mathcal{H}_u)} + \|P_{<0} F\|_{L^2(\mathcal{H}_u)}. \tag{5.6}$$

Note that these Besov spaces in the time foliation correspond to the Besov spaces in the geodesic foliation defined by the norms (4.84) (4.85). The goal of section 5.1 is to prove the following estimates for trχ and $\widehat{\chi}$:

$$\|\nabla\!\!\!/ \mathrm{tr}\chi\|_{\mathcal{B}^0} \lesssim \varepsilon, \tag{5.7}$$

and

$$\nabla\!\!\!/ \widehat{\chi} = \nabla\!\!\!/_L(P) + E \text{ with } \mathcal{N}_1(P) \lesssim \varepsilon \text{ and } \|E\|_{\mathcal{P}^0} \lesssim \varepsilon. \tag{5.8}$$

Note that the corresponding estimates in the geodesic foliation have been proved in the sequence of papers [14] [10] [11]. One may reprove these estimates by adapting the proofs to the context of a time foliation. However, this would be rather lengthy and we suggest here a more elegant solution which consists in identifying the key structure in [14] [10] [11] and showing that the analog structure exists in the time foliation. This will be done in the next section.

We conclude this section with several functional inequalities satisfied by the Besov spaces $\mathcal{B}^a, \mathcal{P}^a$. Note that properties of the Besov spaces on 2-surfaces derived in [10] apply to the Besov spaces $B^a_{2,1}$. Indeed, these properties only depend on the fact that $P_{t,u}$ is a 2-surface satisfying the coordinate system assumption (3.1) and the assumption (4.30) on the gauss curvature K. In particular, the following estimates are immediate consequences on the estimates in [10] for $B^a_{2,1}$ (see also section 5 in [14]):

$$\|f\|_{L^\infty} \lesssim \|f\|_{\mathcal{B}^1} \lesssim \|f\|_{L^\infty_t L^2_{x'}} + \|\nabla\!\!\!/ f\|_{\mathcal{B}^0}, \tag{5.9}$$

where f is a scalar function on $\mathcal{H}_u$,

$$\|F \cdot H\|_{\mathcal{B}^0} \lesssim \big(\|\nabla\!\!\!/ F\|_{L^\infty_t L^2_x} + \|F\|_{L^\infty}\big)\|H\|_{\mathcal{B}^0}, \tag{5.10}$$

where F and H are $P_{t,u}$-tangent tensors, and

$$\|\nabla\!\!\!/ \cdot \mathcal{D}^{-1} F\|_{\mathcal{P}^a} \lesssim \|F\|_{\mathcal{P}^a} \tag{5.11}$$

where $0 \leq a < 1$, F is a $P_{t,u}$ tangent tensor on $\mathcal{H}_u$, and $\mathcal{D}^{-1}$ denotes one of the operators $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$, ${}^*\mathcal{D}_1^{-1}$, ${}^*\mathcal{D}^{-1}$. Also, for $0 \leq a < \frac{1}{2}$ and $\frac{2}{2-a} < p \leq 2$, we have:

$$\|\mathcal{D}^{-1} F\|_{\mathcal{P}^a} \lesssim \|F\|_{L^2_t L^p_{x'}}. \tag{5.12}$$

Finally, we shall make use of the following non sharp embedding and product estimates. For any $P_{t,u}$-tangent tensors F, G, and for any $0 \leq a < \frac{1}{2}$, we have:

$$\|F\|_{\mathcal{B}^a} \lesssim \mathcal{N}_1(F) \tag{5.13}$$

$$\|F \cdot G\|_{\mathcal{P}^a} \lesssim \mathcal{N}_2(F) \cdot \|G\|_{\mathcal{P}^a} \tag{5.14}$$

$$\|F \cdot G\|_{\mathcal{P}^a} \lesssim \mathcal{N}_1(F) \cdot (\|G\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/ G\|_{L^2(\mathcal{H}_u)}). \tag{5.15}$$

5.1.2 Structure of the commutators in the time foliation

As noted at the end of the previous section, the results from the paper [10] on 2-surfaces immediately apply to $P_{t,u}$. We shall now show that results from the paper [11] true in the geodesic foliation apply also to the time foliation due to a similar structure of commutators.

Let A denote $A = n\chi$. Then, the estimates (2.67) for n, (2.70) for $\mathrm{tr}\chi$ and (2.71) for $\widehat{\chi}$ of Theorem 2.21 proved in section 4 imply:

$$\|A\|_{L^\infty_{x'} L^2_t} + \mathcal{N}_1(A) \lesssim \varepsilon. \tag{5.16}$$

In view of (5.16) and the commutator identities (2.48) and (2.49), we have:

$$[\nabla\!\!\!/_{nL}, \nabla\!\!\!/] f = A \cdot \nabla\!\!\!/ f, \tag{5.17}$$

$$[\nabla\!\!\!/_{nL}, \Delta\!\!\!/] f = A \cdot \nabla\!\!\!/^2 f + \nabla\!\!\!/ A \cdot \nabla\!\!\!/ f + A \cdot A \cdot \nabla\!\!\!/ f \tag{5.18}$$

where f is a scalar function on $\mathcal{H}_u$ and:

$$[\nabla\!\!\!/_{nL}, \nabla\!\!\!/] F = A \cdot \nabla\!\!\!/ F + n\beta \cdot F + A \cdot A \cdot F, \tag{5.19}$$

$$[\nabla\!\!\!/_{nL}, \Delta\!\!\!/] F = A \cdot \nabla\!\!\!/^2 F + \nabla\!\!\!/ A \cdot \nabla\!\!\!/ F + A \cdot A \cdot \nabla\!\!\!/ F + n\beta \cdot \nabla\!\!\!/ F + \nabla\!\!\!/(n\beta \cdot \nabla\!\!\!/ F + A \cdot A \cdot F) \tag{5.20}$$

where F is a $P_{t,u}$-tangent tensor on $\mathcal{H}_u$. Note that the structure of the commutators (5.17)-(5.20) together with the estimate (5.16) for A is the same structure as in the case of a geodesic foliation with the correspondence:

$$L' \to nL,\ \chi' \to n\chi \text{ and } \beta' \to n\beta \tag{5.21}$$

where L', χ' and β' have been defined in section 4.6.1.

The proofs of the sharp trace theorems in the paper [11] rely on the following assumptions (see section 3 of [11]) where we translate for the time foliation using the correspondence (5.21):

S1 The two surfaces $P_{t,u}$ satisfy the coordinates system assumption (3.1), the calculus inequalities of section 3.1 and the geometric Littlewood-Paley theory of section 3.2

S2 The Gauss curvature K of $P_{t,u}$ satisfies the bound (4.29) (4.30)

S3 There is A satisfying (5.16) such that we have the commutator structure (5.17)-(5.20)

S4 $n\beta$ satisfies the curvature flux bound $\|n\beta\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon$ (which follows from the curvature bound (2.59) and the estimate (2.67) for n)

Since the proof of the sharp trace theorems in [11] only rely on the structural assumptions **S1-S4**, they immediately extend to the case of a time foliation. In particular, we obtain the following analog of the sharp trace theorems in [11] (see section 4 of [11]):

Proposition 5.1 *Assume that the $P_{t,u}$-tangent tensor U satisfies $U(0) = 0$ and the following transport equation along $\mathcal{H}_u$:*

$$\nabla\!\!\!/_{nL} U + atr\chi U = F_1 \cdot \nabla\!\!\!/_{nL} P + F_2 \cdot W,$$

where a is some positive number. Then,

$$\|U\|_{\mathcal{B}^0} \lesssim \big(\mathcal{N}_1(F_1) + \|F_1\|_{L^\infty_{x'} L^2_t}\big) \cdot \mathcal{N}_1(P) + \big(\mathcal{N}_1(F_2) + \|F_2\|_{L^\infty_{x'} L^2_t}\big) \cdot \|W\|_{\mathcal{P}^0}. \tag{5.22}$$

We also obtain the following useful commutator estimates:

Lemma 5.2 *For a given 1-form F, let w the solution of the scalar transport equation*

$$nL(w) = d\!\!\!/iv(F),\ w = 0 \text{ on } P_{0,u},$$

and let W be a solution of the equation

$$\nabla\!\!\!/_{nL} W - n\chi \cdot W = F,\ W = 0 \text{ on } P_{0,u}.$$

Then, for any $1 \le p \le 2$,

$$\|d\!\!\!/iv(W) - w\|_{L^p_{x'} L^\infty_t} \lesssim \varepsilon \|F\|_{L^{\frac{2p}{2-p}}_{x'} L^1_t}.$$

Lemma 5.3 *For any $P_{t,u}$-tangent tensor F and all $1 \le q < 2$, we have:*

$$\|[P_j, \nabla\!\!\!/_{nL}]F\|_{L^q_t L^2_{x'}} + 2^{-j}\|\nabla\!\!\!/[P_j, \nabla\!\!\!/_{nL}]F\|_{L^q_t L^2_{x'}} \lesssim 2^{-\frac{j}{2}+}\mathcal{N}_1(F) \tag{5.23}$$

(by $2^{-\frac{j}{2}+}$ we mean 2^{-aj} for $a < \frac{1}{2}$ arbitrarily close to $\frac{1}{2}$), while for $q = 1$,

$$\|[P_j, \nabla\!\!\!/_{nL}]F\|_{L^1_t L^2_{x'}} + 2^{-j}\|\nabla\!\!\!/[P_j, \nabla\!\!\!/_{nL}]F\|_{L^1_t L^2_{x'}} \lesssim 2^{-j+}\mathcal{N}_1(F). \tag{5.24}$$

Finally, using Proposition 5.1, we may prove the following version of the sharp classical trace theorem.

Corollary 5.4 *Assume F is an $P_{t,u}$-tangent tensor which admits a decomposition of the form, $\nabla\!\!\!/ F = B\nabla\!\!\!/_{nL} P + E$. Then,*

$$\|F\|_{L^\infty_{x'} L^2_t} \lesssim \mathcal{N}_1(F) + \mathcal{N}_1(P)(\|B\|_{L^\infty} + \|\nabla\!\!\!/ B\|_{L^2_{x'} L^\infty_t} + \|\nabla\!\!\!/_L B\|_{L^2_{x'} L^\infty_t}) + \|E\|_{\mathcal{P}^0}. \tag{5.25}$$

The proof of (5.25) is the analog of the proof of the estimate (4.93) so we skip it.

5.1.3 Structure of the Bianchi identities in the time foliation

In this section, we will show that results from the paper [14] true in the geodesic foliation apply also to the time foliation due to a similar structure of the Bianchi identities. We first enlarge the correspondence (5.21) with the general philosophy that L' should correspond to nL and $\underline{L}'$ to $n^{-1}\underline{L}$:

$$\begin{aligned} &L' \rightarrow nL,\ \underline{L}' \rightarrow n^{-1}\underline{L},\ e'_A \rightarrow e_A \\ &\chi' \rightarrow n\chi,\ \underline{\chi}' \rightarrow n^{-1}\underline{\chi} \\ &\beta' \rightarrow n\beta,\ \rho' \rightarrow \rho,\ \sigma' \rightarrow \sigma \end{aligned} \tag{5.26}$$

where L', $\underline{L}'$, χ', $\underline{\chi}'$, β', ρ' and σ' have been defined in section 4.6.1. Following [14], we define $\check{\rho}, \check{\sigma}$ as:

$$\check{\rho} = \rho - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}},\ \check{\sigma} = \sigma - \frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}}. \tag{5.27}$$

Multiplying the Bianchi identities (2.53) and (2.55) by n together with the null structure equations for χ and $\underline{\chi}$ yields:

$$\begin{aligned} nL(\check{\rho}) + \frac{3}{2}n\mathrm{tr}\chi\check{\rho} &= \not{\mathrm{div}}(n\beta) - \overline{\epsilon}\cdot(n\beta) + \frac{1}{2}(n\widehat{\chi})\cdot\Big(\not{\nabla}\widehat{\otimes}\overline{\epsilon} - \overline{\epsilon}\widehat{\otimes}\overline{\epsilon} \\ &\quad + (n\mathrm{tr}\chi)\cdot(n^{-1}\underline{\widehat{\chi}}) + \frac{1}{2}(n^{-1}\mathrm{tr}\underline{\chi})\cdot(n\widehat{\chi})\Big), \\ nL(\check{\sigma}) + \frac{3}{2}n\mathrm{tr}\chi\check{\sigma} &= -\not{\mathrm{curl}}(n\beta) + \overline{\epsilon}\wedge(n\beta) + \frac{1}{2}(n\widehat{\chi})\wedge\Big(\not{\nabla}\widehat{\otimes}\overline{\epsilon} - \overline{\epsilon}\widehat{\otimes}\overline{\epsilon} \\ &\quad + (n\mathrm{tr}\chi)\cdot(n^{-1}\underline{\widehat{\chi}})\Big). \end{aligned} \tag{5.28}$$

We now denote $A = (n\chi, \overline{\epsilon})$ which together with the estimates (2.67) for n, the estimates (2.68) for $\overline{\epsilon}$, the estimates (2.70) for $\mathrm{tr}\chi$ and (2.71) for $\widehat{\chi}$ of Theorem 2.21 proved in section 4 still imply the estimate (5.16) for A. We also denote $\underline{A} = n\underline{\chi}$ which in view of the estimates (2.67) for n, the estimates (2.68) for k, the estimates (2.70) for $\mathrm{tr}\chi$ and (2.71) for $\widehat{\chi}$ of Theorem 2.21 proved in section 4 satisfies the following estimate:

$$\mathcal{N}_1(\underline{A}) \lesssim \varepsilon. \tag{5.29}$$

In view of the definition of A and $\underline{A}$ together with (5.28), we have:

$$nL(\check{\rho}, -\check{\sigma}) = \mathcal{D}_1(n\beta) + A\cdot(n(\beta,\rho,\sigma) + \not{\nabla}A + A\cdot\underline{A}). \tag{5.30}$$

We now consider a decomposition for $\not{\nabla}\mathcal{D}_2^{-1}\mathcal{D}_1^{-1}L(\check{\rho},\check{\sigma})$ which is the analog of the one derived in section 6 of the paper [14]. It relies on the assumptions **S1-S4** together with the following additional assumptions where we translate for the time foliation using the correspondence (5.26):

S5 $(\check{\rho},\check{\sigma})$ satisfies the curvature flux bound $\|\check{\rho}\|_{L^2(\mathcal{H}_u)} + \|\check{\sigma}\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon$ (which follows from the curvature bound (2.59), the estimate (2.68) for k, and the estimates (2.70) and (2.71) for χ)

S6 $nL(\check{\rho}, -\check{\sigma})$ has the structure (5.30)

S7 The functional inequalities (5.11), (5.12), (5.14) and (5.15) are satisfied

Since the proof of the estimate derived in section 6 of the paper [14] only rely on the structural assumptions **S1-S7**, they immediately extend to the case of a time foliation. In particular, we obtain the following analogs of the decompositions derived in section 6 of the paper [14]:

$$n\beta = \nabla\!\!\!/_{nL} P_1 + E_1 \tag{5.31}$$

and

$$\nabla\!\!\!/ \mathcal{D}_2^{-1}\mathcal{D}_1^{-1} nL(\check{\rho},\check{\sigma}) = \nabla\!\!\!/_{nL} P_2 + E_2 \tag{5.32}$$

where P_1, P_2, E_1 and E satisfy the bounds:

$$\mathcal{N}_1(P_1) + \mathcal{N}_1(P_2) + \|E_1\|_{\mathcal{P}^0} + \|E_2\|_{\mathcal{P}^0} \lesssim \varepsilon. \tag{5.33}$$

5.1.4 Decomposition of $\nabla\!\!\!/(n\widehat{\chi})$

We now in position to prove the decomposition (5.8) for $\nabla\!\!\!/\widehat{\chi}$. We first derive an equation for $n\widehat{\chi}$. Multiplying the Codazzi type equation (2.32) for $\widehat{\chi}$ by n, we obtain:

$$\mathcal{D}_2(n\widehat{\chi}) + \overline{\epsilon}\cdot(n\widehat{\chi}) = \frac{1}{2}(\nabla\!\!\!/(n\mathrm{tr}\chi) + \overline{\epsilon}(n\mathrm{tr}\chi)) - n\beta, \tag{5.34}$$

which yields:

$$\nabla\!\!\!/(n\widehat{\chi}) = \nabla\!\!\!/\mathcal{D}_2^{-1}\bigg(-\overline{\epsilon}\cdot(n\widehat{\chi}) + \frac{1}{2}(\nabla\!\!\!/(n\mathrm{tr}\chi) + \overline{\epsilon}(n\mathrm{tr}\chi)) - n\beta\bigg). \tag{5.35}$$

Now, in view of (5.30), we have:

$$n\beta = \mathcal{D}_1^{-1}\bigg(nL(\check{\rho},-\check{\sigma}) - A\cdot(n(\beta,\rho,\sigma) + \nabla\!\!\!/ A + A\cdot\underline{A})\bigg), \tag{5.36}$$

where A satisfies (5.16) and $\underline{A}$ satisfies (5.29). Injecting (5.36) in (5.35) yields:

$$\begin{aligned}\nabla\!\!\!/(n\widehat{\chi}) \;=\; & -\nabla\!\!\!/\mathcal{D}_2^{-1}\mathcal{D}_1^{-1} nL(\check{\rho},-\check{\sigma}) + \nabla\!\!\!/\mathcal{D}_2^{-1}\mathcal{D}_1^{-1}\bigg(A\cdot(n(\beta,\rho,\sigma) + \nabla\!\!\!/ A + A\cdot\underline{A})\bigg)\\ & + \nabla\!\!\!/\mathcal{D}_2^{-1}\bigg(-\overline{\epsilon}\cdot(n\widehat{\chi}) + \frac{1}{2}(\nabla\!\!\!/(n\mathrm{tr}\chi) + \overline{\epsilon}(n\mathrm{tr}\chi))\bigg).\end{aligned} \tag{5.37}$$

We estimate the second term in the right-hand side of (5.37). Using the embedding (3.58), the estimate (5.11) with $a=0$, and the estimate (5.12) with $a=0$ and $p=\frac{4}{3}$, we have:

$$\begin{aligned} & \bigg\|\nabla\!\!\!/\mathcal{D}_2^{-1}\mathcal{D}_1^{-1}\bigg(A\cdot(n(\beta,\rho,\sigma) + \nabla\!\!\!/ A + A\cdot\underline{A})\bigg)\bigg\|_{\mathcal{P}^0} \qquad (5.38)\\ \lesssim\; & \|A\cdot(n(\beta,\rho,\sigma) + \nabla\!\!\!/ A + A\cdot\underline{A})\|_{L^2_t L^{\frac{4}{3}}_{x'}}\\ \lesssim\; & \|A\|_{L^\infty_t L^4_{x'}}(\|n\|_{L^\infty}(\|\beta\|_{L^2(\mathcal{H}_u)} + \|\rho\|_{L^2(\mathcal{H}_u)} + \|\sigma\|_{L^2(\mathcal{H}_u)}) + \|\nabla\!\!\!/ A\|_{L^2(\mathcal{H}_u)} + \|A\|_{L^\infty_t L^4_{x'}}\|\underline{A}\|_{L^\infty_t L^4_{x'}})\\ \lesssim\; & \mathcal{N}_1(A)(\|n\|_{L^\infty}(\|\beta\|_{L^2(\mathcal{H}_u)} + \|\rho\|_{L^2(\mathcal{H}_u)} + \|\sigma\|_{L^2(\mathcal{H}_u)}) + \mathcal{N}_1(A)(1+\mathcal{N}_1(\underline{A})))\\ \lesssim\; & \varepsilon,\end{aligned}$$

where we used the curvature bound (2.59) for β, the estimate (2.67) for n, the estimate (5.16) for A and the estimate (5.29) for $\underline{A}$.

We estimate the last term in the right-hand side of (5.37). Using the estimate (5.11) and the non sharp product estimates estimates (5.14) and (5.15), we have:

$$\begin{aligned} & \left\| \nabla\!\!\!/ \mathcal{D}_2^{-1} \Big(-\overline{\epsilon} \cdot (n\widehat{\chi}) + \frac{1}{2}(\nabla\!\!\!/(n\mathrm{tr}\chi) + \overline{\epsilon}(n\mathrm{tr}\chi)) \Big) \right\|_{\mathcal{P}^0} \qquad (5.39) \\ \lesssim & \left\| -\overline{\epsilon} \cdot (n\widehat{\chi}) + \frac{1}{2}(\nabla\!\!\!/(n\mathrm{tr}\chi) + \overline{\epsilon}(n\mathrm{tr}\chi)) \right\|_{\mathcal{P}^0} \\ \lesssim & \|\mathrm{tr}\chi \nabla\!\!\!/ n\|_{\mathcal{P}^0} + \mathcal{N}_2(n-1)(\|\nabla\!\!\!/ \mathrm{tr}\chi\|_{\mathcal{P}^0} + \|\overline{\epsilon} \cdot \widehat{\chi}\|_{\mathcal{P}^0} + \|\overline{\epsilon}\mathrm{tr}\chi\|_{\mathcal{P}^0}) \\ \lesssim & \mathcal{N}_2(n-1)(\mathcal{N}_1(\mathrm{tr}\chi) + \|\nabla\!\!\!/ \mathrm{tr}\chi\|_{\mathcal{P}^0} + \mathcal{N}_1(\overline{\epsilon})\mathcal{N}_1(\chi)) \\ \lesssim & \varepsilon + \|\nabla\!\!\!/ \mathrm{tr}\chi\|_{\mathcal{P}^0}, \end{aligned}$$

where we used the estimate (2.67) for n, the estimate (2.68) for $\overline{\epsilon}$ and the estimates (2.70) and (2.71) for χ.

Finally, the decomposition (5.32) for $\nabla\!\!\!/ \mathcal{D}_2^{-1}\mathcal{D}_1^{-1} nL(\check{\rho}, -\check{\sigma})$ together with the estimate (5.33) and(5.37)-(5.39) yields the following decomposition for $\nabla\!\!\!/(n\widehat{\chi})$:

$$\nabla\!\!\!/(n\widehat{\chi}) = \nabla\!\!\!/_{nL} P + E, \qquad (5.40)$$

where P and E satisfy the following estimate:

$$\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} \lesssim \varepsilon + \|\nabla\!\!\!/ \mathrm{tr}\chi\|_{\mathcal{P}^0}. \qquad (5.41)$$

5.1.5 Decomposition of $\nabla\!\!\!/(n\overline{\delta})$

In order to obtain a Besov improvement for $\mathrm{tr}\chi$, we need to derive for $\nabla\!\!\!/(n\overline{\delta})$ the analog of the decomposition for $\nabla\!\!\!/' k_{L'L'}$ derived in (4.130). Recall from (4.116) that $\overline{\delta} = k_{LL}$ with $k_{LL} = -\mathbf{g}(\mathbf{D}_L T, L)$. Thus, we have:

$$\begin{aligned} \nabla\!\!\!/_{e_A}\overline{\delta} &= -\mathbf{D}_{e_A}\mathbf{g}(\mathbf{D}_L T, L) = -\mathbf{g}(\mathbf{D}_{e_A}\mathbf{D}_L T, L) - \mathbf{g}(\mathbf{D}_L T, D_{e_A} L) \\ &= -\mathbf{g}(\mathbf{D}_L\mathbf{D}_{e_A} T, L) - \mathbf{R}_{e_A LTL} - \mathbf{g}(\mathbf{D}_{[e_A,L]} T, L) - \mathbf{g}(-\overline{\delta} N, \chi_{AB} e_B - \epsilon_A L) \\ &= -\nabla\!\!\!/_L \epsilon_A - \frac{1}{2}\beta_A + \chi_{AB}(\epsilon_B + \overline{\epsilon}_B) - n^{-1}\nabla\!\!\!/_A n\overline{\delta}, \end{aligned}$$

which after multiplication by n yields:

$$\nabla\!\!\!/(n\overline{\delta}) = -\nabla\!\!\!/_{nL}\epsilon - \frac{1}{2}\beta + \chi \cdot (\epsilon + \overline{\epsilon}). \qquad (5.42)$$

The estimates (2.67) and (2.68) for ϵ and $\overline{\epsilon}$, the estimates (2.70) and (2.71) for χ, and the non sharp product estimate (5.15) yield:

$$\mathcal{N}_1(\epsilon) + \|\chi \cdot (\epsilon + \overline{\epsilon})\|_{\mathcal{P}^0} \lesssim \varepsilon + \mathcal{N}_1(\chi)(\mathcal{N}_1(\epsilon) + \mathcal{N}_1(\overline{\epsilon})) \lesssim \varepsilon. \qquad (5.43)$$

Finally, (5.42), (5.43) and the decomposition of β given by (5.31) (5.33) yield:

$$\nabla\!\!\!/(n\overline{\delta}) = \nabla\!\!\!/_{nL} P + E, \qquad (5.44)$$

where P and E satisfy:

$$\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} \lesssim \varepsilon. \qquad (5.45)$$

5.1.6 Besov improvement for $\mathrm{tr}\chi$

In view of (5.41), we need an estimate for $\|\nabla\!\!\!/\mathrm{tr}\chi\|_{\mathcal{P}^0}$. We multiply the transport equation (4.170) satisfied by $\nabla\!\!\!/\mathrm{tr}\chi$. We obtain:

$$\nabla\!\!\!/_{nL}\nabla\!\!\!/\mathrm{tr}\chi = -n\left(\frac{3}{2}\mathrm{tr}\chi + \widehat{\chi} + \overline{\delta}\right)\nabla\!\!\!/\mathrm{tr}\chi - 2\widehat{\chi}\nabla\!\!\!/(n\widehat{\chi}) + \nabla\!\!\!/ n(2|\widehat{\chi}|^2 + L(\mathrm{tr}\chi) - \overline{\epsilon}\mathrm{tr}\chi) - \nabla\!\!\!/(n\overline{\delta})\mathrm{tr}\chi. \tag{5.46}$$

Using the decomposition (5.40) for $\nabla\!\!\!/(n\widehat{\chi})$ and the decomposition (5.44) for $\nabla\!\!\!/(n\overline{\delta})$, we obtain:

$$-2\widehat{\chi}\nabla\!\!\!/(n\widehat{\chi}) - \nabla\!\!\!/(n\overline{\delta})\mathrm{tr}\chi = F_1\nabla\!\!\!/_L P + F_2 W \tag{5.47}$$

where in view of (5.41), (5.45) and the estimates (2.70) (2.71) for χ, we have:

$$\mathcal{N}_1(F_1) + \|F_1\|_{L^\infty_{x'}L^2_t} + \mathcal{N}_1(F_2) + \|F_2\|_{L^\infty_{x'}L^2_t} + \mathcal{N}_1(P) + \|W\|_{\mathcal{P}^0} \lesssim \varepsilon + \|\nabla\!\!\!/\mathrm{tr}\chi\|_{\mathcal{P}^0}. \tag{5.48}$$

Also, using the Raychaudhuri equation (2.28), we may rewrite the third term in the right-hand side of (5.46) as:

$$\nabla\!\!\!/ n(2|\widehat{\chi}|^2 + L(\mathrm{tr}\chi) - \overline{\epsilon}\mathrm{tr}\chi) = \chi W_1 \tag{5.49}$$

where in view of the estimate (2.67) for n, the estimate (2.68) for $\overline{\delta}$ and $\overline{\epsilon}$, the estimates (2.70) (2.71) for χ, and the non sharp product estimate (5.15), $W_1 = \nabla\!\!\!/ n \cdot (\chi + \overline{\delta} + \overline{\epsilon})$ satisfies:

$$\|W_1\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(\nabla\!\!\!/ n)(\mathcal{N}_1(\chi) + \mathcal{N}_1(\overline{\epsilon}) + \mathcal{N}_1(\overline{\delta})) \lesssim \varepsilon. \tag{5.50}$$

Using the estimate (2.67) for n, the estimate (2.68) for $\overline{\epsilon}$, the estimates (2.70) (2.71) for χ, we also have:

$$\begin{aligned} & \mathcal{N}_1\left(n\left(\frac{3}{2}\mathrm{tr}\chi + \widehat{\chi} + \overline{\delta}\right)\right) + \left\|n\left(\frac{3}{2}\mathrm{tr}\chi + \widehat{\chi} + \overline{\delta}\right)\right\|_{L^\infty_{x'}L^2_t} \\ \lesssim\ & \mathcal{N}_2(n-1)(\mathcal{N}_1(\chi) + \mathcal{N}_1(\overline{\delta}) + \|\chi\|_{L^\infty_{x'}L^2_t} + \|\overline{\delta}\|_{L^\infty_{x'}L^2_t}) \\ \lesssim\ & \varepsilon. \end{aligned} \tag{5.51}$$

Finally, (5.46)-(5.51) yield:

$$\nabla\!\!\!/_{nL}\nabla\!\!\!/\mathrm{tr}\chi = F_1\nabla\!\!\!/_L P + F_2 W + F_3\nabla\!\!\!/\mathrm{tr}\chi \tag{5.52}$$

where F_1, F_2, F_3, P satisfy:

$$\mathcal{N}_1(F_1) + \|F_1\|_{L^\infty_{x'}L^2_t} + \mathcal{N}_1(F_2) + \|F_2\|_{L^\infty_{x'}L^2_t} + \mathcal{N}_1(F_3) + \|F_3\|_{L^\infty_{x'}L^2_t} + \mathcal{N}_1(P) + \|W\|_{\mathcal{P}^0} \lesssim \varepsilon. \tag{5.53}$$

We now apply Proposition 5.1 and obtain from (5.52) (5.53) the following Besov improvement for $\nabla\!\!\!/\mathrm{tr}\chi$:

$$\|\nabla\!\!\!/\mathrm{tr}\chi\|_{\mathcal{B}^0} \lesssim \varepsilon\|\nabla\!\!\!/\mathrm{tr}\chi\|_{\mathcal{P}^0} + \varepsilon,$$

and the smallness of ε finally yields:

$$\|\nabla\!\!\!/\mathrm{tr}\chi\|_{\mathcal{B}^0} \lesssim \varepsilon. \tag{5.54}$$

Coming back to the decomposition (5.40) (5.41) of $\nabla\!\!\!/(n\widehat{\chi})$ and using (5.54), we obtain:

$$\nabla\!\!\!/(n\widehat{\chi}) = \nabla\!\!\!/_{nL} P + E \text{ with } \mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} \lesssim \varepsilon. \tag{5.55}$$

(5.54) and (5.55) yield the desired estimates (5.7) and (5.8).

5.2 Structure equations for $\underline{L}\underline{L}\mathrm{tr}\chi$ and $\nabla\!\!\!/_{\underline{L}}(\zeta)$

The goal of this section is to prove the following proposition.

Proposition 5.5 *Let $\mu_1 = b\underline{L}(\mu)$. Then, μ_1 satisfies the following transport equation:*

$$\begin{aligned} L(\mu_1) + tr\chi\mu_1 &= -2b\nabla\!\!\!/_{\underline{L}}(\zeta)\cdot\nabla\!\!\!/ tr\chi - 2b\widehat{\chi}\cdot\Big(\nabla\!\!\!/\widehat{\otimes}\nabla\!\!\!/_{\underline{L}}(\zeta) + b^{-1}\nabla\!\!\!/ b\nabla\!\!\!/_{\underline{L}}(\zeta) + 2\nabla\!\!\!/_{\underline{L}}\zeta\widehat{\otimes}\zeta\Big) \\ &\quad -2btr\chi\nabla\!\!\!/_{\underline{L}}\zeta\cdot(2\epsilon - n^{-1}\nabla\!\!\!/ n) + d\!\!\!/iv(F_1) + f_2 \end{aligned} \tag{5.56}$$

where the $P_{t,u}$-tangent vectorfield F_1 and the scalar function f_2 satisfy the estimates:

$$\|F_1\|_{L^2(\mathcal{H}_u)} + \|f_2\|_{L^1(\mathcal{H}_u)} \lesssim \varepsilon. \tag{5.57}$$

Furthermore, $\nabla\!\!\!/_{\underline{L}}\zeta$ satisfies the following Hodge system:

$$\begin{aligned} d\!\!\!/iv(\nabla\!\!\!/_{\underline{L}}\zeta) &= \frac{b^{-1}}{2}\mu_1 - b^{-1}\nabla\!\!\!/ b\cdot\nabla\!\!\!/_{\underline{L}}(\zeta) - 2\zeta\cdot\nabla\!\!\!/_{\underline{L}}(\zeta) + d\!\!\!/iv(\underline{\beta}) + h_1, \\ cu\!\!\!/rl(\nabla\!\!\!/_{\underline{L}}\zeta) &= -b^{-1}\nabla\!\!\!/ b\wedge\nabla\!\!\!/_{\underline{L}}(\zeta) - cu\!\!\!/rl(\underline{\beta}) + h_2, \end{aligned} \tag{5.58}$$

where the scalar functions h_1, h_2 satisfy the estimates:

$$\|h_1\|_{L^1(\mathcal{H}_u)} + \|h_2\|_{L^1(\mathcal{H}_u)} \lesssim \varepsilon. \tag{5.59}$$

Proof We start with the proof of (5.56) (5.57). We differentiate the transport equation (2.39) satisfied by μ with respect to $\underline{L}$. We have:

$$\begin{aligned} \underline{L}(L(\mu)) &= -\mathrm{tr}\chi\underline{L}(\mu) - 2\nabla\!\!\!/_{\underline{L}}(\zeta)\cdot\nabla\!\!\!/\mathrm{tr}\chi + 2(\underline{\zeta}-\zeta)\cdot\nabla\!\!\!/_{\underline{L}}(\nabla\!\!\!/\mathrm{tr}\chi) \\ &\quad -2\widehat{\chi}\cdot\Big(\nabla\!\!\!/_{\underline{L}}(\nabla\!\!\!/\widehat{\otimes}\zeta) + 2\zeta\widehat{\otimes}\nabla\!\!\!/_{\underline{L}}(\zeta)\Big) \\ &\quad -2\mathrm{tr}\chi\big(\underline{L}(\rho) + \nabla\!\!\!/_{\underline{L}}\zeta\cdot(2\epsilon - n^{-1}\nabla\!\!\!/ n) + \widehat{\chi}\cdot\nabla\!\!\!/_{\underline{L}}(\widehat{\eta})\big) + f_2^1, \end{aligned} \tag{5.60}$$

where f_2^1 is given by:

$$\begin{aligned} f_2^1 &= -\underline{L}(\mathrm{tr}\chi)\mu + 2\nabla\!\!\!/_{\underline{L}}\underline{\zeta}\cdot\nabla\!\!\!/\mathrm{tr}\chi - 2\nabla\!\!\!/_{\underline{L}}\widehat{\chi}\cdot\Big(\nabla\!\!\!/\widehat{\otimes}\zeta + \zeta\widehat{\otimes}\zeta - \delta\widehat{\chi}\Big) + 2\widehat{\chi}\cdot(\underline{L}(\delta)\widehat{\chi} + \delta\nabla\!\!\!/_{\underline{L}}\widehat{\chi}) \\ &\quad -2\underline{L}(\mathrm{tr}\chi)\Big(\rho + \zeta\cdot(2\epsilon - n^{-1}\nabla\!\!\!/ n) - n^{-1}\nabla\!\!\!/ n\cdot\epsilon + |\epsilon|^2 + |\widehat{\chi}|^2 + \widehat{\eta}\cdot\widehat{\chi}\Big) \\ &\quad -2\mathrm{tr}\chi\Big(\zeta\cdot(2\nabla\!\!\!/_{\underline{L}}\epsilon - \nabla\!\!\!/_{\underline{L}}(n^{-1}\nabla\!\!\!/ n)) - \nabla\!\!\!/_{\underline{L}}(n^{-1}\nabla\!\!\!/ n)\cdot\epsilon - n^{-1}\nabla\!\!\!/ n\cdot\nabla\!\!\!/_{\underline{L}}\epsilon + 2\epsilon\cdot\nabla\!\!\!/_{\underline{L}}\epsilon \\ &\quad +2\widehat{\chi}\cdot\nabla\!\!\!/_{\underline{L}}\widehat{\chi} + \widehat{\eta}\cdot\nabla\!\!\!/_{\underline{L}}\widehat{\chi}\Big). \end{aligned} \tag{5.61}$$

The curvature bound (2.59) for ρ and the estimates (2.67)-(2.72) obtained in Theorem 2.21 yield:

$$\begin{aligned} \|f_2^1\|_{L^1(\mathcal{H}_u)} &\lesssim \|\nabla\!\!\!/_{\underline{L}}(\chi)\|^2_{L^2(\mathcal{H}_u)} + \mathcal{N}_1(\chi)^2 + \mathcal{N}_1(\zeta)^2 + \mathcal{N}_1(k)^2 + \mathcal{N}_1(\nabla n)^2 + \|\rho\|^2_{L^2(\mathcal{H}_u)} \\ &\quad + \|\nabla^2 n\|^2_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/_{\underline{L}}(\epsilon)\|^2_{L^2(\mathcal{H}_u)} + \|\underline{L}(\delta)\|^2_{L^2(\mathcal{H}_u)} + \|n-1\|_{L^\infty(\mathcal{H}_u)} \\ &\lesssim \varepsilon. \end{aligned} \tag{5.62}$$

We now estimate various terms in (5.60). Note first from the commutator formula (2.46) that we have:

$$\begin{aligned}\underline{L}(L(\mu)) &= L(\underline{L}(\mu)) + [\underline{L}, L](\mu) \\ &= L(\underline{L}(\mu)) - \overline{\delta}\underline{L}(\mu) + (\delta + n^{-1}N(n))L(\mu) + 2(\zeta - \underline{\zeta})\cdot \nabla\!\!\!/\mu \\ &= L(\underline{L}(\mu)) - \overline{\delta}\underline{L}(\mu) + (\delta + n^{-1}N(n))L(\mu) + 2\mathrm{di}\!\!\!/\mathrm{v}\left((\zeta - \underline{\zeta})\mu\right) \\ &\quad -2(\mathrm{di}\!\!\!/\mathrm{v}(\zeta) - \mathrm{di}\!\!\!/\mathrm{v}(\underline{\zeta}))\mu.\end{aligned} \tag{5.63}$$

Using the commutator formula (2.45), we have:

$$\begin{aligned}(\underline{\zeta} - \zeta)\cdot \nabla\!\!\!/_{\underline{L}}(\nabla\!\!\!/\mathrm{tr}\chi) &= (\underline{\zeta} - \zeta)\cdot \nabla\!\!\!/(\underline{L}(\mathrm{tr}\chi)) + (\underline{\zeta} - \zeta)\cdot [\nabla\!\!\!/_{\underline{L}}, \nabla\!\!\!/](\mathrm{tr}\chi) \\ &= \mathrm{di}\!\!\!/\mathrm{v}((\underline{\zeta} - \zeta)\underline{L}(\mathrm{tr}\chi)) - (\mathrm{di}\!\!\!/\mathrm{v}(\underline{\zeta}) - \mathrm{di}\!\!\!/\mathrm{v}(\zeta))\underline{L}(\mathrm{tr}\chi) \\ &\quad +(\underline{\zeta} - \zeta)\cdot(-\underline{\chi}\nabla\!\!\!/\mathrm{tr}\chi + \underline{\xi}L(\mathrm{tr}\chi) + b^{-1}\nabla\!\!\!/ b\underline{L}(\mathrm{tr}\chi)),\end{aligned} \tag{5.64}$$

and

$$\begin{aligned}\nabla\!\!\!/_{\underline{L}}(\nabla\!\!\!/\widehat{\otimes}\zeta) &= \nabla\!\!\!/\widehat{\otimes}(\nabla\!\!\!/_{\underline{L}}\zeta) + [\nabla\!\!\!/_{\underline{L}}, \nabla\!\!\!/]\widehat{\otimes}\zeta \\ &= \nabla\!\!\!/\widehat{\otimes}(\nabla\!\!\!/_{\underline{L}}\zeta) - \underline{\chi}\nabla\!\!\!/\zeta + \underline{\xi}\nabla\!\!\!/_L(\zeta) + b^{-1}\nabla\!\!\!/ b\nabla\!\!\!/_{\underline{L}}(\zeta) + (\chi\underline{\xi} + \underline{\chi}\zeta + \underline{\beta})\zeta.\end{aligned} \tag{5.65}$$

Also, using the Bianchi identity (2.54), we have:

$$\begin{aligned}\mathrm{tr}\chi\underline{L}(\rho) &= -\mathrm{tr}\chi\mathrm{di}\!\!\!/\mathrm{v}\underline{\beta} - \frac{3}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}\rho - \frac{1}{2}\mathrm{tr}\chi\widehat{\chi}\cdot\underline{\alpha} + 2\mathrm{tr}\chi\underline{\xi}\cdot\beta + \mathrm{tr}\chi(\epsilon - 2\zeta)\cdot\underline{\beta} \\ &= -\mathrm{di}\!\!\!/\mathrm{v}(\mathrm{tr}\chi\underline{\beta}) + \nabla\!\!\!/\mathrm{tr}\chi\cdot\underline{\beta} - \frac{3}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}\rho - \frac{1}{2}\mathrm{tr}\chi\widehat{\chi}\cdot\underline{\alpha} + 2\mathrm{tr}\chi\underline{\xi}\cdot\beta + \mathrm{tr}\chi(\epsilon - 2\zeta)\cdot\underline{\beta}.\end{aligned} \tag{5.66}$$

We now consider the term $\mathrm{tr}\chi\widehat{\chi}\cdot\nabla\!\!\!/_{\underline{L}}(\widehat{\eta})$ in the right-hand side of (5.60). We start by computing $\nabla\!\!\!/_{\underline{L}}\eta$. We have:

$$\begin{aligned}\nabla\!\!\!/_{\underline{L}}(\eta)_{AB} &= \underline{L}(k_{AB}) - \eta(\nabla\!\!\!/_{\underline{L}}e_A, e_B) - \eta(e_A, \nabla\!\!\!/_{\underline{L}}e_B) \\ &= -\mathbf{g}(\mathbf{D}_{\underline{L}}\mathbf{D}_{e_A}T, e_B) + \mathbf{g}(\mathbf{D}_{\nabla\!\!\!/_{\underline{L}}e_A}T, e_B) - \mathbf{g}(\mathbf{D}_{e_A}T, \mathbf{D}_{\underline{L}}e_B - \nabla\!\!\!/_{\underline{L}}e_B) \\ &= -\mathbf{g}(\mathbf{D}_{e_A}\mathbf{D}_{\underline{L}}T, e_B) - \mathbf{g}(\mathbf{D}_{[\underline{L},e_A]}T, e_B) + \mathbf{R}_{\underline{L}ATB} + \mathbf{g}(\mathbf{D}_{\nabla\!\!\!/_{\underline{L}}e_A}T, e_B) \\ &\quad -\mathbf{g}(\mathbf{D}_{e_A}T, \mathbf{D}_{\underline{L}}e_B - \nabla\!\!\!/_{\underline{L}}e_B) \\ &= -\nabla\!\!\!/_A\epsilon_B - n^{-1}\nabla\!\!\!/_A\nabla\!\!\!/_B n + n^{-2}\nabla\!\!\!/_A n\nabla\!\!\!/_B n + \mathbf{g}(\mathbf{D}_{\underline{L}}T, \mathbf{D}_{e_A}e_B - \nabla\!\!\!/_{e_A}e_B) \\ &\quad -\mathbf{g}(\mathbf{D}_{\mathbf{D}_{\underline{L}}e_A - \nabla\!\!\!/_{\underline{L}}e_A - \mathbf{D}_{e_A}\underline{L}}T, e_B) + \frac{1}{2}\underline{\alpha}_{AB} - \frac{1}{2}\rho\delta_{AB} + \frac{1}{2}\sigma\in_{AB} \\ &\quad -\mathbf{g}(\mathbf{D}_{e_A}T, \mathbf{D}_{\underline{L}}e_B - \nabla\!\!\!/_{\underline{L}}e_B)\end{aligned}$$

which together with the Ricci equations (2.23) yields:

$$\begin{aligned}\nabla\!\!\!/_{\underline{L}}(\eta)_{AB} &= -n^{-1}\nabla\!\!\!/_A\nabla\!\!\!/_B n + n^{-2}\nabla\!\!\!/_A n\nabla\!\!\!/_B n + \frac{1}{2}\underline{\alpha}_{AB} - \frac{1}{2}\rho\delta_{AB} \\ &\quad -\nabla\!\!\!/_A\epsilon_B - \underline{\chi}_{AC}\eta_{CB} + \underline{\xi}_A\overline{\epsilon}_B + (\epsilon_A - \zeta_A)(n^{-1}\nabla\!\!\!/_B n + \epsilon_B) + (\underline{\xi}_B - \zeta_B)\epsilon_A.\end{aligned} \tag{5.67}$$

In view of (5.67), we have:

$$\begin{aligned}\mathrm{tr}\chi\widehat{\chi}\cdot\nabla\!\!\!/_{\underline{L}}(\widehat{\eta}) &= \mathrm{tr}\chi\widehat{\chi}\cdot\bigg(-n^{-1}\nabla\!\!\!/^2 n+n^{-2}\nabla\!\!\!/ n\nabla\!\!\!/ n+\frac{1}{2}\underline{\alpha}\\ &\quad -\nabla\!\!\!/\epsilon-\underline{\chi}\eta+\underline{\xi}\overline{\epsilon}+(\epsilon-\zeta)(n^{-1}\nabla\!\!\!/ n+\epsilon)+(\underline{\xi}-\zeta)\epsilon\bigg).\end{aligned}\tag{5.68}$$

Now, (5.60) together with (5.63)-(5.66) and (5.68) yields:

$$\begin{aligned}L(\underline{L}(\mu)) &= \overline{\delta}\underline{L}(\mu)-\mathrm{tr}\chi\underline{L}(\mu)-2\nabla\!\!\!/_{\underline{L}}(\zeta)\cdot\nabla\!\!\!/\mathrm{tr}\chi\\ &\quad -2\widehat{\chi}\cdot\Big(\nabla\!\!\!/\widehat{\otimes}\nabla\!\!\!/_{\underline{L}}(\zeta)+b^{-1}\nabla\!\!\!/ b\nabla\!\!\!/_{\underline{L}}(\zeta)+2\zeta\widehat{\otimes}\nabla\!\!\!/_{\underline{L}}(\zeta)\Big)-2\mathrm{tr}\chi\nabla\!\!\!/_{\underline{L}}\zeta\cdot(2\epsilon-n^{-1}\nabla\!\!\!/ n)\\ &\quad +\mathrm{di}\!\!\!/v\bigg(-2(\zeta-\underline{\zeta})\mu+2(\underline{\zeta}-\zeta)\underline{L}(\mathrm{tr}\chi)+2\mathrm{tr}\chi\underline{\beta}\bigg)+f_2^1+f_2^2,\end{aligned}\tag{5.69}$$

where f_2^2 is given by:

$$\begin{aligned}f_2^2 &= -(\delta+n^{-1}N(n))L(\mu)-2(\mathrm{di}\!\!\!/v(\zeta)-\mathrm{di}\!\!\!/v(\underline{\zeta}))\mu-2(\mathrm{di}\!\!\!/v(\underline{\zeta})-\mathrm{di}\!\!\!/v(\zeta))\underline{L}(\mathrm{tr}\chi)\\ &\quad +2(\underline{\zeta}-\zeta)\cdot(-\underline{\chi}\nabla\!\!\!/\mathrm{tr}\chi+\underline{\xi}L(\mathrm{tr}\chi)+b^{-1}\nabla\!\!\!/ b\underline{L}(\mathrm{tr}\chi))\\ &\quad -2\widehat{\chi}\cdot(-\underline{\chi}\nabla\!\!\!/\zeta+\underline{\xi}\nabla\!\!\!/_L(\zeta)+(\chi\underline{\xi}+\underline{\chi}\zeta+\underline{\beta})\zeta)\\ &\quad -2\bigg(\nabla\!\!\!/\mathrm{tr}\chi\cdot\underline{\beta}-\frac{3}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}\rho+2\mathrm{tr}\chi\underline{\xi}\cdot\beta+\mathrm{tr}\chi(\epsilon-2\zeta)\cdot\underline{\beta}\bigg)\\ &\quad -2\mathrm{tr}\chi\widehat{\chi}\cdot\bigg(-n^{-1}\nabla\!\!\!/^2 n+n^{-2}\nabla\!\!\!/ n\nabla\!\!\!/ n-\nabla\!\!\!/\epsilon-\underline{\chi}\eta+\underline{\xi}\overline{\epsilon}\\ &\quad +(\epsilon-\zeta)(n^{-1}\nabla\!\!\!/ n+\epsilon)+(\underline{\xi}-\zeta)\epsilon\bigg).\end{aligned}$$

The curvature bound (2.59) for $\beta,\underline{\beta}$ and the estimates (2.67)-(2.72) obtained in Theorem 2.21 yield:

$$\begin{aligned}\|f_2^2\|_{L^1(\mathcal{H}_u)} &\lesssim \|\mu\|^2_{L^2(\mathcal{H}_u)}+\|L(\mu)\|^2_{L^2_{x'}L^1_t}+\|\nabla\!\!\!/_{\underline{L}}(\chi)\|^2_{L^2(\mathcal{H}_u)}+\mathcal{N}_1(\chi)^2+\mathcal{N}_1(\zeta)^2\\ &\quad +\mathcal{N}_1(k)^2+\mathcal{N}_1(\nabla n)^2+\|\beta\|^2_{L^2(\mathcal{H}_u)}+\|\underline{\beta}\|^2_{L^2(\mathcal{H}_u)}+\|\rho\|^2_{L^2(\mathcal{H}_u)}+\|\sigma\|^2_{L^2(\mathcal{H}_u)}\\ &\quad +\|\nabla^2 n\|^2_{L^2(\mathcal{H}_u)}+\|n-1\|_{L^\infty(\mathcal{H}_u)}\\ &\lesssim \|\mu\|^2_{L^2(\mathcal{H}_u)}+\|L(\mu)\|^2_{L^2_{x'}L^1_t}+\varepsilon.\end{aligned}\tag{5.70}$$

Using the definition of μ (2.35), the formula for $L(\mu)$ given by (2.39), the curvature bound (2.59) for ρ and the estimates (2.67)-(2.72) obtained in Theorem 2.21, we obtain:

$$\|\mu\|^2_{L^2(\mathcal{H}_u)}+\|L(\mu)\|^2_{L^2_{x'}L^1_t}\lesssim\varepsilon$$

which together with (5.70) yields:

$$\|f_2^2\|_{L^1(\mathcal{H}_u)}\lesssim\varepsilon.\tag{5.71}$$

Since $\mu_1=b\underline{L}(\mu)$, we have:

$$L(\mu_1)=L(b)\underline{L}(\mu)+bL(\underline{L}(\mu))=-b\overline{\delta}\underline{L}(\mu)+bL(\underline{L}(\mu))$$

where we used the transport equation (2.27) satisfied by b. Together with (5.69), this yields:

$$\begin{aligned} L(\mu_1) &= -\mathrm{tr}\chi\mu_1 - 2b\nabla\!\!\!/_{\underline{L}}(\zeta)\cdot\nabla\!\!\!/\mathrm{tr}\chi - 2b\widehat{\chi}\cdot\Big(\nabla\!\!\!/\widehat{\otimes}\nabla\!\!\!/_{\underline{L}}(\zeta) + b^{-1}\nabla\!\!\!/ b\nabla\!\!\!/_{\underline{L}}(\zeta) + 2\zeta\widehat{\otimes}\nabla\!\!\!/_{\underline{L}}(\zeta)\Big) \\ &\quad -2b\mathrm{tr}\chi\nabla\!\!\!/_{\underline{L}}\zeta\cdot(2\epsilon - n^{-1}\nabla\!\!\!/ n) + \mathrm{di}\!\!\!/\mathrm{v}\Big(b(-2(\zeta-\underline{\zeta})\mu + 2(\underline{\zeta}-\zeta)\underline{L}(\mathrm{tr}\chi) + 2\mathrm{tr}\chi\underline{\beta})\Big) \\ &\quad -\nabla\!\!\!/ b\cdot(-2(\zeta-\underline{\zeta})\mu + 2(\underline{\zeta}-\zeta)\underline{L}(\mathrm{tr}\chi) + 2\mathrm{tr}\chi\underline{\beta}) + bf_2^1 + bf_2^2, \end{aligned} \tag{5.72}$$

which is the desired transport equation (5.56) for μ_1 with F_1 given by:

$$F_1 = b(-2(\zeta-\underline{\zeta})\mu + 2(\underline{\zeta}-\zeta)\underline{L}(\mathrm{tr}\chi) + 2\mathrm{tr}\chi\underline{\beta})$$

and f_2 given by:

$$f_2 = -\nabla\!\!\!/ b\cdot(-2(\zeta-\underline{\zeta})\mu + 2(\underline{\zeta}-\zeta)\underline{L}(\mathrm{tr}\chi) + 2\mathrm{tr}\chi\underline{\beta}) + bf_2^1 + bf_2^2.$$

Using the curvature bound (2.59) for $\underline{\beta}$ and the estimates (2.67)-(2.72) obtained in Theorem 2.21, we obtain:

$$\begin{aligned} \|F_1\|_{L^2(\mathcal{H}_u)} &\lesssim \|b\|_{L^\infty}(\|\zeta\|^2_{L^\infty_{x'}L^2_t} + \|\underline{\zeta}\|^2_{L^\infty_{x'}L^2_t} + \|\underline{L}(\mathrm{tr}\chi)\|_{L^2_{x'}L^\infty_t} + \|\mu\|_{L^2_{x'}L^\infty_t} \\ &\quad + \|\mathrm{tr}\chi\|_{L^\infty}\|\underline{\beta}\|_{L^2(\mathcal{H}_u)}) \\ &\lesssim \varepsilon, \end{aligned}$$

and:

$$\|f_2\|_{L^1(\mathcal{H}_u)} \lesssim \|b^{-1}\nabla\!\!\!/ b\|_{L^2(\mathcal{H}_u)}\|F_1\|_{L^2(\mathcal{H}_u)} + \|b\|_{L^\infty}(\|f_2^1\|_{L^1(\mathcal{H}_u)} + \|f_2^2\|_{L^1(\mathcal{H}_u)}) \lesssim \varepsilon,$$

where we used the estimate (5.62) for f_2^1 and the estimate (5.71) for f_2^2. This concludes the proof of the estimate (5.57) for F_1 and f_2.

We now turn to the Hodge system satisfied by $\nabla\!\!\!/_{\underline{L}}(\zeta)$. We differentiate the equation (2.36) giving $\mathrm{di}\!\!\!/\mathrm{v}(\zeta)$ with respect to $\underline{L}$:

$$\underline{L}(\mathrm{di}\!\!\!/\mathrm{v}(\zeta)) = \frac{1}{2}\Big(\underline{L}(\mu) - 2\widehat{\chi}\cdot\nabla\!\!\!/_{\underline{L}}(\widehat{\eta}) - 4\zeta\nabla\!\!\!/_{\underline{L}}(\zeta)\Big) - \underline{L}(\rho) + h_1^1, \tag{5.73}$$

where h_1^1 is given by:

$$h_1^1 = \frac{1}{4}\underline{L}(\mathrm{tr}\chi)\mathrm{tr}\underline{\chi} + \frac{1}{4}\mathrm{tr}\chi\underline{L}(\mathrm{tr}\underline{\chi}) - \widehat{\chi}\cdot\nabla\!\!\!/_{\underline{L}}(\widehat{\chi}) - \nabla\!\!\!/_{\underline{L}}(\widehat{\chi})\cdot\widehat{\eta}.$$

The estimates (2.67)-(2.72) obtained in Theorem 2.21 yield:

$$\|h_1^1\|_{L^1(\mathcal{H}_u)} \lesssim \|\nabla\!\!\!/_{\underline{L}}(\chi)\|^2_{L^2(\mathcal{H}_u)} + \mathcal{N}_1(\chi)^2 + \mathcal{N}_1(k)^2 \lesssim \varepsilon. \tag{5.74}$$

Also, using the commutator formula (2.45), we have:

$$\begin{aligned} \underline{L}(\mathrm{di}\!\!\!/\mathrm{v}(\zeta)) &= \mathrm{di}\!\!\!/\mathrm{v}(\nabla\!\!\!/_{\underline{L}}(\zeta)) + [\nabla\!\!\!/_{\underline{L}}, \mathrm{di}\!\!\!/\mathrm{v}]\zeta \\ &= \mathrm{di}\!\!\!/\mathrm{v}(\nabla\!\!\!/_{\underline{L}}(\zeta)) - \underline{\chi}\cdot\nabla\!\!\!/\zeta + \underline{\xi}\cdot\nabla\!\!\!/_L(\zeta) + b^{-1}\nabla\!\!\!/ b\cdot\nabla\!\!\!/_{\underline{L}}(\zeta) + (\chi\underline{\xi} + \underline{\chi}\zeta + \underline{\beta})\zeta. \end{aligned} \tag{5.75}$$

(5.75), (5.67), (5.73) and the Bianchi identity (2.54) yield:

$$\mathrm{div}\!\!\!/(\nabla\!\!\!/_{\underline{L}}(\zeta)) = -b^{-1}\nabla\!\!\!/ b\cdot\nabla\!\!\!/_{\underline{L}}(\zeta) + \frac{1}{2}\Big(b^{-1}\mu_1 - 4\zeta\nabla\!\!\!/_{\underline{L}}(\zeta)\Big) + \mathrm{div}\!\!\!/\underline{\beta} + h_1^1 + h_1^2, \tag{5.76}$$

where h_1^2 is given by:

$$\begin{aligned} h_1^2 &= -\underline{\chi}\cdot\nabla\!\!\!/\zeta + \underline{\xi}\cdot\nabla\!\!\!/_L(\zeta) + (\chi\underline{\xi} + \underline{\chi}\zeta + \underline{\beta})\zeta + \frac{3}{2}\mathrm{tr}\underline{\chi}\rho - 2\underline{\xi}\cdot\beta - (\epsilon - 2\zeta)\cdot\underline{\beta} \\ &\quad -\widehat{\chi}\cdot\bigg(-n^{-1}\nabla\!\!\!/^2 n + n^{-2}\nabla\!\!\!/ n\nabla\!\!\!/ n - \nabla\!\!\!/\epsilon - \underline{\chi}\eta + \underline{\xi}\overline{\epsilon} + (\epsilon-\zeta)(n^{-1}\nabla\!\!\!/ n + \epsilon) + (\underline{\xi}-\zeta)\epsilon\bigg). \end{aligned}$$

The curvature bound (2.59) for β, ρ and $\underline{\beta}$, and the estimates (2.67)-(2.72) obtained in Theorem 2.21 yield:

$$\|h_1^2\|_{L^1(\mathcal{H}_u)} \lesssim \|\beta\|^2_{L^2(\mathcal{H}_u)} + \|\rho\|^2_{L^2(\mathcal{H}_u)} + \|\underline{\beta}\|^2_{L^2(\mathcal{H}_u)} + \mathcal{N}_1(\zeta)^2 + \mathcal{N}_1(\chi)^2 + \mathcal{N}_1(k)^2 + \mathcal{N}_1(\nabla\!\!\!/ n) \lesssim \varepsilon. \tag{5.77}$$

Next, we differentiate the equation (2.37) giving $\mathrm{curl}\!\!\!/(\zeta)$ with respect to $\underline{L}$:

$$\underline{L}(\mathrm{curl}\!\!\!/(\zeta)) = \nabla\!\!\!/_{\underline{L}}(\widehat{\chi})\wedge\widehat{\eta} + \widehat{\chi}\wedge\nabla\!\!\!/_{\underline{L}}(\widehat{\eta}) + \underline{L}(\sigma). \tag{5.78}$$

The commutator formula (2.45), (5.67), (5.78) and the Bianchi identity (2.56) yield:

$$\mathrm{curl}\!\!\!/(\nabla\!\!\!/_{\underline{L}}(\zeta)) = -b^{-1}\nabla\!\!\!/ b\wedge\nabla\!\!\!/_{\underline{L}}(\zeta) - \mathrm{curl}\!\!\!/(\underline{\beta}) + h_2, \tag{5.79}$$

where h_2 is given by:

$$\begin{aligned} h_2 &= \in_{AB}\underline{\chi}_{AC}\nabla\!\!\!/_C\zeta_B - \underline{\xi}\wedge\nabla\!\!\!/_L(\zeta) + (\chi\underline{\xi} + \underline{\chi}\zeta + \underline{\beta})\zeta + \nabla\!\!\!/_{\underline{L}}(\widehat{\chi})\wedge\widehat{\eta} - \frac{3}{2}\mathrm{tr}\underline{\chi}\sigma - 2\underline{\xi}^*\beta + (\epsilon-2\zeta)^*\underline{\beta} \\ &\quad +\widehat{\chi}\wedge\bigg(-n^{-1}\nabla\!\!\!/^2 n + n^{-2}\nabla\!\!\!/ n\nabla\!\!\!/ n - \nabla\!\!\!/\epsilon - \underline{\chi}\eta + \underline{\xi}\overline{\epsilon} + (\epsilon-\zeta)(n^{-1}\nabla\!\!\!/ n + \epsilon) + (\underline{\xi}-\zeta)\epsilon\bigg). \end{aligned}$$

The curvature bound (2.59) for $\beta, \sigma, \underline{\beta}$ and the estimates (2.67)-(2.72) obtained in Theorem 2.21 yield:

$$\|h_2\|_{L^1(\mathcal{H}_u)} \lesssim \|\beta\|^2_{L^2(\mathcal{H}_u)} + \|\sigma\|^2_{L^2(\mathcal{H}_u)} + \|\underline{\beta}\|^2_{L^2(\mathcal{H}_u)} + \mathcal{N}_1(\zeta)^2 + \mathcal{N}_1(\chi)^2 + \mathcal{N}_1(k)^2 + \mathcal{N}_1(\nabla\!\!\!/ n) \lesssim \varepsilon. \tag{5.80}$$

Finally, (5.74)-(5.80) yield (5.58) and (5.59) which concludes the proof of the proposition. ■

5.3 Estimates for $\nabla\!\!\!/_{\underline{L}}(\zeta)$

The goal of this section is to obtain an improvement of the bootstrap assumption (5.2) for $\nabla\!\!\!/_{\underline{L}}(\zeta)$. We will use the following three lemmas.

Lemma 5.6 *Let F a $P_{t,u}$-tangent vectorfield on $\mathcal{H}_u$. Assume there exists two constants $C_1, C_2 > 0$ possibly depending on u such that for all $j \geq 0$, we have:*

$$\|P_j F\|_{L^2(\mathcal{H}_u)} \leq C_1 + 2^{-\frac{j}{2}}C_2. \tag{5.81}$$

Let H a $P_{t,u}$-tangent vectorfield of the same type. Then, for all $j \geq 0$, we have:

$$\|P_j(H\cdot F)\|_{L^2(\mathcal{H}_u)} \lesssim \mathcal{N}_1(H)(2^j C_1 + 2^{\frac{j}{2}}C_2). \tag{5.82}$$

Lemma 5.7 *Let f and h two scalar functions on $\mathcal{H}_u$. Let $2 \leq p \leq +\infty$. Assume there exists two constants $C_1, C_2 > 0$ possibly depending on u such that for all $j \geq 0$, we have:*

$$\|P_j f\|_{L^p_t L^2_{x'}} \leq 2^j C_1 + 2^{\frac{j}{2}} C_2. \tag{5.83}$$

Then, for all $j \geq 0$, we have:

$$\|P_j(hf)\|_{L^p_t L^2_{x'}} \lesssim (\|h\|_{L^\infty} + \|\nabla\!\!\!/ h\|_{\mathcal{B}^0})(2^j C_1 + 2^{\frac{j}{2}} C_2). \tag{5.84}$$

Lemma 5.8 *Let F a $P_{t,u}$-tangent 1-form on $\mathcal{H}_u$. Assume there exists two constants $C_1, C_2 > 0$ such that for all $j \geq 0$, we have:*

$$\|P_j \mathcal{D}_1(F)\|_{L^2(\mathcal{H}_u)} \leq 2^j C_1 + 2^{\frac{j}{2}} C_2. \tag{5.85}$$

Then, for all $j \geq 0$, we have:

$$\|P_j F\|_{L^2(\mathcal{H}_u)} \lesssim C_1 + 2^{-\frac{j}{2}} C_2. \tag{5.86}$$

We also state the following lemmas which will be used in the proof of Lemma 5.6 as well as several places in the paper.

Lemma 5.9 *For any $P_{t,u}$-tangent tensor F on $\mathcal{H}_u$, and for all $j \geq 0$, we have:*

$$\sum_{j \geq 0} 2^j \|P_j F\|^2_{L^\infty_t L^2_{x'}} + 2^{-j} \|\nabla\!\!\!/ P_j F\|^2_{L^\infty_t L^2_{x'}} \lesssim \mathcal{N}_1(F)^2. \tag{5.87}$$

Lemma 5.10 *For any 1-form F on $P_{t,u}$, for any $1 < p \leq 2$ and for all $j \geq 0$, we have:*

$$\|P_j d\!\!\!/ v(F)\|_{L^2(P_{t,u})} \lesssim 2^{\frac{2}{p} j} \|F\|_{L^p(P_{t,u})}. \tag{5.88}$$

We postpone the proof of Lemma 5.6 to section B.1, the proof of Lemma 5.7 to section B.2, the proof of Lemma 5.8 to section B.3, the proof of Lemma 5.9 to sections B.4, and the proof of Lemma 5.10 to section B.5. We show how they improve the bootstrap assumption (5.2). The bootstrap assumption (5.1) together with the definition of μ_1 and μ yields for all $j \geq 0$:

$$\|P_j(b^{-1}\mu_1)\|_{L^2(\mathcal{H}_u)} \lesssim 2^j D\varepsilon + 2^{\frac{j}{2}} D\varepsilon\gamma(u). \tag{5.89}$$

Lemma 5.6 implies:

$$\begin{aligned}
&\left\| P_j \left(\frac{b^{-1}}{2} \mu_1 - b^{-1} \nabla\!\!\!/ b \cdot \nabla\!\!\!/_{\underline{L}}(\zeta) - 2\zeta \cdot \nabla\!\!\!/_{\underline{L}}(\zeta) \right) \right\|_{L^2(\mathcal{H}_u)} + \|P_j(-b^{-1}\nabla\!\!\!/ b \wedge \nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)} \\
\lesssim\ & 2^j D\varepsilon + 2^{\frac{j}{2}} D\gamma(u) + (\mathcal{N}_1(b^{-1}\nabla\!\!\!/ b) + \mathcal{N}_1(\zeta))(2^j D^2\varepsilon + 2^{\frac{j}{2}} D^2\varepsilon\gamma(u)) \\
\lesssim\ & (1 + D\varepsilon)(2^j D\varepsilon + 2^{\frac{j}{2}} D\varepsilon\gamma(u)) \\
\lesssim\ & 2^j D\varepsilon + 2^{\frac{j}{2}} D\varepsilon\gamma(u),
\end{aligned} \tag{5.90}$$

where we used the bootstrap assumptions (5.2) for $\nabla\!\!\!/_{\underline{L}}(\zeta)$, the estimate (5.89) for μ_1 and the estimates (2.69) and (2.72) for b and ζ obtained in Theorem 2.21. Using the

Littlewood-Paley property iii) of Theorem 3.10, and the dual of the sharp Bernstein inequality (4.32) for scalars, we obtain:

$$\begin{aligned} & \|P_j(\not{\mathrm{div}}(\underline{\beta})+h_1)\|_{L^2(\mathcal{H}_u)}+\|P_j(-\not{\mathrm{curl}}(\underline{\beta})+h_2)\|_{L^2(\mathcal{H}_u)} \qquad (5.91)\\ \lesssim\ & 2^j\|\underline{\beta}\|_{L^2(\mathcal{H}_u)}+2^j\|h_1\|_{L^1(\mathcal{H}_u)}+2^j\|h_2\|_{L^1(\mathcal{H}_u)}\\ \lesssim\ & 2^j\varepsilon, \end{aligned}$$

where we used the curvature bound (2.59) for $\underline{\beta}$ and the estimate (5.59) for h_1, h_2. Using the Hodge system (5.58) satisfied by $\not{\nabla}_{\underline{L}}(\zeta)$ and the estimates (5.90) and (5.91), we obtain:

$$\begin{aligned} \|P_j\mathcal{D}_1(\not{\nabla}_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)} &\lesssim \|P_j(\not{\mathrm{div}}(\not{\nabla}_{\underline{L}}(z)))\|_{L^2(\mathcal{H}_u)}+\|P_j(\not{\mathrm{curl}}(\not{\nabla}_{\underline{L}}(\zeta)))\|_{L^2(\mathcal{H}_u)}\\ &\lesssim 2^jD\varepsilon+2^{\frac{j}{2}}D\varepsilon\gamma(u). \end{aligned}$$

which together with Lemma 5.8 yields:

$$\|P_j(\not{\nabla}_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)}\lesssim D\varepsilon+2^{-\frac{j}{2}}D\varepsilon\gamma(u). \qquad (5.92)$$

Note that (5.92) is an improvement of the bootstrap assumption (5.2) for $\not{\nabla}_{\underline{L}}(\zeta)$.

5.4 Estimates for $\underline{LL}\mathrm{tr}\chi$

The goal of this section is to obtain an improvement of the bootstrap assumption (5.1) for $\underline{LL}\mathrm{tr}\chi$. Note first that the bootstrap assumption (5.1) together with Lemma 5.7 with the choice $h=b$ and the definition of μ_1 and μ yields for all $j\geq 0$:

$$\|P_j(\mu_1)\|_{L^2(\mathcal{H}_u)}\lesssim 2^jD\varepsilon+2^{\frac{j}{2}}D\varepsilon\gamma(u). \qquad (5.93)$$

Another application of Lemma 5.7 this time with the choice $h=b^{-1}$ shows that improving on the bootstrap assumption (5.1) is equivalent to improving (5.93). We now focus on improving (5.93). After multiplying the transport equation (5.56) satisfied by μ_1 by n, we have:

$$\begin{aligned} nL(\mu_1)+n\mathrm{tr}\chi\mu_1 &= -2bn\not{\nabla}_{\underline{L}}(\zeta)\cdot\not{\nabla}\mathrm{tr}\chi-2bn\widehat{\chi}\cdot\left(\not{\nabla}\widehat{\otimes}\not{\nabla}_{\underline{L}}(\zeta)+b^{-1}\not{\nabla}b\not{\nabla}_{\underline{L}}(\zeta)+2\not{\nabla}_{\underline{L}}\zeta\widehat{\otimes}\zeta\right)\\ &\quad -2nb\mathrm{tr}\chi\not{\nabla}_{\underline{L}}\zeta\cdot(2\epsilon-n^{-1}\not{\nabla}n)+n\not{\mathrm{div}}(F_1)+nf_2. \end{aligned}$$

which yields:

$$\begin{aligned} & \|P_j(\mu_1)\|_{L^2(\mathcal{H}_u)} \qquad (5.94)\\ \lesssim\ & 2^{\frac{j}{2}}\gamma(u)\varepsilon+\left\|P_j\left(\int_0^t(n\mathrm{tr}\chi\mu_1)d\tau\right)\right\|_{L^2(\mathcal{H}_u)}+\left\|P_j\left(\int_0^t(bn\not{\nabla}_{\underline{L}}(\zeta)\cdot\not{\nabla}\mathrm{tr}\chi)d\tau\right)\right\|_{L^2(\mathcal{H}_u)}\\ & +\left\|P_j\left(\int_0^t(bn\widehat{\chi}\cdot(\not{\nabla}\widehat{\otimes}\not{\nabla}_{\underline{L}}(\zeta)))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}+\left\|P_j\left(\int_0^t(bn\widehat{\chi}\cdot(b^{-1}\not{\nabla}b\not{\nabla}_{\underline{L}}(\zeta)))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}\\ & +\left\|P_j\left(\int_0^t(bn\widehat{\chi}\cdot(\not{\nabla}_{\underline{L}}\zeta\widehat{\otimes}\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}+\left\|P_j\left(\int_0^t(n\mathrm{tr}\chi b(2\epsilon-n^{-1}\not{\nabla}n)\cdot\not{\nabla}_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}\\ & +\left\|P_j\left(\int_0^t(\not{\mathrm{div}}(nF_1))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}+\left\|P_j\left(\int_0^t(\not{\nabla}nF_1)d\tau\right)\right\|_{L^2(\mathcal{H}_u)}\\ & +\left\|P_j\left(\int_0^t(nf_2)d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \end{aligned}$$

where we used the following lemma with $f = \mu_1$:

Lemma 5.11 *Let f a scalar function solution of the following transport equation:*

$$L(f) = 0,\ f = f_0 \text{ on } P_{0,u}.$$

Assume there is a constant $C > 0$ possibly depending on u such that for all $j \geq 0$:

$$\|P_j f_0\|_{L^2(P_{0,u})} \lesssim C 2^{\frac{j}{2}}.$$

Then, we have the following estimate for f:

$$\|P_j f\|_{L^\infty_t L^2_{x'}} \lesssim C 2^{\frac{j}{2}}.$$

The proof of Lemma 5.11 is postponed to section B.6. In order to estimate the right-hand side of (5.93), we will use the following three lemmas, which constitute the core of section 5.

Lemma 5.12 *Let a scalar function f on $\mathcal{H}_u$ such that:*

$$\|f\|_{L^\infty} + \|\nabla\!\!\!/ f\|_{\mathcal{P}^0} \lesssim \varepsilon.$$

Assume that μ_1 satisfies (5.93)*. Then, we have for all $j \geq 0$:*

$$\left\| P_j \left(\int_0^t (f\mu_1) d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \lesssim 2^j D\varepsilon^2 + 2^{\frac{j}{2}} D\varepsilon^2 \gamma(u).$$

Lemma 5.13 *Let a $P_{t,u}$-tangent 2-tensor F on $\mathcal{H}_u$ such that $\nabla\!\!\!/ F$ admits a decomposition of the form:*

$$\nabla\!\!\!/ F = \nabla\!\!\!/_{nL} P + E$$

where P, E are $P_{t,u}$-tangent tensors, and F, P, E satisfy:

$$\mathcal{N}_1(F) + \|F\|_{L^\infty_{x'} L^2_t} + \mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} \lesssim \varepsilon.$$

Assume that $\nabla\!\!\!/_{\underline{L}}(\zeta)$ satisfies the estimate (5.92)*. Then, we have for all $j \geq 0$:*

$$\left\| P_j \left(\int_0^t (F \cdot \nabla\!\!\!/ \nabla\!\!\!/_{\underline{L}}(\zeta)) d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \lesssim 2^j \varepsilon + 2^j D\varepsilon^2 + 2^{\frac{j}{2}} D\varepsilon^2 \gamma(u).$$

Lemma 5.14 *Let a $P_{t,u}$-tangent 1-form F on $\mathcal{H}_u$ such that:*

$$\|F\|_{\mathcal{P}^0} \lesssim \varepsilon.$$

Assume that $\nabla\!\!\!/_{\underline{L}}(\zeta)$ satisfies the estimate (5.92)*. Then, we have for all $j \geq 0$:*

$$\left\| P_j \left(\int_0^t (F \cdot \nabla\!\!\!/_{\underline{L}}(\zeta)) d\tau \right) \right\|_{L^\infty_t L^2_{x'}} \lesssim 2^j D\varepsilon^2 + 2^{\frac{j}{2}} D\varepsilon^2 \gamma(u).$$

We will also need the following three lemmas for the proof of Lemma 5.11, Lemma 5.12, 5.13 and 5.14, as well as various places in this paper.

Lemma 5.15 *Let f a scalar function on $\mathcal{H}_u$ and F a $P_{t,u}$-tangent 2-tensor. For any $j \geq 0$, we have:*

$$\left\| P_j \left(\int_0^t \not{\Delta}(f) d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \lesssim 2^{2j} \|f\|_{L^2_{x'} L^1_t},$$

and

$$\left\| P_j \left(\int_0^t d\!\!\!/ v d\!\!\!/ v(F) d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \lesssim 2^{2j} \|F\|_{L^2_{x'} L^1_t}.$$

Lemma 5.16 *Let F a $P_{t,u}$-tangent 1-form. For any $j \geq 0$ and any $1 < p \leq 2$, we have:*

$$\left\| P_j \left(\int_0^t d\!\!\!/ v(F) d\tau \right) \right\|_{L^\infty_t L^2_{x'}} \lesssim 2^{\frac{2j}{p}} \|F\|_{L^p_{x'} L^1_t}.$$

Lemma 5.17 *The following decomposition holds:*

$$\nabla\!\!\!/(n\rho) + (\nabla\!\!\!/(n\sigma))^* = {}^*\mathcal{D}_1 \cdot J \cdot {}^*\mathcal{D}_1^{-1}(\nabla\!\!\!/_{nL}(\underline{\beta})) + {}^*\mathcal{D}_1(H),$$

where J denotes the involution $(\rho, \sigma) \to (-\rho, \sigma)$ and H is a scalar function on $\mathcal{H}_u$ satisfying the following estimate:

$$\|H\|_{L^2_t L^3_{x'}} \lesssim \varepsilon.$$

We postpone the proof of Lemma 5.12 to section B.7, the proof of Lemma 5.13 to section B.8, the proof of Lemma 5.14 to section B.9, the proof of Lemma 5.15 to section B.10, the proof of Lemma 5.16 to section B.11, and the proof of Lemma 5.17 to section B.12. We show how they improve the estimate (5.93). We estimate each term in the right-hand side of (5.94) starting with the first one.

The scalar function $f = n\mathrm{tr}\chi$ satisfies the following estimate:

$$\begin{aligned} \|f\|_{L^\infty} + \|\nabla\!\!\!/ f\|_{\mathcal{P}^0} &\lesssim \|n\|_{L^\infty} \|\mathrm{tr}\chi\|_{L^\infty} + \|n\nabla\!\!\!/ \mathrm{tr}\chi\|_{\mathcal{P}^0} + \|\mathrm{tr}\chi \nabla\!\!\!/ n\|_{\mathcal{P}^0} \\ &\lesssim \varepsilon + (1 + \mathcal{N}_2(n-1)) \|\nabla\!\!\!/ \mathrm{tr}\chi\|_{\mathcal{P}^0} + \mathcal{N}_1(\mathrm{tr}\chi)\mathcal{N}_1(n-1) \\ &\lesssim \varepsilon, \end{aligned}$$

where we used the estimate (2.70) satisfied by $\mathrm{tr}\chi$, the estimate (2.67) satisfied by n, and the non sharp product estimates (5.14) and (5.15). Thus, in view of Lemma 5.12, we obtain:

$$\left\| P_j \left(\int_0^t (n\mathrm{tr}\chi\mu_1) d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \lesssim 2^j D\varepsilon^2 + 2^{\frac{j}{2}} D\varepsilon^2 \gamma(u). \tag{5.95}$$

We now focus on the third term in the right-hand side of (5.94). We define the 2-tensor $F = bn\widehat{\chi}$. In view of the decomposition (5.55) for $\nabla\!\!\!/(n\widehat{\chi})$, we have:

$$\begin{aligned} \nabla\!\!\!/ F &= b\nabla\!\!\!/(n\widehat{\chi}) + n\widehat{\chi}\nabla\!\!\!/ b \\ &= b(\nabla\!\!\!/_{nL} P + E) + n\widehat{\chi}\nabla\!\!\!/ b \\ &= \nabla\!\!\!/_{nL}(bP) - nL(b)P + bE + n\widehat{\chi}\nabla\!\!\!/ b \\ &= \nabla\!\!\!/_{nL}(bP) + nb\overline{\delta}P + bE + n\widehat{\chi}\nabla\!\!\!/ b \end{aligned}$$

where P and E satisfy:

$$\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} \lesssim \varepsilon.$$

Thus, we set $P_1 = bP$ and $E_1 = nb\overline{\delta}P + bE + n\widehat{\chi}\nabla\!\!\!/ b$ and obtain:

$$\nabla\!\!\!/ F = \nabla\!\!\!/_{nL}(P_1) + E_1.$$

Furthermore, we have:

$$\begin{aligned}
\mathcal{N}_1(P_1) + \|E_1\|_{\mathcal{P}^0} &\lesssim (\|b\|_{L^\infty} + \mathcal{N}_2(b-1))\mathcal{N}_1(P) + \mathcal{N}_2(b-1)\mathcal{N}_2(n-1)\mathcal{N}_1(\overline{\delta})\mathcal{N}_1(P) + \mathcal{N}_2(b-1)\|E\|_{\mathcal{P}^0} \\
&\quad + \mathcal{N}_2(n-1)\mathcal{N}_1(\widehat{\chi})\mathcal{N}_1(\nabla\!\!\!/ b) \\
&\lesssim \mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} + \varepsilon \\
&\lesssim \varepsilon,
\end{aligned}$$

where we used the estimate (2.71) satisfied by $\widehat{\chi}$, the estimate (2.67) satisfied by n, the estimate (2.69) satisfied by b, the estimate (2.68) satisfied by $\overline{\delta}$, and the non sharp product estimates (5.14) and (5.15). Thus, in view of Lemma 5.13, we obtain:

$$\left\| P_j \left(\int_0^t (bn\widehat{\chi} \cdot (\nabla\!\!\!/ \widehat{\otimes} \nabla\!\!\!/_{\underline{L}}(\zeta)) d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \lesssim 2^j D\varepsilon^2 + 2^{\frac{j}{2}} D\varepsilon^2 \gamma(u). \tag{5.96}$$

We consider the second, the fourth, the fifth and the sixth term in the right-hand side of (5.94). We define the 1-forms:

$$F_1 = bn\nabla\!\!\!/ \mathrm{tr}\chi, F_2 = n\widehat{\chi}\nabla\!\!\!/ b, F_3 = bn\widehat{\chi}\zeta \text{ and } F_4 = bn\mathrm{tr}\chi(2\epsilon - n^{-1}\nabla\!\!\!/ n).$$

These 1-forms satisfy the following estimate:

$$\begin{aligned}
&\|F_1\|_{\mathcal{P}^0} + \|F_2\|_{\mathcal{P}^0} + \|F_3\|_{\mathcal{P}^0} + \|F_4\|_{\mathcal{P}^0} \\
\lesssim\ & \mathcal{N}_2(n-1)\mathcal{N}_2(b-1)\|\nabla\!\!\!/ \mathrm{tr}\chi\|_{\mathcal{P}^0} + \mathcal{N}_2(n-1)\mathcal{N}_1(\widehat{\chi})\mathcal{N}_1(\nabla\!\!\!/ b) + \mathcal{N}_2(b-1)\mathcal{N}_2(n-1)\mathcal{N}_1(\widehat{\chi})\mathcal{N}_1(\zeta) \\
&+ \mathcal{N}_2(b-1)\mathcal{N}_2(n-1)\mathcal{N}_1(\mathrm{tr}\chi)(\mathcal{N}_1(\epsilon) + \mathcal{N}_1(n^{-1}\nabla\!\!\!/ n)) \\
\lesssim\ & \varepsilon,
\end{aligned}$$

where we used the estimate (2.70) satisfied by $\mathrm{tr}\chi$, the estimate (2.71) satisfied by $\widehat{\chi}$, the estimate (2.67) satisfied by n, the estimate (2.69) satisfied by b, the estimate (2.72) satisfied by ζ, and the non sharp product estimates (5.14) and (5.15). Thus, in view of Lemma 5.14, we obtain:

$$\begin{aligned}
&\left\| P_j \left(\int_0^t (bn\nabla\!\!\!/ _{\underline{L}}(\zeta) \cdot \nabla\!\!\!/ \mathrm{tr}\chi) d\tau \right) \right\|_{L^2(\mathcal{H}_u)} + \left\| P_j \left(\int_0^t (bn\widehat{\chi} \cdot (b^{-1}\nabla\!\!\!/ b \nabla\!\!\!/ _{\underline{L}}(\zeta)) d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \\
&+ \left\| P_j \left(\int_0^t (bn\widehat{\chi} \cdot (\nabla\!\!\!/ _{\underline{L}}\zeta \widehat{\otimes} \zeta \right) d\tau \right) \right\|_{L^2(\mathcal{H}_u)} + \left\| P_j \left(\int_0^t (n\mathrm{tr}\chi b(2\epsilon - n^{-1}\nabla\!\!\!/ n) \cdot \nabla\!\!\!/ _{\underline{L}}(\zeta)) d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \\
\lesssim\ & 2^j D\varepsilon^2 + 2^{\frac{j}{2}} D\varepsilon^2 \gamma(u).
\end{aligned} \tag{5.97}$$

We consider the seventh term in the right-hand side of (5.94). We define the scalar function w and the the $P_{t,u}$-tangent 1-form W as the solutions of the following transport equations:

$$nL(w) = \mathrm{div}\!\!\!/(nF_1),\ w = 0 \text{ on } P_{0,u}, \text{ and } \nabla\!\!\!/ _{nL}W - n\chi \cdot W = nF_1,\ W = 0 \text{ on } P_{0,u}.$$

We have:

$$\begin{aligned}\left\|P_j\left(\int_0^t(\not{\mathrm{div}}(nF_1))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} &= \|P_j w\|_{L^2(\mathcal{H}_u)} \\ &\lesssim \|P_j(w-\not{\mathrm{div}}(W))\|_{L^2(\mathcal{H}_u)}+\|P_j\not{\mathrm{div}}(W)\|_{L^2(\mathcal{H}_u)} \\ &\lesssim 2^j\|w-\not{\mathrm{div}}(W)\|_{L^2_tL^1_{x'}}+2^j\|W\|_{L^2(\mathcal{H}_u)}\end{aligned} \tag{5.98}$$

where we used the dual of the sharp Bernstein inequality (4.32) and the finite band property of the Littlewood-Paley projection P_j. We estimate the two terms in the right-hand side of (5.98). Using Lemma 5.2, we have:

$$\|w-\not{\mathrm{div}}(W)\|_{L^2_tL^1_{x'}}\lesssim\varepsilon\|nF_1\|_{L^2(\mathcal{H}_u)}\lesssim\varepsilon^2 \tag{5.99}$$

where we used the estimate (5.57) on F_1 and the L^∞ bound for n given by (2.67). Also, using the estimate (3.66) for transport equations, we have:

$$\begin{aligned}\|W\|_{L^2(\mathcal{H}_u)} &\lesssim \|\chi W\|_{L^2_{x'}L^1_t}+\|nF_1\|_{L^2(\mathcal{H}_u)} \\ &\lesssim \|\chi\|_{L^\infty_{x'}L^2_t}\|W\|_{L^2(\mathcal{H}_u)}+\varepsilon \\ &\lesssim \varepsilon\|W\|_{L^2(\mathcal{H}_u)}+\varepsilon,\end{aligned} \tag{5.100}$$

where we used the estimate (5.57) on F_1, the trace bound on χ given by (2.70) (2.71), and the L^∞ bound for n given by (2.67). (5.100) yields:

$$\|W\|_{L^2(\mathcal{H}_u)}\lesssim\varepsilon$$

which together with (5.98) and (5.99) implies:

$$\left\|P_j\left(\int_0^t(\not{\mathrm{div}}(nF_1))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}\lesssim 2^j\varepsilon. \tag{5.101}$$

Finally, we consider the last two terms in the right-hand side of (5.94). Using the dual of the sharp Bernstein inequality (4.32) and the estimate (3.66) for transport equations, we have:

$$\begin{aligned}&\left\|P_j\left(\int_0^t(\not{\nabla}nF_1))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}+\left\|P_j\left(\int_0^t(nf_2)d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \\ \lesssim\ & 2^j\left\|\int_0^t(\not{\nabla}nF_1))d\tau\right\|_{L^2_tL^1_{x'}}+2^j\left\|\int_0^t(nf_2)d\tau\right\|_{L^2_tL^1_{x'}} \\ \lesssim\ & 2^j\|\not{\nabla}nF_1\|_{L^1(\mathcal{H}_u)}+2^j\|nf_2\|_{L^1(\mathcal{H}_u)} \\ \lesssim\ & 2^j\|\not{\nabla}n\|_{L^2(\mathcal{H}_u)}\|F_1\|_{L^2(\mathcal{H}_u)}+2^j\|n\|_{L^\infty}\|f_2\|_{L^1(\mathcal{H}_u)} \\ \lesssim\ & 2^j\varepsilon,\end{aligned} \tag{5.102}$$

where we used the estimate (5.57) on F_1 and f_2, and the L^∞ bound for n given by (2.67).

Finally, (5.94)-(5.98), (5.101) and (5.102) yield:

$$\|P_j(\mu_1)\|_{L^2(\mathcal{H}_u)}\lesssim D\varepsilon^2 2^j+D\varepsilon^2 2^{\frac{j}{2}}\gamma(u)+2^j\varepsilon \tag{5.103}$$

which is an improvement of (5.93). (5.103) together with Lemma 5.7 with the choice $h = b^{-1}$ and the definition of μ_1 yields for all $j \geq 0$:

$$\|P_j(\underline{L}\mu)\|_{L^2(\mathcal{H}_u)} \lesssim 2^j D\varepsilon^2 + 2^{\frac{j}{2}} D\varepsilon^2\gamma(u) + 2^j\varepsilon$$

which in view of the definition of μ implies for all $j \geq 0$:

$$\|P_j(\underline{L}\underline{L}\mathrm{tr}\chi)\|_{L^2(\mathcal{H}_u)} \lesssim 2^j D\varepsilon^2 + 2^{\frac{j}{2}} D\varepsilon^2\gamma(u) + 2^j\varepsilon. \tag{5.104}$$

(5.92) and (5.104) improve the bootstrap assumptions (5.1) (5.2). Thus, there exists a universal constant $D > 0$ such that (5.1) (5.2) hold. This yields (2.73) (2.74).

5.5 Estimates for $\underline{L}\underline{L}b$

The goal of this section is to prove the estimate (2.75) for $\underline{L}\underline{L}b$ and to conclude the proof of Theorem 2.23.

5.5.1 Structure equation for $\underline{L}\underline{L}b$

The goal of this section is to prove the following proposition.

Proposition 5.18 *Let $b_1 = b\underline{L}\underline{L}b - b^2(\underline{L}(\delta) + \underline{L}(n^{-1}N(n)))$. Then, b_1 satisfies the following transport equation:*

$$L(b_1) = -(2b\nabla\!\!\!/ b + 2b^2(2\epsilon - n^{-1}\nabla\!\!\!/ n)) \cdot \nabla\!\!\!/_{\underline{L}}(\zeta) + b^2\widehat{\chi}\underline{\alpha} + d\!\!\!/iv(F_1) + f_2, \tag{5.105}$$

where the $P_{t,u}$-tangent vectorfield F_1 and the scalar function f_2 satisfy the estimates:

$$\|F_1\|_{L^2(\mathcal{H}_u)} + \|f_2\|_{L^1(\mathcal{H}_u)} \lesssim \varepsilon. \tag{5.106}$$

Proof We differentiate the transport equation (4.157) satisfied by $\underline{L}b$ with respect to $\underline{L}$. We obtain:

$$\begin{aligned} L(\underline{L}\underline{L}(b)) &= [L, \underline{L}](\underline{L}(b)) + \underline{L}(L(\underline{L}(b))) \\ &= \overline{\delta}\underline{L}\underline{L}(b) - (\delta + n^{-1}N(n))L(\underline{L}(b)) - 2(\zeta - \underline{\zeta})\nabla\!\!\!/\underline{L}(b) + (\underline{L}(\delta) + \underline{L}(n^{-1}N(n)))\overline{\delta}b \\ &\quad +(\delta + n^{-1}N(n))\underline{L}(\overline{\delta})b + (\delta + n^{-1}N(n))\overline{\delta}\underline{L}(b) - 2(\nabla\!\!\!/_{\underline{L}}(\zeta) - \nabla\!\!\!/_{\underline{L}}(\underline{\zeta})) \cdot \nabla\!\!\!/ b \\ &\quad -2(\zeta - \underline{\zeta}) \cdot \nabla\!\!\!/_{\underline{L}}\nabla\!\!\!/ b - \underline{L}(b)\underline{L}(\overline{\delta}) - b\underline{L}\underline{L}(\overline{\delta}), \end{aligned} \tag{5.107}$$

where we used in the last equality the commutator formula (2.46).

In view of (5.107), we need to compute $\underline{L}\underline{L}(\overline{\delta})$. Differentiating the formula (2.43) for $\underline{L}(\overline{\delta})$ with respect to $\underline{L}$, we obtain:

$$\begin{aligned} \underline{L}\underline{L}(\overline{\delta}) &= -L(\underline{L}(\delta) + \underline{L}(n^{-1}N(n))) - [\underline{L}, L](\delta + n^{-1}N(n)) + 2\underline{L}(\rho) \\ &\quad +4\epsilon \cdot \nabla\!\!\!/_{\underline{L}}(\epsilon) + 4\delta\underline{L}(\delta) + 2\nabla\!\!\!/_{\underline{L}}\zeta \cdot (2\epsilon - n^{-1}\nabla\!\!\!/ n) + 2\zeta \cdot (2\nabla\!\!\!/_{\underline{L}}\epsilon - \nabla\!\!\!/_{\underline{L}}(n^{-1}\nabla\!\!\!/ n)) \\ &\quad -2\nabla\!\!\!/_{\underline{L}}(n^{-1}\nabla\!\!\!/ n) \cdot \epsilon - 2n^{-1}\nabla\!\!\!/ n \cdot \nabla\!\!\!/_{\underline{L}}\epsilon - 4n^{-1}N(n)\underline{L}(n^{-1}N(n)) \\ &= -L(\underline{L}(\delta) + \underline{L}(n^{-1}N(n))) - [\underline{L}, L](\delta + n^{-1}N(n)) - 2d\!\!\!/iv(\underline{\beta}) - \widehat{\chi} \cdot \underline{\alpha} - 3\mathrm{tr}\underline{\chi}\rho \\ &\quad +4\underline{\xi} \cdot \beta + 2(\epsilon - 2\zeta) \cdot \underline{\beta} + 4\epsilon \cdot \nabla\!\!\!/_{\underline{L}}(\epsilon) + 4\delta\underline{L}(\delta) + 2\nabla\!\!\!/_{\underline{L}}\zeta \cdot (2\epsilon - n^{-1}\nabla\!\!\!/ n) \\ &\quad +2\zeta \cdot (2\nabla\!\!\!/_{\underline{L}}\epsilon - \nabla\!\!\!/_{\underline{L}}(n^{-1}\nabla\!\!\!/ n)) - 2\nabla\!\!\!/_{\underline{L}}(n^{-1}\nabla\!\!\!/ n) \cdot \epsilon - 2n^{-1}\nabla\!\!\!/ n \cdot \nabla\!\!\!/_{\underline{L}}\epsilon \\ &\quad -4n^{-1}N(n)\underline{L}(n^{-1}N(n)), \end{aligned} \tag{5.108}$$

where we used the Bianchi identity (2.54) for $\underline{L}(\rho)$ in the last equality. Now, (5.107), (5.108), the transport equation (2.27) satisfied by b, and the definition of b_1 yield:

$$\begin{aligned} L(b_1) &= bL(\underline{L}\underline{L}b) + L(b)\underline{L}\underline{L}b - b^2L(\underline{L}(\delta) + \underline{L}(n^{-1}N(n))) - 2bL(b)(\underline{L}(\delta) + \underline{L}(n^{-1}N(n))) \\ &= bL(\underline{L}\underline{L}b) - b\overline{\delta}\underline{L}\underline{L}b - bL(\underline{L}(\delta) + \underline{L}(n^{-1}N(n))) + 2b\overline{\delta}(\underline{L}(\delta) + \underline{L}(n^{-1}N(n))) \\ &= -(2b\nabla\!\!\!/ b + 2b^2(2\epsilon - n^{-1}\nabla\!\!\!/ n)) \cdot \nabla\!\!\!/_{\underline{L}}(\zeta) + b^2\widehat{\chi}\underline{\alpha} + \mathrm{div}\!\!\!/(F_1) + f_2, \end{aligned} \tag{5.109}$$

where the $P_{t,u}$-tangent vectorfield F_1 is given by:

$$F_1 = -4b(\zeta - \underline{\zeta})\underline{L}(b) + 2b^2\underline{\beta}, \tag{5.110}$$

and the scalar function f_2 is given by:

$$\begin{aligned} f_2 &= -b(\delta + n^{-1}N(n))L(\underline{L}(b)) + 4b(\nabla\!\!\!/\zeta - \nabla\!\!\!/\underline{\zeta})\underline{L}(b) + 4\nabla\!\!\!/ b \cdot (\zeta - \underline{\zeta})\underline{L}(b) \\ &\quad + b(\underline{L}(\delta) + \underline{L}(n^{-1}N(n)))\overline{\delta}b + b(\delta + n^{-1}N(n))\underline{L}(\overline{\delta})b + b(\delta + n^{-1}N(n))\overline{\delta}\underline{L}(b) \\ &\quad + 2b\nabla\!\!\!/_{\underline{L}}(\underline{\zeta}) \cdot \nabla\!\!\!/ b - 2b(\zeta - \underline{\zeta}) \cdot [\nabla\!\!\!/_{\underline{L}}, \nabla\!\!\!/](b) - b\underline{L}(b)\underline{L}(\overline{\delta}) - 2bL(b)(\underline{L}(\delta) + \underline{L}(n^{-1}N(n))) \\ &\quad + b^2[\underline{L}, L](\delta + n^{-1}N(n)) - 4b\nabla\!\!\!/(b) \cdot \underline{\beta} + 3b^2\mathrm{tr}\underline{\chi}\rho - 4b^2\underline{\xi} \cdot \beta - 2b^2(\epsilon - 2\zeta) \cdot \underline{\beta} - 4b^2\epsilon \cdot \nabla\!\!\!/_{\underline{L}}(\epsilon) \\ &\quad - 4b^2\delta\underline{L}(\delta) - 2b^2\zeta \cdot (2\nabla\!\!\!/_{\underline{L}}\epsilon - \nabla\!\!\!/_{\underline{L}}(n^{-1}\nabla\!\!\!/ n)) + 2b^2\nabla\!\!\!/_{\underline{L}}(n^{-1}\nabla\!\!\!/ n) \cdot \epsilon + 2b^2n^{-1}\nabla\!\!\!/ n \cdot \nabla\!\!\!/_{\underline{L}}\epsilon \\ &\quad + 4b^2n^{-1}N(n)\underline{L}(n^{-1}N(n)) + 2b\overline{\delta}(\underline{L}(\delta) + \underline{L}(n^{-1}N(n))). \end{aligned} \tag{5.111}$$

In view of the definition (5.110) of F_1, we have:

$$\begin{aligned} \|F_1\|_{L^2(\mathcal{H}_u)} &\lesssim \|b\|_{L^\infty}(\|\zeta\|_{L^\infty_{x'}L^2_t} + \|\underline{\zeta}\|_{L^\infty_{x'}L^2_t})\|\underline{L}(b)\|_{L^2_{x'}L^\infty_t} + \|b\|^2_{L^\infty}\|\underline{\beta}\|_{L^2(\mathcal{H}_u)} \\ &\lesssim \varepsilon, \end{aligned} \tag{5.112}$$

where we used in the last inequality the curvature bound (2.59) for $\underline{\beta}$, and the estimates (2.67)-(2.72) for $b, \underline{\zeta}$ and ζ.

Next, we estimate f_2. In view of (5.111), we have:

$$
\begin{aligned}
&\|f_2\|_{L^1(\mathcal{H}_u)} \qquad (5.113)\\
\lesssim\ & \|b\|_{L^\infty}\Big(\|\delta+n^{-1}N(n)\|_{L^2(\mathcal{H}_u)}\|L(\underline{L}(b))\|_{L^2(\mathcal{H}_u)}+(\|\nabla\!\!\!/\zeta-\nabla\!\!\!/\underline{\zeta}\|_{L^2(\mathcal{H}_u)}\\
&+\|b^{-1}\nabla\!\!\!/ b\|_{L^4(\mathcal{H}_u)}\|\zeta-\underline{\zeta}\|_{L^4(\mathcal{H}_u)})\|\underline{L}(b)\|_{L^2(\mathcal{H}_u)}+\|\nabla\!\!\!/_{\underline{L}}(\underline{\zeta})\|_{L^2(\mathcal{H}_u)}\|\nabla\!\!\!/ b\|_{L^2(\mathcal{H}_u)}\\
&+\|\zeta-\underline{\zeta}\|_{L^4(\mathcal{H}_u)}\|[\nabla\!\!\!/_{\underline{L}},\nabla\!\!\!/](b)\|_{L^2_tL^{\frac{4}{3}}_{x'}}+\|\underline{L}(b)\|_{L^2(\mathcal{H}_u)}\|\underline{L}(\overline{\delta})\|_{L^2(\mathcal{H}_u)}\\
&+\|L(b)\|_{L^2(\mathcal{H}_u)}\|\underline{L}(\delta)+\underline{L}(n^{-1}N(n))\|_{L^2(\mathcal{H}_u)}\Big)\\
&+\|b\|^2_{L^\infty}\Big(\|\underline{L}(\delta)+\underline{L}(n^{-1}N(n))\|_{L^2(\mathcal{H}_u)}\|\overline{\delta}\|_{L^2(\mathcal{H}_u)}+(\|\delta\|_{L^2(\mathcal{H}_u)}\\
&+\|n^{-1}N(n)\|_{L^2(\mathcal{H}_u)})\|\underline{L}(\overline{\delta})\|_{L^2(\mathcal{H}_u)}+(\|\delta\|_{L^4(\mathcal{H}_u)}+\|n^{-1}N(n)\|_{L^4(\mathcal{H}_u)})\|\overline{\delta}\|_{L^4(\mathcal{H}_u)}\\
&\times\|\underline{L}(b)\|_{L^2(\mathcal{H}_u)}+\|[\underline{L},L](\delta+n^{-1}N(n))\|_{L^1(\mathcal{H}_u)}+\|b^{-1}\nabla\!\!\!/(b)\|_{L^2(\mathcal{H}_u)}\|\underline{\beta}\|_{L^2(\mathcal{H}_u)}\\
&+\|\mathrm{tr}\underline{\chi}\|_{L^2(\mathcal{H}_u)}\|\rho\|_{L^2(\mathcal{H}_u)}+\|\underline{\xi}\|_{L^2(\mathcal{H}_u)}\|\beta\|_{L^2(\mathcal{H}_u)}+\|\epsilon-2\zeta\|_{L^2(\mathcal{H}_u)}\|\underline{\beta}\|_{L^2(\mathcal{H}_u)}\\
&+b\|\epsilon\|_{L^2(\mathcal{H}_u)}\|\nabla\!\!\!/_{\underline{L}}(\epsilon)\|_{L^2(\mathcal{H}_u)}+\|\delta\|_{L^2(\mathcal{H}_u)}\|\underline{L}(\delta)\|_{L^2(\mathcal{H}_u)}\\
&+\|\zeta\|_{L^2(\mathcal{H}_u)}\|2\nabla\!\!\!/_{\underline{L}}(\epsilon)-\nabla\!\!\!/_{\underline{L}}(n^{-1}\nabla\!\!\!/ n)\|_{L^2(\mathcal{H}_u)}+\|\nabla\!\!\!/_{\underline{L}}(\epsilon)\|_{L^2(\mathcal{H}_u)}\|n^{-1}\nabla\!\!\!/ n\|_{L^2(\mathcal{H}_u)}\\
&+\|\epsilon\|_{L^2(\mathcal{H}_u)}\|\nabla\!\!\!/_{\underline{L}}(n^{-1}\nabla\!\!\!/ n)\|_{L^2(\mathcal{H}_u)}+\|n^{-1}N(n)\|_{L^2(\mathcal{H}_u)}\|\underline{L}(n^{-1}N(n))\|_{L^2(\mathcal{H}_u)}\\
&+\|\overline{\delta}\|_{L^2(\mathcal{H}_u)}(\|\underline{L}(\delta)\|_{L^2(\mathcal{H}_u)}+\|\underline{L}(n^{-1}N(n))\|_{L^2(\mathcal{H}_u)})\Big)\\
\lesssim\ & \varepsilon+\varepsilon\|L(\underline{L}(b))\|_{L^2(\mathcal{H}_u)}+\varepsilon\|[\nabla\!\!\!/_{\underline{L}},\nabla\!\!\!/](b)\|_{L^2_tL^{\frac{4}{3}}_{x'}}+\|[\underline{L},L](\delta+n^{-1}N(n))\|_{L^1(\mathcal{H}_u)},
\end{aligned}
$$

where we used in the last inequality the curvature bound (2.59) for β, ρ and $\underline{\beta}$, and the estimates (2.67)-(2.72) for $n, b, \epsilon, \delta, \overline{\delta}, \underline{\zeta}, \underline{\xi}$ and ζ. Now, we evaluate the right-hand side of (5.113). Using the estimate (4.158) for $\|L(\underline{L}(b))\|_{L^2(\mathcal{H}_u)}$ and the commutator formulas (2.45) and (2.46), we have:

$$
\begin{aligned}
&\|L(\underline{L}(b))\|_{L^2(\mathcal{H}_u)}+\|[\nabla\!\!\!/_{\underline{L}},\nabla\!\!\!/](b)\|_{L^2_tL^{\frac{4}{3}}_{x'}}+\|[\underline{L},L](\delta+n^{-1}N(n))\|_{L^1(\mathcal{H}_u)}\\
\lesssim\ & \varepsilon+\|\widehat{\underline{\chi}}\|_{L^\infty_tL^4_{x'}}\|\nabla\!\!\!/ b\|_{L^2(\mathcal{H}_u)}+\|\underline{\xi}\|_{L^\infty_tL^4_{x'}}\|L(b)\|_{L^2(\mathcal{H}_u)}+\|b^{-1}\nabla\!\!\!/ b\|_{L^\infty_tL^4_{x'}}\|\underline{L}(b)\|_{L^2(\mathcal{H}_u)}\\
&+\|\overline{\delta}\|_{L^2(\mathcal{H}_u)}\|\underline{L}(\delta+n^{-1}N(n))\|_{L^2(\mathcal{H}_u)}+\|\delta+n^{-1}N(n)\|_{L^2(\mathcal{H}_u)}\|L(\delta+n^{-1}N(n))\|_{L^2(\mathcal{H}_u)}\\
&+\|\zeta-\underline{\zeta}\|_{L^2(\mathcal{H}_u)}\|\nabla\!\!\!/(\delta+n^{-1}N(n))\|_{L^2(\mathcal{H}_u)}\\
\lesssim\ & \varepsilon,
\end{aligned}
$$

where we used in the last inequality the estimates (2.67)-(2.72) for $n, b, \delta, \overline{\delta}, \zeta, \underline{\xi}, \widehat{\underline{\chi}}$ and ζ. Together with (5.113), this yields:

$$\|f_2\|_{L^1(\mathcal{H}_u)}\lesssim\varepsilon. \qquad (5.114)$$

In view of (5.109), (5.112) and (5.114), this concludes the proof of Proposition 5.18. ■

5.5.2 Estimates for $\underline{L}\underline{L}(b)$

After multiplying the transport equation (5.105) satisfied by b_1 by n, we have:

$$nL(b_1) = -(2nb\nabla\!\!\!/ b + 2nb^2(2\epsilon - n^{-1}\nabla\!\!\!/ n)) \cdot \nabla\!\!\!/_{\underline{L}}(\zeta) + nb^2\widehat{\chi}\underline{\alpha} + \mathrm{div}\!\!\!/(nF_1) - \nabla\!\!\!/ nF_1 + f_2,$$

which together with Lemma 5.11 yields:

$$\begin{aligned}
\|P_j(b_1)\|_{L^\infty_t L^2_{x'}} \lesssim\ & 2^{\frac{j}{2}}\gamma(u)\varepsilon + \left\| P_j\left(\int_0^t ((2nb\nabla\!\!\!/ b + 2nb^2(2\epsilon - n^{-1}\nabla\!\!\!/ n)) \cdot \nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^\infty_t L^2_{x'}} \\
& + \left\| P_j\left(\int_0^t (nb^2\widehat{\chi}\cdot\underline{\alpha})d\tau\right)\right\|_{L^\infty_t L^2_{x'}} + \left\| P_j\left(\int_0^t (\mathrm{div}\!\!\!/(nF_1))d\tau\right)\right\|_{L^\infty_t L^2_{x'}} \\
& + \left\| P_j\left(\int_0^t (\nabla\!\!\!/ nF_1)d\tau\right)\right\|_{L^\infty_t L^2_{x'}} + \left\| P_j\left(\int_0^t (nf_2)d\tau\right)\right\|_{L^\infty_t L^2_{x'}}. \qquad (5.115)
\end{aligned}$$

Next, we evaluate the right-hand side of (5.115). Using the nonsharp product estimates (5.14) and (5.15), we have:

$$\|2nb\nabla\!\!\!/ b + 2nb^2(2\epsilon - n^{-1}\nabla\!\!\!/ n)\|_{\mathcal{P}^0} \lesssim \mathcal{N}_2(b-1)\mathcal{N}_1(n)(\mathcal{N}_1(\nabla\!\!\!/ b) + \mathcal{N}_2(b-1)(\mathcal{N}_1(\epsilon) + \mathcal{N}_1(n^{-1}\nabla\!\!\!/ n))) \lesssim \varepsilon,$$

where we used in the last inequality the estimates (2.67)-(2.69) for n, ϵ and b. Together with Lemma 5.14, this yields the following estimate for the second term in the right-hand side of (5.115):

$$\left\| P_j\left(\int_0^t ((2nb\nabla\!\!\!/ b + 2nb^2(2\epsilon - n^{-1}\nabla\!\!\!/ n)) \cdot \nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^\infty_t L^2_{x'}} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u). \qquad (5.116)$$

Using Lemma 5.16 with $p = 2$, we have the following estimate for the second term in the right-hand side of (5.115):

$$\left\| P_j\left(\int_0^t (\mathrm{div}\!\!\!/(nF_1))d\tau\right)\right\|_{L^\infty_t L^2_{x'}} \lesssim 2^j\|nF_1\|_{L^2(\mathcal{H}_u)} \lesssim 2^j\|n\|_{L^\infty}\|F_1\|_{L^2(\mathcal{H}_u)} \lesssim 2^j\varepsilon, \qquad (5.117)$$

where we used in the last inequality the estimate (2.67) for n and the estimate (5.106) for F_1. Also, using the dual of the sharp Bernstein inequality for scalars (4.32) and the L^2 boundedness of P_j, and the estimate for transport equations (3.66), we have the following estimate for the remaining terms in the right-hand side of (5.115):

$$\begin{aligned}
& \left\| P_j\left(\int_0^t (nb^2\widehat{\chi}\cdot\underline{\alpha})d\tau\right)\right\|_{L^\infty_t L^2_{x'}} + \left\| P_j\left(\int_0^t (\nabla\!\!\!/ nF_1)d\tau\right)\right\|_{L^\infty_t L^2_{x'}} \\
& + \left\| P_j\left(\int_0^t (nf_2)d\tau\right)\right\|_{L^\infty_t L^2_{x'}} \\
\lesssim\ & 2^{\frac{j}{2}}\left\|nb^2\widehat{\chi}\cdot\underline{\alpha}\right\|_{L^2_{x'}L^1_t} + 2^j\left\|\nabla\!\!\!/ nF_1\right\|_{L^1(\mathcal{H}_u)} + 2^j\left\|nf_2\right\|_{L^1(\mathcal{H}_u)} \\
\lesssim\ & \|n\|_{L^\infty}\|b\|^2_{L^\infty}\|\widehat{\chi}\|_{L^\infty_{x'}L^2_t}\|\underline{\alpha}\|_{L^2(\mathcal{H}_u)} + 2^j\|\nabla\!\!\!/ n\|_{L^\infty_t L^2_{x'}}\|F_1\|_{L^2(\mathcal{H}_u)} + 2^j\|n\|_{L^\infty}\|f_2\|_{L^1(\mathcal{H}_u)} \\
\lesssim\ & 2^j\varepsilon + \varepsilon\gamma(u), \qquad (5.118)
\end{aligned}$$

where we used in the last inequality the curvature bound $\|\underline{\alpha}\|_{L^2(\mathcal{H}_u)} \lesssim \gamma(u)\varepsilon$ provided by (2.59), the estimates (2.67)-(2.71) for n, b and $\widehat{\chi}$, and the estimate (5.106) for F_1 and f_2.

Finally, in view of (5.115)-(5.118), we have:

$$\|P_j(b_1)\|_{L^\infty_t L^2_{x'}} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u). \tag{5.119}$$

Now, in view of the definition of b_1 in Proposition 5.18, we have:

$$\begin{aligned}\|P_j(b\underline{L}\underline{L}(b))\|_{L^\infty_t L^2_{x'}} &\lesssim \|P_j(b_1)\|_{L^\infty_t L^2_{x'}} + \|P_j(b^2(\underline{L}(\delta) + \underline{L}(n^{-1}N(n))))\|_{L^\infty_t L^2_{x'}} \\ &\lesssim \|P_j(b_1)\|_{L^\infty_t L^2_{x'}} + \|P_j(b^2(\underline{L}(\delta))\|_{L^\infty_t L^2_{x'}} + \|\underline{L}(n^{-1}N(n)))\|_{L^\infty_t L^2_{x'}} \\ &\lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u) + \|P_j(b^2(\underline{L}(\delta))\|_{L^\infty_t L^2_{x'}},\end{aligned} \tag{5.120}$$

where we used in the last inequality the estimate (2.67) for n, and the estimate (5.119) for b_1. Now, we have in view of (4.62) and (4.63):

$$\underline{L}(\delta) = \rho + \mathrm{div}\!\!\!/\epsilon + h, \tag{5.121}$$

where the scalar h is given by

$$h = -n^{-1}\nabla_N^2 n + \delta^2 - \zeta\underline{\zeta} + \zeta\epsilon - \underline{\zeta}\epsilon + \frac{3}{2}\delta\mathrm{tr}\theta - \widehat{\eta}\widehat{\theta} + 2b^{-1}\nabla\!\!\!/_A b\epsilon_A.$$

In view of the definition of h, we have

$$\begin{aligned}\|h\|_{L^\infty_t L^2_{x'}} &\lesssim \|n^{-1}\nabla_N^2 n\|_{L^\infty_t L^2_{x'}} + \Big(\|\delta\|_{L^\infty_t L^4_{x'}} + \|\zeta\|_{L^\infty_t L^4_{x'}} + \|\underline{\zeta}\|_{L^\infty_t L^4_{x'}} + \|\epsilon\|_{L^\infty_t L^4_{x'}} \\ &\quad + \|\theta\|_{L^\infty_t L^4_{x'}} + \|\widehat{\eta}\|_{L^\infty_t L^4_{x'}} + \|b^{-1}\nabla\!\!\!/ b\|_{L^\infty_t L^4_{x'}}\Big)^2 \\ &\lesssim \varepsilon,\end{aligned} \tag{5.122}$$

where we used in the last inequality the estimates (2.67)-(2.72) for $n, \delta, \zeta, \underline{\zeta}, \epsilon, \theta, \widehat{\eta}$ and b. Also, using the finite band property for P_j and the estimate (2.68) for ϵ, we have

$$\|P_j(\mathrm{div}\!\!\!/\epsilon)\|_{L^\infty_t L^2_{x'}} \lesssim 2^j\|\epsilon\|_{L^\infty_t L^2_{x'}} \lesssim 2^j\varepsilon. \tag{5.123}$$

We will obtain in Lemma 6.21 the following estimate for ρ

$$\|P_j\rho\|_{L^\infty_t L^2_{x'}} \lesssim 2^{\frac{j}{2}}\varepsilon. \tag{5.124}$$

Finally, (5.121)-(5.124) imply

$$\|P_j(\underline{L}(\delta))\|_{L^\infty_t L^2_{x'}} \lesssim 2^j\varepsilon. \tag{5.125}$$

Together with Lemma 5.7 with the choice $h = b^2$, this yields:

$$\|P_j(b^2\underline{L}(\delta))\|_{L^\infty_t L^2_{x'}} \lesssim 2^j\varepsilon.$$

Together with (5.120), we obtain

$$\|P_j(b\underline{L}\underline{L}(b))\|_{L^\infty_t L^2_{x'}} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u).$$

Together with Lemma 5.7 with the choice $h = b^{-1}$, this yields:

$$\|P_j(\underline{L}\underline{L}(b))\|_{L^\infty_t L^2_{x'}} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u),$$

which implies the estimate (2.75) for $\underline{L}\underline{L}(b)$. Together with the estimates (2.73) and (2.74) which were obtained in section 5.3 and section 5.4, this concludes the proof of Theorem 2.23.

6 First order derivatives with respect to ω

The goal of this section is to prove Theorem 2.24. In section 6.1, we derive commutator formulas involving ∂_ω. In section 6.2, we prove the estimates (2.76) and (2.77) for $\partial_\omega N$, $\partial_\omega b$, $\partial_\omega \chi$ and $\partial_\omega \zeta$. In section 6.3, we prove the estimate (2.78) for $\nabla\!\!\!/_{\underline{L}} \Pi(\partial_\omega \chi)$. In section 6.4, we derive the decomposition (2.79)-(2.82) for $\widehat{\chi}$. In section 6.5, we derive Besov improvements for $\partial_\omega N$ and $\partial_\omega \chi$. Finally, we prove the lower bound (2.83) for $N(\cdot,\omega) - N(\cdot,\omega')$ in section 6.6.

Remark 6.1 *Recall that not all quantities depend on ω. Indeed, only the quantities tied to $u = u(\cdot,\omega)$ depend on ω. More precisely, the following quantities depend on ω*

- *The optical function u.*
- *The null lapse b and the vectorfield N.*
- *The null frame (e_4, e_3, e_1, e_2).*
- *The Ricci coefficients χ, ζ.*

On the other hand, the curvature tensor $\mathbf{R}$, the vectorfield T, the time lapse n and the second fundamental for k are all independent of ω.

6.1 Commutator formulas

In this section, we derive several formulas involving commutators with ∂_ω. We start with some useful identities.

Lemma 6.2 *For any 1-form F, we have the following identity:*

$$F_{\partial_\omega e_A} e_A + F_A \partial_\omega e_A = -F_N \partial_\omega N - F_{\partial_\omega N} N. \tag{6.1}$$

For any symmetric 2-tensor H, we have the following identity:

$$H_{A\partial_\omega e_C} H_{CB} + H_{AC} H_{\partial_\omega e_C B} = -H_{AN} H_{\partial_\omega N B} - H_{A\partial_\omega N} H_{NB}. \tag{6.2}$$

For any 2-tensor H and any 1-form F, we have the following identity:

$$F_{\partial_\omega e_B} H_{BA} + F_B H_{\partial_\omega e_B A} = -F_N H_{\partial_\omega N A} - F_{\partial_\omega N} H_{NA}. \tag{6.3}$$

Proof The identities (6.1), (6.2) and (6.3) are easy consequences of the identities:

$$\mathbf{g}(\partial_\omega e_1, e_1) = 0,\ \mathbf{g}(\partial_\omega e_2, e_2) = 0,\ \mathbf{g}(\partial_\omega e_1, e_2) = -\mathbf{g}(\partial_\omega e_2, e_1),\ \mathbf{g}(\partial_\omega e_A, N) = -\mathbf{g}(\partial_\omega N, e_A), \tag{6.4}$$

which follow from the fact that (e_1, e_2, N) is orthonormal. ∎

We first consider commutators for scalar functions.

Proposition 6.3 *Let f a scalar function on $\mathcal{M}$. We have:*

$$[\partial_\omega, L]f = \partial_\omega N(f),\ [\partial_\omega, \underline{L}]f = -\partial_\omega N(f) \tag{6.5}$$

and:

$$[\partial_\omega, \not\nabla]f = -\not\nabla_{\partial_\omega N} f N - \nabla_N f \partial_\omega N. \tag{6.6}$$

Proof Differentiating $\mathbf{g}(T, N) = 0$ and $\mathbf{g}(N, N) = 1$, and using the fact that T is independent of ω, we obtain

$$\mathbf{g}(T, \partial_\omega N) = 0 \text{ and } \mathbf{g}(N, \partial_\omega N) = 0$$

which shows that $\partial_\omega N$ is tangent to $P_{t,u}$. Furthermore, since T is independent of ω, and since $L = T + N$ and $\underline{L} = T - N$, we have

$$\partial_\omega L = \partial_\omega N \text{ and } \partial_\omega \underline{L} = -\partial_\omega N, \tag{6.7}$$

which immediately yields (6.5). Furthermore, we have:

$$\not\nabla f = \mathbf{D}f + \frac{1}{2}\mathbf{g}(\mathbf{D}f, L)\underline{L} + \frac{1}{2}\mathbf{g}(\mathbf{D}f, \underline{L})L$$

where $\mathbf{D}f = -\mathbf{g}^{\alpha\beta}\partial_\alpha(f)\partial_\beta$ denotes the space-time gradient of f. Together with (6.5) and the fact that $[\partial_\omega, \mathbf{D}] = 0$, this implies (6.6). This concludes the proof of the proposition. ■

Next, we consider commutators for $P_{t,u}$-tangent vectorfields. Recall the definition of the projection Π of vectorfields on Σ_t onto vectorfields tangent to $P_{t,u}$:

$$\Pi X = X - g(X, N)N.$$

We have the following proposition:

Proposition 6.4 *Let X a $P_{t,u}$-tangent vectorfield. We have:*

$$\partial_\omega \not\nabla_L X - \not\nabla_L(\Pi(\partial_\omega X)) = \not\nabla_{\partial_\omega N} X - \mathbf{g}(\not\nabla_L(X), \partial_\omega N)N + \overline{\epsilon}_X \partial_\omega N - g(X, \partial_\omega N)\overline{\epsilon}_A e_A, \tag{6.8}$$

and:

$$\begin{aligned} \partial_\omega \not\nabla_{\underline{L}} X - \not\nabla_{\underline{L}}(\Pi(\partial_\omega X)) \;=\; & -\not\nabla_{\partial_\omega N} X - \mathbf{g}(\not\nabla_{\underline{L}}(X), \partial_\omega N)N + (\zeta_X - \underline{\xi}_X)\partial_\omega N \\ & -g(X, \partial_\omega N)(\zeta_A - \underline{\xi}_A)e_A. \end{aligned} \tag{6.9}$$

Proof We start with $\partial_\omega \not\nabla_L X - \not\nabla_L(\Pi(\partial_\omega X))$. By the definition of $\not\nabla_L$, we have:

$$\not\nabla_L(X) = \mathbf{g}(\mathbf{D}_L X, e_A)e_A.$$

Differentiating with respect to ω and using (6.7), we obtain:

$$\partial_\omega(\not\nabla_L(X)) = \mathbf{g}(\mathbf{D}_L(\partial_\omega X), e_A)e_A + \mathbf{g}(\mathbf{D}_{\partial_\omega N} X, e_A)e_A + \mathbf{g}(\mathbf{D}_L X, \partial_\omega e_A)e_A + \mathbf{g}(\mathbf{D}_L X, e_A)\partial_\omega e_A$$

which together with (6.1) yields:

$$\partial_\omega \nabla\!\!\!/_L X = \mathbf{g}(\mathbf{D}_L(\partial_\omega X), e_A)e_A + \nabla\!\!\!/_{\partial_\omega N} X - \mathbf{g}(\mathbf{D}_L X, N)\partial_\omega N - \mathbf{g}(\mathbf{D}_L X, \partial_\omega N)N. \quad (6.10)$$

Since X is tangent to $P_{t,u}$, we have:

$$\partial_\omega X = \Pi(\partial_\omega X) + g(\partial_\omega X, N)N = \Pi(\partial_\omega X) - g(X, \partial_\omega N)N \quad (6.11)$$

which yields:

$$\mathbf{g}(\mathbf{D}_L(\partial_\omega X), e_A)e_A = \mathbf{g}(\mathbf{D}_L(\Pi(\partial_\omega X)), e_A)e_A - g(X, \partial_\omega N)\mathbf{g}(\mathbf{D}_L N, e_A)e_A. \quad (6.12)$$

Now, using the Ricci equations (2.23) for $\mathbf{D}_L N$ and the fact that X is $P_{t,u}$-tangent, we have:

$$\mathbf{g}(\mathbf{D}_L N, e_A) = \overline{\epsilon}_A \text{ and } \mathbf{g}(\mathbf{D}_L X, N) = -\mathbf{g}(X, \mathbf{D}_L N) = -\overline{\epsilon}_X.$$

Together with (6.10) and (6.12), this yields (6.8).

Next, we consider $\partial_\omega \nabla\!\!\!/_{\underline{L}} X - \nabla\!\!\!/_{\underline{L}}(\Pi(\partial_\omega X))$. Similarly as before, we obtain the analog of (6.10):

$$\partial_\omega \nabla\!\!\!/_{\underline{L}} X = \mathbf{g}(\mathbf{D}_{\underline{L}}(\partial_\omega X), e_A)e_A - \nabla\!\!\!/_{\partial_\omega N} X - \mathbf{g}(\mathbf{D}_{\underline{L}} X, N)\partial_\omega N - \mathbf{g}(\mathbf{D}_{\underline{L}} X, \partial_\omega N)N. \quad (6.13)$$

and the analog of (6.12):

$$\mathbf{g}(\mathbf{D}_{\underline{L}}(\partial_\omega X), e_A)e_A = \mathbf{g}(\mathbf{D}_{\underline{L}}(\Pi(\partial_\omega X)), e_A)e_A - g(X, \partial_\omega N)\mathbf{g}(\mathbf{D}_{\underline{L}} N, e_A)e_A. \quad (6.14)$$

Now, using the Ricci equations (2.23) for $\mathbf{D}_{\underline{L}} N$ and the fact that X is $P_{t,u}$-tangent, we have:

$$\mathbf{g}(\mathbf{D}_{\underline{L}} N, e_A) = (\zeta_A - \underline{\xi}_A) \text{ and } \mathbf{g}(\mathbf{D}_{\underline{L}} X, N) = -\mathbf{g}(X, \mathbf{D}_{\underline{L}} N) = -(\zeta_X - \underline{\xi}_X).$$

Together with (6.13) and (6.14), this yields (6.9). This concludes the proof of the proposition. ∎

Next, we consider commutators for $P_{t,u}$-tangent tensors. Let F a m-covariant tensor tangent to the surfaces $P_{t,u}$. Then, $\partial_\omega F$ is not tangent to $P_{t,u}$. We denote by ΠG the $P_{t,u}$-tangent part of G. We have the following proposition:

Proposition 6.5 *Let $F_{\underline{A}}$ be an m-covariant tensor tangent to the surfaces $P_{t,u}$. Then,*

$$\partial_\omega \nabla\!\!\!/_L F_{\underline{A}} - \nabla\!\!\!/_L \Pi(\partial_\omega F)_{\underline{A}} = \nabla\!\!\!/_{\partial_\omega N} F_{\underline{A}} - \overline{\epsilon}_{A_i}(\partial_\omega N)_C F_{A_1..\check{C}..A_m} + g(e_{A_i}, \partial_\omega N)\overline{\epsilon}_C F_{A_1..\check{C}..A_m}, \quad (6.15)$$

and:

$$\begin{aligned} \partial_\omega \nabla\!\!\!/_{\underline{L}} F_{\underline{A}} - \nabla\!\!\!/_{\underline{L}} \Pi(\partial_\omega F)_{\underline{A}} \;=\; & -\nabla\!\!\!/_{\partial_\omega N} F_{\underline{A}} - (\zeta_{A_i} - \underline{\xi}_{A_i})(\partial_\omega N)_C F_{A_1..\check{C}..A_m} \\ & + g(e_{A_i}, \partial_\omega N)(\zeta_C - \underline{\xi}_C) F_{A_1..\check{C}..A_m}. \end{aligned} \quad (6.16)$$

Proof For simplicity give the proof for a $P_{t,u}$-tangent 1-form F, the general case being similar. We start with $\partial_\omega \nabla\!\!\!/_L F - \nabla\!\!\!/_L \Pi(\partial_\omega F)$. By definition, we have:

$$\nabla\!\!\!/_L F_A = L(F_A) - F_{\nabla\!\!\!/_L e_A}.$$

Differentiating with respect to ω and using (6.7), we obtain:

$$\begin{aligned}\partial_\omega(\nabla\!\!\!/_L F)_A + \nabla\!\!\!/_L F_{\partial_\omega e_A} &= \partial_\omega N(F_A) + L(\partial_\omega F_A) + L(F_{\partial_\omega e_A}) - \partial_\omega F_{\nabla\!\!\!/_L(e_A)} - F_{\partial_\omega(\nabla\!\!\!/_L e_A)} \\ &= \partial_\omega N(F_A) + \nabla\!\!\!/_L(\Pi(\partial_\omega F))_A + L(F_{\partial_\omega e_A}) - F_{\partial_\omega(\nabla\!\!\!/_L e_A)},\end{aligned}$$

which together with (6.11) with $X = e_A$, (6.8), and the fact that F and $\nabla\!\!\!/_L F$ are $P_{t,u}$-tangent yields:

$$\begin{aligned}\partial_\omega(\nabla\!\!\!/_L F)_A + \nabla\!\!\!/_L F_{\Pi(\partial_\omega e_A)} &= \partial_\omega N(F_A) + \nabla\!\!\!/_L(\Pi(\partial_\omega F))_A + L(F_{\Pi(\partial_\omega e_A)}) \\ &\quad - F_{\nabla\!\!\!/_L(\Pi(\partial_\omega e_A))} - F_{\nabla\!\!\!/_{\partial_\omega N} e_A} - \bar{\epsilon}_A F_{\partial_\omega N} + g(e_A, \partial_\omega N) F \cdot \bar{\epsilon} \\ &= \nabla\!\!\!/_{\partial_\omega N} F_A + \nabla\!\!\!/_L(\Pi(\partial_\omega F))_A + \nabla\!\!\!/_L F_{\Pi(\partial_\omega e_A)} \\ &\quad - \bar{\epsilon}_A F_{\partial_\omega N} + g(e_A, \partial_\omega N) F \cdot \bar{\epsilon}.\end{aligned}$$

This concludes the proof of (6.15). The proof of (6.16) is similar and left to the reader. ■

Finally, we consider the commutator of ∂_ω with $\mathcal{D}_2$.

Proposition 6.6 *Let H a symmetric $P_{t,u}$-tangent 2-tensor. Then, we have:*

$$\begin{aligned}\partial_\omega(div\!\!\!/(H))_A - div\!\!\!/(\Pi(\partial_\omega H))_A &= -\nabla\!\!\!/_N H_{A\partial_\omega N} + g(\partial_\omega N, e_A)\theta \cdot H \\ &\quad + \theta_{\partial_\omega N B} H_{BA} - \theta_{AB} H_{B\partial_\omega N} - tr\theta H_{A\partial_\omega N},\end{aligned} \tag{6.17}$$

where θ is the second fundamental form of $P_{t,u}$ in Σ_t (i.e. $\theta_{AB} = g(\nabla_A N, e_B)$).

Proof We first derive a formula for $\partial_\omega(\nabla\!\!\!/_B e_A)$. We have:

$$\begin{aligned}\partial_\omega(\nabla\!\!\!/_B e_A) &= \partial_\omega(g(\mathbf{D}_B e_A, e_C)e_C) \\ &= g(\mathbf{D}_{\partial_\omega e_B} e_A, e_C)e_C + g(\mathbf{D}_B(\partial_\omega e_A), e_C)e_C + g(\mathbf{D}_B e_A, \partial_\omega e_C)e_C \\ &\quad + g(\mathbf{D}_B e_A, e_C)\partial_\omega e_C.\end{aligned}$$

Now, using (6.11) to decompose $\partial_\omega e_A$, we have:

$$g(\mathbf{D}_B \partial_\omega e_A, e_C) = g(\mathbf{D}_B(\Pi(\partial_\omega e_A)), e_C) - g(\partial_\omega N, e_A) g(\mathbf{D}_B N, e_C).$$

Furthermore, the analog of (6.1) for 2-tensors yields:

$$g(\mathbf{D}_B e_A, \partial_\omega e_C)e_C + g(\mathbf{D}_B e_A, e_C)\partial_\omega e_C = -g(\partial_\omega N, e_C)(g(\mathbf{D}_B e_A, N)e_C + g(\mathbf{D}_B e_A, e_C)N).$$

Thus, we obtain:

$$\begin{aligned}\partial_\omega(\nabla\!\!\!/_B e_A) &= \nabla\!\!\!/_{\partial_\omega e_B}(e_A) + \nabla\!\!\!/_B(\Pi(\partial_\omega e_A)) - g(\partial_\omega N, e_A) g(\mathbf{D}_B N, e_C) e_C \\ &\quad - g(\partial_\omega N, e_C)(g(\mathbf{D}_B e_A, N)e_C + g(\mathbf{D}_B e_A, e_C)N) \\ &= \nabla\!\!\!/_{\partial_\omega e_B}(e_A) + \nabla\!\!\!/_B(\Pi(\partial_\omega e_A)) - g(\partial_\omega N, e_A)\theta_{BC} e_C + \theta_{AB}\partial_\omega N \\ &\quad - g(\mathbf{D}_B e_A, \partial_\omega N)N.\end{aligned} \tag{6.18}$$

We now compute $\partial_\omega(\nabla\!\!\!/_C H_{AB})$. We have:

$$\begin{aligned}\partial_\omega(\nabla\!\!\!/_C H_{AB}) &= \partial_\omega e_C(H_{AB}) + e_C(\partial_\omega H_{AB}) + e_C(H_{\partial_\omega e_A B}) + e_C(H_{A\partial_\omega e_B}) \\ &\quad -\partial_\omega H_{\nabla\!\!\!/_C e_A B} - \partial_\omega H_{A\nabla\!\!\!/_C e_B} - H_{\partial_\omega(\nabla\!\!\!/_C e_A)B} - H_{\nabla\!\!\!/_C e_A \partial_\omega(e_B)} \\ &\quad -H_{A\partial_\omega(\nabla\!\!\!/_C e_B)} - H_{\partial_\omega(e_A)\nabla\!\!\!/_C e_B} \\ &= \nabla\!\!\!/_C(\Pi(\partial_\omega H))_{AB} + \partial_\omega e_C(H_{AB}) + e_C(H_{\partial_\omega e_A B}) + e_C(H_{A\partial_\omega e_B}) \\ &\quad -H_{\partial_\omega(\nabla\!\!\!/_C e_A)B} - H_{\nabla\!\!\!/_C e_A \partial_\omega(e_B)} - H_{A\partial_\omega(\nabla\!\!\!/_C e_B)} - H_{\partial_\omega(e_A)\nabla\!\!\!/_C e_B}\end{aligned} \tag{6.19}$$

Using (6.18), we have:

$$\begin{aligned}H_{\partial_\omega(\nabla\!\!\!/_C e_A)B} + H_{A\partial_\omega(\nabla\!\!\!/_C e_B)} &= H_{\nabla\!\!\!/_{\partial_\omega e_C} e_A B} + H_{A\nabla\!\!\!/_{\partial_\omega e_C} e_B} + H_{\nabla\!\!\!/_C(\Pi(\partial_\omega e_A))B} \\ &\quad +H_{A\nabla\!\!\!/_C(\Pi(\partial_\omega e_B))} - g(\partial_\omega N, e_A)\theta_{CD}H_{DB} \\ &\quad -g(\partial_\omega N, e_B)\theta_{CD}H_{AD} + \theta_{AC}H_{\partial_\omega N B} + \theta_{BC}H_{A\partial_\omega N},\end{aligned}$$

which together with (6.19) yields:

$$\begin{aligned}\partial_\omega(\nabla\!\!\!/_C H_{AB}) &= \nabla\!\!\!/_C(\Pi(\partial_\omega H))_{AB} + \nabla\!\!\!/_{\partial_\omega e_C} H_{AB} + \nabla\!\!\!/_C H_{\Pi(\partial_\omega e_A)B} + \nabla\!\!\!/_C H_{A\Pi(\partial_\omega e_B)} \\ &\quad +g(\partial_\omega N, e_A)\theta_{CD}H_{DB} + g(\partial_\omega N, e_B)\theta_{CD}H_{AD} - \theta_{AC}H_{\partial_\omega N B} - \theta_{BC}H_{A\partial_\omega N}.\end{aligned} \tag{6.20}$$

Contracting (6.20), we obtain:

$$\begin{aligned}\partial_\omega(\mathrm{di}\!\!\!/\mathrm{v} H_A) &= \mathrm{di}\!\!\!/\mathrm{v}(\Pi(\partial_\omega H))_A + \nabla\!\!\!/_{\partial_\omega e_C} H_{AC} + \nabla\!\!\!/_C H_{A\partial_\omega e_C} + \nabla\!\!\!/_C H_{\partial_\omega e_A C} \\ &\quad +g(\partial_\omega N, e_A)\theta_{BC}H_{CB} + g(\partial_\omega N, e_C)\theta_{CB}H_{AB} - \theta_{AC}H_{\partial_\omega N C} - \mathrm{tr}\theta H_{A\partial_\omega N}.\end{aligned} \tag{6.21}$$

Now, the analog of (6.1)-(6.3) yields:

$$\nabla\!\!\!/_{\partial_\omega e_C} H_{AC} + \nabla\!\!\!/_C H_{A\partial_\omega e_C} = -\nabla\!\!\!/_N H_{A\partial_\omega N}$$

which together with (6.21) implies (6.17). This concludes the proof. ∎

6.2 Control of $\partial_\omega N$, $\partial_\omega b$, $\partial_\omega \chi$ and $\partial_\omega \zeta$

6.2.1 Derivatives of $\partial_\omega N$ with respect to the null frame

We first compute the derivatives of $\partial_\omega N$ with respect to the null frame.

Lemma 6.7

$$\mathbf{D}_L(\partial_\omega N) = -\chi_{\partial_\omega N B} e_B - \overline{\delta}\partial_\omega N - \overline{\epsilon}_{\partial_\omega N} L, \tag{6.22}$$

$$\begin{aligned}\mathbf{D}_{\underline{L}}(\partial_\omega N) &= 2\partial_\omega \zeta_A e_A + \chi_{\partial_\omega N B} e_B + (\delta + n^{-1}N(n))\partial_\omega N \\ &\quad +(\epsilon_{\partial_\omega N} + n^{-1}\nabla_{\partial_\omega N} n)L - 2\zeta_{\partial_\omega N} N,\end{aligned} \tag{6.23}$$

$$\mathbf{D}_A(\partial_\omega N) = \partial_\omega \chi_{AB} e_B - g(\partial_\omega N, e_A)\zeta_B e_B - \epsilon_A \partial_\omega N - \eta_{A\partial_\omega N} L - \chi_{A\partial_\omega N} N. \tag{6.24}$$

Proof We start with $\mathbf{D}_L(\partial_\omega N)$. Using the Ricci equation for $\mathbf{D}_L L$ and the fact that $\partial_\omega L = \partial_\omega N$, we have:

$$\mathbf{D}_L \partial_\omega N + \mathbf{D}_{\partial_\omega N} L = -\partial_\omega(\overline{\delta}) L - \overline{\delta}\partial_\omega N. \tag{6.25}$$

Now, we have:

$$\partial_\omega \delta = 2\epsilon_{\partial_\omega N},\ \partial_\omega \overline{\delta} = 2\epsilon_{\partial_\omega N} - n^{-1}\nabla_{\partial_\omega N} n. \tag{6.26}$$

Also, the Ricci equations (2.23) and the fact that $\partial_\omega N$ is $P_{t,u}$-tangent imply:

$$\mathbf{D}_{\partial_\omega N} L = \chi_{\partial_\omega N B} e_B - \epsilon_{\partial_\omega N} L$$

which together with (6.25) and (6.26) yield (6.22).

Next we consider $\mathbf{D}_{\underline{L}}(\partial_\omega N)$. Using the Ricci equation for $\mathbf{D}_L L$ and the fact that $\partial_\omega L = \partial_\omega N$ and $\partial_\omega \underline{L} = -\partial_\omega N$, we have:

$$\mathbf{D}_{\underline{L}}\partial_\omega N - \mathbf{D}_{\partial_\omega N} L = 2\partial_\omega \zeta_A e_A + 2\zeta_{\partial_\omega e_A} e_A + 2\zeta_A \partial_\omega e_A + (\partial_\omega \delta + n^{-1}\nabla_{\partial_\omega N} n) L + (\delta + n^{-1} N(n))\partial_\omega N,$$

which together with the Ricci equations (2.23), (6.1) and (6.26) yields (6.23).

Finally, we consider $\mathbf{D}_A(\partial_\omega N)$. Using the Ricci equation for $\mathbf{D}_A L$ and the fact that $\partial_\omega L = \partial_\omega N$, we have:

$$\mathbf{D}_A \partial_\omega N + \mathbf{D}_{\partial_\omega e_A} L = \partial_\omega \chi_{AB} e_B + \chi_{\partial_\omega e_A B} e_B + \chi_{A\partial_\omega e_B} e_B + \chi_{AB}\partial_\omega e_B - k_{\partial_\omega N A} L - k_{N\partial_\omega e_A} L - \epsilon_A \partial_\omega N.$$

Using (6.11) with $X = e_A$, we obtain:

$$\begin{aligned} \mathbf{D}_A \partial_\omega N - g(\partial_\omega N, e_A)\mathbf{D}_N L &= \partial_\omega \chi_{AB} e_B + \chi_{A\partial_\omega e_B} e_B + \chi_{AB}\partial_\omega e_B - k_{\partial_\omega N A} L \\ &\quad + g(\partial_\omega N, e_A)\delta L - \epsilon_A \partial_\omega N, \end{aligned}$$

which together with the Ricci equations (2.23) and (6.1) yields (6.24). This concludes the proof of the lemma. ∎

6.2.2 Transport equations for $\partial_\omega \chi$ and $\partial_\omega \zeta$

Lemma 6.8 *$\partial_\omega \chi$ and $\partial_\omega \zeta$ satisfy the following transport equations:*

$$\begin{aligned} \nabla\!\!\!/_L(\Pi(\partial_\omega \chi))_{AB} &= -\nabla\!\!\!/_{\partial_\omega N}\chi_{AB} - (\partial_\omega \chi)_{AC}\chi_{CB} - \chi_{AC}(\partial_\omega \chi)_{CB} - \overline{\delta}\partial_\omega \chi_{AB} \\ &\quad + \overline{\epsilon}_A \chi_{\partial_\omega N B} + \overline{\epsilon}_B \chi_{A\partial_\omega N} - (\partial_\omega N)_A \chi_{CB}\overline{\epsilon}_C - (\partial_\omega N)_B \chi_{AC}\overline{\epsilon}_C \\ &\quad - (2\epsilon_{\partial_\omega N} - n^{-1}\nabla_{\partial_\omega N} n)\chi_{AB} + (\partial_\omega N)_C(\in_{AC} {}^*\beta_B + \in_{BC} {}^*\beta_A), \end{aligned} \tag{6.27}$$

$$\begin{aligned} \nabla\!\!\!/_L(\Pi(\partial_\omega \zeta))_A &= -\nabla\!\!\!/_{\partial_\omega N}\zeta_A + \overline{\epsilon}_A \zeta_{\partial_\omega N} - (\partial_\omega N)_A \overline{\epsilon}\cdot\zeta - (k_{B\partial_\omega N} + \partial_\omega \zeta_B)\chi_{AB} \\ &\quad - (\overline{\epsilon}_B + \zeta_B)\partial_\omega \chi_{AB} - \frac{(\partial_\omega N)_B}{2}(-\alpha_{AB} + \rho\delta_{AB} + 3\sigma \in_{AB}). \end{aligned} \tag{6.28}$$

Proof We start with the proof of (6.27). Note first from the definition of α, β and the fact that $\partial_\omega L = \partial_\omega N$:

$$\partial_\omega \alpha_{AB} = -(\partial_\omega N)_C(\in_{AC} {}^*\beta_B + \in_{BC} {}^*\beta_A). \tag{6.29}$$

Now, differentiating the transport equation (2.31) with respect to ω, we obtain:

$$\begin{aligned}\partial_\omega(\nabla\!\!\!/_L\chi)_{AB} &= -(\partial_\omega\chi)_{AC}\chi_{CB} - \chi_{AC}(\partial_\omega\chi)_{CB} - \chi_{A\partial_\omega e_C}\chi_{CB} - \chi_{AC}\chi_{\partial_\omega e_C B}\\ &\quad -\partial_\omega(\overline{\delta})\chi_{AB} - \overline{\delta}\partial_\omega\chi_{AB} - \partial_\omega\alpha_{AB}.\end{aligned}$$

Together with (6.2), the commutator formula (6.15), (6.26) and (6.29), we obtain (6.27).

Next, we prove (6.28). Note first from the definition of $\alpha, \beta, \rho, \sigma$ and the fact that $\partial_\omega L = \partial_\omega N, \partial_\omega \underline{L} = -\partial_\omega N$:

$$\partial_\omega \beta_A = \frac{(\partial_\omega N)_B}{2}(-\alpha_{AB} + \rho\delta_{AB} + 3\sigma \in_{AB}). \tag{6.30}$$

Now, differentiating the transport equation (2.30) with respect to ω, we obtain:

$$\begin{aligned}\partial_\omega(\nabla\!\!\!/_L\zeta)_A &= -(\partial_\omega\overline{\epsilon}_B + \partial_\omega\zeta_B)\chi_{AB} - (\overline{\epsilon}_B + \zeta_B)\partial_\omega\chi_{AB} - (\overline{\epsilon}_{\partial_\omega e_B} + \zeta_{\partial_\omega e_B})\chi_{AB}\\ &\quad -(\overline{\epsilon}_B + \zeta_B)\chi_{A\partial_\omega e_B} - \partial_\omega\beta_A.\end{aligned}$$

Together with (6.3), the commutator formula (6.15), and (6.30), we obtain (6.28). This concludes the proof of the lemma. ∎

6.2.3 Estimates for $\partial_\omega N, \partial_\omega b, \partial_\omega\chi$ and $\partial_\omega\zeta$

We first derive the L^∞ bound (2.76) for $\partial_\omega N$. In view of the formula (6.22) for $\mathbf{D}_L(\partial_\omega N)$, we have:

$$\begin{aligned}\|\mathbf{D}_L(\partial_\omega N)\|_{L^\infty_{x'}L^2_t} &\lesssim \|-\chi_{\partial_\omega N B}e_B - \overline{\delta}\partial_\omega N + \overline{\epsilon}_{\partial_\omega N}L\|_{L^\infty_{x'}L^2_t}\\ &\lesssim (\|\chi\|_{L^\infty_{x'}L^2_t} + \|\overline{\delta}\|_{L^\infty_{x'}L^2_t} + \|\overline{\epsilon}\|_{L^\infty_{x'}L^2_t})\|\partial_\omega N\|_{L^\infty}\\ &\lesssim \varepsilon\|\partial_\omega N\|_{L^\infty},\end{aligned} \tag{6.31}$$

where we used the estimates (2.70) (2.71) for χ and the estimate (2.68) for $\overline{\delta}$ and $\overline{\epsilon}$ in the last inequality. The estimate for transport equations (3.66) and (6.31) yield:

$$\|\partial_\omega N\|_{L^\infty} \lesssim \|\partial_\omega N(0)\|_{L^\infty(P_{0,u})} + \|\nabla\!\!\!/_L\partial_\omega N\|_{L^\infty_{x'}L^2_t} \lesssim 1 + \varepsilon\|\partial_\omega N\|_{L^\infty}$$

where we used the control of the initial data of $\partial_\omega N$ provided by (2.64). This yields the L^∞ bound (2.76) for $\partial_\omega N$:

$$\|\partial_\omega N\|_{L^\infty} \lesssim 1. \tag{6.32}$$

Next, we derive an estimate for $\partial_\omega\chi$. First, the fact that χ is a $P_{t,u}$-tangent 2-tensor yields for any vectorfields X, Y on Σ_t:

$$\partial_\omega\chi_{XY} = \Pi(\partial_\omega\chi)_{\Pi(X)\Pi(Y)} - g(N,X)\chi_{\partial_\omega N\Pi(Y)} - g(N,Y)\chi_{\partial_\omega N\Pi(X)}$$

which implies:

$$\|\partial_\omega\chi\|_{L^2_{x'}L^\infty_t} \lesssim \|\Pi(\partial_\omega\chi)\|_{L^2_{x'}L^\infty_t} + \|\chi\|_{L^2_{x'}L^\infty_t}\|\partial_\omega N\|_{L^\infty} \lesssim \|\Pi(\partial_\omega\chi)\|_{L^2_{x'}L^\infty_t} + \varepsilon, \tag{6.33}$$

where we used the estimate (6.32) for $\partial_\omega N$, and the estimates (2.70) (2.71) for χ. In view of (6.33), we have to estimate $\|\Pi(\partial_\omega\chi)\|_{L^2_{x'}L^\infty_t}$. The formula (6.27) for $\mathbf{D}_L(\Pi(\partial_\omega\chi))$ implies:

$$\begin{aligned}
&\|\mathbf{D}_L(\Pi(\partial_\omega\chi))\|_{L^2(\mathcal{H}_u)} \qquad (6.34)\\
\lesssim\ & \|\nabla\!\!\!/_{\partial_\omega N}\chi\|_{L^2(\mathcal{H}_u)} + \|\partial_\omega(\chi)\chi\|_{L^2(\mathcal{H}_u)} + \|\overline{\delta}\partial_\omega\chi\|_{L^2(\mathcal{H}_u)} + \|\overline{\epsilon}\chi\partial_\omega N\|_{L^2(\mathcal{H}_u)} + \|\partial_\omega N\chi\epsilon\|_{L^2(\mathcal{H}_u)}\\
&+\|(2\epsilon_{\partial_\omega N} - n^{-1}\nabla_{\partial_\omega N}n)\chi\|_{L^2(\mathcal{H}_u)} + \|\partial_\omega N(\in_{AC} {}^*\beta_B + \in_{BC} {}^*\beta_A)\|_{L^2(\mathcal{H}_u)}\\
\lesssim\ & (1+\|\partial_\omega N\|_{L^\infty})\bigg(\|\nabla\!\!\!/\chi\|_{L^2(\mathcal{H}_u)} + \|\partial_\omega(\chi)\|_{L^2_{x'}L^\infty_t}(\|\chi\|_{L^\infty_{x'}L^2_t} + \|\overline{\delta}\|_{L^\infty_{x'}L^2_t})\\
&+\mathcal{N}_1(\chi)(\mathcal{N}_1(\epsilon)+\mathcal{N}_2(n-1)) + \|\beta\|_{L^2(\mathcal{H}_u)}\bigg)\\
\lesssim\ & \varepsilon + \varepsilon\|\partial_\omega(\chi)\|_{L^2_{x'}L^\infty_t},
\end{aligned}$$

where we used in the last inequality the estimate (6.32) for $\partial_\omega N$, the curvature bound (2.59) for β, and the estimates (2.67)-(2.71) for $n, \epsilon, \overline{\epsilon}, \overline{\delta}$ and χ. The estimate for transport equations (3.66) and (6.34) yield:

$$\|\Pi(\partial_\omega\chi)\|_{L^2_{x'}L^\infty_t} \lesssim \|\Pi(\partial_\omega\chi)(0)\|_{L^2(P_{0,u})} + \|\nabla\!\!\!/_L\Pi(\partial_\omega\chi))\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon + \varepsilon\|\partial_\omega(\chi)\|_{L^2_{x'}L^\infty_t}$$

where we used the control on $\partial_\omega\chi(0)$ provided by (2.62). Together with (6.33), this yields:

$$\|\partial_\omega\chi\|_{L^2_{x'}L^\infty_t} \lesssim \varepsilon. \tag{6.35}$$

We now derive an estimate for $\partial_\omega\zeta$. First, the fact that ζ is a $P_{t,u}$-tangent 1-form yields for any vectorfields X on Σ_t:

$$\partial_\omega\zeta_X = \Pi(\partial_\omega\zeta)_{\Pi(X)} - g(N,X)\zeta_{\partial_\omega N}$$

which implies:

$$\|\partial_\omega\zeta\|_{L^2_{x'}L^\infty_t} \lesssim \|\Pi(\partial_\omega\zeta)\|_{L^2_{x'}L^\infty_t} + \|\zeta\|_{L^2_{x'}L^\infty_t}\|\partial_\omega N\|_{L^\infty} \lesssim \|\Pi(\partial_\omega\zeta)\|_{L^2_{x'}L^\infty_t} + \varepsilon, \tag{6.36}$$

where we used the estimate (6.32) for $\partial_\omega N$, and the estimate (2.72) for ζ. In view of (6.36), we have to estimate $\|\Pi(\partial_\omega\zeta)\|_{L^2_{x'}L^\infty_t}$. The formula (6.28) for $\mathbf{D}_L(\Pi(\partial_\omega\zeta))$ implies:

$$\begin{aligned}
&\|\mathbf{D}_L(\Pi(\partial_\omega\zeta))\|_{L^2(\mathcal{H}_u)} \qquad (6.37)\\
\lesssim\ & \|\nabla\!\!\!/_{\partial_\omega N}\zeta\|_{L^2(\mathcal{H}_u)} + \|\overline{\epsilon}\zeta_{\partial_\omega N}\|_{L^2(\mathcal{H}_u)} + \|\partial_\omega N\overline{\epsilon}\cdot\zeta\|_{L^2(\mathcal{H}_u)} + \|(k_{B\partial_\omega N} + \partial_\omega\zeta_B)\chi\|_{L^2(\mathcal{H}_u)}\\
&+\|(\overline{\epsilon}+\zeta)\partial_\omega\chi\|_{L^2(\mathcal{H}_u)} + \|\partial_\omega N(-\alpha_{AB} + \rho\delta_{AB} + 3\sigma\in_{AB}\|_{L^2(\mathcal{H}_u)}\\
\lesssim\ & (1+\|\partial_\omega N\|_{L^\infty})\bigg(\|\nabla\!\!\!/\zeta\|_{L^2(\mathcal{H}_u)} + \mathcal{N}_1(\overline{\epsilon})\mathcal{N}_1(\zeta) + (\|k\|_{L^2_{x'}L^\infty_t} + \|\zeta\|_{L^2_{x'}L^\infty_t})\|\chi\|_{L^2_{x'}L^\infty_t}\\
&+(\|\overline{\epsilon}\|_{L^\infty_{x'}L^2_t} + \|\zeta\|_{L^\infty_{x'}L^2_t})\|\partial_\omega\chi\|_{L^2_{x'}L^\infty_t} + \|\alpha\|_{L^2(\mathcal{H}_u)} + \|\rho\|_{L^2(\mathcal{H}_u)} + \|\sigma\|_{L^2(\mathcal{H}_u)}\bigg)\\
\lesssim\ & \varepsilon + \varepsilon\|\partial_\omega(\zeta)\|_{L^2_{x'}L^\infty_t},
\end{aligned}$$

where we used in the last inequality the estimate (6.32) for $\partial_\omega N$, the curvature bound (2.59) for α, ρ and σ, and the estimates (2.67)-(2.72) for $\overline{\epsilon}, k, \chi$ and ζ. The estimate for transport equations (3.66) and (6.37) yield:

$$\|\Pi(\partial_\omega\zeta)\|_{L^2_{x'}L^\infty_t} \lesssim \|\nabla\!\!\!/_L \Pi(\partial_\omega\zeta))\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon + \varepsilon\|\partial_\omega(\zeta)\|_{L^2_{x'}L^\infty_t}$$

which together with (6.36) yields:

$$\|\partial_\omega\zeta\|_{L^2_{x'}L^\infty_t} \lesssim \varepsilon. \tag{6.38}$$

We now estimate $\partial_\omega b$. Differentiating the transport equation (2.27) for b with respect to ω and using the commutator formula (6.5), we obtain:

$$L(\partial_\omega b) = -\nabla\!\!\!/_{\partial_\omega N} b - \partial_\omega(b)\overline{\delta} - \partial_\omega(\overline{\delta})b = -\nabla\!\!\!/_{\partial_\omega N} b - \partial_\omega(b)\overline{\delta} - (2\epsilon_{\partial_\omega N} - n^{-1}\nabla_{\partial_\omega N} n)b,$$

where we used (6.26) in the last equality. Since, $\nabla\!\!\!/ b = b(\zeta - \epsilon)$ from (2.26), we obtain:

$$L(\partial_\omega b) = -b\zeta_{\partial_\omega N} - \partial_\omega(b)\overline{\delta} - \overline{\epsilon}_{\partial_\omega N} b. \tag{6.39}$$

This yields:

$$\begin{aligned} & \|L(\partial_\omega b)\|_{L^\infty_{x'}L^2_t} \qquad (6.40)\\ \lesssim\ & \|b\zeta_{\partial_\omega N}\|_{L^\infty_{x'}L^2_t} + \|\partial_\omega(b)\overline{\delta}\|_{L^\infty_{x'}L^2_t} + \|\overline{\epsilon}_{\partial_\omega N} b\|_{L^\infty_{x'}L^2_t} \\ \lesssim\ & (1 + \|\partial_\omega N\|_{L^\infty})\bigg(\|b\|_{L^\infty(\mathcal{H}_u)}(\|\zeta\|_{L^\infty_{x'}L^2_t} + \|\overline{\epsilon}\|_{L^\infty_{x'}L^2_t}) + \|\partial_\omega b\|_{L^\infty(\mathcal{H}_u)}\|\overline{\delta}\|_{L^\infty_{x'}L^2_t}\bigg) \\ \lesssim\ & \varepsilon + \varepsilon\|\partial_\omega(b)\|_{L^\infty(\mathcal{H}_u)}, \end{aligned}$$

where we used in the last inequality the estimate (6.32) for $\partial_\omega N$, and the estimates (2.67)-(2.69) for $n, \epsilon, \overline{\delta}$ and b. The estimate for transport equations (3.66) and (6.40) yield:

$$\|\partial_\omega b\|_{L^\infty(\mathcal{H}_u)} \lesssim \|\partial_\omega b(0)\|_{L^\infty(P_{u,0})} + \|L(\partial_\omega b)\|_{L^\infty_{x'}L^2_t} \lesssim \varepsilon + \varepsilon\|\partial_\omega b\|_{L^\infty(\mathcal{H}_u)}$$

where we used the control on $\partial_\omega b(0)$ provided by (2.62). This implies:

$$\|\partial_\omega b\|_{L^\infty(\mathcal{H}_u)} + \|L(\partial_\omega b)\|_{L^\infty_{x'}L^2_t} \lesssim \varepsilon. \tag{6.41}$$

Next, we estimate $\nabla\!\!\!/\partial_\omega b$. Recall from (2.26) that $\nabla\!\!\!/ b = b(\zeta - \epsilon)$. Differentiating with respect to ω and using the commutator formula (6.6), we obtain:

$$\nabla\!\!\!/_A \partial_\omega b = \nabla\!\!\!/_{\partial_\omega N} bN + \nabla_N b\partial_\omega N + \partial_\omega b(\zeta_A - \epsilon_A) + b(\partial_\omega\zeta_A - k_{\partial_\omega N A})$$

which yields the estimate:

$$\begin{aligned} \|\nabla\!\!\!/ b\|_{L^2_{x'}L^\infty_t} \lesssim\ & (1 + \|\partial_\omega N\|_{L^\infty})\bigg(\|\nabla b\|_{L^2_{x'}L^\infty_t} + \mathcal{N}_1(\partial_\omega b)(\mathcal{N}_1(\zeta) + \mathcal{N}_1(\underline{\zeta})) + \|\partial_\omega\zeta\|_{L^2_{x'}L^\infty_t} \\ & + \|k\|_{L^2_{x'}L^\infty_t}\bigg) \\ \lesssim\ & \varepsilon + \mathcal{N}_1(\partial_\omega b)\varepsilon, \end{aligned} \tag{6.42}$$

where we used in the last inequality the estimate (6.32) for $\partial_\omega N$, the estimates (2.67)-(2.72) for $k, b, \underline{\zeta}$ and ζ, and the estimate (6.38) for $\partial_\omega \zeta$. Now, (6.41) and (6.42) yield:

$$\|\nabla\!\!\!/ \partial_\omega b\|_{L^2_{x'} L^\infty_t} \lesssim \varepsilon. \tag{6.43}$$

Finally, we estimate $\mathbf{D}\partial_\omega N$. In view of (6.31) and (6.32), we have:

$$\|\mathbf{D}_L(\partial_\omega N)\|_{L^\infty_{x'} L^2_t} \lesssim \varepsilon. \tag{6.44}$$

Then, using the formula for $\mathbf{D}_{\underline{L}}\partial_\omega N$ and $\mathbf{D}_A \partial_\omega N$ given respectively by (6.23) and (6.24), we obtain:

$$\begin{aligned} & \|\mathbf{D}_{\underline{L}}(\partial_\omega N)\|_{L^2_{x'} L^\infty_t} + \|\mathbf{D}_A(\partial_\omega N)\|_{L^2_{x'} L^\infty_t} \\ \lesssim\ & \|\partial_\omega \zeta\|_{L^2_{x'} L^\infty_t} + \|\partial_\omega \chi\|_{L^2_{x'} L^\infty_t} + \|\partial_\omega N\|_{L^\infty} \Big(\|\chi\|_{L^2_{x'} L^\infty_t} + \|\delta\|_{L^2_{x'} L^\infty_t} \\ & + \|n^{-1}\nabla n\|_{L^2_{x'} L^\infty_t} + \|\epsilon\|_{L^2_{x'} L^\infty_t} + \|\eta\|_{L^2_{x'} L^\infty_t} + \|\zeta\|_{L^2_{x'} L^\infty_t} \Big) \\ \lesssim\ & \varepsilon, \end{aligned} \tag{6.45}$$

where we used the estimate (6.32) for $\partial_\omega N$, the estimates (2.67)-(2.72) for $n, \delta, \epsilon, \eta, \chi$ and ζ, the estimate (6.35) for $\partial_\omega \chi$, and the estimate (6.38) for $\partial_\omega \zeta$.

Finally, (6.32) yields the desired L^∞ bound (2.76) for $\partial_\omega N$, while (6.35), (6.38), (6.41), (6.43), (6.44) and (6.45) yields the desired estimate (2.77).

6.3 Control of $\nabla\!\!\!/_{\underline{L}} \Pi(\partial_\omega \chi)$

The goal of this section is to prove the estimate (2.78) for $\nabla\!\!\!/_{\underline{L}} \Pi(\partial_\omega \chi)$. We will use the following lemmas.

Lemma 6.9 *$\nabla\!\!\!/_{\underline{L}} \Pi(\partial_\omega \chi)$ satisfies the following transport equation*

$$\nabla\!\!\!/_L(\nabla\!\!\!/_{\underline{L}} \Pi(\partial_\omega \chi)) = -\nabla\!\!\!/_{\underline{L}} \Pi(\partial_\omega \chi) \cdot \chi - \chi \cdot \nabla\!\!\!/_{\underline{L}} \Pi(\partial_\omega \chi) + \nabla\!\!\!/ F_1 + F_2, \tag{6.46}$$

where F_1 and F_2 are $P_{t,u}$-tangent tensors satisfying the following estimate

$$\|F_1\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|F_2\|_{L^2_t L^1_{x'}} \lesssim \varepsilon. \tag{6.47}$$

Lemma 6.10 *Recall that γ denotes the metric induced by $\mathbf{g}$ on $P_{t,u}$. Let M the $P_{t,u}$-tangent 2-tensor defined as the solution of the following transport equation:*

$$\nabla\!\!\!/_L M_{AB} = M_{AC}\chi_{CB},\ M_{AB} = \gamma_{AB} \text{ on } P_{0,u}, \tag{6.48}$$

Then, M_{AB} satisfies the following estimate:

$$\|M - \gamma\|_{L^\infty} + \|\nabla\!\!\!/ M\|_{\mathcal{B}^0} \lesssim \varepsilon. \tag{6.49}$$

Lemma 6.11 *Recall that γ denotes the metric induced by $\mathbf{g}$ on $P_{t,u}$. Let $\widetilde{M}$ the $P_{t,u}$-tangent 2-tensor defined as the solution of the following transport equation:*

$$\nabla\!\!\!/_L \widetilde{M}_{AB} = \chi_{AC}\widetilde{M}_{CB},\ \widetilde{M}_{AB} = \gamma_{AB} \text{ on } P_{0,u}, \tag{6.50}$$

Then, $\widetilde{M}_{AB}$ satisfies the following estimate:

$$\|\widetilde{M} - \gamma\|_{L^\infty} + \|\nabla\!\!\!/\widetilde{M}\|_{\mathcal{B}^0} \lesssim \varepsilon. \tag{6.51}$$

Lemma 6.12 *Let F a $P_{t,u}$-tangent tensor. Then, for any $1 \leq p < q \leq +\infty$ and for any $j \geq 0$, we have:*

$$\|P_j F\|_{L^p_t L^\infty_{x'}} \lesssim 2^j \|F\|_{L^q_t L^2_{x'}}. \tag{6.52}$$

Also, taking the dual, for any $1 \leq p < q \leq +\infty$ and for any $j \geq 0$, we have

$$\|P_j F\|_{L^p_t L^2_{x'}} \lesssim 2^j \|F\|_{L^q_t L^1_{x'}}. \tag{6.53}$$

Lemma 6.13 *Let F a $P_{t,u}$-tangent tensor. Then, for any $j \geq 0$ and for any $2 \leq p < +\infty$, we have:*

$$\left\| P_j \left(\int_0^t \nabla\!\!\!/(F) dt \right) \right\|_{L^p_t L^2_{x'}} \lesssim 2^j \|F\|_{L^1_t L^2_{x'}}.$$

Lemma 6.14 *Let F a $P_{t,u}$-tangent 1-form and $2 < p \leq +\infty$ such that for all $j \geq 0$:*

$$\|P_j F\|_{L^p_t L^2_{x'}} \lesssim 2^j \varepsilon + 2^{\frac{j}{2}} \varepsilon \gamma(u),$$

and let M such that:

$$\|M - \gamma\|_{L^\infty} + \|\nabla\!\!\!/ M\|_{\mathcal{B}^0} \lesssim \varepsilon.$$

Then, we have for any $2 \leq q < p$ and all $j \geq 0$:

$$\|P_j(M^{-1}F)\|_{L^q_t L^2_{x'}} \lesssim 2^j \varepsilon + 2^{\frac{j}{2}} \varepsilon \gamma(u).$$

Lemma 6.15 *Let F a $P_{t,u}$-tangent 1-form and $2 \leq p \leq +\infty$ such that for all $j \geq 0$:*

$$\|P_j F\|_{L^p_t L^2_{x'}} \lesssim 2^j \varepsilon + 2^{\frac{j}{2}} \varepsilon \gamma(u),$$

and let $\widetilde{M}$ such that:

$$\|\widetilde{M} - \gamma\|_{L^\infty} + \|\nabla\!\!\!/\widetilde{M}\|_{\mathcal{B}^0} \lesssim \varepsilon.$$

Then, we have for any $2 \leq q < p$ and all $j \geq 0$:

$$\|P_j(F\widetilde{M}^{-1})\|_{L^q_t L^2_{x'}} \lesssim 2^j \varepsilon + 2^{\frac{j}{2}} \varepsilon \gamma(u).$$

The proof of Lemma 6.9 is postponed to section C.1. The proof of Lemma 6.10 is postponed to section C.2. The proof of Lemma 6.11 is completely analogous to the one of Lemma 6.10 and left to the reader. The proof of Lemma 6.12 is postponed to section C.3. The proof of Lemma 6.13 is postponed to section C.4. The proof of Lemma 6.14 is postponed to section C.5. Finally, The proof of Lemma 6.15 is completely analogous to the one of Lemma 6.14, and left to the reader. We are now in position to derive the estimate for $\not\nabla_{\underline{L}}\Pi(\partial_\omega\chi)$. Using the transport equation (6.46) for $\not\nabla_{\underline{L}}\Pi(\partial_\omega\chi)$, the transport equation (6.48) for M and the transport equation (6.50) for $\widetilde{M}$ allows us to get rid of the first two terms in the left-hand side of (6.46):

$$\begin{aligned}
& \not\nabla_L(M\cdot\not\nabla_{\underline{L}}\Pi(\partial_\omega\chi)\cdot\widetilde{M}) \\
= \;& \not\nabla_L(M)\cdot\not\nabla_{\underline{L}}\Pi(\partial_\omega\chi)\cdot\widetilde{M} + M\cdot\not\nabla_L(\not\nabla_{\underline{L}}\Pi(\partial_\omega\chi))\cdot\widetilde{M} + M\cdot\not\nabla_{\underline{L}}\Pi(\partial_\omega\chi)\cdot\not\nabla_L(\widetilde{M}) \\
= \;& M\cdot(\not\nabla F_1 + F_2)\cdot\widetilde{M} \\
= \;& \not\nabla(M\cdot F_1\cdot\widetilde{M}) - \not\nabla(M)\cdot F_1\cdot\widetilde{M} - M\cdot F_1\cdot\not\nabla(\widetilde{M}) + M\cdot F_2\cdot\widetilde{M}.
\end{aligned}$$

Let $2\leq p<q<+\infty$. This yields:

$$\begin{aligned}
& \|P_j(M\cdot\not\nabla_{\underline{L}}\Pi(\partial_\omega\chi)\cdot\widetilde{M})\|_{L^q_tL^2_{x'}} \qquad (6.54)\\
\lesssim \;& 2^{\frac{j}{2}}\gamma(u) + \left\|P_j\left(\int_0^t\not\nabla(M\cdot F_1\cdot\widetilde{M})\right)\right\|_{L^q_tL^2_{x'}} + \left\|P_j\left(\int_0^t\not\nabla(M)\cdot F_1\cdot\widetilde{M}dt\right)\right\|_{L^q_tL^2_{x'}} \\
& + \left\|P_j\left(\int_0^t M\cdot F_1\cdot\not\nabla(\widetilde{M})dt\right)\right\|_{L^q_tL^2_{x'}} + \left\|P_j\left(\int_0^t M\cdot F_2\cdot\widetilde{M}dt\right)\right\|_{L^q_tL^2_{x'}},
\end{aligned}$$

where the term $2^{\frac{j}{2}}\gamma(u)$ comes from the initial data term at $t=0$. Next, we estimate the various terms in the right-hand side of (6.54).

We consider the first term in the right-hand side of (6.54). Using Lemma 6.13, we have:

$$\begin{aligned}
\left\|P_j\left(\int_0^t\not\nabla(M\cdot F_1\cdot\widetilde{M})dt\right)\right\|_{L^q_tL^2_{x'}} &\lesssim 2^j\|M\cdot F_1\cdot\widetilde{M}\|_{L^2(\mathcal{H}_u)} \qquad (6.55)\\
&\lesssim 2^j\|M\|_{L^\infty}\|F_1\|_{L^2(\mathcal{H}_u)}\|\widetilde{M}\|_{L^\infty}\\
&\lesssim 2^j\varepsilon,
\end{aligned}$$

where we used in the last inequality the estimate (6.47) for F_1, the estimate (6.49) for M, and the estimate (6.51) for $\widetilde{M}$.

Next, we consider the last three terms in the right-hand side of (6.54). Using the dual sharp Bernstein inequality for tensors (6.53) and the estimate (3.66) for transport

equations, we have:

$$\begin{aligned}
&\left\|P_j\left(\int_0^t \nabla\!\!\!/(M)\cdot F_1\cdot\widetilde{M}dt\right)\right\|_{L^q_tL^2_{x'}}+\left\|P_j\left(\int_0^t M\cdot F_1\cdot\nabla\!\!\!/(\widetilde{M})dt\right)\right\|_{L^q_tL^2_{x'}} \qquad (6.56)\\
&+\left\|P_j\left(\int_0^t M\cdot F_2\cdot\widetilde{M}dt\right)\right\|_{L^q_tL^2_{x'}}\\
\lesssim\ & 2^j\left\|P_j\left(\int_0^t \nabla\!\!\!/(M)\cdot F_1\cdot\widetilde{M}dt\right)\right\|_{L^\infty_tL^1_{x'}}+2^j\left\|P_j\left(\int_0^t M\cdot F_1\cdot\nabla\!\!\!/(\widetilde{M})dt\right)\right\|_{L^\infty_tL^1_{x'}}\\
&+2^j\left\|P_j\left(\int_0^t M\cdot F_2\cdot\widetilde{M}dt\right)\right\|_{L^\infty_tL^1_{x'}}\\
\lesssim\ & 2^j\|\nabla\!\!\!/(M)\cdot F_1\cdot\widetilde{M}\|_{L^1(\mathcal{H}_u)}+2^j\|M\cdot F_1\cdot\nabla\!\!\!/(\widetilde{M})\|_{L^1(\mathcal{H}_u)}+2^j\|M\cdot F_2\cdot\widetilde{M}\|_{L^1(\mathcal{H}_u)}\\
\lesssim\ & 2^j\|\nabla\!\!\!/(M)\|_{L^2(\mathcal{H}_u)}\|F_1\|_{L^2(\mathcal{H}_u)}\|\widetilde{M}\|_{L^\infty}+2^j\|M\|_{L^\infty}\|F_1\|_{L^2(\mathcal{H}_u)}\|\nabla\!\!\!/\widetilde{M}\|_{L^2(\mathcal{H}_u)}\\
&+2^j\|M\|_{L^\infty}\|F_2\|_{L^1(\mathcal{H}_u)}\|\widetilde{M}\|_{L^\infty}\\
\lesssim\ & 2^j\varepsilon,
\end{aligned}$$

where we used in the last inequality the estimate (6.47) for F_1 and F_2, the estimate (6.49) for M, and the estimate (6.51) for $\widetilde{M}$. Finally, (6.54), (6.55) and (6.56) imply

$$\|P_j(M\cdot\nabla\!\!\!/_{\underline{L}}\Pi(\partial_\omega\chi)\cdot\widetilde{M})\|_{L^q_tL^2_{x'}}\lesssim 2^j\varepsilon+2^{\frac{j}{2}}\gamma(u). \qquad (6.57)$$

Now, since we have chosen $p<q$, (6.57) together with Lemma 6.14 and Lemma 6.15 yields:

$$\|P_j\nabla\!\!\!/_{\underline{L}}\Pi(\partial_\omega\chi)\|_{L^p_tL^2_{x'}}\lesssim 2^j\varepsilon+2^{\frac{j}{2}}\gamma(u),$$

for any $2\leq p<+\infty$ which is the desired estimate (2.78) for $\nabla\!\!\!/_{\underline{L}}\Pi(\partial_\omega\chi)$.

6.4 Proof of the decomposition (2.79) for $\widehat{\chi}$

To conclude the proof of Theorem 2.24, we still need to prove the existence of a decomposition (2.79) for $\widehat{\chi}$. In view of the Codazzi-type equation (2.32) for $\widehat{\chi}$, we have:

$$\widehat{\chi}=\mathcal{D}_2^{-1}\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi-\beta\right),$$

and we choose the following decomposition:

$$\widehat{\chi}=\chi_1+\chi_2 \text{ where } \chi_1=\mathcal{D}_2^{-1}\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right) \text{ and } \chi_2=-\mathcal{D}_2^{-1}\beta. \qquad (6.58)$$

Note that, by definition of $\mathcal{D}_2^{-1}$, χ_1 and χ_2 are both traceless symmetric $P_{t,u}$ tangent 2-tensors.

6.4.1 Estimates for χ_1

Estimate for $\|\nabla\!\!\!/\chi_1\|_{L^2(\mathcal{H}_u)}$. We start by estimating $\nabla\!\!\!/\chi_1$. Using the estimate (3.50) satisfied by $\mathcal{D}_2^{-1}$ and the definition (6.58) of χ_1, we have:

$$\begin{aligned}\|\nabla\!\!\!/\chi_1\|_{L^\infty_t L^2_{x'}} &\lesssim \left\|\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi - \epsilon\cdot\chi\right\|_{L^\infty_t L^2_{x'}} \\ &\lesssim \|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^\infty_t L^2_{x'}} + \mathcal{N}_1(\epsilon)\mathcal{N}_1(\chi) \\ &\lesssim \varepsilon,\end{aligned} \tag{6.59}$$

where we used the estimate (2.68) for ϵ, and the estimates (2.70) (2.71) for χ in the last inequality.

Estimate for $\|\nabla\!\!\!/_L\chi_1\|_{L^\infty_t L^2_{x'}+L^2_t L^q_{x'}}$. Next, we estimate $\nabla\!\!\!/_L\chi_1$ and $\nabla\!\!\!/_{\underline{L}}\chi_1$. Note first that for any vectorfield X on $\mathcal{M}$, we have:

$$[\nabla\!\!\!/_X, \mathcal{D}_2^{-1}] = -\mathcal{D}_2^{-1}[\nabla\!\!\!/_X, \mathcal{D}_2]\mathcal{D}_2^{-1}$$

which together with the definition of χ_1 implies:

$$\nabla\!\!\!/_X\chi_1 = \mathcal{D}_2^{-1}\left(\nabla\!\!\!/_X\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi - \epsilon\cdot\chi\right)\right) - \mathcal{D}_2^{-1}[\nabla\!\!\!/_X, \mathcal{D}_2]\chi_1. \tag{6.60}$$

Let $2\leq q<4$. Applying (6.60) with $X = nL$, we obtain:

$$\begin{aligned}&\|\nabla\!\!\!/_{nL}\chi_1\|_{L^\infty_t L^2_{x'}+L^2_t L^q_{x'}} \\ \lesssim\ & \|\mathcal{D}_2^{-1}\left(\nabla\!\!\!/_{nL}\nabla\!\!\!/\mathrm{tr}\chi\right)\|_{L^\infty_t L^2_{x'}} + \|\mathcal{D}_2^{-1}\left(\nabla\!\!\!/_{nL}\left(\epsilon\cdot\chi\right)\right)\|_{L^2_t L^q_{x'}} + \|\mathcal{D}_2^{-1}[\nabla\!\!\!/_{nL}, \mathcal{D}_2]\chi_1\|_{L^2_t L^q_{x'}}.\end{aligned} \tag{6.61}$$

We estimate the three terms in the right-hand side of (6.61) starting with the first one. Using the commutator formula (2.48) for $[\nabla\!\!\!/_{nL}, \nabla\!\!\!/]\mathrm{tr}\chi$, and Remark 3.15 and the dual of (3.50) for $\mathcal{D}_2^{-1}$, we obtain:

$$\begin{aligned}\|\mathcal{D}_2^{-1}\left(\nabla\!\!\!/_{nL}\nabla\!\!\!/\mathrm{tr}\chi\right)\|_{L^\infty_t L^2_{x'}} &\lesssim \|\mathcal{D}_2^{-1}\left([\nabla\!\!\!/_{nL}, \nabla\!\!\!/]\mathrm{tr}\chi\right)\|_{L^\infty_t L^2_{x'}} + \|\mathcal{D}_2^{-1}\left(\nabla\!\!\!/\nabla\!\!\!/_{nL}\mathrm{tr}\chi\right)\|_{L^\infty_t L^2_{x'}} \\ &\lesssim \|[\nabla\!\!\!/_{nL}, \nabla\!\!\!/]\mathrm{tr}\chi\|_{L^\infty_t L^{\frac{4}{3}}_{x'}} + \|nL(\mathrm{tr}\chi)\|_{L^\infty_t L^2_{x'}} \\ &\lesssim \|n\chi\nabla\!\!\!/\mathrm{tr}\chi\|_{L^\infty_t L^{\frac{4}{3}}_{x'}} + \varepsilon \\ &\lesssim \|n\|_{L^\infty}\|\chi\|_{L^\infty_t L^4_{x'}}\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^\infty_t L^2_{x'}} + \varepsilon \\ &\lesssim \varepsilon,\end{aligned} \tag{6.62}$$

where we used the estimate (2.67) for n, and the estimates (2.70) (2.71) for χ.

Next, we estimate the second term in the right-hand side of (6.61). Using Lemma 3.17, and since $2\leq q<4$, we obtain:

$$\begin{aligned}\|\mathcal{D}_2^{-1}\left(\nabla\!\!\!/_{nL}\left(\epsilon\cdot\chi\right)\right)\|_{L^2_t L^q_{x'}} &\lesssim \|\mathcal{D}_2^{-1}\left(\nabla\!\!\!/_{nL}(\epsilon)\cdot\chi\right)\|_{L^2_t L^q_{x'}} + \|\mathcal{D}_2^{-1}\left(\epsilon\cdot\nabla\!\!\!/_{nL}(\chi)\right)\|_{L^2_t L^q_{x'}} \\ &\lesssim \|\nabla\!\!\!/_{nL}(\epsilon)\cdot\chi\|_{L^2_t L^{\frac{4}{3}}_{x'}} + \|\epsilon\cdot\nabla\!\!\!/_{nL}(\chi)\|_{L^2_t L^{\frac{4}{3}}_{x'}} \\ &\lesssim \|\nabla\!\!\!/_{nL}(\epsilon)\|_{L^2(\mathcal{H}_u)}\|\chi\|_{L^\infty_t L^4_{x'}} + \|\epsilon\|_{L^\infty_t L^4_{x'}}\|\nabla\!\!\!/_{nL}(\chi)\|_{L^2(\mathcal{H}_u)} \\ &\lesssim \|n\|_{L^\infty}\mathcal{N}_1(\epsilon)\mathcal{N}_1(\chi) \\ &\lesssim \varepsilon,\end{aligned} \tag{6.63}$$

where we used the estimate (2.67) for n, the estimate (2.68) for ϵ, and the estimates (2.70) (2.71) for χ.

Finally, we estimate the third term in the right-hand side of (6.61). Using the commutator formula (2.48) for $[\nabla\!\!\!/_{nL}, \mathcal{D}_2]\mathrm{tr}\chi$, using Lemma 3.17, and since $2 \leq q < 4$, we obtain:

$$\begin{aligned}
\|\mathcal{D}_2^{-1}[\nabla\!\!\!/_{nL}, \mathcal{D}_2]\chi_1\|_{L_t^2 L_{x'}^q} &\lesssim \|[\nabla\!\!\!/_{nL}, \mathcal{D}_2]\chi_1\|_{L_t^2 L_{x'}^{\frac{4}{3}}} \\
&\lesssim \|n\chi\nabla\!\!\!/\chi_1\|_{L_t^2 L_{x'}^{\frac{4}{3}}} + \|n\chi\epsilon\chi_1\|_{L_t^2 L_{x'}^{\frac{4}{3}}} + \|n\beta\chi_1\|_{L_t^2 L_{x'}^{\frac{4}{3}}} \\
&\lesssim \|n\|_{L^\infty}\Big(\|\chi\|_{L_t^\infty L_{x'}^4}\|\nabla\!\!\!/\chi_1\|_{L^2(\mathcal{H}_u)} + \|\chi\|_{L_t^\infty L_{x'}^4}\|\epsilon\|_{L^4(\mathcal{H}_u)}\|\chi_1\|_{L^4(\mathcal{H}_u)} \\
&\quad + \|\beta\|_{L^2(\mathcal{H}_u)}\|\chi_1\|_{L_t^\infty L_{x'}^4}\Big) \\
&\lesssim \varepsilon\mathcal{N}_1(\chi_1), \qquad (6.64)
\end{aligned}$$

where we used the curvature bound (2.59) for β, the estimate (2.67) for n, the estimate (2.68) for ϵ, and the estimates (2.70) (2.71) for χ. Now, (6.61)-(6.64) yield:

$$\|\nabla\!\!\!/_{nL}\chi_1\|_{L_t^\infty L_{x'}^2 + L_t^2 L_{x'}^q} \lesssim \varepsilon + \varepsilon\mathcal{N}_1(\chi_1),$$

which together with the bound (2.67) on n and the bound (6.59) on $\nabla\!\!\!/\chi_1$ yields:

$$\|\nabla\!\!\!/_L\chi_1\|_{L_t^\infty L_{x'}^2 + L_t^2 L_{x'}^q} \lesssim \varepsilon. \qquad (6.65)$$

Estimate for $\|\nabla\!\!\!/_{\underline{L}}\chi_1\|_{L_t^\infty L_{x'}^2 + L_t^2 L_{x'}^q}$. Next, we estimate $\nabla\!\!\!/_{\underline{L}}\chi_1$. Let $2 \leq q < 4$. Applying (6.60) with $X = bN$, we obtain:

$$\begin{aligned}
&\|\nabla\!\!\!/_{bN}\chi_1\|_{L_t^\infty L_{x'}^2 + L_t^2 L_{x'}^q} \qquad (6.66) \\
\lesssim\ &\|\mathcal{D}_2^{-1}\left(\nabla\!\!\!/_{bN}\nabla\!\!\!/\mathrm{tr}\chi\right)\|_{L_t^\infty L_{x'}^2} + \|\mathcal{D}_2^{-1}\left(\nabla\!\!\!/_{bN}\left(\epsilon\cdot\chi\right)\right)\|_{L_t^2 L_{x'}^q} + \|\mathcal{D}_2^{-1}[\nabla\!\!\!/_{bN}, \mathcal{D}_2]\chi_1\|_{L_t^2 L_{x'}^q}.
\end{aligned}$$

We estimate the three terms in the right-hand side of (6.66) starting with the first one. Using the commutator formula (2.50) for $[\nabla\!\!\!/_{bN}, \nabla\!\!\!/]\mathrm{tr}\chi$, and Remark 3.15 and the dual of (3.50) for $\mathcal{D}_2^{-1}$, we obtain:

$$\begin{aligned}
\|\mathcal{D}_2^{-1}\left(\nabla\!\!\!/_{bN}\nabla\!\!\!/\mathrm{tr}\chi\right)\|_{L_t^\infty L_{x'}^2} &\lesssim \|\mathcal{D}_2^{-1}\left([\nabla\!\!\!/_{bN}, \nabla\!\!\!/]\mathrm{tr}\chi\right)\|_{L_t^\infty L_{x'}^2} + \|\mathcal{D}_2^{-1}\left(\nabla\!\!\!/\nabla\!\!\!/_{bN}\mathrm{tr}\chi\right)\|_{L_t^\infty L_{x'}^2} \\
&\lesssim \|[\nabla\!\!\!/_{bN}, \nabla\!\!\!/]\mathrm{tr}\chi\|_{L_t^\infty L_{x'}^{\frac{4}{3}}} + \|\nabla\!\!\!/_{bN}\mathrm{tr}\chi\|_{L_t^\infty L_{x'}^2} \\
&\lesssim \|n(\chi+\eta)\nabla\!\!\!/\mathrm{tr}\chi\|_{L_t^\infty L_{x'}^{\frac{4}{3}}} + \varepsilon \\
&\lesssim \|n\|_{L^\infty}(\|\chi\|_{L_t^\infty L_{x'}^4} + \|k\|_{L_t^\infty L_{x'}^4})\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L_t^\infty L_{x'}^2} + \varepsilon \\
&\lesssim \varepsilon, \qquad (6.67)
\end{aligned}$$

where we used the estimate (2.67) for n, the estimate (2.68) for k, and the estimates (2.70) (2.71) for χ.

Next, we estimate the second term in the right-hand side of (6.66). Using Lemma 3.17, and since $2 \leq q < 4$, we obtain:

$$\begin{aligned}
\|\mathcal{D}_2^{-1}\left(\nabla\!\!\!/_{bN}\left(\epsilon \cdot \chi\right)\right)\|_{L_t^2 L_{x'}^q} &\lesssim \|\mathcal{D}_2^{-1}\left(\nabla\!\!\!/_{bN}(\epsilon) \cdot \chi\right)\|_{L_t^2 L_{x'}^q} + \|\mathcal{D}_2^{-1}\left(\epsilon \cdot \nabla\!\!\!/_{bN}(\chi)\right)\|_{L_t^2 L_{x'}^q} \\
&\lesssim \|\nabla\!\!\!/_{bN}(\epsilon) \cdot \chi\|_{L_t^2 L_{x'}^{\frac{4}{3}}} + \|\epsilon \cdot \nabla\!\!\!/_{bN}(\chi)\|_{L_t^2 L_{x'}^{\frac{4}{3}}} \\
&\lesssim \|\nabla\!\!\!/_{bN}(\epsilon)\|_{L^2(\mathcal{H}_u)}\|\chi\|_{L_t^\infty L_{x'}^4} + \|\epsilon\|_{L_t^\infty L_{x'}^4}\|\nabla\!\!\!/_{bN}(\chi)\|_{L^2(\mathcal{H}_u)} \\
&\lesssim \|b\|_{L^\infty}(\mathcal{N}_1(\epsilon)\|\nabla\!\!\!/_N(\chi)\|_{L^2(\mathcal{H}_u)} + \mathcal{N}_1(\chi)\|\nabla\!\!\!/_N(\epsilon)\|_{L^2(\mathcal{H}_u)}) \\
&\lesssim \varepsilon, \qquad (6.68)
\end{aligned}$$

where we used the estimate (2.68) for ϵ, the estimate (2.69) for b, and the estimates (2.70) (2.71) for χ.

Finally, we estimate the third term in the right-hand side of (6.66). Using the commutator formula (2.50) for $[\nabla\!\!\!/_{bN}, \mathcal{D}_2]\mathrm{tr}\chi$, and using Lemma 3.17, and since $2 \leq q < 4$, we obtain:

$$\begin{aligned}
&\|\mathcal{D}_2^{-1}[\nabla\!\!\!/_{bN}, \mathcal{D}_2]\chi_1\|_{L_t^2 L_{x'}^q} \qquad (6.69) \\
\lesssim\ &\|[\nabla\!\!\!/_{bN}, \mathcal{D}_2]\chi_1\|_{L_t^2 L_{x'}^{\frac{4}{3}}} \\
\lesssim\ &\|b(\chi+\eta)\nabla\!\!\!/\chi_1\|_{L_t^2 L_{x'}^{\frac{4}{3}}} + \|b\chi(\overline{\epsilon}+\underline{\xi})\chi_1\|_{L_t^2 L_{x'}^{\frac{4}{3}}} + \|b\underline{\chi}\zeta\chi_1\|_{L_t^2 L_{x'}^{\frac{4}{3}}} + \|b\beta\chi_1\|_{L_t^2 L_{x'}^{\frac{4}{3}}} \\
&+\|b\underline{\beta}\chi_1\|_{L_t^2 L_{x'}^{\frac{4}{3}}} \\
\lesssim\ &\|b\|_{L^\infty}\Big(\|\chi+\eta\|_{L_t^\infty L_{x'}^4}\|\nabla\!\!\!/\chi_1\|_{L^2(\mathcal{H}_u)} + \|\chi\|_{L_t^\infty L_{x'}^4}\|\overline{\epsilon}+\underline{\xi}\|_{L^4(\mathcal{H}_u)}\|\chi_1\|_{L^4(\mathcal{H}_u)} \\
&+\|\underline{\chi}\|_{L_t^\infty L_{x'}^4}\|\zeta\|_{L^4(\mathcal{H}_u)}\|\chi_1\|_{L^4(\mathcal{H}_u)} + \|\beta\|_{L^2(\mathcal{H}_u)}\|\chi_1\|_{L_t^\infty L_{x'}^4} + \|\underline{\beta}\|_{L^2(\mathcal{H}_u)}\|\chi_1\|_{L_t^\infty L_{x'}^4}\Big) \\
\lesssim\ &\varepsilon + \varepsilon\mathcal{N}_1(\chi_1),
\end{aligned}$$

where we used the curvature bound (2.59) for β and $\underline{\beta}$, and the estimates (2.67)-(2.72) for $b, \overline{\epsilon}, \eta, \chi, \underline{\xi}$ and ζ. Now, (6.66)-(6.69) yield:

$$\|\nabla\!\!\!/_{bN}\chi_1\|_{L_t^\infty L_{x'}^2 + L_t^2 L_{x'}^q} \lesssim \varepsilon + \varepsilon\mathcal{N}_1(\chi_1),$$

which together with the bound (2.69) on b, the fact that $\underline{L} = L - 2N$, and the bound (6.59) and (6.65) on $\nabla\!\!\!/\chi_1$ yields:

$$\|\nabla\!\!\!/_{\underline{L}}\chi_1\|_{L_t^\infty L_{x'}^2 + L_t^2 L_{x'}^q} \lesssim \varepsilon. \qquad (6.70)$$

Estimate for $\|\chi_1\|_{L^p_t L^\infty_{x'}}$. Using the property (3.16) of the Littlewood-Paley projections, we have:

$$\begin{aligned} & \|\chi_1\|_{L^\infty(P_{t,u})} \\ \lesssim & \sum_{j,l} \left\| P_l \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^\infty(P_{t,u})} \\ \lesssim & \sum_{j,l} 2^l (1 + 2^{-\frac{l}{q}} \|K\|^{\frac{1}{q}}_{L^2(P_{t,u})} + 2^{-\frac{l}{q-1}} \|K\|^{\frac{1}{q-1}}_{L^2(P_{t,u})}) \left\| P_l \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} \\ \lesssim & (1 + \|K\|^{\frac{1}{q-1}}_{L^2(P_{t,u})}) \sum_{j,l} 2^l \left\| P_l \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} \end{aligned} \tag{6.71}$$

where $2 \leq q < +\infty$ will be chosen later, and where we used the sharp Bernstein inequality (4.37) for tensors. Next, we estimate the right-hand side of (6.71). Using the finite band property for P_j, and the inequality (3.50) for ${}^*\mathcal{D}_2$, we have:

$$\|\mathcal{D}_2^{-1} P_j\|_{\mathcal{L}(L^2(P_{t,u}))} = \|P_j {}^*\mathcal{D}_2^{-1}\|_{\mathcal{L}(L^2(P_{t,u}))} \lesssim 2^{-j} \|\nabla\!\!\!/^* \mathcal{D}_2^{-1}\|_{\mathcal{L}(L^2(P_{t,u}))} \lesssim 2^{-j} \tag{6.72}$$

which together with the boundedness on L^2 of P_l yields:

$$\begin{aligned} \left\| P_l \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} & \lesssim \left\| \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} \\ & \lesssim 2^{-j} \left\| P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})}. \end{aligned} \tag{6.73}$$

We now derive a second estimate for $\left\| P_l \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})}$. Using the finite band property for P_l, we have:

$$\left\| P_l \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} \lesssim 2^{-2l} \left\| \Delta\!\!\!/ \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})}. \tag{6.74}$$

Next, we estimate the right-hand side of (6.74). In view of the identity (3.40) for $\mathcal{D}_2$, we have:

$$\begin{aligned} & \left\| \Delta\!\!\!/ \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} \\ \lesssim & \left\| {}^*\mathcal{D}_2 P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} + \left\| K \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})}. \end{aligned} \tag{6.75}$$

We now estimate both terms in the right-hand side of (6.75) starting with the first one. Using the L^2 boundedness for P_l and the finite band property for P_j, we have:

$$\left\| {}^*\mathcal{D}_2 P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} \lesssim 2^j \left\| P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})}. \tag{6.76}$$

Next, we estimate the second term in the right-hand side of (6.75). We have:

$$\left\|K\mathcal{D}_2^{-1}P_j\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right)\right\|_{L^2(P_{t,u})}\lesssim\|K\|_{L^2(P_{t,u})}\left\|\mathcal{D}_2^{-1}P_j\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right)\right\|_{L^\infty(P_{t,u})}. \tag{6.77}$$

In order to estimate the $L^\infty(P_{t,u})$ norm in the right-hand side of (6.77), we use the estimate (3.38). This yields

$$\begin{aligned}
&\left\|\mathcal{D}_2^{-1}P_j\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right)\right\|_{L^\infty(P_{t,u})} \qquad (6.78)\\
\lesssim\ &\left\|\nabla\!\!\!/^2\mathcal{D}_2^{-1}P_j\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right)\right\|^{\frac{1}{2}}_{L^2(P_{t,u})}\left\|\mathcal{D}_2^{-1}P_j\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right)\right\|^{\frac{1}{2}}_{L^2(P_{t,u})}\\
&+\left\|\nabla\!\!\!/\mathcal{D}_2^{-1}P_j\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right)\right\|_{L^2(P_{t,u})}\\
\lesssim\ &2^{-\frac{j}{2}}\left\|\nabla\!\!\!/^2\mathcal{D}_2^{-1}P_j\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right)\right\|^{\frac{1}{2}}_{L^2(P_{t,u})}\left\|P_j\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right)\right\|^{\frac{1}{2}}_{L^2(P_{t,u})}\\
&+\left\|P_j\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right)\right\|_{L^2(P_{t,u})},
\end{aligned}$$

where we used in the last inequality (6.72), the estimate (3.50) for $\nabla\!\!\!/\mathcal{D}_2^{-1}$, and the boundedness of P_j on $L^2(P_{t,u})$. In order to estimate the first term in the right-hand side of (6.78), we use the Bochner inequality for tensors (3.7). This yields

$$\begin{aligned}
&\left\|\nabla\!\!\!/^2\mathcal{D}_2^{-1}P_j\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right)\right\|_{L^2(P_{t,u})} \qquad (6.79)\\
\lesssim\ &\left\|\Delta\!\!\!/\mathcal{D}_2^{-1}P_j\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right)\right\|_{L^2(P_{t,u})}+\|K\|_{L^2(P_{t,u})}\left\|\nabla\!\!\!/\mathcal{D}_2^{-1}P_j\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right)\right\|_{L^2(P_{t,u})}\\
&+\|K\|^2_{L^2(P_{t,u})}\left\|\mathcal{D}_2^{-1}P_j\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right)\right\|_{L^2(P_{t,u})}\\
\lesssim\ &\left\|\Delta\!\!\!/\mathcal{D}_2^{-1}P_j\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right)\right\|_{L^2(P_{t,u})}+\|K\|_{L^2(P_{t,u})}\left\|P_j\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right)\right\|_{L^2(P_{t,u})}\\
&+2^{-j}\|K\|^2_{L^2(P_{t,u})}\left\|P_j\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right)\right\|_{L^2(P_{t,u})},
\end{aligned}$$

where we used in the last inequality (6.72) and the estimate (3.50) for $\nabla\!\!\!/\mathcal{D}_2^{-1}$. Now, (6.75), (6.76), (6.77), (6.78) and (6.79) imply

$$\begin{aligned}
&\left\|\nabla\!\!\!/^2\mathcal{D}_2^{-1}P_j\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right)\right\|_{L^2(P_{t,u})} \qquad (6.80)\\
\lesssim\ &\left(2^j+\|K\|_{L^2(P_{t,u})}+2^{-j}\|K\|^2_{L^2(P_{t,u})}\right)\left\|P_j\left(\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\cdot\chi\right)\right\|_{L^2(P_{t,u})}.
\end{aligned}$$

Then, (6.74) and (6.80) yield

$$\begin{aligned} &\left\| P_l \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} \\ \lesssim\ & 2^{-2l} \left(2^j + \|K\|_{L^2(P_{t,u})} + 2^{-j} \|K\|^2_{L^2(P_{t,u})} \right) \left\| P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})}. \end{aligned} \tag{6.81}$$

Also, using the finite band property P_l and the estimate (3.50) for $\nabla\!\!\!/ \mathcal{D}_2^{-1}$, we have

$$\left\| P_l \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} \lesssim 2^{-l} \left\| P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})}.$$

Interpolating with (6.81), we obtain for any $0 \leq \delta \leq 1$

$$\begin{aligned} &\left\| P_l \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} \\ \lesssim\ & 2^{-l} 2^{-\delta(l-j)} \left(1 + \|K\|^2_{L^2(P_{t,u})} \right)^{\delta} \left\| P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})}. \end{aligned} \tag{6.82}$$

In view of (6.71), and using (6.73) for $l \leq j$ and (6.82) for $l > j$, we obtain for any $2 \leq q < +\infty$ and any $0 < \delta \leq 1$

$$\begin{aligned} \|\chi_1\|_{L^\infty(P_{t,u})} &\lesssim (1 + \|K\|_{L^2(P_{t,u})})^{\frac{1}{q-1}+2\delta} \sum_{j,l} 2^{-\delta|l-j|} \left\| P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} \\ &\lesssim (1 + \|K\|_{L^2(P_{t,u})})^{\frac{1}{q-1}+2\delta} \left\| \frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right\|_{B^0_{2,1}(P_{t,u})} \end{aligned}$$

where we used in the last inequality the fact that $\delta > 0$ and the definition (5.4) for the Besov space $B^0_{2,1}(P_{t,u})$. This yields

$$\begin{aligned} \|\chi_1\|_{L^\infty(P_{t,u})} &\lesssim (1 + \|K\|_{L^2(P_{t,u})})^{\frac{1}{q-1}+2\delta} (\|\nabla\!\!\!/ \mathrm{tr}\chi\|_{\mathcal{B}^0} + \|\epsilon \cdot \chi\|_{B^0_{2,1}(P_{t,u})}) \\ &\lesssim (1 + \|K\|_{L^2(P_{t,u})})^{\frac{1}{q-1}+2\delta} (\varepsilon + \|\epsilon \cdot \chi\|_{B^0_{2,1}(P_{t,u})}) \end{aligned} \tag{6.83}$$

where we used the Besov improvement (5.54) for $\nabla\!\!\!/ \mathrm{tr}\chi$. Let $2 \leq p < +\infty$. We choose $2 \leq q < +\infty$ and $0 < \delta \leq 1$ such that

$$\frac{1}{q-1} + 2\delta = \frac{2}{p}.$$

Then, (6.83) implies:

$$\begin{aligned} \|\chi_1\|_{L^p_t L^\infty_{x'}} &\lesssim \left(1 + \|K\|^{\frac{2}{p}}_{L^2(\mathcal{H}_u)} \right) (\varepsilon + \|\epsilon \cdot \chi\|_{L^{2p}_t B^0_{2,1}(P_{t,u})}) \\ &\lesssim \varepsilon + \|\epsilon \cdot \chi\|_{L^{2p}_t B^0_{2,1}(P_{t,u})}, \end{aligned} \tag{6.84}$$

where we used the estimate (4.29) for the Gauss curvature K. We now conclude using the following lemma:

Lemma 6.16 *Let F, H two $P_{t,u}$-tangent tensors. For any $2 \leq r < +\infty$, we have:*

$$\|F \cdot H\|_{L^r_t B^0_{2,1}(P_{t,u})} \lesssim \mathcal{N}_1(F)\mathcal{N}_1(H). \tag{6.85}$$

The proof of Lemma 6.16 is postponed to section C.6. We now derive the estimate for $\|\chi_1\|_{L^p_t L^\infty_{x'}}$. (6.84), and (6.85) with $r = 2p$, $F = \epsilon$ and $H = \chi$ yield:

$$\begin{aligned} \|\chi_1\|_{L^p_t L^\infty_{x'}} &\lesssim \varepsilon + \mathcal{N}_1(\epsilon)\mathcal{N}_1(\chi) \\ &\lesssim \varepsilon, \end{aligned} \tag{6.86}$$

where we used the estimate (2.68) for ϵ and the estimates (2.70) (2.71) for χ. (6.86) is the desired estimate for $\|\chi_1\|_{L^p_t L^\infty_{x'}}$.

Estimate for $\|\nabla\!\!\!/\chi_1\|_{L^\infty_u L^p_t B^0_{2,1}(P_{t,u})}$. We will need later on an estimate for $\nabla\!\!\!/\chi_1$ in $L^\infty_u L^p_t B^0_{2,1}(P_{t,u})$. We proceed as for the estimate of χ_1 in $L^p_t L^\infty_{x'}$. In view of the definition (5.4) of the Besov space $B^0_{2,1}(P_{t,u})$, we have

$$\|\nabla\!\!\!/\chi_1\|_{B^0_{2,1}(P_{t,u})} \lesssim \sum_{j,l} \left\| P_l \nabla\!\!\!/ \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})}. \tag{6.87}$$

Next, we estimate the right-hand side of (6.87). The finite band property for P_l together with the estimate (6.72) yields

$$\begin{aligned} \left\| P_l \nabla\!\!\!/ \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} &\lesssim 2^l \left\| \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} \\ &\lesssim 2^{l-j} \left\| P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})}. \end{aligned} \tag{6.88}$$

We now derive second estimate for $\left\| P_l \nabla\!\!\!/ \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})}$. Using the finite band property for P_l and the estimate (6.80), we have:

$$\begin{aligned} &\left\| P_l \nabla\!\!\!/ \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} \\ \lesssim\ & 2^{-l} \left\| \nabla\!\!\!/^2 \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} \\ \lesssim\ & 2^{-l+j} (1 + \|K\|^2_{L^2(P_{t,u})}) \left\| P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})}. \end{aligned} \tag{6.89}$$

Also, using the boundedness of P_l on $L^2(P_{t,u})$ and the estimate (3.50) for $\nabla\!\!\!/ \mathcal{D}_2^{-1}$, we have

$$\left\| P_l \nabla\!\!\!/ \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} \lesssim \left\| P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})}.$$

Interpolating with (6.81), we obtain for any $0 \leq \delta \leq 1$

$$\begin{aligned} & \left\| P_l \nabla\!\!\!/ \mathcal{D}_2^{-1} P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} \\ \lesssim \; & 2^{-\delta(l-j)} \left(1 + \| K \|^2_{L^2(P_{t,u})} \right)^{\delta} \left\| P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} . \end{aligned} \tag{6.90}$$

In view of (6.87), and using (6.88) for $l \leq j$ and (6.90) for $l > j$, we obtain for any $0 < \delta \leq 1$

$$\begin{aligned} \| \nabla\!\!\!/ \chi_1 \|_{B^0_{2,1}(P_{t,u})} & \lesssim (1 + \| K \|_{L^2(P_{t,u})})^{2\delta} \sum_{j,l} 2^{-\delta|l-j|} \left\| P_j \left(\frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right) \right\|_{L^2(P_{t,u})} \\ & \lesssim (1 + \| K \|_{L^2(P_{t,u})})^{2\delta} \left\| \frac{1}{2} \nabla\!\!\!/ \mathrm{tr}\chi - \epsilon \cdot \chi \right\|_{B^0_{2,1}(P_{t,u})} \end{aligned}$$

where we used in the last inequality the fact that $\delta > 0$ and the definition (5.4) for the Besov space $B^0_{2,1}(P_{t,u})$. This yields

$$\begin{aligned} \| \nabla\!\!\!/ \chi_1 \|_{B^0_{2,1}(P_{t,u})} & \lesssim (1 + \| K \|_{L^2(P_{t,u})})^{2\delta} (\| \nabla\!\!\!/ \mathrm{tr}\chi \|_{\mathcal{B}^0} + \| \epsilon \cdot \chi \|_{B^0_{2,1}(P_{t,u})}) \\ & \lesssim (1 + \| K \|_{L^2(P_{t,u})})^{2\delta} (\varepsilon + \| \epsilon \cdot \chi \|_{B^0_{2,1}(P_{t,u})}) \end{aligned} \tag{6.91}$$

where we used the Besov improvement (5.54) for $\nabla\!\!\!/ \mathrm{tr}\chi$. Let $2 \leq p < +\infty$. We choose $0 < \delta \leq 1$ such that

$$2\delta = \frac{1}{p}.$$

Then, (6.91) implies:

$$\begin{aligned} \| \nabla\!\!\!/ \chi_1 \|_{L^p_t B^0_{2,1}(P_{t,u})} & \lesssim \left(1 + \| K \|^{\frac{1}{p}}_{L^2(\mathcal{H}_u)} \right) (\varepsilon + \| \epsilon \cdot \chi \|_{L^{2p}_t B^0_{2,1}(P_{t,u})}) \\ & \lesssim \varepsilon + \mathcal{N}_1(\epsilon) \mathcal{N}_1(\chi), \end{aligned}$$

where we used in the last inequality the estimate (4.29) for the Gauss curvature K, and the estimate (6.85). Together with the estimate (2.68) for ϵ and the estimates (2.70) (2.71) for χ, we finally obtain

$$\| \nabla\!\!\!/ \chi_1 \|_{L^p_t B^0_{2,1}(P_{t,u})} \lesssim \varepsilon, \tag{6.92}$$

for any $2 \leq p < +\infty$.

6.4.2 Estimates for χ_2

In view of the decomposition (6.58), the estimates (2.70) (2.71) for χ, and the estimates (6.59), (6.65) and (6.70) for χ_1, we have:

$$\mathcal{N}_1(\chi_2) + \| \nabla_{\underline{L}} \chi_2 \|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{6.93}$$

We now compute $\partial_\omega \chi_2$. We have:

$$[\partial_\omega, \mathcal{D}_2^{-1}] = -\mathcal{D}_2^{-1}[\partial_\omega, \mathcal{D}_2]\mathcal{D}_2^{-1}$$

which together with the definition of χ_2 implies:

$$\begin{aligned}
\partial_\omega \chi_2 &= -\mathcal{D}_2^{-1}\Pi(\partial_\omega \beta + \partial_\omega \epsilon \cdot \chi + \epsilon \cdot \partial_\omega \chi) - \mathcal{D}_2^{-1}[\Pi\partial_\omega, \mathcal{D}_2]\chi_2 \qquad (6.94)\\
&= -\mathcal{D}_2^{-1}\bigg(\frac{(\partial_\omega N)_B}{2}(-\alpha_{AB} + \rho\delta_{AB} + 3\sigma \in_{AB}) + \partial_\omega \epsilon \cdot \chi + \epsilon \cdot \partial_\omega \chi\bigg)\\
&\quad -\mathcal{D}_2^{-1}\bigg(- \nabla\!\!\!/_N(\chi_2)_{A\partial_\omega N} + g(\partial_\omega N, e_A)\theta \cdot \chi_2 + \theta_{\partial_\omega N B}(\chi_2)_{BA} - \theta_{AB}(\chi_2)_{B\partial_\omega N}\\
&\quad -\mathrm{tr}\theta(\chi_2)_{A\partial_\omega N}\bigg),
\end{aligned}$$

where we used the formula (6.30) for $\partial_\omega \beta$ and the commutator formula (6.17) for $[\Pi\partial_\omega, \mathcal{D}_2]$. In particular, using the property (3.50) of $\mathcal{D}_2^{-1}$, we have the following estimate for $\nabla\!\!\!/\partial_\omega \chi_2$:

$$\begin{aligned}
&\|\nabla\!\!\!/\partial_\omega \chi_2\|_{L^2(\mathcal{H}_u)} \qquad (6.95)\\
\lesssim\ & \bigg\|\frac{(\partial_\omega N)_B}{2}(-\alpha_{AB} + \rho\delta_{AB} + 3\sigma \in_{AB}) + \partial_\omega \epsilon \cdot \chi + \epsilon \cdot \partial_\omega \chi\bigg\|_{L^2(\mathcal{H}_u)} + \| - \nabla\!\!\!/_N(\chi_2)_{A\partial_\omega N}\\
&+g(\partial_\omega N, e_A)\theta \cdot \chi_2 + \theta_{\partial_\omega N B}(\chi_2)_{BA} - \theta_{AB}(\chi_2)_{B\partial_\omega N} - \mathrm{tr}\theta(\chi_2)_{A\partial_\omega N}\|_{L^2(\mathcal{H}_u)}\\
\lesssim\ & \|\partial_\omega N\|_{L^\infty}\bigg(\|\alpha\|_{L^2(\mathcal{H}_u)} + \|\rho\|_{L^2(\mathcal{H}_u)} + \|\sigma\|_{L^2(\mathcal{H}_u)} + \|\partial_\omega \epsilon\|_{L^2_{x'}L^\infty_t}\|\chi\|_{L^\infty_{x'}L^2_t}\\
&+\|\partial_\omega \chi\|_{L^2_{x'}L^\infty_t}\|\epsilon\|_{L^\infty_{x'}L^2_t} + \|\nabla\!\!\!/_N \chi_2\|_{L^2(\mathcal{H}_u)} + \mathcal{N}_1(\theta)\mathcal{N}_1(\chi_2)\bigg)\\
\lesssim\ & \varepsilon
\end{aligned}$$

where we used the curvature bound (2.59) for α, ρ and σ, the estimates (2.68) (2.70) (2.71) for ϵ, χ and θ, the estimate (2.76) for $\partial_\omega N$, the estimate (2.77) for $\partial_\omega \chi$ and the estimate (6.93) for χ_2.

Next, we plan to estimate the $L^p_t L^{4-}_{x'}$-norm of $\partial_\omega \chi_2$ for $2 < p < +\infty$. Our goal will be first to show that the terms involving α in $\partial_\omega \chi_2$ cancel each other. Applying (6.60) to χ_2 with the choice $X = bN$ yields:

$$\nabla\!\!\!/_{bN}\chi_2 = -\mathcal{D}_2^{-1}\left(\nabla\!\!\!/_{bN}(\chi \cdot \epsilon + \beta)\right) - \mathcal{D}_2^{-1}[\nabla\!\!\!/_{bN}, \mathcal{D}_2]\chi_2. \qquad (6.96)$$

In view of (6.96), we need to evaluate $\nabla\!\!\!/_N(\chi \cdot \epsilon + \beta)$. We have:

$$\nabla\!\!\!/_N(\chi \cdot \epsilon + \beta) = \chi \cdot \nabla\!\!\!/_N(\epsilon) + \frac{1}{2}\left(\nabla\!\!\!/_L(\chi) \cdot \epsilon + \nabla\!\!\!/_L \beta - \nabla\!\!\!/_{\underline{L}}(\chi) \cdot \epsilon - \nabla\!\!\!/_{\underline{L}}\beta\right)$$

which together with the equation (2.31) and (2.40) for χ, the Bianchi identities (2.51) and (2.52) for β, and the last equation of (4.62) for $\nabla\!\!\!/_N \epsilon$ yields:

$$\begin{aligned}
2\nabla\!\!\!/_N(\chi \cdot \epsilon + \beta) &= \mathrm{div}\!\!\!/\alpha + b^{-1}\nabla\!\!\!/ b \cdot \alpha - \nabla\!\!\!/\rho - (\nabla\!\!\!/\sigma)^* + \chi \cdot \beta + (\mathrm{tr}\chi - 2\mathrm{tr}\underline{\chi} - 2\delta)\beta\\
&\quad -(\epsilon + 3\zeta)\rho + (\epsilon - 3^*\zeta)\sigma - (\chi + 2\underline{\widehat{\chi}})\underline{\beta} + 2\nabla\!\!\!/\delta \cdot \chi - 2\epsilon \cdot \nabla\!\!\!/\zeta + 3\overline{\delta}^{-1}\nabla\!\!\!/ b \cdot \chi\\
&\quad -2b^{-1}\nabla\!\!\!/ b\widehat{\eta}\chi - 2\theta\epsilon\chi - \epsilon\chi(\delta + n^{-1}N(n)) - \epsilon\zeta \otimes \zeta + \epsilon\underline{\chi}\chi. \qquad (6.97)
\end{aligned}$$

(6.96) and (6.97) yield:

$$\begin{aligned}2\nabla\!\!\!/_N\chi_2 &= -\alpha + b^{-1}\mathcal{D}_2^{-1}b\Big(\nabla\!\!\!/\rho + (\nabla\!\!\!/\sigma)^* - \chi\cdot\beta - (\mathrm{tr}\chi - 2\mathrm{tr}\underline{\chi} - 2\delta)\beta + (\epsilon+3\zeta)\rho\\ &\quad -(\epsilon - 3^*\zeta)\sigma + (\chi + 2\widehat{\underline{\chi}})\underline{\beta} - 2\nabla\!\!\!/\delta\cdot\chi + 2\epsilon\cdot\nabla\!\!\!/\zeta - 3\overline{\delta}^{-1}\nabla\!\!\!/ b\cdot\chi + 2b^{-1}\nabla\!\!\!/ b\widehat{\eta}\chi + 2\theta\epsilon\chi\\ &\quad +\epsilon\chi(\delta + n^{-1}N(n)) + \epsilon\zeta\otimes\zeta - \epsilon\underline{\chi}\chi\Big) - 2b^{-1}\mathcal{D}_2^{-1}[\nabla\!\!\!/_{bN},\mathcal{D}_2]\chi_2. \qquad (6.98)\end{aligned}$$

Now, in view of (6.94) and (6.98), the terms in α cancel each other, and we finally obtain:

$$\begin{aligned}\partial_\omega\chi_2 &= -\mathcal{D}_2^{-1}\left(\frac{(\partial_\omega N)_B}{2}(\rho\delta_{AB} + 3\sigma\in_{AB}) + \partial_\omega\epsilon\cdot\chi + \epsilon\cdot\partial_\omega\chi\right) \qquad (6.99)\\ &\quad +\mathcal{D}_2^{-1}\Bigg(-\frac{1}{2}b^{-1}\mathcal{D}_2^{-1}b\Big(\nabla\!\!\!/\rho + (\nabla\!\!\!/\sigma)^* - \chi\cdot\beta - (\mathrm{tr}\chi - 2\mathrm{tr}\underline{\chi} - 2\delta)\beta + (\epsilon+3\zeta)\rho\\ &\quad -(\epsilon - 3^*\zeta)\sigma + (\chi + 2\widehat{\underline{\chi}})\underline{\beta} - 2\nabla\!\!\!/\delta\cdot\chi + 2\epsilon\cdot\nabla\!\!\!/\zeta - 3\delta b^{-1}\nabla\!\!\!/ b\cdot\chi + 2b^{-1}\nabla\!\!\!/ b\widehat{\eta}\chi + 2\theta\epsilon\chi\\ &\quad +\epsilon\chi(\delta + n^{-1}N(n)) + \epsilon\zeta\otimes\zeta - \epsilon\underline{\chi}\chi\Big)_{A\partial_\omega N} + b^{-1}\mathcal{D}_2^{-1}[\nabla\!\!\!/_{bN},\mathcal{D}_2](\chi_2)_{A\partial_\omega N}\\ &\quad +g(\partial_\omega N, e_A)\theta\cdot\chi_2 + \theta_{\partial_\omega N B}(\chi_2)_{BA} - \theta_{AB}(\chi_2)_{B\partial_\omega N} - \mathrm{tr}\theta(\chi_2)_{A\partial_\omega N}\Bigg).\end{aligned}$$

We will use the following four Lemmas.

Lemma 6.17 *Let F a $P_{t,u}$-tangent 1-form and let h a scalar function which denotes a curvature term among (ρ,σ). Then, we have the following estimate:*

$$\|\mathcal{D}_2^{-1}(Fh)\|_{L_t^\infty L_{x'}^{4-}} \lesssim (\|F\|_{L^\infty} + \|\nabla\!\!\!/ F\|_{L_t^\infty L_{x'}^2})\varepsilon. \qquad (6.100)$$

Lemma 6.18 *Let h a scalar function which denotes a curvature term among (ρ,σ). Then, for any $2\leq p<+\infty$, we have the following estimate:*

$$\|\mathcal{D}_2^{-1}b^{-1}\mathcal{D}_2^{-1}(b\nabla\!\!\!/ h)\|_{L_t^p L_{x'}^{4-}} \lesssim \varepsilon. \qquad (6.101)$$

Lemma 6.19 *Let F a $P_{t,u}$-tangent tensors and let H denote a term among $(\rho,\sigma,\beta,\underline{\beta})$ and G is a $P_{t,u}$-tensor satisfying $\mathcal{N}_1(G)\lesssim\varepsilon$. Then, for any $2\leq p<+\infty$, we have the following estimate:*

$$\|\mathcal{D}_2^{-1}b^{-1}(\mathcal{D}_2^{-1}(F\cdot H))\|_{L_t^p L_{x'}^{4-}} + \|\mathcal{D}_2^{-1}b^{-1}(\mathcal{D}_2^{-1}(F\cdot\nabla\!\!\!/ G))\|_{L_t^p L_{x'}^{4-}} \lesssim \mathcal{N}_1(F)\varepsilon. \qquad (6.102)$$

Lemma 6.20 *Let F, G and H three $P_{t,u}$-tangent tensors. Then, we have the following estimate:*

$$\|\mathcal{D}_2^{-1}(FGH)\|_{L_t^\infty L_{x'}^{4-}} \lesssim \mathcal{N}_1(F)\mathcal{N}_1(G)\mathcal{N}_1(H). \qquad (6.103)$$

We also state the following lemma which will be necessary for the proof of Lemma 6.17 as well as several places in this paper.

Lemma 6.21 *Let H denote a curvature term among $(\rho, \sigma, \beta, \underline{\beta})$. Then, for any $j \geq 0$, we have the following estimate:*

$$\|P_j H\|_{L^\infty_t L^2_{x'}} \lesssim 2^{\frac{j}{2}} \varepsilon. \tag{6.104}$$

The proof of Lemma 6.17 is postponed to section C.7, the proof of Lemma 6.18 to section C.8, the proof of Lemma 6.19 to section C.9, the proof of Lemma 6.20 to section C.10, and the proof of Lemma 6.21 to section C.11. We now derive the estimate for the $L^p_t L^{4-}_{x'}$-norm of $\partial_\omega \chi_2$. We consider the various terms in the right-hand side of (6.99). Lemma 6.17 and Lemma 6.19 yield[19]:

$$\begin{aligned}
&\left\| \mathcal{D}_2^{-1} \left(\frac{(\partial_\omega N)_B}{2} (\rho \delta_{AB} + 3\sigma \in_{AB}) \right) \right\|_{L^\infty_t L^{4-}_{x'}} + \left\| \mathcal{D}_2^{-1} \Big(- b^{-1} \mathcal{D}_2^{-1} b \Big(- \chi \cdot \beta \right. \\
&\left. -(\mathrm{tr}\chi - 2\mathrm{tr}\underline{\chi} - 2\delta)\beta + (\epsilon + 3\zeta)\rho - (\epsilon - 3{}^*\zeta)\sigma + (\chi + 2\widehat{\underline{\chi}})\underline{\beta} \Big)_{A \partial_\omega N} \Big) \right\|_{L^p_t L^{4-}_{x'}} \\
\lesssim\ & \varepsilon (\|\partial_\omega N\|_{L^\infty} + \|\nabla\!\!\!/ \partial_\omega N\|_{L^\infty_t L^2_{x'}} + \mathcal{N}_1(\chi) + \mathcal{N}_1(\delta) + \mathcal{N}_1(\epsilon) + \mathcal{N}_1(\zeta) + \mathcal{N}_1(\underline{\chi})) \\
\lesssim\ & \varepsilon,
\end{aligned} \tag{6.105}$$

where we used the estimates (2.68)-(2.72) for $\delta, \epsilon, \chi, \underline{\chi}$ and ζ, and the estimates (2.76) (2.77) for $\partial_\omega N$.

Using the commutator formula (2.50) together with Remark 3.15 for $\mathcal{D}_2^{-1}$ and Lemma 6.19 and Lemma 6.20, we obtain:

$$\begin{aligned}
&\left\| \mathcal{D}_2^{-1} \left(b^{-1} \mathcal{D}_2^{-1} [\nabla\!\!\!/_{bN}, \mathcal{D}_2] (\chi_2)_{A \partial_\omega N} \right) \right\|_{L^p_t L^{4-}_{x'}} \\
\lesssim\ & \left\| \mathcal{D}_2^{-1} \left(b^{-1} \mathcal{D}_2^{-1} \nabla\!\!\!/ (b(\chi + \eta)\chi_2)_{A \partial_\omega N} \right) \right\|_{L^p_t L^{4-}_{x'}} + \left\| \mathcal{D}_2^{-1} \left(b^{-1} \mathcal{D}_2^{-1} \nabla\!\!\!/ (b(\chi + \eta))\chi_2)_{A \partial_\omega N} \right) \right\|_{L^p_t L^{4-}_{x'}} \\
&+ \left\| \mathcal{D}_2^{-1} \left(b^{-1} \mathcal{D}_2^{-1} ((\chi(\overline{\epsilon} + \underline{\xi} + \underline{\chi}\zeta)\chi_2)_{A \partial_\omega N} \right) \right\|_{L^p_t L^{4-}_{x'}} + \left\| \mathcal{D}_2^{-1} \left(b^{-1} \mathcal{D}_2^{-1} ((\beta + \underline{\beta})\chi_2)_{A \partial_\omega N} \right) \right\|_{L^p_t L^{4-}_{x'}} \\
\lesssim\ & (\mathcal{N}_2(b-1)(\mathcal{N}_1(\chi) + \mathcal{N}_1(\eta))\mathcal{N}_1(\chi_2) + \mathcal{N}_1(\chi)(\mathcal{N}_1(\overline{\epsilon}) + \mathcal{N}_1(\underline{\xi}))\mathcal{N}_1(\chi_2) \\
&+ \mathcal{N}_1(\underline{\chi})\mathcal{N}_1(\zeta)\mathcal{N}_1(\chi_2) + \varepsilon \mathcal{N}_1(\chi_2) \\
\lesssim\ & \varepsilon,
\end{aligned} \tag{6.106}$$

where we used the estimates (2.67)-(2.72) for $b, \eta, \overline{\epsilon}, \chi, \underline{\chi}, \underline{\xi}$ and ζ, the estimate (2.76) for $\partial_\omega N$, and the estimate (6.93) for χ_2.

[19]Note that we apply Lemma 6.17 respectively with $(h = \rho, F_A = (\partial_\omega N)_A)$ and with $(h = \sigma, F_A = \in_{AB} (\partial_\omega N)_B)$.

Using Remark 3.15 for $\mathcal{D}_2^{-1}$ and Lemma 6.19 and Lemma 6.20, we obtain:

$$\begin{aligned}
&\left\|\mathcal{D}_2^{-1}\left(b^{-1}\mathcal{D}_2^{-1}b\Big(-2\nabla\!\!\!/\delta\cdot\chi+2\epsilon\cdot\nabla\!\!\!/\zeta-3\delta b^{-1}\nabla\!\!\!/ b\cdot\chi+2b^{-1}\nabla\!\!\!/ b\widehat{\eta}\chi+2\theta\epsilon\chi\Big)_{A\partial_\omega N}\right)\right\|_{L_t^pL_{x'}^{4-}}\\
&+\left\|\mathcal{D}_2^{-1}\left(b^{-1}\mathcal{D}_2^{-1}b\Big(\epsilon\chi(\delta+n^{-1}N(n))+\epsilon\zeta\otimes\zeta-\epsilon\underline{\chi}\chi\Big)_{A\partial_\omega N}\right)\right\|_{L_t^\infty L_{x'}^{4-}}\\
&+\left\|\mathcal{D}_2^{-1}\left(g(\partial_\omega N,e_A)\theta\cdot\chi_2+\theta_{\partial_\omega NB}(\chi_2)_{BA}-\theta_{AB}(\chi_2)_{B\partial_\omega N}-\mathrm{tr}\theta(\chi_2)_{A\partial_\omega N}\right)\right\|_{L_t^\infty L_{x'}^{4-}}\\
\lesssim\;&\mathcal{N}_2(b-1)\big(\mathcal{N}_1(\delta)\mathcal{N}_1(\chi)+\mathcal{N}_1(\epsilon)\mathcal{N}_1(\zeta)+\mathcal{N}_1(\eta)\mathcal{N}_1(\chi)+(\mathcal{N}_1(\theta)+\mathcal{N}_1(\delta)+\mathcal{N}_1(n^{-1}N(n))\\
&+\mathcal{N}_1(\underline{\chi}))\mathcal{N}_1(\epsilon)\mathcal{N}_1(\chi)+\mathcal{N}_1(\epsilon)\mathcal{N}_1(\zeta)^2\big)+\|\partial_\omega N\|_{L^\infty}\mathcal{N}_1(\theta)\mathcal{N}_1(\chi_2)\\
\lesssim\;&\varepsilon,
\end{aligned}\tag{6.107}$$

where we used the estimates (2.67)-(2.72) for $n, b, \eta, \epsilon, \chi, \underline{\chi}, \theta$ and ζ, and the estimate (2.76) for $\partial_\omega N$, and the estimate (6.93) for χ_2.

Using the analog of Lemma 3.17 for $\mathcal{D}_2^{-1}$, we obtain:

$$\begin{aligned}
&\|\mathcal{D}_2^{-1}\left(\partial_\omega\epsilon\cdot\chi+\epsilon\cdot\partial_\omega\chi\right)\|_{L_t^\infty L_{x'}^{4-}}\\
\lesssim\;&\|\partial_\omega\epsilon\cdot\chi+\epsilon\cdot\partial_\omega\chi\|_{L_t^\infty L_{x'}^{\frac{4}{3}}}\\
\lesssim\;&\|\partial_\omega\epsilon\|_{L_t^\infty L_{x'}^2}\|\chi\|_{L_t^\infty L_{x'}^4}+\|\epsilon\|_{L_t^\infty L_{x'}^4}\|\partial_\omega\chi\|_{L_t^\infty L_{x'}^2}\\
\lesssim\;&\varepsilon,
\end{aligned}\tag{6.108}$$

where we used the estimates (2.67)-(2.71) for ϵ and χ, and the estimate (2.77) for $\partial_\omega\chi$.

Finally, (6.99), Lemma 6.18, and (6.105)-(6.108) yield for all $2\leq p<+\infty$:

$$\|\partial_\omega\chi_2\|_{L_t^pL_{x'}^{4-}}\lesssim\varepsilon.$$

Using the Gagliardo Nirenberg inequality (3.3), (6.95) and interpolation, we obtain:

$$\|\partial_\omega\chi_2\|_{L^{6-}(\mathcal{H}_u)}\lesssim\varepsilon.$$

Together with the estimates (6.59), (6.65), (6.70) and (6.86) for χ_1, and the estimates (6.93), (6.95) for χ_2, we obtain the desired decomposition (2.79)-(2.82) for $\widehat{\chi}$

6.5 Besov improvement for $\partial_\omega N$ and $\partial_\omega\chi$

The goal of this section in to prove the following proposition.

Proposition 6.22 *We have the following estimate:*

$$\|\nabla\!\!\!/\partial_\omega N\|_{\mathcal{B}^0}+\|\Pi(\partial_\omega\chi)\|_{\mathcal{B}^0}\lesssim\varepsilon.\tag{6.109}$$

Proof The formula (6.24) for $\mathbf{D}_A(\partial_\omega N)$ yields:

$$\nabla\!\!\!/_A \partial_\omega N = \partial_\omega \chi_{AB} e_B - g(\partial_\omega N, e_A)\zeta_B e_B - \epsilon_A \partial_\omega N,$$

which together with the estimate (5.10) and the non sharp embedding (5.13) yields:

$$\begin{aligned} \|\nabla\!\!\!/ \partial_\omega N\|_{\mathcal{B}^0} &\lesssim \|\Pi(\partial_\omega \chi)\|_{\mathcal{B}^0} + \|\partial_\omega N \cdot (\zeta, \epsilon)\|_{\mathcal{B}^0} \\ &\lesssim \|\Pi(\partial_\omega \chi)\|_{\mathcal{B}^0} + (\|\nabla\!\!\!/ \partial_\omega N\|_{L^\infty_t L^2_{x'}} + \|\partial_\omega N\|_{L^\infty})\|(\zeta, \epsilon)\|_{\mathcal{B}^0} \\ &\lesssim \|\Pi(\partial_\omega \chi)\|_{\mathcal{B}^0} + (\|\nabla\!\!\!/ \partial_\omega N\|_{L^\infty_t L^2_{x'}} + \|\partial_\omega N\|_{L^\infty})(\mathcal{N}_1(\zeta) + \mathcal{N}_1(\epsilon)) \\ &\lesssim \|\Pi(\partial_\omega \chi)\|_{\mathcal{B}^0} + \varepsilon, \end{aligned} \tag{6.110}$$

where we used in the last inequality the estimate (2.72) for ζ, the estimate (2.68) for ϵ, and the estimates (2.76) and (2.77) for $\partial_\omega N$.

In view of (6.110), it remains to estimate $\|\Pi(\partial_\omega \chi)\|_{\mathcal{B}^0}$. We recall the structure of the transport equation (6.27) satisfied by $\Pi(\partial_\omega \chi)$:

$$\nabla\!\!\!/_L(\Pi(\partial_\omega \chi)) = -\nabla\!\!\!/_{\partial_\omega N}\chi - (2\chi + \overline{\delta})\cdot\Pi(\partial_\omega \chi) + (4\overline{\epsilon} - 2\epsilon + n^{-1}\nabla n)\cdot\chi\cdot\partial_\omega N + \partial_\omega N \cdot \beta. \tag{6.111}$$

Recall from (5.31) and (5.55) the following decompositions:

$$n\beta = \nabla\!\!\!/_{nL} P_1 + E_1,\ \nabla\!\!\!/(n\widehat{\chi}) = \nabla\!\!\!/_{nL} P_2 + E_2 \text{ where } \mathcal{N}_1(P_j) + \|E_j\|_{\mathcal{P}^0} \lesssim \varepsilon \text{ for } j = 1, 2.$$

Together with (6.111), this yields:

$$\nabla\!\!\!/_{nL}(\Pi(\partial_\omega \chi)) = -(2\chi + \overline{\delta})\cdot\Pi(\partial_\omega \chi) + F \cdot \nabla\!\!\!/_{nL}(P) + F \cdot E, \tag{6.112}$$

where F, P and E are given respectively by:

$$F = n\partial_\omega N,\ P = -P_1 + P_2,$$

and

$$E = -E_1 + E_2 + (4\overline{\epsilon} - 2\epsilon + n^{-1}\nabla n)\cdot\chi.$$

F satisfies:

$$\mathcal{N}_1(F) + \|F\|_{L^\infty_{x'} L^2_t} \lesssim (\|n\|_{L^\infty} + \mathcal{N}_1(n))(\|\partial_\omega N\|_{L^\infty} + \mathcal{N}_1(\partial_\omega N)) \lesssim \varepsilon, \tag{6.113}$$

where we used in the last inequality the estimate (2.67) for n and the estimates (2.76) and (2.77) for $\partial_\omega N$. P satisfies:

$$\mathcal{N}_1(P) \lesssim \mathcal{N}_1(P_1) + \mathcal{N}_1(P_2) \lesssim \varepsilon. \tag{6.114}$$

Finally, using the non sharp product estimate (5.15), E satisfies:

$$\|E\|_{\mathcal{P}^0} \lesssim \|E_1\|_{\mathcal{P}^0} + \|E_2\|_{\mathcal{P}^0} + (\mathcal{N}_1(\overline{\epsilon}) + \mathcal{N}_1(\epsilon) + \mathcal{N}_1(n^{-1}\nabla n))\mathcal{N}_1(\chi) \lesssim \varepsilon, \tag{6.115}$$

where we used in the last inequality the estimates (2.67)-(2.71) for $n, \epsilon, \overline{\epsilon}$ and χ. Now, (6.112)-(6.115) together with the sharp trace theorem estimate (5.22) yields:

$$\begin{aligned} \|\Pi(\partial_\omega \chi)\|_{\mathcal{B}^0} &\lesssim \big(\mathcal{N}_1(\chi) + \mathcal{N}_1(\overline{\delta}) + \|\chi\|_{L^\infty_{x'} L^2_t} + \|\overline{\delta}\|_{L^\infty_{x'} L^2_t}\big)\cdot\|\Pi(\partial_\omega \chi)\|_{\mathcal{P}^0} \\ &\quad + \big(\mathcal{N}_1(F) + \|F\|_{L^\infty_{x'} L^2_t}\big)\cdot\mathcal{N}_1(P) + \big(\mathcal{N}_1(F) + \|F\|_{L^\infty_{x'} L^2_t}\big)\cdot\|E\|_{\mathcal{P}^0} \\ &\lesssim \varepsilon\|\Pi(\partial_\omega \chi)\|_{\mathcal{P}^0} + \varepsilon, \end{aligned} \tag{6.116}$$

where we used the estimate (2.67)-(2.71) for $\overline{\delta}$ and χ in the last estimate.

Finally, (6.110) and (6.116) yield (6.109) which concludes the proof of the proposition. ∎

6.6 Estimate for $N(\cdot,\omega) - N(\cdot,\omega')$

The goal of this section is to prove (2.83). The following lemmas will be useful.

Lemma 6.23 *We have:*

$$\|Q_{>1}(N)\|_{L^\infty} \lesssim \varepsilon, \tag{6.117}$$

where Q_j is the geometric Littlewood-Paley decomposition on Σ_t introduced in section 3.6.

Lemma 6.24 *Let ω and ω' in $\mathbb{S}^2$. Let $N' = N(\cdot,\omega')$, and let $\mathcal{B}^0$ the Besov space defined with respect to $u(\cdot,\omega)$. We have:*

$$\|\nabla\!\!\!/ Q_{\leq 1}(N')\|_{\mathcal{B}^0} \lesssim \varepsilon. \tag{6.118}$$

Lemma 6.25 *Let ω and ω' in $\mathbb{S}^2$. Let $N' = N(\cdot,\omega')$, and let $L^2(\mathcal{H}_u)$ defined with respect to $u(\cdot,\omega)$. We have:*

$$\|\mathbf{D}_L(N')\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{6.119}$$

The proof of Lemma 6.23 is postponed to section C.12, the proof of Lemma 6.24 is postponed to section C.13, and the proof of Lemma 6.25 is postponed to section C.14. We now prove (2.83).

Let us define the angle $\omega_1 \in \mathbb{S}^2$ as:

$$\omega_1 = \frac{\omega - \omega'}{|\omega - \omega'|},$$

and let $N_1 = N(\cdot,\omega_1)$. In view of Lemma 6.23, we have:

$$\begin{aligned}
\|g(\partial_\omega N, N_1) - 1\|_{L^\infty} &\lesssim \|g(\partial_\omega N, Q_{\leq 1}(N_1)) - 1\|_{L^\infty} + \|g(\partial_\omega N, Q_{>1}(N_1))\|_{L^\infty} \\
&\lesssim \|g(\partial_\omega N, Q_{\leq 1}(N_1)) - 1\|_{L^\infty} + \|\partial_\omega N\|_{L^\infty}\|Q_{>1}(N_1)\|_{L^\infty} \\
&\lesssim \|g(\partial_\omega N, Q_{\leq 1}(N_1)) - 1\|_{L^\infty} + \varepsilon,
\end{aligned} \tag{6.120}$$

where we used the estimate (2.76) for $\partial_\omega N$.

Since $g(\partial_\omega N, Q_{\leq 1}(N_1)) - 1$ is a scalar function, we may estimate its L^∞ norm using (5.9):

$$\begin{aligned}
&\|g(\partial_\omega N, Q_{\leq 1}(N_1)) - 1\|_{L^\infty} \\
\lesssim\ &\|g(\partial_\omega N, Q_{\leq 1}(N_1)) - 1\|_{L^\infty_t L^2_{x'}} + \|\nabla\!\!\!/(g(\partial_\omega N, Q_{\leq 1}(N_1)) - 1)\|_{\mathcal{B}^0} \\
\lesssim\ &\|g(\partial_\omega N, N_1) - 1\|_{L^\infty_t L^2_{x'}} + \|\partial_\omega N\|_{L^\infty_t L^2_{x'}}\|Q_{>1}(N_1)\|_{L^\infty} + \|\nabla\!\!\!/ g(\partial_\omega N, Q_{\leq 1}(N_1))\|_{\mathcal{B}^0} \\
\lesssim\ &\|g(\partial_\omega N, N_1) - 1\|_{L^\infty_t L^2_{x'}} + \varepsilon + \|\nabla\!\!\!/ g(\partial_\omega N, Q_{\leq 1}(N_1))\|_{\mathcal{B}^0},
\end{aligned} \tag{6.121}$$

where we used the estimate (2.76) for $\partial_\omega N$ and Lemma 6.23 in the last inequality.

Next, we estimate the right-hand side of (6.121) starting with the last term. Using the estimate (5.10), we have:

$$\begin{aligned}
\|\nabla\!\!\!/ g(\partial_\omega N, Q_{\leq 1}(N_1))\|_{\mathcal{B}^0} &\lesssim (\|\nabla\!\!\!/ Q_{\leq 1}(N_1)\|_{L^\infty_t L^2_{x'}} + \|Q_{\leq 1}(N_1)\|_{L^\infty})\|\nabla\!\!\!/\partial_\omega N\|_{\mathcal{B}^0} \\
&\quad + (\|\nabla\!\!\!/\partial_\omega N\|_{L^\infty_t L^2_{x'}} + \|\partial_\omega N\|_{L^\infty})\|\nabla\!\!\!/ Q_{\leq 1}(N_1)\|_{\mathcal{B}^0} \\
&\lesssim \varepsilon
\end{aligned} \tag{6.122}$$

where we used in the last inequality the estimates (2.76) (2.77) for $\partial_\omega N$ and the estimate of Lemma 6.24 for $Q_{\leq 1}(N_1)$.

We consider the last term in the right-hand side of (6.121). Let $\omega'' \in \mathbb{S}^2$ on the arc joining ω and ω', and let $N = N(\cdot, \omega'')$. Then, with our choice for N_1, we have at $t = 0$ (see [21]):

$$\|g(\partial_\omega N, N_1) - 1\|_{L^2(P_{0,u})} \lesssim \varepsilon + |\omega - \omega'|,$$

which together with the estimate (3.66) for transport equations yields:

$$\begin{aligned} & \|g(\partial_\omega N, N_1) - 1\|_{L^\infty_t L^2_{x'}} \\ \lesssim\ & \|\nabla\!\!\!/_L(g(\partial_\omega N, N_1))\|_{L^2(\mathcal{H}_u)} + \varepsilon + |\omega - \omega'| \\ \lesssim\ & \|\mathbf{D}_L(\partial_\omega N)\|_{L^2(\mathcal{H}_u)} + \|\mathbf{D}_L(N_1)\|_{L^2(\mathcal{H}_u)} \|\partial_\omega N\|_{L^\infty} + \varepsilon + |\omega - \omega'| \\ \lesssim\ & \varepsilon + |\omega - \omega'|, \end{aligned} \tag{6.123}$$

where we used in the last inequality the estimates (2.76) and (2.77) for $\partial_\omega N$, and Lemma 6.25 for N_1.

Finally, (6.121)-(6.123) yield:

$$\|g(\partial_\omega N, N_1) - 1\|_{L^\infty} \lesssim \varepsilon + |\omega - \omega'|,$$

for any $N = N(\cdot, \omega'')$ with $\omega'' \in \mathbb{S}^2$ on the arc joining ω and ω'. This yields:

$$|g(N - N', N_1) - |\omega - \omega'|| \lesssim |\omega - \omega'|(\varepsilon + |\omega - \omega'|).$$

Therefore, we have:

$$|N - N'| \geq |g(N - N', N_1)| \geq |\omega - \omega'|(1 - O(\varepsilon) - O(|\omega - \omega'|)) \gtrsim |\omega - \omega'|,$$

which implies the desired estimate (2.83). This concludes the proof of Theorem 2.24.

7 Second order derivatives with respect to ω

The goal of this section is to prove Theorem 2.27.

7.1 Equation for $\mathbf{D}_L\partial^2_\omega N, \mathbf{D}_A\partial^2_\omega N, \mathbf{D}_{\underline{L}}\partial^2_\omega N$, $\partial^2_\omega \zeta$ and $\partial^2_\omega b$

The following lemma provides the formulas satisfied by $\mathbf{D}_L\partial^2_\omega N, \mathbf{D}_A\partial^2_\omega N$ and $\mathbf{D}_{\underline{L}}\partial^2_\omega N$.

Lemma 7.1 *$\partial^2_\omega N$ satisfies the following formulas:*

$$\begin{aligned} \mathbf{D}_L(\partial^2_\omega N) \ =\ & -2(\partial_\omega \chi)_{\partial_\omega N B} e_B - \chi_{\Pi(\partial^2_\omega N)B} e_B + 2\chi_{\partial_\omega N \partial_\omega N} N - (|\partial_\omega N|^2 n^{-1} N(n) \\ & + \eta_{\partial_\omega N \partial_\omega N} + \overline{\epsilon}_{\Pi(\partial^2_\omega N)}) L - \overline{\delta}\partial^2_\omega N - \overline{\epsilon}_{\partial_\omega N}\partial_\omega N + |\partial_\omega N|^2 \zeta_B e_B, \end{aligned} \tag{7.1}$$

$$\begin{aligned} \mathbf{D}_A(\partial^2_\omega N) \ =\ & (\partial^2_\omega \chi)_{AB} e_B - (\partial^2_\omega N)_A \zeta_B e_B - \chi_{A\Pi(\partial^2_\omega N)} N - \partial_\omega \chi_{A\partial_\omega N} N - 2\chi_{A\partial_\omega N}\partial_\omega N \\ & - (\partial_\omega N)_A \left(2\partial_\omega \zeta_B e_B - 2\zeta_{\partial_\omega N} N + \left(\epsilon_{\partial_\omega N} + n^{-1}\nabla_{\partial_\omega N} n\right) L + \delta \partial_\omega N\right) \\ & - \epsilon_A \partial^2_\omega N - 2\eta_{A\partial_\omega N}\partial_\omega N - \eta_{A\Pi(\partial^2_\omega N)} L + \epsilon_A |\partial_\omega N|^2 N, \end{aligned} \tag{7.2}$$

and

$$\begin{aligned}\mathbf{D}_{\underline{L}}(\partial_\omega^2 N) &= 2\partial_\omega^2\zeta_B e_B - 4\partial_\omega\zeta_{\partial_\omega N}N - 2\zeta_{\Pi(\partial_\omega^2 N)}N - |\partial_\omega N|^2\zeta_B e_B + 2\partial_\omega\chi_{\partial_\omega N B}e_B \\ &\quad +\chi_{\Pi(\partial_\omega^2 N)B}e_B - 2\chi_{\partial_\omega N\partial_\omega N}N + (\delta + n^{-1}N(n))\partial_\omega^2 N + (3\epsilon_{\partial_\omega N} \\ &\quad +n^{-1}\nabla_{\partial_\omega N}n)\partial_\omega N + (-|\partial_\omega N|^2\delta + \epsilon_{\Pi(\partial_\omega^2 N)} + n^{-1}\nabla_{\partial_\omega^2 N}n)L.\end{aligned} \tag{7.3}$$

Proof We first derive (7.1). We differentiate the equation (6.22) satisfied by $\partial_\omega N$ with respect to ω. Using the fact that $\partial_\omega L = \partial_\omega N$, we obtain:

$$\begin{aligned}&\mathbf{D}_L\partial_\omega^2 N + \mathbf{D}_{\partial_\omega N}\partial_\omega N \\ =\ & -\partial_\omega\chi_{\partial_\omega N B}e_B - \chi_{\Pi(\partial_\omega^2 N)B}e_B - \chi_{\partial_\omega N\partial_\omega e_B}e_B - \chi_{\partial_\omega N B}\partial_\omega e_B - \partial_\omega(\overline{\delta})\partial_\omega N - \overline{\delta}\partial_\omega^2 N \\ & -(k_{\partial_\omega N\partial_\omega N} + k_{N\partial_\omega^2 N} - n^{-1}\nabla_{\partial_\omega^2 N}n)L - \overline{\epsilon}_{\partial_\omega N}\partial_\omega N.\end{aligned} \tag{7.4}$$

We compute the various term in the right-hand side of (7.4). Using (6.1), we have:

$$\chi_{\partial_\omega N\partial_\omega e_B}e_B + \chi_{\partial_\omega N B}\partial_\omega e_B = -\chi_{\partial_\omega N\partial_\omega N}N. \tag{7.5}$$

Also, the formula (6.23) for $\mathbf{D}_A(\partial_\omega N)$ yields:

$$\mathbf{D}_{\partial_\omega N}\partial_\omega N = \partial_\omega\chi_{\partial_\omega N B}e_B - |\partial_\omega N|^2\zeta_B e_B - |\partial_\omega N|^2\delta L - \chi_{\partial_\omega N\partial_\omega N}N. \tag{7.6}$$

Now, differentiating twice $g(N,N) = 1$ with respect to ω yields:

$$\partial_\omega^2 N = \Pi(\partial_\omega^2 N) - |\partial_\omega N|^2 N. \tag{7.7}$$

Finally, (7.4), (7.5), (7.7) and the formula (6.26) for $\partial_\omega\overline{\delta}$ yields (7.1).

Next, we derive (7.2). We differentiate the equation (6.24) satisfied by $\mathbf{D}_A(\partial_\omega N)$ with respect to ω. Using the fact that $\partial_\omega L = \partial_\omega N$, we obtain:

$$\begin{aligned}&\mathbf{D}_A(\partial_\omega^2 N) + g(\partial_\omega e_A, N)\mathbf{D}_N(\partial_\omega N) \\ =\ & \partial_\omega^2\chi_{AB}e_B + g(\partial_\omega e_A, N)\partial_\omega\chi_{NB}e_B + \partial_\omega\chi_{A\partial_\omega e_B}e_B + \partial_\omega\chi_{AB}\partial_\omega e_B \\ & -g(\partial_\omega^2 N, e_A)\zeta_B e_B - g(\partial_\omega N, e_A)(\partial_\omega\zeta_B e_B + \zeta_{\partial_\omega B}e_B + \zeta_B\partial_\omega e_B) \\ & -\epsilon_A\partial_\omega^2 N - 2\eta_{A\partial_\omega N}\partial_\omega N - k_{A\partial_\omega^2 N}L - \partial_\omega\chi_{A\partial_\omega N}N - \chi_{A\Pi(\partial_\omega^2 N)} - \chi_{A\partial_\omega N}\partial_\omega N.\end{aligned} \tag{7.8}$$

We compute the various term in the right-hand side of (7.8). Using (6.1), we have:

$$\chi_{A\partial_\omega e_B}e_B + \chi_{AB}\partial_\omega e_B = -\chi_{A\partial_\omega N}N, \tag{7.9}$$

and

$$\zeta_{\partial_\omega B}e_B + \zeta_B\partial_\omega e_B = -\zeta_{\partial_\omega N}N. \tag{7.10}$$

Using the equations (7.1) and (7.2) respectively for $\mathbf{D}_L(\partial_\omega N)$ and $\mathbf{D}_{\underline{L}}(\partial_\omega N)$ together with the fact that $N = \frac{1}{2}(L - \underline{L})$ yields:

$$\mathbf{D}_N(\partial_\omega N) = -\partial_\omega\zeta_B e_B - \chi_{\partial_\omega N B}e_B - \delta\partial_\omega N - \left(\epsilon_{\partial_\omega N} + n^{-1}\nabla_{\partial_\omega N}n\right)L + \zeta_{\partial_\omega N}N. \tag{7.11}$$

Finally, (7.8)-(7.11) together with the fact that $g(\partial_\omega e_A, N) = -g(\partial_\omega N, e_A)$ and $\partial_\omega\chi_{NB} = -\chi_{\partial_\omega N B}$ yields (7.2).

Last, we derive (7.3). We differentiate the equation (6.23) satisfied by $\mathbf{D}_{\underline{L}}(\partial_\omega N)$ with respect to ω. Using the fact that $\partial_\omega L = \partial_\omega N$ and $\partial_\omega \underline{L} = -\partial_\omega N$, we obtain:

$$\begin{aligned}
&\mathbf{D}_{\underline{L}}(\partial_\omega^2 N) - \mathbf{D}_{\partial_\omega N}(\partial_\omega N) \qquad (7.12)\\
=\ & 2(\partial_\omega^2\zeta)_A e_A + 2\partial_\omega\zeta_{\partial_\omega e_A} e_A + 2\partial_\omega\zeta_A\partial_\omega e_A + \partial_\omega\chi_{\partial_\omega NA}e_A + \chi_{\Pi(\partial_\omega^2 N)A}e_A + \chi_{\partial_\omega N\partial_\omega e_A}e_A\\
&+\chi_{\partial_\omega NA}\partial_\omega e_A + (\delta + n^{-1}N(n))\partial_\omega^2 N + (\partial_\omega(\delta) + n^{-1}\nabla_{\partial_\omega N}n)\partial_\omega N\\
&+(\eta_{\partial_\omega N\partial_\omega N} + k_{\partial_\omega^2 NN} + n^{-1}\nabla_{\partial_\omega^2 N}n)L + (\epsilon_{\partial_\omega N} + n^{-1}\nabla_{\partial_\omega N}n)\partial_\omega N\\
&-2\partial_\omega\zeta_{\partial_\omega N}N - 2\zeta_{\Pi(\partial_\omega^2 N)}N - 2\zeta_{\partial_\omega N}\partial_\omega N.
\end{aligned}$$

We compute the various term in the right-hand side of (7.12). Using (6.1), we have:

$$\partial_\omega\zeta_{\partial_\omega A}e_A + \partial_\omega\zeta_A\partial_\omega e_A = -\partial_\omega\zeta_N\partial_\omega N - \partial_\omega\zeta_{\partial_\omega N}N = \zeta_{\partial_\omega N}\partial_\omega N - \partial_\omega\zeta_{\partial_\omega N}N, \qquad (7.13)$$

where we used the fact that $\partial_\omega\zeta_N = -\zeta_{\partial_\omega N}$. Also, contracting (7.9) with $\partial_\omega N$ yields:

$$\chi_{\partial_\omega N\partial_\omega e_A}e_A + \chi_{\partial_\omega NA}\partial_\omega e_A = -\chi_{\partial_\omega N\partial_\omega N}N. \qquad (7.14)$$

Finally, (7.12)-(7.14) together with (6.26) for $\partial_\omega(\delta)$, (7.6) and (7.7) yields (7.3). This concludes the proof of Lemma 7.1. ∎

The following lemma provides the transport equation satisfied by $\Pi(\partial_\omega^2\zeta)$.

Lemma 7.2 *$\Pi(\partial_\omega^2\zeta)$ satisfies the following transport equation:*

$$\begin{aligned}
&\nabla\!\!\!/_L(\Pi(\partial_\omega^2\zeta))_A \qquad (7.15)\\
=\ & -\chi_{AB}\partial_\omega^2\zeta_B - (\overline{\epsilon}_B + \zeta_B)\partial_\omega^2\chi_{AB} - \nabla\!\!\!/_{\partial_\omega^2 N}\zeta_A - \frac{(\partial_\omega^2 N)_B}{2}(-\alpha_{AB} + \rho\delta_{AB} + 3\sigma\in_{AB})\\
&+\overline{\epsilon}_A\zeta_{\Pi(\partial_\omega^2 N)} - \chi_{AB}\eta_{B\Pi(\partial_\omega^2 N)} - (\partial_\omega^2 N)_A\overline{\epsilon}\cdot\zeta - 2\nabla\!\!\!/_{\partial_\omega N}(\Pi(\partial_\omega\zeta))_A\\
&+(\partial_\omega N)_A(\nabla\!\!\!/_L\zeta_{\partial_\omega N} - \chi_{\partial_\omega NB}\zeta_B - \overline{\delta}\zeta_{\partial_\omega N} - 2\overline{\epsilon}\cdot\partial_\omega\zeta - \theta_{\partial_\omega NB}\zeta_B - \eta_{\partial_\omega NB}\zeta_B)\\
&-2(\eta_{B\partial_\omega N} + \partial_\omega\zeta_B)\partial_\omega\chi_{AB} + \overline{\delta}\partial_\omega\chi_{A\partial_\omega N} - 2\overline{\epsilon}_A\partial_\omega\zeta_{\partial_\omega N} + (-3\zeta_{\partial_\omega N} + \epsilon_{\partial_\omega N} - \overline{\epsilon}_{\partial_\omega N})\chi_{A\partial_\omega N}\\
&+|\partial_\omega N|^2\epsilon_B\chi_{AB} + (\theta_{A\partial_\omega N} + \eta_{\partial_\omega NA} - (\partial_\omega N)_A\overline{\delta})\zeta_{\partial_\omega N} + \frac{|\partial_\omega N|^2}{2}\beta_A\\
&+\frac{(\partial_\omega N)_B}{2}\Bigg((\partial_\omega N)_C(\in_{AC}{}^*\beta_B + \in_{BC}{}^*\beta_A) - \delta_{AB}(\beta_{\partial_\omega N} + \underline{\beta}_{\partial_\omega N})\\
&+\frac{3}{2}\in_{AB}({}^*\beta_{\partial_\omega N} - {}^*\underline{\beta}_{\partial_\omega N})\Bigg).
\end{aligned}$$

Proof We differentiate the equation (6.28) satisfied by $\mathbf{D}_{\underline{L}}(\partial_\omega N)$ with respect to ω:

$$\begin{aligned}
&\partial_\omega(\nabla\!\!\!/_L(\Pi(\partial_\omega\zeta)))_A \qquad (7.16)\\
=\ & -\partial_\omega(\nabla\!\!\!/_{\partial_\omega N}\zeta)_A + (\eta_{\partial_\omega NA} + g(\partial_\omega e_A, N)\overline{\delta})\zeta_{\partial_\omega N} + \overline{\epsilon}_A\partial_\omega\zeta_{\partial_\omega N} + \overline{\epsilon}_A\zeta_{\Pi(\partial_\omega^2 N)} - (\partial_\omega^2 N)_A\overline{\epsilon}\cdot\zeta\\
&-(\partial_\omega N)_A\partial_\omega(\overline{\epsilon}\cdot\zeta) - \partial_\omega(k_{B\partial_\omega N} + \partial_\omega\zeta_B)\chi_{AB} - (k_{B\partial_\omega N} + \partial_\omega\zeta_B)(\partial_\omega\chi_{AB} + \chi_{A\partial_\omega e_B})\\
&-\partial_\omega(\overline{\epsilon}_B + \zeta_B)\partial_\omega\chi_{AB} - (\overline{\epsilon}_B + \zeta_B)(\partial_\omega^2\chi_{AB} + \partial_\omega\chi_{A\partial_\omega e_B})\\
&-\partial_\omega\left(\frac{(\partial_\omega N)_B}{2}(-\alpha_{\cdot B} + \rho\delta_{\cdot B} + 3\sigma\in_{\cdot B})\right)_A
\end{aligned}$$

We compute the various term in the right-hand side of (7.16). We have:

$$\partial_\omega^2\zeta_A = \partial_\omega(\Pi(\partial_\omega\zeta))_A - \zeta_{\partial_\omega N}(\partial_\omega N)_A,$$

which yields:

$$\begin{aligned}\nabla\!\!\!/_L(\Pi(\partial_\omega(\Pi(\partial_\omega\zeta))))_A &= \nabla\!\!\!/_L(\Pi(\partial_\omega^2\zeta))_A - (\nabla\!\!\!/_L\zeta)_{\partial_\omega N}(\partial_\omega N)_A \\ &\quad -\zeta_{\nabla\!\!\!/_{L(\partial_\omega N)}}(\partial_\omega N)_A - \zeta_{\partial_\omega N}g(\nabla\!\!\!/_L(\partial_\omega N), e_A) \\ &= \nabla\!\!\!/_L(\Pi(\partial_\omega^2\zeta))_A - (\nabla\!\!\!/_L\zeta)_{\partial_\omega N}(\partial_\omega N)_A \\ &\quad +\chi_{\partial_\omega N B}\zeta_B(\partial_\omega N)_A + \overline{\delta}\zeta_{\partial_\omega N}(\partial_\omega N)_A + \zeta_{\partial_\omega N}\chi_{\partial_\omega N A} + \overline{\delta}\zeta_{\partial_\omega N}(\partial_\omega N)_A,\end{aligned} \tag{7.17}$$

where we used the formula (6.22) for $\mathbf{D}_L(\partial_\omega N)$ in the last equality. Using the commutator formula (6.15) together with (7.17) yields:

$$\begin{aligned}&\partial_\omega(\nabla\!\!\!/_L(\Pi(\partial_\omega\zeta)))_A \\ =\ & \nabla\!\!\!/_L(\Pi\partial_\omega(\Pi(\partial_\omega\zeta)))_A + \nabla\!\!\!/_{\partial_\omega N}\Pi(\partial_\omega\zeta)_A - \overline{\epsilon}_A\partial_\omega\zeta_{\partial\omega N} + (\partial_\omega N)_A\overline{\epsilon}\cdot\partial_\omega\zeta \\ =\ & \nabla\!\!\!/_L(\Pi(\partial_\omega^2\zeta))_A - (\nabla\!\!\!/_L\zeta)_{\partial_\omega N}(\partial_\omega N)_A + \chi_{\partial_\omega N B}\zeta_B(\partial_\omega N)_A + \overline{\delta}\zeta_{\partial_\omega N}(\partial_\omega N)_A \\ & +\zeta_{\partial_\omega N}\chi_{\partial_\omega N A} + \overline{\delta}\zeta_{\partial_\omega N}(\partial_\omega N)_A + \nabla\!\!\!/_{\partial_\omega N}\Pi(\partial_\omega\zeta)_A - \overline{\epsilon}_A\partial_\omega\zeta_{\partial\omega N} + (\partial_\omega N)_A\overline{\epsilon}\cdot\partial_\omega\zeta.\end{aligned} \tag{7.18}$$

Next, we compute the term $\partial_\omega(\nabla\!\!\!/_{\partial_\omega N}\zeta)_A$. We have:

$$\begin{aligned}\partial_\omega(\nabla\!\!\!/_{\partial_\omega N}\zeta_A) &= \partial_\omega(\mathbf{D}_{\partial_\omega N}(\zeta_A) - \zeta_{\nabla\!\!\!/_{\partial_\omega N e_A}}) \\ &= \mathbf{D}_{\partial_\omega^2 N}(\zeta_A) + \mathbf{D}_{\partial_\omega N}(\partial_\omega\zeta_A) + \mathbf{D}_{\partial_\omega N}(\zeta_{\Pi(\partial_\omega e_A)}) - \partial_\omega\zeta_{\nabla\!\!\!/_{\partial_\omega N e_A}} - \zeta_{\partial_\omega(\nabla\!\!\!/_{\partial_\omega N e_A})} \\ &= \mathbf{D}_{\partial_\omega^2 N}(\zeta_A) + \nabla\!\!\!/_{\partial_\omega N}(\Pi(\partial_\omega\zeta))_A + \mathbf{D}_{\partial_\omega N}(\zeta_{\Pi(\partial_\omega e_A)}) - \zeta_{\partial_\omega(\nabla\!\!\!/_{\partial_\omega N e_A})}\end{aligned} \tag{7.19}$$

Now, (6.18) implies:

$$\begin{aligned}\partial_\omega(\nabla\!\!\!/_{\partial_\omega N}e_A) &= \partial_\omega((\partial_\omega N)_B\nabla\!\!\!/_B e_A) \\ &= (\partial_\omega^2 N)_B\nabla\!\!\!/_B(e_A) + g(\partial_\omega N, \partial_\omega e_B)\nabla\!\!\!/_B(e_A) + (\partial_\omega N)_B(\nabla\!\!\!/_{\partial_\omega e_B}(e_A) \\ &\quad +\nabla\!\!\!/_B(\Pi(\partial_\omega e_A)) - (\partial_\omega N)_A\theta_{BC}e_C + \theta_{AB}\partial_\omega N - (\mathbf{D}_B e_A, \partial_\omega N)N) \\ &= \nabla\!\!\!/_{\Pi(\partial_\omega^2 N)}(e_A) - |\partial_\omega N|^2\nabla\!\!\!/_N(e_A) + \nabla\!\!\!/_{\partial_\omega N}(\Pi(\partial_\omega e_A)) - (\partial_\omega N)_A\theta_{\partial_\omega N C}e_C \\ &\quad +\theta_{A\partial_\omega N}\partial_\omega N - (\mathbf{D}_{\partial_\omega N}e_A, \partial_\omega N)N) \\ &= \nabla\!\!\!/_{\partial_\omega^2 N}(e_A) + \nabla\!\!\!/_{\partial_\omega N}(\Pi(\partial_\omega e_A)) - (\partial_\omega N)_A\theta_{\partial_\omega N C}e_C + \theta_{A\partial_\omega N}\partial_\omega N \\ &\quad -(\mathbf{D}_{\partial_\omega N}e_A, \partial_\omega N)N),\end{aligned}$$

where we used (7.7) in the last equality. Together with (7.19), we obtain:

$$\begin{aligned}\partial_\omega(\nabla\!\!\!/_{\partial_\omega N}\zeta_A) &= \nabla\!\!\!/_{\partial_\omega^2 N}(\zeta)_A + \nabla\!\!\!/_{\partial_\omega N}(\Pi(\partial_\omega\zeta))_A + \nabla\!\!\!/_{\partial_\omega N}(\zeta)_{\Pi(\partial_\omega e_A)} \\ &\quad +(\partial_\omega N)_A\theta_{\partial_\omega N B}\zeta_B - \theta_{A\partial_\omega N}\zeta_{\partial_\omega N},\end{aligned}$$

which yields:

$$\partial_\omega(\nabla\!\!\!/_{\partial_\omega N}\zeta)_A = \nabla\!\!\!/_{\partial_\omega^2 N}(\zeta)_A + \nabla\!\!\!/_{\partial_\omega N}(\Pi(\partial_\omega\zeta))_A + (\partial_\omega N)_A\theta_{\partial_\omega N B}\zeta_B - \theta_{A\partial_\omega N}\zeta_{\partial_\omega N}. \tag{7.20}$$

Next, we compute $\partial_\omega(\overline{\epsilon}\cdot\zeta)$. Using (6.1), we have:

$$\begin{aligned}\partial_\omega(\overline{\epsilon}\cdot\zeta) &= (\eta_{\partial_\omega N B}+k_{N\partial_\omega e_B}-n^{-1}\nabla_{\partial_\omega e_B}n)\zeta_B+\overline{\epsilon}_B(\partial_\omega\zeta_B+\zeta_{\partial_\omega e_B}) \\ &= \eta_{\partial_\omega N e_B}\zeta_B-\overline{\delta}\zeta_{\partial_\omega N}+\overline{\epsilon}\cdot\partial_\omega\zeta.\end{aligned} \tag{7.21}$$

Using again (6.1), we also obtain:

$$\begin{aligned}&-\partial_\omega(k_{B\partial_\omega N}+\partial_\omega\zeta_B)\chi_{AB}-(k_{B\partial_\omega N}+\partial_\omega\zeta_B)(\partial_\omega\chi_{AB}+\chi_{A\partial_\omega e_B}) \\ &-\partial_\omega(\overline{\epsilon}_B+\zeta_B)\partial_\omega\chi_{AB}-(\overline{\epsilon}_B+\zeta_B)(\partial_\omega^2\chi_{AB}+\partial_\omega\chi_{A\partial_\omega e_B}) \\ =\ &-(k_{B\partial_\omega^2 N}+\partial_\omega^2\zeta_B)\chi_{AB}+(k_{N\partial_\omega N}+\partial_\omega\zeta_N)\chi_{A\partial_\omega N}-(\eta_{B\partial_\omega N}+\partial_\omega\zeta_B)\partial_\omega\chi_{AB} \\ &-(k_{\partial_\omega N B}+\partial_\omega\zeta_B)\partial_\omega\chi_{AB}+(k_{NN}-n^{-1}N(n))\partial_\omega\chi_{A\partial_\omega N}+(\overline{\epsilon}_{\partial_\omega N}+\zeta_{\partial_\omega N})\partial_\omega\chi_{AN} \\ &-(\overline{\epsilon}_B+\zeta_B)\partial_\omega^2\chi_{AB} \\ =\ &-(\eta_{B\Pi(\partial_\omega^2 N)}-|\partial_\omega N|^2\epsilon_B+\partial_\omega^2\zeta_B)\chi_{AB}+(\epsilon_{\partial_\omega N}-\zeta_{\partial_\omega N})\chi_{A\partial_\omega N} \\ &-2(\eta_{B\partial_\omega N}+\partial_\omega\zeta_B)\partial_\omega\chi_{AB}+\overline{\delta}\partial_\omega\chi_{A\partial_\omega N}-(\overline{\epsilon}_{\partial_\omega N}+\zeta_{\partial_\omega N})\chi_{A\partial_\omega N}-(\overline{\epsilon}_B+\zeta_B)\partial_\omega^2\chi_{AB},\end{aligned} \tag{7.22}$$

where we used the fact that $\partial_\omega\zeta_N=-\zeta_{\partial_\omega N}$, $\partial_\omega\chi_{AN}=-\chi_{A\partial_\omega N}$ and the decomposition of $\partial_\omega^2 N$ (7.7) in the last inequality.

Finally, we consider the last term in the right-hand side of (7.16). From the definition of β, ρ, σ, and the fact that $\partial_\omega L=\partial_\omega N$ and $\partial_\omega\underline{L}=-\partial_\omega N$, we have:

$$\partial_\omega\rho=-\beta_{\partial_\omega N}-\underline{\beta}_{\partial_\omega N},\ \partial_\omega(\in\sigma)_{AB}=\frac{1}{2}\in_{AB}({}^*\beta_{\partial_\omega N}-{}^*\underline{\beta}_{\partial_\omega N}),$$

which together with the formula (6.29) for $\partial_\omega\alpha$ yields:

$$\begin{aligned}&\partial_\omega\left(\frac{(\partial_\omega N)_B}{2}(-\alpha_{.B}+\rho\delta_{.B}+3\sigma\in_{.B})\right)_A \\ =\ &\frac{1}{2}(\partial_\omega^2 N)_B(-\alpha_{AB}+\rho\delta_{AB}+3\sigma\in_{AB})-\frac{|\partial_\omega N|^2}{2}\beta_A+\frac{(\partial_\omega N)_B}{2}\Big((\partial_\omega N)_C(\in_{AC}{}^*\beta_B \\ &+\in_{BC}{}^*\beta_A)-\delta_{AB}(\beta_{\partial_\omega N}+\underline{\beta}_{\partial_\omega N})+\frac{3}{2}\in_{AB}({}^*\beta_{\partial_\omega N}-{}^*\underline{\beta}_{\partial_\omega N})\Big).\end{aligned} \tag{7.23}$$

Using (7.16)-(7.23) yields (7.15) which concludes the proof of Lemma 7.2. ■

Finally, the following lemma provides the transport equation satisfied by $\partial_\omega^2 b$.

Lemma 7.3 *$\partial_\omega^2 b$ satisfies the following transport equation:*

$$\begin{aligned}L(\partial_\omega^2 b) &= -\nabla\!\!\!/_{\partial_\omega N}(\partial_\omega b)-b\partial_\omega\zeta_{\partial_\omega N}-b\zeta_{\Pi(\partial_\omega^2 N)}-\partial_\omega(b)\zeta_{\partial_\omega N}-\partial_\omega^2(b)\overline{\delta} \\ &\quad -\partial_\omega(b)(2\epsilon_{\partial_\omega N}-n^{-1}\nabla\!\!\!/_{\partial_\omega N}n)-k_{\partial_\omega N\partial_\omega N}b-\overline{\epsilon}_{\Pi(\partial_\omega^2 N)}b-\overline{\epsilon}_{\partial_\omega N}\partial_\omega b,\end{aligned} \tag{7.24}$$

Proof Recall the transport equation (6.39) satisfied by $\partial_\omega b$

$$L(\partial_\omega b)=-b\zeta_{\partial_\omega N}-\partial_\omega(b)\overline{\delta}-\overline{\epsilon}_{\partial_\omega N}b.$$

Differentiating with respect to ω yields (7.24). This concludes the proof of the Lemma. ■

7.2 Estimates for $\partial_\omega^2 N, \partial_\omega^2 b, \partial_\omega^2 \chi$ and $\partial_\omega^2 \zeta$

7.2.1 Estimates for $\partial_\omega^2 N$

In view of the formula (7.1) for $\mathbf{D}_L(\partial_\omega^2 N)$, we have:

$$\begin{aligned} & \|\mathbf{D}_L(\partial_\omega^2 N)\|_{L^2(\mathcal{H}_u)} \qquad (7.25)\\ \lesssim\ & \|\partial_\omega \chi\|_{L^2(\mathcal{H}_u)}\|\partial_\omega N\|_{L^\infty} + (\|\chi\|_{L^\infty_{x'}L^2_t} + \|\overline{\epsilon}\|_{L^\infty_{x'}L^2_t} + \|\overline{\delta}\|_{L^\infty_{x'}L^2_t})\|\Pi(\partial_\omega^2 N)\|_{L^2_{x'}L^\infty_t}\\ & +(\|\chi\|_{L^2(\mathcal{H}_u)} + \|n^{-1}\nabla n\|_{L^2(\mathcal{H}_u)} + \|\eta\|_{L^2(\mathcal{H}_u)} + \|\overline{\epsilon}\|_{L^2(\mathcal{H}_u)} + \|\zeta\|_{L^2(\mathcal{H}_u)})\|\partial_\omega N\|^2_{L^\infty}\\ \lesssim\ & \varepsilon + \varepsilon\|\Pi(\partial_\omega^2 N)\|_{L^2_{x'}L^\infty_t}, \end{aligned}$$

where we used in the last inequality the estimates (2.67)-(2.72) for $n, \eta, \overline{\epsilon}, \overline{\delta}, \chi$ and ζ, the estimate (2.76) for $\partial_\omega N$ and the estimate (2.77) for $\partial_\omega \chi$. Now, the decomposition (7.7) for $\partial_\omega^2 N$ yields:

$$\mathbf{D}_L(\partial_\omega^2 N) = \mathbf{D}_L(\Pi(\partial_\omega^2 N)) - |\partial_\omega N|^2 \mathbf{D}_L N - 2g(\partial_\omega N, \mathbf{D}_L(\partial_\omega N))N,$$

which together with (7.25) and the estimates (2.76) (2.77) for $\partial_\omega N$ yields:

$$\|\not\nabla_L(\Pi(\partial_\omega^2 N))\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon + \varepsilon\|\Pi(\partial_\omega^2 N)\|_{L^2_{x'}L^\infty_t}.$$

Together with the estimate (3.66) for transport equations, this implies:

$$\|\Pi(\partial_\omega^2 N)\|_{L^2_{x'}L^\infty_t} \lesssim 1,$$

and using again the decomposition (7.7) for $\partial_\omega^2 N$ and the estimate (2.76) for $\partial_\omega N$, we obtain:

$$\|\partial_\omega^2 N\|_{L^2_{x'}L^\infty_t} \lesssim 1. \qquad (7.26)$$

Finally, (7.25) and (7.26) imply:

$$\|\mathbf{D}_L(\partial_\omega^2 N)\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon. \qquad (7.27)$$

7.2.2 estimate for $\partial_\omega^2 b$

In view of the transport equation (7.24), we have

$$L(\partial_\omega^2 b) = f, \qquad (7.28)$$

where the scalar f is given by

$$\begin{aligned} f \ =\ & -\not\nabla_{\partial_\omega N}(\partial_\omega b) - b\partial_\omega \zeta_{\partial_\omega N} - b\zeta_{\Pi(\partial_\omega^2 N)} - \partial_\omega(b)\zeta_{\partial_\omega N} - \partial_\omega^2(b)\overline{\delta}\\ & -\partial_\omega(b)(2\epsilon_{\partial_\omega N} - n^{-1}\not\nabla_{\partial_\omega N} n) - k_{\partial_\omega N \partial_\omega N} b - \overline{\epsilon}_{\Pi(\partial_\omega^2 N)} b - \overline{\epsilon}_{\partial_\omega N}\partial_\omega b. \end{aligned}$$

In view of the definition of f, we have

$$\begin{aligned} \|f\|_{L^2_{x'}L^1_t} \ \lesssim\ & (1 + \|\partial_\omega N\|_{L^\infty} + \|b\|_{L^\infty} + \|\partial_\omega b\|_{L^\infty})^3 (1 + \|\zeta\|_{L^\infty_{x'}L^2_t} + \|\epsilon\|_{L^\infty_{x'}L^2_t})\\ & \times(\|\not\nabla \partial_\omega b\|_{L^2(\mathcal{H}_u)} + \|\partial_\omega \zeta\|_{L^2(\mathcal{H}_u)} + \|\partial_\omega^2 N\|_{L^2(\mathcal{H}_u)} + \|k\|_{L^2(\mathcal{H}_u)}\\ & +\|n^{-1}\nabla n\|_{L^2(\mathcal{H}_u)} + \|\zeta\|_{L^2(\mathcal{H}_u)}) + \|\overline{\delta}\|_{L^\infty_{x'}L^2_t}\|\partial_\omega^2 b\|_{L^2(\mathcal{H}_u)}\\ \lesssim\ & \varepsilon + \varepsilon\|\partial_\omega^2 b\|_{L^2(\mathcal{H}_u)}, \qquad (7.29) \end{aligned}$$

where we used in the last inequality the estimates (2.67) and (2.68) for n, k, ϵ and $\overline{\delta}$, the estimate (2.69) for b, the estimate (2.72) for ζ, the estimate (2.76) for $\partial_\omega N$, the estimate (2.77) for $\partial_\omega b$ and $\partial_\omega \zeta$, and the estimate (7.26) for $\partial_\omega^2 N$. (7.28), (7.29) together with the estimate for transport equations (3.66) yield

$$\|\partial_\omega^2 b\|_{L^\infty_{x'} L^2_t} \lesssim \varepsilon + \varepsilon \|\partial_\omega^2 b\|_{L^2(\mathcal{H}_u)},$$

which implies

$$\|\partial_\omega^2 b\|_{L^\infty_{x'} L^2_t} \lesssim \varepsilon. \tag{7.30}$$

7.2.3 Estimates for $\partial_\omega^2 \chi$

In view of (7.2), we have:

$$g(\mathbf{D}_A(\partial_\omega^2 N), e_B) = (\partial_\omega^2 \chi)_{AB} + F_{AB}, \tag{7.31}$$

where the $P_{t,u}$-tangent 2-tensor F is given by:

$$F_{AB} = -(\partial_\omega^2 N)_A \zeta_B e_B - 2\chi_{A\partial_\omega N}(\partial_\omega N)_B - (\partial_\omega N)_A \left(2\partial_\omega \zeta_B + 2\delta(\partial_\omega N)_B\right).$$

F satisfies the following estimate:

$$\begin{aligned} & \|F\|_{L^\infty_t L^{\frac{4}{3}}_{x'}} \qquad (7.32)\\ \lesssim\ & \|\partial_\omega^2 N\|_{L^\infty_t L^2_{x'}} \|\zeta\|_{L^\infty_t L^4_{x'}} + (\|\chi\|_{L^\infty_t L^2_{x'}} + \|\delta\|_{L^\infty_t L^2_{x'}}) \|\partial_\omega N\|^2_{L^\infty} + \|\partial_\omega N\|_{L^\infty} \|\partial_\omega \zeta\|_{L^\infty_t L^2_{x'}} \\ \lesssim\ & \varepsilon, \end{aligned}$$

where we used in the last inequality the estimates (2.68)-(2.72) for δ, χ and ζ, the estimates (2.76) (2.77) for $\partial_\omega N$ and $\partial_\omega \zeta$, and the estimate (7.26) for $\partial_\omega^2 N$.

Using the decomposition (7.7), we have:

$$\mathbf{D}_A(\partial_\omega^2 N) = \mathbf{D}_A(\Pi(\partial_\omega^2 N)) - 2g(\partial_\omega N, \mathbf{D}_A \partial_\omega N) N - |\partial_\omega N|^2 \mathbf{D}_A N$$

which together with the fact that $\mathbf{D}_A N = \theta_{AB} e_B$ yields:

$$g(\nabla\!\!\!/_A(\Pi(\partial_\omega^2 N)), e_B) = g(\mathbf{D}_A(\partial_\omega^2 N), e_B) + |\partial_\omega N|^2 \theta_{AB}.$$

Together with (7.31), this yields:

$$\nabla\!\!\!/(\Pi(\partial_\omega^2 N)) = \Pi(\partial_\omega^2 \chi) + \tilde{F} \tag{7.33}$$

where $\widetilde{F} = F + |\partial_\omega N|^2 \theta$. In view of (7.32) and the estimates (2.68)-(2.71) for $\theta = \chi + \eta$, we have:

$$\|\widetilde{F}\|_{L^\infty_t L^{\frac{4}{3}}_{x'}} \lesssim \|F\|_{L^\infty_t L^{\frac{4}{3}}_{x'}} + \|\theta\|_{L^\infty_t L^2_{x'}} \|\partial_\omega N\|^2_{L^\infty} \lesssim \varepsilon. \tag{7.34}$$

Using (7.33) together with the finite band property and the weak Bernstein inequality for P_j, we have:

$$\begin{aligned} \|P_j \Pi(\partial_\omega^2 \chi)\|_{L^\infty_t L^2_{x'}} \lesssim\ & \|P_j \nabla\!\!\!/(\Pi(\partial_\omega^2 N))\|_{L^\infty_t L^2_{x'}} + \|P_j \tilde{F}\|_{L^\infty_t L^2_{x'}} \qquad (7.35)\\ \lesssim\ & 2^j \|\Pi(\partial_\omega^2 N)\|_{L^\infty_t L^2_{x'}} + 2^{\frac{j}{2}} \|\tilde{F}\|_{L^\infty_t L^{\frac{4}{3}}_{x'}} \\ \lesssim\ & 2^j \varepsilon, \end{aligned}$$

where we used the estimate (7.26) for $\partial_\omega^2 N$, and the estimate (7.34) for $\widetilde{F}$. (7.35) is the desired estimate for $\partial_\omega^2\chi$.

Remark 7.4 *While $\widetilde{F}$ satisfies* (7.34)*, we may also derive a second estimate. We have:*

$$\begin{aligned}\|\widetilde{F}\|_{L^1_{x'}L^\infty_t} &\lesssim \|\partial_\omega^2 N\|_{L^2_{x'}L^\infty_t}\|\zeta\|_{L^2_{x'}L^\infty_t} + (\|\chi\|_{L^2_{x'}L^\infty_t} + \|\delta\|_{L^2_{x'}L^\infty_t} + \|\theta\|_{L^2_{x'}L^\infty_t})\|\partial_\omega N\|^2_{L^\infty} \\ &\quad + \|\partial_\omega N\|_{L^\infty}\|\partial_\omega\zeta\|_{L^2_t L^\infty_{x'}} \\ &\lesssim \varepsilon, \end{aligned} \tag{7.36}$$

where we used in the last inequality the estimates (2.68)-(2.72) *for δ, χ, θ and ζ, the estimates* (2.76) (2.77) *for $\partial_\omega N$ and $\partial_\omega\zeta$, and the estimate* (7.26) *for $\partial_\omega^2 N$.*

7.2.4 estimate for $\partial_\omega^2\zeta$

In view of the formula (7.15) for $\nabla\!\!\!/_L(\Pi(\partial_\omega^2\zeta))$, the decomposition (7.7) for $\partial_\omega^2 N$, and the decomposition (7.33) for $\partial_\omega^2\chi$, we have:

$$\nabla\!\!\!/_L(\Pi(\partial_\omega^2\zeta)) = -\chi\cdot\Pi(\partial_\omega^2\zeta) + \nabla\!\!\!/(F_1) + F_2 - \frac{|\partial_\omega N|^2}{2}\nabla\!\!\!/_{\underline{L}}(\zeta), \tag{7.37}$$

where the $P_{t,u}$-tangent tensors F_1 and F_2 are respectively given by:

$$F_1 = -(\overline{\epsilon}+\zeta)\cdot\Pi(\partial_\omega^2 N) - 2\partial_\omega N\cdot\Pi(\partial_\omega\zeta)$$

and

$$\begin{aligned}(F_2)_A &= (\nabla\!\!\!/(\overline{\epsilon}) + \nabla\!\!\!/(\zeta))\cdot\Pi(\partial_\omega^2 N) + (\overline{\epsilon}+\zeta)\cdot\widetilde{F} - \nabla\!\!\!/_{\Pi(\partial_\omega^2 N)}\zeta_A + \frac{|\partial_\omega N|^2}{2}\nabla\!\!\!/_L(\zeta) \\ &\quad -\frac{(\partial_\omega^2 N)_B}{2}(-\alpha_{AB} + \rho\delta_{AB} + 3\sigma\in_{AB}) \\ &\quad +\overline{\epsilon}_A\zeta_{\Pi(\partial_\omega^2 N)} - \chi_{AB}\eta_{B\Pi(\partial_\omega^2 N)} - (\partial_\omega^2 N)_A\overline{\epsilon}\cdot\zeta + 2\mathrm{div}\!\!\!/(\partial_\omega N)(\Pi(\partial_\omega\zeta))_A \\ &\quad +(\partial_\omega N)_A(\nabla\!\!\!/_L\zeta_{\partial_\omega N} - \chi_{\partial_\omega N B}\zeta_B - \overline{\delta}\zeta_{\partial_\omega N} - 2\overline{\epsilon}\cdot\partial_\omega\zeta - \theta_{\partial_\omega N B}\zeta_B - \eta_{\partial_\omega N B}\zeta_B) \\ &\quad -2(\eta_{B\partial_\omega N} + \partial_\omega\zeta_B)\partial_\omega\chi_{AB} + \overline{\delta}\partial_\omega\chi_{A\partial_\omega N} - 2\overline{\epsilon}_A\partial_\omega\zeta_{\partial_\omega N} + (-3\zeta_{\partial_\omega N} + \epsilon_{\partial_\omega N} \\ &\quad -\overline{\epsilon}_{\partial_\omega N})\chi_{A\partial_\omega N} + |\partial_\omega N|^2\epsilon_B\chi_{AB} + (\theta_{A\partial_\omega N} + \eta_{\partial_\omega N A} - (\partial_\omega N)_A\overline{\delta})\zeta_{\partial_\omega N} + \frac{|\partial_\omega N|^2}{2}\beta_A \\ &\quad +\frac{(\partial_\omega N)_B}{2}\bigg((\partial_\omega N)_C(\in_{AC}{}^*\beta_B + \in_{BC}{}^*\beta_A) - \delta_{AB}(\beta_{\partial_\omega N} + \underline{\beta}_{\partial_\omega N}) \\ &\quad +\frac{3}{2}\in_{AB}({}^*\beta_{\partial_\omega N} - {}^*\underline{\beta}_{\partial_\omega N})\bigg).\end{aligned}$$

We estimate F_1 and F_2. For F_1, we have:

$$\begin{aligned}\|F_1\|_{L^2(\mathcal{H}_u)} &\lesssim (\|\overline{\epsilon}\|_{L^\infty_{x'}L^2_t} + \|\zeta\|_{L^\infty_{x'}L^2_t})\|\Pi(\partial_\omega^2 N)\|_{L^2_{x'}L^\infty_t} + \|\partial_\omega N\|_{L^\infty}\|\Pi(\partial_\omega\zeta)\|_{L^2(\mathcal{H}_u)} \\ &\lesssim \varepsilon, \end{aligned} \tag{7.38}$$

where we used the estimates (2.67)-(2.72) for $\overline{\epsilon}$ and ζ, the estimates (2.76) (2.77) for $\partial_\omega N$ and $\partial_\omega \zeta$, and the estimate (7.26) for $\partial_\omega^2 N$. For F_2, we have:

$$\begin{aligned}
&\|F_2\|_{L^1_{x'}L^2_t} \qquad (7.39)\\
\lesssim\ & (\|\nabla\!\!\!/\overline{\epsilon}\|_{L^2(\mathcal{H}_u)} + (\|\overline{\epsilon}\|L^\infty_{x'}L^2_t + \|\zeta\|_{L^\infty_{x'}L^2_t})\|\widetilde{F}\|_{L^1_{x'}L^\infty_t} + \|\partial_\omega^2 N\|_{L^2_{x'}L^\infty_t}\bigg(\|\nabla\!\!\!/\zeta\|_{L^2(\mathcal{H}_u)}\\
&+\|\alpha\|_{L^2(\mathcal{H}_u)} + \|\rho\|_{L^2(\mathcal{H}_u)} + \|\sigma\|_{L^2(\mathcal{H}_u)} + \mathcal{N}_1(\overline{\epsilon})\mathcal{N}_1(\zeta) + \mathcal{N}_1(\chi)\mathcal{N}_1(\eta)\bigg)\\
&+\|\nabla\!\!\!/\partial_\omega N\|_{L^2_{x'}L^\infty_t}\|\partial_\omega\zeta\|_{L^2_{x'}L^\infty_t} + \|\partial_\omega N\|^2_{L^\infty}\bigg(\|\nabla\!\!\!/_L(\zeta)\|_{L^2(\mathcal{H}_u)} + (\mathcal{N}_1(\chi) + \mathcal{N}_1(\delta) + \mathcal{N}_1(\theta)\\
&+\mathcal{N}_1(\eta) + \mathcal{N}_1(\theta) + \mathcal{N}_1(\overline{\delta}))\mathcal{N}_1(\zeta) + (\mathcal{N}_1(\epsilon) + \mathcal{N}_1(\overline{\epsilon}))\mathcal{N}_1(\chi) + \|\beta\|_{L^2(\mathcal{H}_u)} + \|\underline{\beta}\|_{L^2(\mathcal{H}_u)}\bigg)\\
&+\|\partial_\omega N\|_{L^\infty}\bigg(\|\overline{\epsilon}\|_{L^2(\mathcal{H}_u)}\|\partial_\omega\zeta\|_{L^2_{x'}L^\infty_t} + (\|\eta\|_{L^2(\mathcal{H}_u)} + \|\overline{\delta}\|_{L^2(\mathcal{H}_u)})\|\partial_\omega\chi\|_{L^2_{x'}L^\infty_t}\bigg)\\
&+\|\partial_\omega\zeta\|_{L^2_{x'}L^\infty_t}\|\partial_\omega\chi\|_{L^2_{x'}L^\infty_t}\\
\lesssim\ & \varepsilon,
\end{aligned}$$

where we used in the last inequality the curvature bound (2.59) for $\alpha, \beta, \rho, \sigma, \beta$ and $\underline{\beta}$, the estimates (2.67)-(2.72) for $\epsilon, \overline{\epsilon}, \eta, \delta, \overline{\delta}, \chi, \theta$ and ζ, the estimates (2.76) (2.77) for $\partial_\omega N, \partial_\omega \chi$ and $\partial_\omega \zeta$, the estimate (7.26) for $\partial_\omega^2 N$, and the estimate (7.36) for $\widetilde{F}$.

We are now in position to derive the estimate for $\partial_\omega^2\zeta$. Using the transport equation (7.37) for $\Pi(\partial_\omega^2\zeta)$ and the transport equation (6.48), for M allows us to get rid of the troublesome term $\chi \cdot \Pi(\partial_\omega^2\zeta)$:

$$\begin{aligned}
\nabla\!\!\!/_L(M\cdot\Pi(\partial_\omega^2\zeta)) &= \nabla\!\!\!/_L(M)\cdot\Pi(\partial_\omega^2\zeta) + M\cdot\nabla\!\!\!/_L(\Pi(\partial_\omega^2\zeta))\\
&= M\cdot\nabla\!\!\!/(F_1) + M\cdot F_2 - \frac{|\partial_\omega N|^2}{2}M\cdot\nabla\!\!\!/_{\underline{L}}(\zeta)\\
&= \nabla\!\!\!/(M\cdot F_1) - \nabla\!\!\!/(M)\cdot F_1 + M\cdot F_2 - \frac{|\partial_\omega N|^2}{2}M\cdot\nabla\!\!\!/_{\underline{L}}(\zeta),
\end{aligned}$$

Let $2 \leq p < q < +\infty$. This yields:

$$\begin{aligned}
&\|P_j(M\cdot\Pi(\partial_\omega^2\zeta))\|_{L^q_t L^2_{x'}} \qquad (7.40)\\
\lesssim\ & \left\|P_j\left(\int_0^t \nabla\!\!\!/(M\cdot F_1)dt\right)\right\|_{L^q_t L^2_{x'}} + \left\|P_j\left(\int_0^t \nabla\!\!\!/(M)\cdot F_1 dt\right)\right\|_{L^q_t L^2_{x'}}\\
&+\left\|P_j\left(\int_0^t M\cdot F_2 dt\right)\right\|_{L^q_t L^2_{x'}} + \left\|P_j\left(\int_0^t \frac{|\partial_\omega N|^2}{2}M\cdot\nabla\!\!\!/_{\underline{L}}(\zeta)dt\right)\right\|_{L^q_t L^2_{x'}},
\end{aligned}$$

Next, we estimate the various terms in the right-hand side of (7.40).

We consider the first term in the right-hand side of (7.40). Using Lemma 6.13, we

have:

$$\begin{aligned}\left\|P_j\left(\int_0^t \nabla\!\!\!/(M\cdot F_1)dt\right)\right\|_{L^q_t L^2_{x'}} &\lesssim 2^j\|M\cdot F_1\|_{L^2(\mathcal{H}_u)} \\ &\lesssim 2^j\|M\|_{L^\infty}\|F_1\|_{L^2(\mathcal{H}_u)} \\ &\lesssim 2^j\varepsilon,\end{aligned} \tag{7.41}$$

where we used in the last inequality the estimate (7.38) for F_1 and the estimate (6.49) for M.

Next, we consider the second and the third term in the right-hand side of (7.40). Using the dual sharp Bernstein inequality for tensors (6.53) and the estimate (3.66) for transport equations, we have:

$$\begin{aligned}&\left\|P_j\left(\int_0^t \nabla\!\!\!/(M)\cdot F_1 dt\right)\right\|_{L^q_t L^2_{x'}} + \left\|P_j\left(\int_0^t M\cdot F_2 dt\right)\right\|_{L^q_t L^2_{x'}} \\ \lesssim\ & 2^j\left\|\int_0^t \nabla\!\!\!/(M)\cdot F_1 dt\right\|_{L^\infty_t L^1_{x'}} + 2^j\left\|\int_0^t M\cdot F_2 dt\right\|_{L^\infty_t L^1_{x'}} \\ \lesssim\ & 2^j\|\nabla\!\!\!/(M)\cdot F_1\|_{L^1(\mathcal{H}_u)} + 2^j\|M\cdot F_2\|_{L^1(\mathcal{H}_u)} \\ \lesssim\ & 2^j\|\nabla\!\!\!/(M)\|_{L^2(\mathcal{H}_u)}\|F_1\|_{L^2(\mathcal{H}_u)} + 2^j\|M\|_{L^\infty}\|F_2\|_{L^1(\mathcal{H}_u)} \\ \lesssim\ & 2^j\varepsilon,\end{aligned} \tag{7.42}$$

where we used in the last inequality the estimate (7.38) for F_1, the estimate (7.39) for F_2, and the estimate (6.49) for M.

Finally, we consider the last term in the right-hand side of (7.40). Using Lemma 5.14, we have:

$$\left\|P_j\left(\int_0^t \frac{|\partial_\omega N|^2}{2} M\cdot \nabla\!\!\!/_{\underline{L}}(\zeta)dt\right)\right\|_{L^\infty_t L^2_{x'}} \lesssim \||\partial_\omega N|^2 M\|_{\mathcal{P}^0}(2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u)). \tag{7.43}$$

Now, using the non sharp product estimate (5.15), we have:

$$\begin{aligned}\||\partial_\omega N|^2 M\|_{\mathcal{P}^0} &\lesssim \mathcal{N}_1(\partial_\omega N)(\|M\partial_\omega N\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/(M\partial_\omega N)\|_{L^2(\mathcal{H}_u)} \\ &\lesssim \mathcal{N}_1(\partial_\omega N)(\|M\|_{L^\infty}\mathcal{N}_1(\partial_\omega N) + \|\partial_\omega N\|_{L^\infty}\|\nabla\!\!\!/ M\|_{L^2(\mathcal{H}_u)}) \\ &\lesssim 1,\end{aligned}$$

where we used in the last inequality the estimates (2.76) (2.77) for $\partial_\omega N$, and the estimate (6.49) for M. Together with (7.43), this yields:

$$\left\|P_j\left(\int_0^t \frac{|\partial_\omega N|^2}{2} M\cdot \nabla\!\!\!/_{\underline{L}}(\zeta)dt\right)\right\|_{L^\infty_t L^2_{x'}} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u),$$

which together with (7.40), (7.41) and (7.42) implies:

$$\|P_j(M\cdot\Pi(\partial^2_\omega\zeta))\|_{L^q_t L^2_{x'}} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u). \tag{7.44}$$

Now, since we have chosen $p < q$, (7.44) and Lemma 6.14 yield:

$$\|P_j(\Pi(\partial^2_\omega\zeta))\|_{L^p_t L^2_{x'}} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u), \tag{7.45}$$

for any $2 \le p < +\infty$ which is the desired estimate for $\partial^2_\omega\zeta$.

7.2.5 Estimate for $\nabla\!\!\!/_{\underline{L}}(\Pi(\partial_\omega^2 N))$

In view of the decomposition (7.7) for $\partial_\omega^2 N$, we have:

$$\mathbf{D}_{\underline{L}}(\Pi(\partial_\omega^2 N)) = \mathbf{D}_{\underline{L}}(\partial_\omega^2 N) + 2g(\partial_\omega N, \mathbf{D}_{\underline{L}}(\partial_\omega N))N + |\partial_\omega N|^2 \mathbf{D}_{\underline{L}} N$$

which yields:

$$\begin{aligned} \nabla\!\!\!/_{\underline{L}}(\Pi(\partial_\omega^2 N)) &= \Pi(\mathbf{D}_{\underline{L}}(\partial_\omega^2 N)) + |\partial_\omega N|^2 \Pi(\mathbf{D}_{\underline{L}} N) \\ &= \Pi(\mathbf{D}_{\underline{L}}(\partial_\omega^2 N)) + |\partial_\omega N|^2 (\zeta_A - \underline{\xi}_A) e_A. \end{aligned} \tag{7.46}$$

where we used the Ricci equations (2.23) for $\mathbf{D}_{\underline{L}} N$ in the last equality. The formula (7.3) for $\mathbf{D}_{\underline{L}}(\partial_\omega^2 N)$ and (7.46) imply:

$$\begin{aligned} \nabla\!\!\!/_{\underline{L}}(\Pi(\partial_\omega^2 N)) &= 2\partial_\omega^2 \zeta_B e_B - |\partial_\omega N|^2 \zeta_B e_B + 2\partial_\omega \chi_{\partial_\omega N B} e_B + \chi_{\Pi(\partial_\omega^2 N) B} e_B + (\delta + n^{-1} N(n)) \\ &\quad \times \Pi(\partial_\omega^2 N) + (3\epsilon_{\partial_\omega N} + n^{-1} \nabla_{\partial_\omega N} n) \partial_\omega N + |\partial_\omega N|^2 (\zeta_A - \underline{\xi}_A) e_A. \end{aligned} \tag{7.47}$$

Now, let $2 \leq p < +\infty$. (7.47), the estimate (7.45) for $\|P_j(\Pi(\partial_\omega^2 \zeta))\|_{L_t^p L_{x'}^2}$, together with the L^2 boundedness and the weak Bernstein inequality for P_j, yields:

$$\begin{aligned} & \|P_j \nabla\!\!\!/_{\underline{L}}(\Pi(\partial_\omega^2 N))\|_{L_t^p L_{x'}^2} \\ \lesssim\ & \|P_j \Pi(\partial_\omega^2 \zeta)\|_{L_t^p L_{x'}^2} + \|P_j(|\partial_\omega N|^2 \zeta)\|_{L_t^\infty L_{x'}^2} + \|P_j(\partial_\omega \chi \partial_\omega N)\|_{L_t^\infty L_{x'}^2} \\ & + \|P_j(\chi \Pi(\partial_\omega^2 N))\|_{L_t^\infty L_{x'}^2} + \|P_j((\delta + n^{-1} N(n)) \Pi(\partial_\omega^2 N))\|_{L_t^\infty L_{x'}^2} \\ & + \|P_j((4\epsilon_{\partial_\omega N} + n^{-1} \nabla_{\partial_\omega N} n) \partial_\omega N\|_{L_t^\infty L_{x'}^2} + \|P_j(|\partial_\omega N|^2 (\zeta - \underline{\xi})\|_{L_t^\infty L_{x'}^2} \\ \lesssim\ & 2^j \varepsilon + 2^{\frac{j}{2}} \varepsilon \gamma(u) + \||\partial_\omega N|^2 \zeta\|_{L_t^\infty L_{x'}^2} + \|\partial_\omega \chi \partial_\omega N\|_{L_t^\infty L_{x'}^2} + 2^{\frac{j}{2}} \|\chi \Pi(\partial_\omega^2 N)\|_{L_t^\infty L_{x'}^{\frac{4}{3}}} \\ & + 2^{\frac{j}{2}} \|(\delta + n^{-1} N(n)) \Pi(\partial_\omega^2 N)\|_{L_t^\infty L_{x'}^{\frac{4}{3}}} + \|(4\epsilon_{\partial_\omega N} + n^{-1} \nabla_{\partial_\omega N} n) \partial_\omega N\|_{L_t^\infty L_{x'}^2} \\ & + \||\partial_\omega N|^2 (\zeta - \underline{\xi})\|_{L_t^\infty L_{x'}^2} \\ \lesssim\ & 2^j \varepsilon + 2^{\frac{j}{2}} \varepsilon \gamma(u) + \|\partial_\omega N\|_{L^\infty}^2 (\|\zeta\|_{L_t^\infty L_{x'}^2} + \|\epsilon\|_{L_t^\infty L_{x'}^2} + \|n^{-1} \nabla n\|_{L_t^\infty L_{x'}^2} + \|\underline{\xi}\|_{L_t^\infty L_{x'}^2}) \\ & + \|\partial_\omega N\|_{L^\infty} \|\partial_\omega \chi\|_{L_t^\infty L_{x'}^2} + 2^{\frac{j}{2}} (\|\chi\|_{L_t^\infty L_{x'}^4} + \|\delta\|_{L_t^\infty L_{x'}^4} + \|n^{-1} \nabla n\|_{L_t^\infty L_{x'}^4}) \|\Pi(\partial_\omega^2 N)\|_{L_t^\infty L_{x'}^2} \\ \lesssim\ & 2^j \varepsilon + 2^{\frac{j}{2}} \varepsilon \gamma(u), \end{aligned} \tag{7.48}$$

where we used in the last inequality the estimates (2.67)-(2.72) for $n, \delta, \epsilon, \chi, \underline{\xi}$ and ζ, the estimates (2.76) and (2.77) for $\partial_\omega N$ and $\partial_\omega \chi$, and the estimate (7.26) for $\partial_\omega^2 N$. (7.48) is the desired estimate for $\nabla\!\!\!/_{\underline{L}}(\Pi(\partial_\omega^2 N))$.

In view of the estimates (7.26), (7.27), (7.30), (7.35), (7.45) and (7.48), this concludes the proof of Theorem 2.27.

8 Dependance of the norm $L_u^\infty L^2(\mathcal{H}_u)$ on $\omega \in \mathbb{S}^2$

The goal of this section is to derive the various decompositions of section 2.8. In section 8.1, we derive the basic estimates, first for scalars, and then for tensors using a scalarization procedure. In section 8.2, we obtain the desired decompositions for $\partial_\omega N$, $\mathrm{tr}\chi$ and b^p.

In section 8.3, we provide variants of the results in section 8.1. In section 8.4, we obtain the desired decompositions for χ, $\widehat{\chi}^2$ and $\widehat{\chi}^3$. In section 8.5, we provide further variants of the results in section 8.1. Finally, the desired decompositions for ζ, $\nabla\!\!\!/ b$ and $\partial_\omega b$ are derived in section 8.6.

8.1 The basic estimates

The goal of this section in to prove the following proposition.

Proposition 8.1 *Let $f(\cdot,\omega)$ a scalar function depending on a parameter $\omega \in \mathbb{S}^2$ such that:*

$$\|f\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\mathbf{D}f\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\partial_\omega f\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon.$$

Assume also that the existence of a function γ in $L^2(\mathbb{R})$ such that for all $j \geq 0$, we have:

$$\|P_j(L(\partial_\omega f))\|_{L^2(\mathcal{H}_u)} + \|P_j(\underline{L}(\partial_\omega f))\|_{L^2(\mathcal{H}_u)} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\gamma(u)\varepsilon.$$

Let ω and ω' in $\mathbb{S}^2$. Let $u = u(t,x,\omega)$ and $u' = u(t,x,\omega')$. Then, for any ω'' in $\mathbb{S}^2$ on the arc joining ω and ω', and for any $j \geq 0$, we have the following decomposition for $f(\cdot,\omega'')$:

$$f(\cdot,\omega'') = P_{\leq \frac{j}{2}}(f(\cdot,\omega')) + f_2^j$$

and where f_2^j satisfies:

$$\|f_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}}\varepsilon + |\omega-\omega'|^{\frac{3}{2}}2^{\frac{j}{4}}\varepsilon.$$

As a corollary of Proposition 8.1, we obtain:

Corollary 8.2 *Let $F(\cdot,\omega)$ a tensor depending on a parameter $\omega \in \mathbb{S}^2$ such that:*

$$\|F\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\mathbf{D}F\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\partial_\omega F\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon, \quad \|F\|_{L^2_t L^\infty_{x'}} \lesssim 1.$$

Assume also that the existence of a function γ in $L^2(\mathbb{R})$ such that for all $j \geq 0$ and for some $2 < p \leq +\infty$, we have:

$$\|P_j(\nabla\!\!\!/_L(\partial_\omega F))\|_{L^p_t L^2_{x'}} + \|P_j(\nabla\!\!\!/_{\underline{L}}(\partial_\omega F))\|_{L^p_t L^2_{x'}} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\gamma(u)\varepsilon.$$

Let ω and ω' in $\mathbb{S}^2$. Let $u = u(t,x,\omega)$ and $u' = u(t,x,\omega')$. Then, for any ω'' in $\mathbb{S}^2$ on the arc joining ω and ω', and for any $j \geq 0$, we have the following decomposition for $F(\cdot,\omega'')$:

$$F(\cdot,\omega'') = F_1^j + F_2^j$$

where F_1^j does not depend on ω and satisfies:

$$\|F_1^j\|_{L^\infty(P_{t,u'})} \lesssim \|F\|_{L^\infty(P_{t,u'})},$$

and where F_2^j satisfies:

$$\|F_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}}\varepsilon + |\omega-\omega'|^{\frac{3}{2}}2^{\frac{j}{4}}\varepsilon.$$

The following lemmas will be useful for the proof of Proposition 8.1 and Corollary 8.2.

Lemma 8.3 *Let ω and ω' in $\mathbb{S}^2$. Let $u = u(t,x,\omega)$ and $u' = u(t,x,\omega')$. Then, for any tensor F, we have:*

$$\|F\|_{L^\infty_{u'}L^2(\mathcal{H}_{u'})} \lesssim \|F\|_{L^\infty_u L^2(\mathcal{H}_u)} + |\omega-\omega'|^{\frac{1}{4}} \|F\|^{\frac{1}{2}}_{L^\infty_u L^2(\mathcal{H}_u)} \left(\sup_u \left(\int_u^{u+|\omega-\omega'|} \|\mathbf{D}F\|^2_{L^2(\mathcal{H}_\tau)} d\tau \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}.$$

Lemma 8.4 *Let f a scalar function and ω, ω' in $\mathbb{S}^2$. Then, for any $l \geq 0$, we have:*

$$\|P_l f\|_{L^\infty_{u'}L^2(\mathcal{H}_{u'})} \lesssim (2^{-l} + |\omega-\omega'|^{\frac{1}{2}} 2^{-\frac{l}{2}})(\|f\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\mathbf{D}f\|_{L^\infty_u L^2(\mathcal{H}_u)}).$$

Lemma 8.5 *Let f a scalar function and ω, ω' in $\mathbb{S}^2$. Then, for any $l \geq 0$, we have:*

$$\begin{aligned} & \|P_{\leq l} f\|_{L^\infty_{u'}L^2(\mathcal{H}_{u'})} \\ \lesssim\ & (1 + |\omega-\omega'|^{\frac{1}{2}} 2^{\frac{l}{2}}) \|f\|_{L^\infty_u L^2(\mathcal{H}_u)} + |\omega-\omega'|^{\frac{1}{4}} \|f\|^{\frac{1}{2}}_{L^\infty_u L^2(\mathcal{H}_u)} \\ & \times \left(\sup_u \sum_{q \leq l} \left(\int_u^{u+|\omega-\omega'|} (\|P_q(nL(f))\|^2_{L^2(\mathcal{H}_\tau)} + \|P_q(bN(f))\|^2_{L^2(\mathcal{H}_\tau)}) \right)^{\frac{1}{2}} d\tau \right)^{\frac{1}{2}}. \end{aligned}$$

Lemma 8.6 *Let f a scalar function and ω, ω' in $\mathbb{S}^2$. Then, for any $l \geq 0$, we have:*

$$\|[\partial_\omega, P_{\leq l}] f\|_{L^\infty_{u'}L^2(\mathcal{H}_{u'})} \lesssim \|\mathbf{D}f\|_{L^\infty_u L^2(\mathcal{H}_u)}.$$

Lemma 8.7 *We have:*

$$\|\mathbf{D}_T Q_{\leq 1}(N)\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla \mathbf{D}_T Q_{\leq 1}(N)\|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon.$$

Lemma 8.8 *Let $N_j = N(\cdot, \omega_j), j = 1,2,3$ where $\omega_j \in \mathbb{S}^2$ are given respectively by $\omega_1 = (1,0,0)$, $\omega_2 = (0,1,0)$ and $\omega_3 = (0,0,1)$. Then, $Q_{\leq 1}(N_1), Q_{\leq 1}(N_2)$ and $Q_{\leq 1}(N_3)$ form a basis of the tangent space of Σ_t.*

We also state the following lemma which will be used in the proof of Lemma 8.6. Note this lemma, together with Lemma 8.3, is at the core of all decompositions of section 8.

Lemma 8.9 *Let ω and ω' in $\mathbb{S}^2$. Let $u = u(t,x,\omega)$ and $u' = u(t,x,\omega')$. Then, for any tensor F and any $2 \leq p < +\infty$, we have:*

$$\|F\|_{L^\infty_{u'}L^p(\mathcal{H}_{u'})} \lesssim \|F\|^{1-\frac{1}{p}}_{L^\infty_u L^{2(p-1)}(\mathcal{H}_u)} \|\nabla\!\!\!/ F\|^{\frac{1}{p}}_{L^\infty_u L^2(\mathcal{H}_u)}.$$

The proof of Corollary 8.2 is postponed to section 8.1.2, the proof of Lemma 8.3 is postponed to section D.1, the proof of Lemma 8.4 is postponed to section D.2, the proof of Lemma 8.5 is postponed to section D.3, the proof of Lemma 8.6 is postponed to section D.4, the proof of Lemma 8.7 is postponed to section D.5, the proof of Lemma 8.8 is postponed to section D.6, and the proof of Lemma 8.9 is postponed to section D.7. We now conclude the proof of Proposition 8.1.

8.1.1 Proof of Proposition 8.1

We decompose $f(\cdot,\omega'')$ as:

$$\begin{aligned} f(\cdot,\omega'') &= P''_{\leq \frac{j}{2}}(f(\cdot,\omega'')) + \sum_{l>\frac{j}{2}} P''_l(f(\cdot,\omega'') \qquad (8.1)\\ &= P'_{\leq \frac{j}{2}}(f(\cdot,\omega')) + \int_{[\omega',\omega'']} \partial_\omega P'''_{\leq \frac{j}{2}}(f(\cdot,\omega'''))d\omega'''(\omega'-\omega'') + \sum_{l>\frac{j}{2}} P''_l(f(\cdot,\omega'')) \\ &= P'_{\leq \frac{j}{2}}(f(\cdot,\omega')) + \int_{[\omega',\omega'']} (P'''_{\leq \frac{j}{2}}(\partial_\omega f)(\cdot,\omega''') + [\partial_\omega, P'''_{\leq \frac{j}{2}}]f(\cdot,\omega'''))d\omega'''(\omega'-\omega'') \\ &\quad + \sum_{l>\frac{j}{2}} P''_l(f(\cdot,\omega'')), \end{aligned}$$

where ω''' denotes an angle in $\mathbb{S}^2$ on the arc joining ω' and ω''.

Next, we estimate the last two terms in the right-hand side of (8.1). Using Lemma 8.4, we have:

$$\begin{aligned} \sum_{l>\frac{j}{2}} \|P''_l(f(\cdot,\omega''))\|_{L^\infty_u L^2(\mathcal{H}_u)} &\lesssim \sum_{l>\frac{j}{2}} (2^{-l} + |\omega''-\omega|^{\frac{1}{2}} 2^{-\frac{l}{2}})(\|f\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\mathbf{D}f\|_{L^\infty_u L^2(\mathcal{H}_u)}) \\ &\lesssim (2^{-\frac{j}{2}} + |\omega''-\omega|^{\frac{1}{2}} 2^{-\frac{j}{4}})\varepsilon, \qquad (8.2) \end{aligned}$$

where we used the assumptions on f in the last inequality.

Using Lemma 8.5, we have:

$$\begin{aligned} &\|P'''_{\leq \frac{j}{2}}(\partial_\omega f)(\cdot,\omega''')\|_{L^\infty_u L^2(\mathcal{H}_u)} \qquad (8.3)\\ \lesssim\ & (1+|\omega'''-\omega|^{\frac{1}{2}} 2^{\frac{j}{4}})\|\partial_\omega f\|_{L^\infty_u L^2(\mathcal{H}_u)} + |\omega'''-\omega|^{\frac{1}{4}} \|\partial_\omega f\|^{\frac{1}{2}}_{L^\infty_u L^2(\mathcal{H}_u)} \\ &\times \left(\sup_u \sum_{q\leq \frac{j}{2}} \left(\int_u^{u+|\omega-\omega'''|} (\|P_q(nL(\partial_\omega f))\|^2_{L^2(\mathcal{H}_\tau)} + \|P_q(bN(\partial_\omega f))\|^2_{L^2(\mathcal{H}_\tau)})d\tau \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\ \lesssim\ & (1+|\omega'''-\omega|^{\frac{1}{2}} 2^{\frac{j}{4}})\varepsilon + |\omega-\omega'''|^{\frac{1}{4}}\varepsilon^{\frac{1}{2}} \\ &\times \left(\sup_u \sum_{q\leq \frac{j}{2}} \left(\int_u^{u+|\omega-\omega'''|} (\|P_q(nL(\partial_\omega f))\|^2_{L^2(\mathcal{H}_\tau)} + \|P_q(bN(\partial_\omega f))\|^2_{L^2(\mathcal{H}_\tau)})d\tau \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}, \end{aligned}$$

where we used the assumptions on $\partial_\omega f$ in the last inequality. Now, the assumption on $L(\partial_\omega f)$ and $\underline{L}(\partial_\omega f)$ together with Lemma 5.12 yields:

$$\begin{aligned} &\|P_q(nL(\partial_\omega f))\|^2_{L^2(\mathcal{H}_\tau)} + \|P_q(bN(\partial_\omega f))\|^2_{L^2(\mathcal{H}_\tau)} \\ \lesssim\ & (\|n\|_{L^\infty} + \|\nabla\!\!\!/ n\|_{\mathcal{P}^0} + \|b\|_{L^\infty} + \|\nabla\!\!\!/ b\|_{\mathcal{P}^0})^2 (2^{2q}\varepsilon^2 + 2^q\gamma(u)^2) \\ \lesssim\ & 2^{2q}\varepsilon^2 + 2^q\gamma(u)^2, \end{aligned}$$

where we used in the last inequality the estimate (2.67) for n and the estimate (2.69) for b. Together with (8.3), this implies:

$$\begin{aligned}\|P'''_{\leq \frac{j}{2}}(\partial_\omega f)(\cdot,\omega''')\|_{L^\infty_u L^2(\mathcal{H}_u)} &\lesssim (1+|\omega'''-\omega|^{\frac{1}{4}}2^{\frac{j}{8}}+|\omega'''-\omega|^{\frac{1}{2}}2^{\frac{j}{4}})\varepsilon \\ &\lesssim (1+|\omega'''-\omega|^{\frac{1}{2}}2^{\frac{j}{4}})\varepsilon.\end{aligned} \tag{8.4}$$

Using Lemma 8.6, we have:

$$\|[\partial_\omega, P'''_{\leq \frac{j}{2}}]f(\cdot,\omega'''))\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \|\mathbf{D}f\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon, \tag{8.5}$$

where we used the assumptions on f in the last inequality.

In view of (8.1), we have $f(\cdot,\omega'') = f_j^1 + f_2^j$ where f_1^j is defined as:

$$f_j^1 = P'_{\leq \frac{j}{2}}(f(\cdot,\omega')), \tag{8.6}$$

and f_j^2 is defined as:

$$f_j^2 = \int_{[\omega',\omega'']} (P'''_{\leq \frac{j}{2}}(\partial_\omega f)(\cdot,\omega''') + [\partial_\omega, P'''_{\leq \frac{j}{2}}]f(\cdot,\omega'''))d\omega'''(\omega'-\omega'') + \sum_{l>\frac{j}{2}} P''_l(f(\cdot,\omega'')). \tag{8.7}$$

Using (8.2), (8.4) and (8.5), and the fact that ω'' is on the arc of $\mathbb{S}^2$ joining ω and ω', we have the following estimate for f_j^2:

$$\begin{aligned}\|f_j^2\|_{L^\infty_u L^2(\mathcal{H}_u)} &\lesssim \int_{[\omega',\omega'']} (1+|\omega'''-\omega|^{\frac{1}{2}}2^{\frac{j}{4}})\varepsilon d\omega'''|\omega'-\omega''| + (2^{-\frac{j}{2}}+|\omega''-\omega|^{\frac{1}{2}}2^{-\frac{j}{4}})\varepsilon \\ &\lesssim (1+|\omega'-\omega|^{\frac{1}{2}}2^{\frac{j}{4}})|\omega'-\omega|\varepsilon + (2^{-\frac{j}{2}}+|\omega'-\omega|^{\frac{1}{2}}2^{-\frac{j}{4}})\varepsilon \\ &\lesssim 2^{-\frac{j}{2}}\varepsilon + |\omega'-\omega|^{\frac{3}{2}}2^{\frac{j}{4}}\varepsilon.\end{aligned} \tag{8.8}$$

This concludes the proof of Proposition 8.1.

8.1.2 Proof of Corollary 8.2

Using Lemma 8.8, it suffices to prove the decomposition of Corollary 8.2 where $F(\omega'',.)$ is replaced by $g(F(\cdot,\omega''),Q_{\leq 1}(N_l))$ for $l=1,2,3$. Since the proof is identical for $l=1,2,3$, we simply take $l=1$. Therefore, it remains to prove that the following decomposition holds $g(F(\cdot,\omega''),Q_{\leq 1}(N_1))$:

$$g(F(\cdot,\omega''),Q_{\leq 1}(N_1)) = P_{\leq \frac{j}{2}}(g(F(\cdot,\omega'),Q_{\leq 1}(N_1))) + f_2^j, \tag{8.9}$$

where the scalar function f_2^j satisfies:

$$\|f_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}}\varepsilon + |\omega-\omega'|^{\frac{3}{2}}2^{\frac{j}{4}}\varepsilon. \tag{8.10}$$

In particular, F_1^j is connected to the first term in the right-hand side of (8.9), which does not depend on ω and satisfies the following estimate

$$\left\| P_{\leq \frac{j}{2}}(g(F(\cdot,\omega'),Q_{\leq 1}(N_1)))\right\|_{L^\infty(P_{t,u'})} \lesssim \|F\|_{L^\infty(P_{t,u'})}\|Q_{\leq 1}(N_1)\|_{L^\infty} \lesssim \|F\|_{L^\infty(P_{t,u'})},$$

where we used the fact that $P_{\leq \frac{j}{2}}$ is bounded on $L^\infty(P_{t,u'})$ and the fact that $Q_{\leq 1}$ is bounded on L^∞.

Let $f = g(F(\cdot,\omega), Q_{\leq 1}(N_1))$. In order to prove the decomposition (8.9) (8.10) for $g(F(\cdot,\omega''), Q_{\leq 1}(N_1))$, it suffices to show that f satisfies the assumptions of Proposition 8.1. First, we estimate $\mathbf{D}f$. We have:

$$\begin{aligned}\|\mathbf{D}f\|_{L^\infty_u L^2(\mathcal{H}_u)} &\lesssim \|\mathbf{D}F\|_{L^\infty_u L^2(\mathcal{H}_u)}\|Q_{\leq 1}(N_1)\|_{L^\infty} + \|F\|_{L^2_t L^\infty_{x'}}\|\mathbf{D}Q_{\leq 1}(N_1)\|_{L^\infty_t L^2_{x'}}\\ &\lesssim \varepsilon + \|\mathbf{D}Q_{\leq 1}(N_1)\|_{L^\infty_t L^2_{x'}}\end{aligned}$$

where we used in the last inequality the assumptions on F, and the fact that $Q_{\leq 1}$ is bounded on L^∞. Using the functional inequality (3.74), we obtain:

$$\begin{aligned}&\|\mathbf{D}Q_{\leq 1}(N_1)\|_{L^\infty_t L^2_{x'}} \qquad (8.11)\\ \lesssim\ &\|\mathbf{D}Q_{\leq 1}(N_1)\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla\mathbf{D}Q_{\leq 1}(N_1)\|_{L^\infty_t L^2(\Sigma_t)}\\ \lesssim\ &\|\nabla Q_{\leq 1}(N_1)\|_{L^\infty_t L^2(\Sigma_t)} + \|\mathbf{D}_T Q_{\leq 1}(N_1)\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla^2 Q_{\leq 1}(N_1)\|_{L^\infty_t L^2(\Sigma_t)}\\ &+\|\nabla\mathbf{D}_T Q_{\leq 1}(N_1)\|_{L^\infty_t L^2(\Sigma_t)}\\ \lesssim\ &\|\nabla Q_{\leq 1}(N_1)\|_{L^\infty_t L^2(\Sigma_t)} + \|\mathbf{D}_T Q_{\leq 1}(N_1)\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla\mathbf{D}_T Q_{\leq 1}(N_1)\|_{L^\infty_t L^2(\Sigma_t)}\\ \lesssim\ &\varepsilon,\end{aligned}$$

where we used the Bochner identity on Σ_t (3.81), the finite band property for $Q_{\leq 1}$, and Lemma 8.7. Finally, we obtain:

$$\|\mathbf{D}f\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon. \qquad (8.12)$$

Next, we estimate $\partial_\omega f$. We have:

$$\partial_\omega f = g(\partial_\omega F, Q_{\leq 1}(N_1)),$$

which yields:

$$\|\partial_\omega f\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \|\partial_\omega F\|_{L^\infty_u L^2(\mathcal{H}_u)}\|Q_{\leq 1}(N_1)\|_{L^\infty} \lesssim \varepsilon \qquad (8.13)$$

where we used in the last inequality the assumptions on $\partial_\omega F$, and the fact that $Q_{\leq 1}$ is bounded on L^∞.

Finally, we estimate $L(\partial_\omega f)$ and $\underline{L}(\partial_\omega f)$. The estimate for $L(\partial_\omega f)$ being similar, we focus on $\underline{L}(\partial_\omega f)$. We have:

$$\mathbf{D}_{\underline{L}}(\partial_\omega f) = g(\mathbf{D}_{\underline{L}}(\partial_\omega F), Q_{\leq 1}(N_1)) + g(\partial_\omega F, \mathbf{D}_{\underline{L}} Q_{\leq 1}(N_1)). \qquad (8.14)$$

The estimate (6.118) yields:

$$\|\not{\nabla} Q_{\leq 1}(N_1)\|_{\mathcal{B}^0} \lesssim \varepsilon$$

which together with Lemma 6.14 and the assumption for $\mathbf{D}_{\underline{L}}\partial_\omega F$ yields:

$$\|P_j(g(\mathbf{D}_{\underline{L}}(\partial_\omega^2 N), Q_{\leq 1}(N_1)))\|_{L^2(\mathcal{H}_u)} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u). \qquad (8.15)$$

Furthermore, using the dual of the sharp Bernstein inequality (4.32), we obtain:

$$\begin{aligned}\|P_j(g(\partial_\omega F, \mathbf{D}_{\underline{L}} Q_{\leq 1}(N_1)))\|_{L^2(\mathcal{H}_u)} &\lesssim 2^j\|g(\partial_\omega F, \mathbf{D}_{\underline{L}} Q_{\leq 1}(N_1))\|_{L^2_t L^1_{x'}} \qquad (8.16)\\ &\lesssim 2^j\|\partial_\omega F\|_{L^\infty_u L^2(\mathcal{H}_u)}\|\mathbf{D}_{\underline{L}} Q_{\leq 1}(N_1)))\|_{L^\infty_t L^2_{x'}}\\ &\lesssim 2^j\varepsilon\end{aligned}$$

where we used in the last inequality the assumptions for $\partial_\omega F$, and the estimate (8.11). Now, (8.14)-(8.16) yield:

$$\|P_j(\underline{L}(\partial_\omega f))\|_{L^2(\mathcal{H}_u)} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u).$$

The corresponding estimate for $L(\partial_\omega f)$ may be obtained in the same way and is actually easier. Thus, we obtain:

$$\|P_j(L(\partial_\omega f))\|_{L^2(\mathcal{H}_u)} + \|P_j(\underline{L}(\partial_\omega f))\|_{L^2(\mathcal{H}_u)} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u). \tag{8.17}$$

In view of (8.12), (8.13), and (8.17), f satisfies the assumptions of Proposition 8.1, which in turn yields the decomposition (8.9)-(8.10) for $g(F(\cdot,\omega''), Q_{\leq 1}(N_1))$. This concludes the proof of Corollary 8.2.

8.2 Decompositions involving $\partial_\omega N$, $\mathrm{tr}\chi$ and b^p

In this section, we obtain the proof of Proposition 2.31, Proposition 2.32 and Proposition 2.33 as a consequence of Proposition 8.1, Corollary 8.2, and Lemma 8.9.

8.2.1 Proof of Proposition 2.31

We have:

$$N - N' = \int_{[\omega,\omega']} \partial_\omega N(\cdot,\omega'')d\omega''(\omega - \omega'). \tag{8.18}$$

We denote $\partial_\omega N'' = \partial_\omega N(\cdot,\omega'')$. Now, in view of the estimates (2.76) and (2.77) for $\partial_\omega N$, and (2.84), (2.85) and (2.87) for $\partial_\omega^2 N$, $\partial_\omega N$ satisfies the assumptions of Corollary 8.2, up to the fact that some of the estimates are controlled by $\lesssim 1$ instead of $\lesssim \varepsilon$. Thus, the conclusions of Corollary 8.2 hold with ε being replaced by 1, i.e. we have the following decomposition for $\partial_\omega N''$

$$\partial_\omega N'' = F_1^j + F_2^j, \tag{8.19}$$

where the vectorfield F_1^j only depends on ω' and satisfies:

$$\|F_1^j\|_{L^\infty} \lesssim \|\partial_\omega N'\|_{L^\infty} \lesssim 1 \tag{8.20}$$

in view of (2.76), and where the vectorfield F_2^j satisfies:

$$\|F_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}}. \tag{8.21}$$

Injecting the decomposition (8.19) in (8.18), and in view of (8.20) (8.21), we obtain the desired decomposition for $N - N'$. This concludes the proof of the proposition.

8.2.2 Proof of Proposition 2.32

In view of the estimates (2.70), (2.77) and (2.78) for $\mathrm{tr}\chi$, $f = \mathrm{tr}\chi$ satisfies the assumption of Proposition 8.1. Thus, in view of Proposition 8.1, $\mathrm{tr}\chi(\cdot,\omega)$ satisfies the desired decomposition with

$$f_1^j = P_{\leq \frac{j}{2}}(\mathrm{tr}\chi(\cdot,\omega')).$$

There remains to prove the L^∞ estimate for f_1^j which is an immediate consequence of the estimate (2.70) for $\mathrm{tr}\chi$ and the fact that $P_{\leq \frac{j}{2}}$ is bounded on $L^\infty(P_{t,u})$. This concludes the proof of the proposition.

8.2.3 Proof of Proposition 2.33

We have

$$\begin{aligned}\|b^p(\cdot,\omega)-b^p(\cdot,\omega')\|_{L^\infty_u L^2(\mathcal{H}_u)} &\lesssim \left(\int_{[\omega,\omega']}\|b^{p-1}(\cdot,\omega'')\partial_\omega b(\cdot,\omega'')\|_{L^\infty_u L^2(\mathcal{H}_u)}d\omega''\right)|\omega-\omega'|\\ &\lesssim \left(\int_{[\omega,\omega']}\|\partial_\omega b(\cdot,\omega'')\|_{L^\infty_u L^2(\mathcal{H}_u)}d\omega''\right)|\omega-\omega'|, \qquad (8.22)\end{aligned}$$

where we used in the last inequality the estimate (2.69) for b. Now, using Lemma 8.9 with $p=2$, we have

$$\|\partial_\omega b(\cdot,\omega'')\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \|\partial_\omega b\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/\partial_\omega b\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon$$

where we used the estimate (2.77) for $\partial_\omega b$ in the last inequality. Together with (8.22), we obtain

$$\|b^p(.\omega)-b^p(\cdot,\omega')\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim |\omega-\omega'|\varepsilon,$$

which concludes the proof of the proposition.

8.3 A first variant of Proposition 8.1

We start with the following refinement of Lemma 8.9:

Corollary 8.10 *Let ω and ω' in $\mathbb{S}^2$. Let $u=u(t,x,\omega)$ and $u'=u(t,x,\omega')$. Then, for any tensor F, and for any $2\le p,q\le+\infty$, we have:*

$$\|F\|_{L^\infty_{u'}L^2(\mathcal{H}_{u'})} \lesssim \|F\|^{\frac12}_{L^p_t L^q_{x'}}\|\nabla\!\!\!/ F\|^{\frac12}_{L_t^{\frac{p}{p-1}}L_{x'}^{\frac{q}{q-1}}}.$$

Proof For $\omega\in\mathbb{S}^2$, let $\Phi_{t,\omega}:\Sigma_t\to\mathbb{R}^3$ defined by:

$$\Phi_{t,\omega}(t,x)=u(t,x,\omega)\omega+\partial_\omega u(t,x,\omega)$$

which is introduced in (D.1). See the beginning of section D.1 concerning the fact that it is a global C^1 diffeomorphism from Σ_t to $\mathbb{R}^3$ and satisfies (D.2), i.e.

$$\|d\Phi^{-1}_{t,\omega}\|_{L^\infty}\lesssim 1,\quad \||\det(\mathrm{Jac}\Phi_{t,\omega})|-1\|_{L^\infty}\lesssim\varepsilon.$$

Let f a scalar. Then, using a standard estimate in $\mathbb{R}^2$, we have the analog of (D.63)

$$\begin{aligned}&\int_{y_1}\sup_{y_2}|f(\Phi^{-1}_{t,\omega}(u,y_1,y_2))|^2dy_1\\ \lesssim\ &\left(\int_y|f(\Phi^{-1}_{t,\omega}(u,y_1,y_2))|^q dy_1dy_2\right)^{\frac1q}\left(\int_y|\partial_{y_2}f(\Phi^{-1}_{t,\omega}(u,y_1,y_2))|^{\frac{q}{q-1}}dy_1dy_2\right)^{1-\frac1q}\\ \lesssim\ &\left(\int_{P_{t,u}}|f|^q d\mu_{t,u}\right)^{\frac1q}\left(\int_{P_{t,u}}|\nabla\!\!\!/ f|^{1-\frac1q}d\mu_{t,u}\right)^{\frac12}.\end{aligned}$$

Together with (D.62), this yields:

$$\begin{aligned}\|f\|^2_{L^2(\mathcal{H}_{u'=u_0})} &\lesssim \frac{1}{|\omega-\omega'|}\left(\int_0^1\int_{u_0-|\omega-\omega'|}^{u_0+|\omega-\omega'|}\left(\int_{P_{t,u}}|f|^q d\mu_{t,u}\right)^{\frac{1}{q}}\left(\int_{P_{t,u}}|\nabla\!\!\!/ f|^{\frac{q}{q-1}}d\mu_{t,u}\right)^{1-\frac{1}{q}}dudt\right)\\ &\lesssim \|f\|_{L^p_t L^q_{x'}}\|\nabla\!\!\!/ f\|_{L_t^{\frac{p}{p-1}}L_{x'}^{\frac{q}{q-1}}}.\end{aligned}$$

Since this holds for any real number u_0, we take the supremum which yields:

$$\|f\|^2_{L^\infty_{u'}L^2(\mathcal{H}_{u'})}\lesssim \|f\|_{L^p_t L^q_{x'}}\|\nabla\!\!\!/ f\|_{L_t^{\frac{p}{p-1}}L_{x'}^{\frac{q}{q-1}}}.$$

Finally, let F a tensor. Applying the previous inequality to $f=|F|$, we obtain

$$\|F\|^2_{L^\infty_{u'}L^2(\mathcal{H}_{u'})}\lesssim \|F\|_{L^p_t L^q_{x'}}\|\nabla\!\!\!/ F\|_{L_t^{\frac{p}{p-1}}L_{x'}^{\frac{q}{q-1}}}.$$

This concludes the proof of the corollary. ∎

We will need the following refinement of Corollary 8.2:

Corollary 8.11 *Let $F(\cdot,\omega)$ a tensor depending on a parameter $\omega\in\mathbb{S}^2$ such that for any $2\leq p<+\infty$:*

$$\|F\|_{L^\infty_u L^2(\mathcal{H}_u)}+\|F\|_{L^p_t L^\infty_{x'}}+\|\mathbf{D}F\|_{L^\infty_u L^2(\mathcal{H}_u)}+\|\partial_\omega F\|_{L^\infty_u L^2(\mathcal{H}_u)}\lesssim\varepsilon.$$

Assume that there exists two tensors H_1 and H_2 such that

$$\partial_\omega F = H_1+H_2,$$

such that we have

$$\|H_1\|_{L^\infty_u L^2(\mathcal{H}_u)}+\|H_2\|_{L^\infty_u L^2(\mathcal{H}_u)}\lesssim\varepsilon,$$

and there exists a function γ in $L^2(\mathbb{R})$ such that for all $j\geq 0$ and for some $2<q\leq+\infty$, we have:

$$\|P_j(\nabla\!\!\!/_L(H_1))\|_{L^q_t L^2_{x'}}+\|P_j(\nabla\!\!\!/_{\underline{L}}(H_1))\|_{L^q_t L^2_{x'}}\lesssim 2^j\varepsilon+2^{\frac{j}{2}}\gamma(u)\varepsilon,$$

and such that H_2 satisfies for some $2\leq q<+\infty$

$$\|H_2\|_{L^q_t L^{\frac{8}{3}}_{x'}}+\|\nabla\!\!\!/ H_2\|_{L_t^{\frac{q}{q-1}}L_{x'}^{\frac{8}{5}}}\lesssim\varepsilon.$$

Let ω and ω' in $\mathbb{S}^2$. Let $u=u(t,x,\omega)$ and $u'=u(t,x,\omega')$. Then, for any ω'' in $\mathbb{S}^2$ on the arc joining ω and ω', and for any $j\geq 0$, we have the following decomposition for $F(\cdot,\omega'')$:

$$F(\cdot,\omega'')=F_1^j+F_2^j$$

where F_1^j does not depend on ω and satisfies such that for any $2\leq p<+\infty$:

$$\|F_1^j\|_{L^\infty_{u\omega'}L^p_t L^\infty(P_{t,u'})}\lesssim\varepsilon,$$

and where F_2^j satisfies:

$$\|F_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)}\lesssim 2^{-\frac{j}{2}}\varepsilon+|\omega-\omega'|^{\frac{3}{2}}2^{\frac{j}{4}}\varepsilon.$$

Proof Using Lemma 8.8, it suffices to prove the decomposition of Corollary 8.11 where $F(\omega'', .)$ is replaced by $g(F(\cdot, \omega''), Q_{\leq 1}(N_l))$ for $l = 1, 2, 3$. Since the proof is identical for $l = 1, 2, 3$, we simply take $l = 1$. Therefore, it remains to prove that the following decomposition holds $g(F(\cdot, \omega''), Q_{\leq 1}(N_1))$:

$$g(F(\cdot, \omega''), Q_{\leq 1}(N_1)) = f_1^j + f_2^j, \tag{8.23}$$

where f_1^j does not depend on ω and satisfies such that for any $2 \leq p < +\infty$:

$$\|f_1^j\|_{L^\infty_{u'} L^p_t L^\infty(P_{t,u'})} \lesssim \varepsilon, \tag{8.24}$$

and where the vectorfields f_2^j satisfies:

$$\|f_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}}\varepsilon + |\omega - \omega'|^{\frac{3}{2}} 2^{\frac{j}{4}}\varepsilon. \tag{8.25}$$

Let $f(\cdot, \omega) = g(F(\cdot, \omega), Q_{\leq 1}(N_1))$. Arguing as in the proof of Corollary 8.2, and using the assumptions for F, we have the analog of (8.12) and (8.13):

$$\|\mathbf{D} f\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{8.26}$$

and

$$\|\partial_\omega f\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{8.27}$$

Also, in view of the assumptions for F and the fact that $Q_{\leq 1}$ is bounded on L^∞, we have

$$\|f(\cdot, \omega')\|_{L^\infty_{u'} L^p_t L^\infty(P_{t,u'})} \lesssim \|F(\cdot, \omega')\|_{L^\infty_{u'} L^p_t L^\infty(P_{t,u'})} \|Q_{\leq 1}(N_1)\|_{L^\infty} \lesssim \varepsilon. \tag{8.28}$$

In order to prove the decomposition (8.23) (8.24) (8.25) for $g(F(\cdot, \omega''), Q_{\leq 1}(N_1))$, we follow the proof of Proposition 8.1. In particular, we recall the decomposition (8.1) of $f(\cdot, \omega'')$:

$$\begin{aligned} f(\cdot, \omega'') &= P'_{\leq \frac{j}{2}}(f(\cdot, \omega')) + \int_{[\omega', \omega'']} (P'''_{\leq \frac{j}{2}}(\partial_\omega f)(\cdot, \omega''') + [\partial_\omega, P'''_{\leq \frac{j}{2}}] f(\cdot, \omega'''))\, d\omega'''(\omega' - \omega'') \\ &\quad + \sum_{l > \frac{j}{2}} P''_l(f(\cdot, \omega'')), \end{aligned} \tag{8.29}$$

where ω''' denotes an angle in $\mathbb{S}^2$ on the arc joining ω' and ω''. Also, in view of the estimate (8.26), we have the analog of the estimate (8.2)

$$\sum_{l > \frac{j}{2}} \|P''_l(f(\cdot, \omega''))\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim (2^{-\frac{j}{2}} + |\omega'' - \omega|^{\frac{1}{2}} 2^{-\frac{j}{4}})\varepsilon, \tag{8.30}$$

and the analog of the estimate (8.5)

$$\|[\partial_\omega, P'''_{\leq \frac{j}{2}}] f(\cdot, \omega'''))\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{8.31}$$

Also, using (8.28) and the fact that $P'_{\leq \frac{j}{2}}$ is bounded on $L^\infty(P_{t,u'})$, we have for any $2 \leq p < +\infty$:

$$\left\| P'_{\leq \frac{j}{2}}(f(\cdot, \omega')) \right\|_{L^\infty_{u'} L^p_t L^\infty(P_{t,u'})} \lesssim \varepsilon. \tag{8.32}$$

In view of (8.29), we have $f(\cdot,\omega'') = f_j^1 + f_2^j$ where f_1^j is defined as:

$$f_j^1 = P'_{\leq \frac{j}{2}}(f(\cdot,\omega')),$$

and f_j^2 is defined as:

$$f_j^2 = f_{2,1}^j + f_{2,2}^j \tag{8.33}$$

with

$$f_{2,1}^j = \int_{[\omega',\omega'']} (P'''_{\leq \frac{j}{2}}(\partial_\omega f)(\cdot,\omega''')(\omega' - \omega'') \tag{8.34}$$

and

$$f_{2,2}^j = \int_{[\omega',\omega'']} [\partial_\omega, P'''_{\leq \frac{j}{2}}](\partial_\omega f)(\cdot,\omega'''))d\omega'''(\omega' - \omega'') + \sum_{l > \frac{j}{2}} P''_l(f(\cdot,\omega'')).$$

In view of the definition of f_1^j, $f_{2,2}^j$ and the estimates (8.30), (8.31) and (8.32), f_1^j does not depend on ω and satisfies for any $2 \leq p < +\infty$:

$$\|f_1^j\|_{L^\infty_{u'} L^p_t L^\infty(P_{t,u'})} \lesssim \varepsilon, \tag{8.35}$$

while f_2^j satisfies:

$$\|f_{2,2}^j\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}}\varepsilon + |\omega - \omega'|^{\frac{3}{2}} 2^{\frac{j}{4}}\varepsilon. \tag{8.36}$$

We still need to estimate $f_{2,1}^j$. We have:

$$\partial_\omega f = g(\partial_\omega F, Q_{\leq 1}(N_1))$$

and thus

$$\partial_\omega f = h_1 + h_2 \tag{8.37}$$

where

$$h_j = g(H_j, Q_{\leq 1}(N_1)),\ j = 1, 2,$$

Since the assumptions for H_1 in Corollary 8.11 are the same as the assumptions for $\partial_\omega F$ in Corollary 8.2, we obtain the analog of (8.13) and (8.17) for h_1:

$$\|h_1\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon,$$

and

$$\|P_j(L(h_1))\|_{L^2(\mathcal{H}_u)} + \|P_j(\underline{L}(h_1))\|_{L^2(\mathcal{H}_u)} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u).$$

Thus, the estimates for h_1 in Corollary 8.11 are the same as the assumptions for $\partial_\omega f$ in Proposition 8.1, and we obtain the analog of (8.4)

$$\|P'''_{\leq \frac{j}{2}} h_1(\cdot,\omega''')\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim (1 + |\omega''' - \omega|^{\frac{1}{2}} 2^{\frac{j}{4}})\varepsilon. \tag{8.38}$$

Next, let $2 \leq q < +\infty$. We have in view of Corollary 8.10

$$\begin{aligned} \|P'''_{\leq \frac{j}{2}} h_2(\cdot,\omega''')\|_{L^\infty_u L^2(\mathcal{H}_u)} &\lesssim \|P_{\leq \frac{j}{2}} h_2\|^{\frac{1}{2}}_{L^q_t L^{\frac{8}{3}}_{x'}} \|\nabla\!\!\!/ P_{\leq \frac{j}{2}} h_2\|^{\frac{1}{2}}_{L^{\frac{q}{q-1}}_t L^{\frac{8}{5}}_{x'}} \\ &\lesssim \|h_2\|^{\frac{1}{2}}_{L^q_t L^{\frac{8}{3}}_{x'}} \|\nabla\!\!\!/ h_2\|^{\frac{1}{2}}_{L^{\frac{q}{q-1}}_t L^{\frac{8}{5}}_{x'}}, \end{aligned} \tag{8.39}$$

where we used in the last inequality the finite band property and the boundedness on $L^2(P_{t,u})$ of $P_{\leq \frac{j}{2}}$. Now, in view of the definition of h_2, we have

$$\begin{aligned} & \|h_2\|_{L_t^q L_{x'}^{\frac{8}{3}}} + \|\nabla\!\!\!/ h_2\|_{L_t^{\frac{q}{q-1}} L_{x'}^{\frac{8}{5}}} \\ \lesssim & \ \|Q_{\leq 1}(N_1)\|_{L^\infty}(\|H_2\|_{L_t^q L_{x'}^{\frac{8}{3}}} + \|\nabla\!\!\!/ H_2\|_{L_t^q L_{x'}^{\frac{8}{3}}}) + \|H_2\|_{L_t^q L_{x'}^{\frac{8}{3}}} \|\nabla\!\!\!/ Q_{\leq 1}(N_1)\|_{L_t^\infty L_{x'}^4} \\ \lesssim & \ \varepsilon + \varepsilon \|\nabla\!\!\!/ Q_{\leq 1}(N_1)\|_{L_t^\infty L_{x'}^4}, \end{aligned} \tag{8.40}$$

where we used in the last inequality the assumptions on H_2 and the fact that $Q_{\leq 1}$ is bounded on L^∞. In order to estimate the right-hand side of (8.40), we use the estimate (3.80). We obtain

$$\begin{aligned} \|\nabla\!\!\!/ Q_{\leq 1}(N_1)\|_{L_t^\infty L_{x'}^4} & \lesssim \|\nabla \nabla\!\!\!/ Q_{\leq 1}(N_1)\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla\!\!\!/ Q_{\leq 1}(N_1)\|_{L_t^\infty L^2(\Sigma_t)} \\ & \lesssim 1, \end{aligned} \tag{8.41}$$

where we used in the last inequality the estimate (C.96). Together with (8.40), this yields

$$\|h_2\|_{L_t^q L_{x'}^{\frac{8}{3}}} + \|\nabla\!\!\!/ h_2\|_{L_t^{\frac{q}{q-1}} L_{x'}^{\frac{8}{5}}} \lesssim \varepsilon.$$

In view of (8.39) we deduce

$$\|P'''_{\leq \frac{j}{2}} h_2(\cdot, \omega''')\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{8.42}$$

Now, (8.34), (8.37) and (8.42) imply:

$$\|f_{2,1}^j\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim |\omega - \omega'|\varepsilon. \tag{8.43}$$

Finally, (8.33), (8.36) and (8.43) imply

$$\|f_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}}\varepsilon + |\omega - \omega'|^{\frac{3}{2}} 2^{\frac{j}{4}}\varepsilon,$$

which together with the decomposition $f(\cdot, \omega'') = f_j^1 + f_2^j$ and the estimate (8.35) yields the conclusion of the corollary. ∎

8.4 Decompositions involving χ

The goal of this section is to prove the decompositions of Lemma 2.34, Proposition 2.35, Proposition 2.36, Proposition 2.37 and Proposition 2.38. The proof of Lemma 2.34 is given in section 8.4.1, the proof of Proposition 2.35 is given in section 8.4.2, the proof of Proposition 2.36 is given in section 8.4.5, the proof of Proposition 2.37 is given in section 8.4.6, and the proof of Proposition 2.38 is given in section 8.4.8.

We will need the following product lemma.

Lemma 8.12 *Let F and H $P_{t,u}$-tangent tensors on $\mathcal{H}_u$ such that for any $2 \leq r < +\infty$ we have*

$$\|F\|_{L_t^r L_{x'}^\infty} + \|\nabla\!\!\!/ F\|_{L_t^r B_{2,1}^0(P_{t,u})} + \|H\|_{L_t^r L_{x'}^\infty} + \|\nabla\!\!\!/ H\|_{L_t^r B_{2,1}^0(P_{t,u})} \lesssim \varepsilon.$$

Then, we have for any $2 \leq r < +\infty$ we have

$$\|FH\|_{L_t^r L_{x'}^\infty} + \|\nabla\!\!\!/(FH)\|_{L_t^r B_{2,1}^0(P_{t,u})} \lesssim \varepsilon.$$

We will also need the following consequence of Corollary 8.11 and Lemma 8.12.

Corollary 8.13 *Let ω and ω' in $\mathbb{S}^2$. For any $j \geq 0$ and any integer $l \geq 2$, we have the following decomposition for $\chi_1(\cdot,\omega)^l$:*

$$\chi_1(\cdot,\omega)^l = F_1^j + F_2^j$$

where F_1^j does not depend on ω and satisfies for any $2 \leq p < +\infty$:

$$\|F_1^j\|_{L^\infty_{u'} L^p_t L^\infty(P_{t,u'})} \lesssim \varepsilon,$$

where F_2^j satisfies:

$$\|F_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}} + |\omega - \omega'|^{\frac{3}{2}} 2^{\frac{j}{4}} \varepsilon.$$

We will need the following consequence of Lemma 2.34 and Corollary 8.13:

Corollary 8.14 *Let ω and ω' in $\mathbb{S}^2$. For any $j \geq 0$ and any integer $l \geq 1$, we have the following decomposition for $\chi_1(\cdot,\omega)^l \chi_2(\cdot,\omega)$:*

$$\chi_1(\cdot,\omega)^l \chi_2(\cdot,\omega) = \chi_2(\cdot,\omega') F_1^j + \chi_2(\cdot,\omega') F_2^j + F_3^j$$

where F_1^j does not depend on ω and satisfies for any $2 \leq p < +\infty$:

$$\|F_1^j\|_{L^\infty_{u'} L^p_t L^\infty(P_{t,u'})} \lesssim \varepsilon,$$

where F_2^j and F_3^j satisfy:

$$\|F_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|F_3^j\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}} + |\omega - \omega'|^{\frac{3}{2}} 2^{\frac{j}{4}} \varepsilon.$$

Finally, we will need the following consequence in particular of Lemma 2.34:

Corollary 8.15 *Let ω and ω' in $\mathbb{S}^2$. For any $j \geq 0$, we have the following decomposition for $\chi_1(\cdot,\omega)\chi_2(\cdot,\omega)^2$:*

$$\chi_1(\cdot,\omega)\chi_2(\cdot,\omega)^2 = \chi_2(\cdot,\omega')^2 F_1^j + \chi_2(\cdot,\omega')^2 F_2^j + \chi_2(\cdot,\omega') F_3^j + F_4^j$$

where F_1^j does not depend on ω and satisfies:

$$\|F_1^j\|_{L^\infty_{u'} L^2_t L^\infty(P_{t,u'})} \lesssim \varepsilon,$$

where F_2^j and F_3^j satisfy:

$$\|F_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|F_3^j\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}} + |\omega - \omega'|^{\frac{3}{2}} 2^{\frac{j}{4}} \varepsilon,$$

and where F_4^j satisfies

$$\|F_4^j\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{-j}.$$

The proof of Lemma 8.12 is postponed to section D.8, the proof of Corollary 8.13 is postponed to section 8.4.4, the proof of Corollary 8.14 is postponed to section 8.4.5 and the proof of Corollary 8.15 is postponed to section 8.4.7.

8.4.1 Proof of Lemma 2.34

We have

$$\|\chi_2(.\omega) - \chi_2(\cdot,\omega')\|_{L^\infty_u L^{4-}(\mathcal{H}_u)} \lesssim \left(\int_{[\omega,\omega']} \|\partial_\omega \chi_2(\cdot,\omega'')\|_{L^\infty_u L^{4-}(\mathcal{H}_u)} d\omega''\right) |\omega - \omega'|. \tag{8.44}$$

Now, using Lemma 8.9 with $p = 4_-$, we have

$$\|\partial_\omega \chi_2(\cdot,\omega'')\|_{L^\infty_u L^{4-}(\mathcal{H}_u)} \lesssim \|\partial_\omega \chi_2\|_{L^\infty_u L^{6-}(\mathcal{H}_u)} + \|\nabla\!\!\!/ \partial_\omega \chi_2\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon$$

where we used the estimate (2.81) for $\partial_\omega \chi_2$ in the last inequality. Together with (8.44), we obtain

$$\|\chi_2(.\omega) - \chi_2(\cdot,\omega')\|_{L^\infty_u L^{4-}(\mathcal{H}_u)} \lesssim |\omega - \omega'|\varepsilon,$$

which concludes the proof of the lemma.

8.4.2 Proof of Proposition 2.35

In view of the decomposition (2.15) of χ in its trace part $\mathrm{tr}\chi$ and traceless part $\widehat{\chi}$, in view of the decomposition (2.79) of $\widehat{\chi}$ in the sum of χ_1 and χ_2, and in view of the decomposition of Corollary 2.32 for $\mathrm{tr}\chi$, it suffices to obtain the following decomposition for χ_1

$$\chi_1(\cdot,\omega) = F_1^j + F_2^j, \tag{8.45}$$

where the vectorfield F_1^j only depends on (t, x, ω') and satisfies for any $2 \leq p < +\infty$:

$$\|F_1^j\|_{L^\infty_{u'} L^p_t L^\infty(P_{t,u'})} \lesssim \varepsilon \tag{8.46}$$

and where the vectorfield F_2^j satisfies:

$$\|F_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}}\varepsilon. \tag{8.47}$$

Now, in view of the estimates (2.80), (2.81) and (2.82) for χ_1, $F = \chi_1$ satisfies for any $2 \leq p < +\infty$:

$$\|F\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|F\|_{L^p_t L^\infty_{x'}} + \|\mathbf{D}F\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\partial_\omega F\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon.$$

Also, we have

$$\partial_\omega F = H_1 + H_2 \text{ with } H_1 = \partial_\omega \widehat{\chi} \text{ and } H_2 = -\partial_\omega \chi_2,$$

and H_1 and H_2 satisfy the assumption of Corollary 8.11 in view of the estimates (2.77) and (2.78) for $\partial_\omega \widehat{\chi}$ and the estimate (2.81) for $\partial_\omega \chi_2$. Thus, in view of Corollary 8.11, $\chi_1(\cdot,\omega)$ satisfies the decomposition (8.45) and the estimates (8.46) (8.47). This concludes the proof of the proposition.

8.4.3 Proof of Proposition 2.36

In view of Corollary 2.35, it suffices to prove the decomposition for χ_2:

$$\chi_2(\cdot,\omega) = F_1^j + F_2^j$$

where F_1^j does not depend on ω and satisfies:

$$\|F_1^j\|_{L^\infty_{u'}L^\infty(P_{t,u'})L^2_t} \lesssim \varepsilon,$$

and where F_2^j satisfies:

$$\|F_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}}.$$

We choose

$$F_1^j = \chi_2(\cdot,\omega') \text{ and } F_2^j = \chi_2(\cdot,\omega) - \chi_2(\cdot,\omega').$$

Then, the estimates for F_1^j and F_2^j follow from the estimate (2.80) and the Lemma 2.34 for χ_2. This concludes the proof of the proposition.

8.4.4 Proof of Corollary 8.13

In view of the estimates (2.80), (2.81) and (2.82) for χ_1, $F = \chi_1^l$ satisfies for any $2 \leq p < +\infty$:

$$\|F\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|F\|_{L^p_t L^\infty_{x'}} + \|\mathbf{D}F\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\partial_\omega F\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon.$$

Also, we have

$$\partial_\omega F = H_1 + H_2 \text{ with } H_1 = l\chi_1^{l-1}\partial_\omega\widehat{\chi} \text{ and } H_2 = -l\chi_1^{l-1}\partial_\omega\chi_2.$$

Lemma 8.12, together with the estimates (2.81) and (6.92) for χ_1 yields for any $2 \leq r < +\infty$

$$\|\chi_1^{l-1}\|_{L^r_t L^\infty_{x'}} + \|\nabla\!\!\!/(\chi_1^{l-1})\|_{L^r_t B^0_{2,1}(P_{t,u})} \lesssim \varepsilon.$$

Together with Lemma C.1 and the estimates (2.77) and (2.78) for $\partial_\omega\widehat{\chi}$, we obtain:

$$\|P_j(\nabla\!\!\!/_L(H_1))\|_{L^q_t L^2_{x'}} + \|P_j(\nabla\!\!\!/_{\underline{L}}(H_1))\|_{L^q_t L^2_{x'}} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\gamma(u)\varepsilon.$$

Also, H_2 satisfies the following estimate

$$\begin{aligned}
& \|H_2\|_{L^2_t L^{\frac{8}{3}}_{x'}} + \|\nabla\!\!\!/ H_2\|_{L^2_t L^{\frac{8}{5}}_{x'}} \\
\lesssim\ & \|\chi_1^{l-1}\|_{L^4_t L^\infty_{x'}}\|\partial_\omega\chi_2\|_{L^4_t L^{\frac{8}{3}}_{x'}} + \|\chi_1^{l-1}\|_{L^\infty_t L^8_{x'}}\|\nabla\!\!\!/\partial_\omega\chi_2\|_{L^\infty_u L^2(\mathcal{H}_u)} \\
& + \|\chi_1^{l-2}\|_{L^\infty_t L^{16}_{x'}}\|\nabla\!\!\!/\chi_1\|_{L^\infty_t L^2_{x'}}\|\partial_\omega\chi_2\|_{L^2_t L^{16}_{x'}} \\
\lesssim\ & \varepsilon,
\end{aligned}$$

where we used in the last inequality the estimate (2.81) for χ_1 and $\partial_\omega\chi_2$.

Finally, we have proved that F, H_1 and H_2 satisfy the assumption of Corollary 8.11. Thus, we may apply Corollary 8.11 to obtain the desired decomposition $\chi_1^l(\cdot,\omega)$. This concludes the proof of the corollary.

8.4.5 Proof of Corollary 8.14

We decompose $\chi_1(\cdot,\omega)^l\chi_2(\cdot,\omega)$ as

$$\chi_1(\cdot,\omega)^l\chi_2(\cdot,\omega) = \chi_1(\cdot,\omega)^l\chi_2(\cdot,\omega') + \chi_1(\cdot,\omega)^l(\chi_2(\cdot,\omega) - \chi_2(\cdot,\omega')). \tag{8.48}$$

In view of Lemma 2.34 and the estimate (2.81) for χ_1, we have

$$\begin{aligned} & \|\chi_1(\cdot,\omega)^l(\chi_2(\cdot,\omega) - \chi_2(\cdot,\omega'))\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ \lesssim\ & \|\chi_1(\cdot,\omega)\|^l_{L^\infty_u L^{6l}(\mathcal{H}_u)}\|\chi_2(\cdot,\omega) - \chi_2(\cdot,\omega')\|_{L^\infty_u L^3(\mathcal{H}_u)} \\ \lesssim\ & |\omega-\omega'|\varepsilon. \end{aligned} \tag{8.49}$$

Finally, in view of the decomposition for $\chi_1(\cdot,\omega)^l$ provided by Corollary 8.13, (8.48) and (8.49), we obtain the desired decomposition for $\chi_1(\cdot,\omega)^l\chi_2(\cdot,\omega)$ with F_1^j and F_2^j defined in the statement of Corollary 8.13, and

$$F_3^j = \chi_1(\cdot,\omega)^l(\chi_2(\cdot,\omega) - \chi_2(\cdot,\omega')).$$

This concludes the proof of the corollary.

8.4.6 Proof of Proposition 2.37

In view of the decomposition (2.80) for $\widehat{\chi}$, we decompose $\widehat{\chi}(\cdot,\omega)^2$ as

$$\widehat{\chi}(\cdot,\omega)^2 = \chi_1(\cdot,\omega)^2 + 2\chi_1(\cdot,\omega)\chi_2(\cdot,\omega) + \chi_2(\cdot,\omega)^2. \tag{8.50}$$

We have

$$\chi_2(\cdot,\omega)^2 = \chi_2(\cdot,\omega')^2 + 2\chi_2(\cdot,\omega')(\chi_2(\cdot,\omega) - \chi_2(\cdot,\omega')) + (\chi_2(\cdot,\omega) - \chi_2(\cdot,\omega'))^2. \tag{8.51}$$

Now, we have in view of Lemma 2.34 and the estimate (2.81) for χ_2:

$$\|\chi_2(\cdot,\omega) - \chi_2(\cdot,\omega')\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim |\omega-\omega'|\varepsilon \text{ and } \|(\chi_2(\cdot,\omega) - \chi_2(\cdot,\omega'))^2\|_{L^2(\mathcal{M})} \lesssim |\omega-\omega'|^2\varepsilon. \tag{8.52}$$

Finally, in view of (8.50), Corollary 8.13 with $l=2$, Corollary 8.14 with $l=1$, (8.51) and (8.52), we obtain the desired decomposition for $\widehat{\chi}^2$.

8.4.7 Proof of Corollary 8.15

We decompose $\chi_1(\cdot,\omega)\chi_2(\cdot,\omega)^2$ as

$$\begin{aligned} \chi_1(\cdot,\omega)\chi_2(\cdot,\omega)^2 \ =\ & \chi_1(\cdot,\omega)\chi_2(\cdot,\omega')^2 + 2\chi_1(\cdot,\omega)\chi_2(\cdot,\omega')(\chi_2(\cdot,\omega) - \chi_2(\cdot,\omega')) \\ & +\chi_1(\cdot,\omega)(\chi_2(\cdot,\omega) - \chi_2(\cdot,\omega'))^2. \end{aligned} \tag{8.53}$$

In view of Lemma 2.34 and the estimate (2.81) for χ_1, we have

$$\begin{aligned} & \|\chi_1(\cdot,\omega)(\chi_2(\cdot,\omega) - \chi_2(\cdot,\omega'))\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ \lesssim\ & \|\chi_1(\cdot,\omega)\|_{L^\infty_u L^6(\mathcal{H}_u)}\|\chi_2(\cdot,\omega) - \chi_2(\cdot,\omega')\|_{L^\infty_u L^3(\mathcal{H}_u)} \\ \lesssim\ & |\omega-\omega'|\varepsilon. \end{aligned} \tag{8.54}$$

Also, in view of the estimate (2.80) and (2.81) for χ_1 and χ_2, we have[20]

$$\begin{aligned}\|\chi_1(\cdot,\omega)(\chi_2(\cdot,\omega)-\chi_2(\cdot,\omega'))^2\|_{L^2(\mathcal{M})} &\lesssim \|\chi_1(\cdot,\omega)\|_{L^{6_+}(\mathcal{M})}\|\chi_2(\cdot,\omega)-\chi_2(\cdot,\omega')\|^2_{L^{6_-}(\mathcal{M})}\\ &\lesssim (\mathcal{N}_1(\chi_1)+\|\chi_1\|_{L_t^{6_+}L^\infty_{x'}})\|\partial_\omega\chi_2\|^2_{L^{6_-}(\mathcal{M})}|\omega-\omega'|^2\\ &\lesssim |\omega-\omega'|^2\varepsilon. \end{aligned} \tag{8.55}$$

Finally, in view of the decomposition for $\chi_1(\cdot,\omega)$ provided by (8.45) (8.46) (8.47), (8.53), (8.54) and (8.55), we obtain the desired decomposition for $\chi_1(\cdot,\omega)\chi_2(\cdot,\omega)^2$ with F_1^j and F_2^j defined in (8.45),

$$F_3^j = \chi_1(\cdot,\omega)^l(\chi_2(\cdot,\omega)-\chi_2(\cdot,\omega')),$$

and

$$F_4^j = \chi_1(\cdot,\omega)(\chi_2(\cdot,\omega)-\chi_2(\cdot,\omega'))^2.$$

This concludes the proof of the corollary.

8.4.8 Proof of Proposition 2.38

In view of the decomposition (2.80) for $\widehat{\chi}$, we decompose $\widehat{\chi}(\cdot,\omega)^3$ as

$$\widehat{\chi}(\cdot,\omega)^3 = \chi_1(\cdot,\omega)^3 + 3\chi_1(\cdot,\omega)\chi_2(\cdot,\omega)^2 + 3\chi_1(\cdot,\omega)^2\chi_2(\cdot,\omega) + \chi_2(\cdot,\omega)^3. \tag{8.56}$$

We have

$$\begin{aligned}\chi_2(\cdot,\omega)^3 &= \chi_2(\cdot,\omega')^3 + 3\chi_2(\cdot,\omega')^2(\chi_2(\cdot,\omega)-\chi_2(\cdot,\omega')) + 3\chi_2(\cdot,\omega')(\chi_2(\cdot,\omega)-\chi_2(\cdot,\omega'))^2\\ &\quad +(\chi_2(\cdot,\omega)-\chi_2(\cdot,\omega'))^3. \end{aligned} \tag{8.57}$$

Now, we have in view of Lemma 2.34 and the estimate (2.81) for χ_2:

$$\begin{aligned}&\|\chi_2(\cdot,\omega)-\chi_2(\cdot,\omega')\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim |\omega-\omega'|\varepsilon,\ \|(\chi_2(\cdot,\omega)-\chi_2(\cdot,\omega'))^2\|_{L^2(\mathcal{M})} \lesssim |\omega-\omega'|^2\varepsilon,\\ &\text{and } \|(\chi_2(\cdot,\omega)-\chi_2(\cdot,\omega'))^3\|_{L^{2_-}(\mathcal{M})} \lesssim |\omega-\omega'|^3\varepsilon. \end{aligned} \tag{8.58}$$

Finally, in view of (8.56), Corollary 8.13 with $l=3$, Corollary 8.14 with $l=2$, Corollary 8.15, (8.57) and (8.58), we obtain the desired decomposition for $\widehat{\chi}^3$.

8.5 A second variant of Proposition 8.1

We have the following variant of Proposition 8.1

Proposition 8.16 *Let $f(\cdot,\omega)$ a scalar function depending on a parameter $\omega\in\mathbb{S}^2$ such that:*

$$\|f\|_{L^\infty_u L^2(\mathcal{H}_u)} + \mathcal{N}_1(f) + \|\Lambda^{-1}(\nabla_{bN}f)\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\partial_\omega f\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon.$$

[20]Note that we first choose 6_- such that $\partial_\omega\chi_2\in L^{6-}(\mathcal{H}_u)$ in view of (2.81), then 6_+ such that

$$\frac{1}{6_+}+\frac{2}{6_-}=\frac{1}{2}$$

and finally, we choose $p=6_+$ in (2.81) so that $\chi_1\in L_t^{6+}L^\infty_{x'}$.

Let ω and ω' in $\mathbb{S}^2$. Let $u = u(t,x,\omega)$ and $u' = u(t,x,\omega')$. Then, for any ω'' in $\mathbb{S}^2$ on the arc joining ω and ω', and for any $j \geq 0$, we have the following decomposition for $f(\cdot,\omega'')$:

$$f(\cdot,\omega'') = P_{\leq \frac{j}{2}}(f(\cdot,\omega')) + f_2^j$$

and where f_2^j satisfies:

$$\|f_2^j\|_{L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{4}}\varepsilon + 2^{\frac{j}{4}}|\omega - \omega'|\varepsilon.$$

As a corollary of Proposition 8.16, we obtain:

Corollary 8.17 *Let $F(\cdot,\omega)$ a tensor depending on a parameter $\omega \in \mathbb{S}^2$ such that:*

$$\|F\|_{L^\infty_u L^2(\mathcal{H}_u)} + \mathcal{N}_1(F) + \|\partial_\omega F\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon.$$

Also, assume the existence of tensors H_1 and H_2 such that

$$\nabla\!\!\!/_{bN} F = \nabla\!\!\!/ H_1 + H_2 \text{ with } \|H_1\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|H_2\|_{L^2_t L^{\frac{4}{3}}_{x'}} \lesssim \varepsilon.$$

Let ω and ω' in $\mathbb{S}^2$. Let $u = u(t,x,\omega)$ and $u' = u(t,x,\omega')$. Then, for any ω'' in $\mathbb{S}^2$ on the arc joining ω and ω', and for any $j \geq 0$, we have the following decomposition for $F(\cdot,\omega'')$:

$$F(\cdot,\omega'') = F_1^j + F_2^j$$

where F_1^j does not depend on ω and satisfies for any $2 \leq q \leq +\infty$:

$$\|F_1^j\|_{L^q(P_{t,u'})} \lesssim \|F\|_{L^q(P_{t,u'})},$$

and where F_2^j satisfies:

$$\|F_2^j\|_{L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{4}}\varepsilon + 2^{\frac{j}{4}}|\omega - \omega'|\varepsilon.$$

The following lemma will be useful for the proof of Proposition 8.16 and Corollary 8.17.

Lemma 8.18 *Let f a scalar function and ω, ω' in $\mathbb{S}^2$. Assume that f satisfies*

$$\|f\|_{L^\infty_u L^2(\mathcal{H}_u)} + \mathcal{N}_1(f) + \|\Lambda^{-1}(\nabla_{bN} f)\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon.$$

Then, for any $l \geq 0$, we have:

$$\|[\partial_\omega, P_{\leq l}]f\|_{L^2(\mathcal{H}_{u'})} \lesssim 2^{\frac{l}{2}}\varepsilon.$$

The proof of Corollary 8.17 is postponed to section 8.5.2 and the proof of Lemma 8.18 is postponed to section D.9. We now conclude the proof of Proposition 8.16.

8.5.1 Proof of Proposition 8.16

We decompose $f(\cdot,\omega'')$ as:

$$\begin{aligned} f(\cdot,\omega'') &= P''_{\leq \frac{j}{2}}(f(\cdot,\omega'')) + \sum_{l>\frac{j}{2}} P''_l(f(\cdot,\omega'')) \qquad (8.59)\\ &= P'_{\leq \frac{j}{2}}(f(\cdot,\omega')) + \int_{[\omega',\omega'']} \partial_\omega P'''_{\leq \frac{j}{2}}(f(\cdot,\omega'''))d\omega'''(\omega'-\omega'') + \sum_{l>\frac{j}{2}} P''_l(f(\cdot,\omega''))\\ &= P'_{\leq \frac{j}{2}}(f(\cdot,\omega')) + \int_{[\omega',\omega'']} (P'''_{\leq \frac{j}{2}}(\partial_\omega f)(\cdot,\omega''') + [\partial_\omega, P'''_{\leq \frac{j}{2}}]f(\cdot,\omega'''))d\omega'''(\omega'-\omega'')\\ &\quad + \sum_{l>\frac{j}{2}} P''_l(f(\cdot,\omega'')), \end{aligned}$$

where ω''' denotes an angle in $\mathbb{S}^2$ on the arc joining ω' and ω''.

Next, we estimate the last two terms in the right-hand side of (8.59). Using Lemma 8.9 with $p=2$, we have:

$$\begin{aligned} \sum_{l>\frac{j}{2}} \|P''_l(f(\cdot,\omega''))\|_{L^\infty_u L^2(\mathcal{H}_u)} &\lesssim \sum_{l>\frac{j}{2}} \|P_l f\|^{\frac{1}{2}}_{L^\infty_u L^2(\mathcal{H}_u)} \|\not\nabla P_l f\|^{\frac{1}{2}}_{L^\infty_u L^2(\mathcal{H}_u)}\\ &\lesssim \left(\sum_{l>\frac{j}{2}} 2^{-\frac{l}{2}}\right) \|\not\nabla f\|_{L^\infty_u L^2(\mathcal{H}_u)}\\ &\lesssim 2^{-\frac{j}{4}}\varepsilon, \qquad (8.60) \end{aligned}$$

where we used the finite band property for P_l and the assumptions on f. Also, using Lemma 8.9 with $p=2$, we have:

$$\begin{aligned} \|P'''_{\leq \frac{j}{2}}(\partial_\omega f)(\cdot,\omega''')\|_{L^\infty_u L^2(\mathcal{H}_u)} &\lesssim \|P_{\leq \frac{j}{2}}\partial_\omega f\|^{\frac{1}{2}}_{L^\infty_u L^2(\mathcal{H}_u)} \|\not\nabla P_{\leq \frac{j}{2}}\partial_\omega f\|^{\frac{1}{2}}_{L^\infty_u L^2(\mathcal{H}_u)} \qquad (8.61)\\ &\lesssim 2^{\frac{j}{4}}\|\partial_\omega f\|_{L^\infty_u L^2(\mathcal{H}_u)}\\ &\lesssim 2^{\frac{j}{4}}\varepsilon, \end{aligned}$$

where we used the finite band property for $P_{\leq \frac{j}{2}}$ and the assumptions on f.

Using Lemma 8.18 together with the assumptions on f, we have:

$$\|[\partial_\omega, P'''_{\leq \frac{j}{2}}]f(\cdot,\omega'''))\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{\frac{j}{4}}\varepsilon. \qquad (8.62)$$

In view of (8.59), we have $f(\cdot,\omega'') = f_j^1 + f_2^j$ where f_1^j is defined as:

$$f_j^1 = P'_{\leq \frac{j}{2}}(f(\cdot,\omega')), \qquad (8.63)$$

and f_j^2 is defined as:

$$f_j^2 = \int_{[\omega',\omega'']} (P'''_{\leq \frac{j}{2}}(\partial_\omega f)(\cdot,\omega''') + [\partial_\omega, P'''_{\leq \frac{j}{2}}]f(\cdot,\omega'''))d\omega'''(\omega'-\omega'') + \sum_{l>\frac{j}{2}} P''_l(f(\cdot,\omega'')). \qquad (8.64)$$

Using (8.60), (8.61) and (8.62), and the fact that ω'' is on the arc of $\mathbb{S}^2$ joining ω and ω', we have the following estimate for f_j^2:

$$\|f_j^2\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{4}}\varepsilon + 2^{\frac{j}{4}}|\omega-\omega'|\varepsilon.$$

This concludes the proof of Proposition 8.16.

8.5.2 Proof of Corollary 8.17

Using Lemma 8.8, it suffices to prove the decomposition of Corollary 8.17 where $F(\omega'',.)$ is replaced by $g(F(\cdot,\omega''),Q_{\leq 1}(N_l))$ for $l=1,2,3$. Since the proof is identical for $l=1,2,3$, we simply take $l=1$. Therefore, it remains to prove that the following decomposition holds $g(F(\cdot,\omega''),Q_{\leq 1}(N_1))$:

$$g(F(\cdot,\omega''),Q_{\leq 1}(N_1)) = P_{\leq \frac{j}{2}}(g(F(\cdot,\omega'),Q_{\leq 1}(N_1))) + f_2^j, \tag{8.65}$$

where the scalar function f_2^j satisfies:

$$\|f_2^j\|_{L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{4}}\varepsilon + 2^{\frac{j}{4}}|\omega-\omega'|\varepsilon. \tag{8.66}$$

In particular, F_1^j is connected to the first term in the right-hand side of (8.65), which does not depend on ω and satisfies the following estimate for any $2\leq q\leq +\infty$:

$$\Big\|P_{\leq \frac{j}{2}}(g(F(\cdot,\omega'),Q_{\leq 1}(N_1)))\Big\|_{L^q(P_{t,u'})} \lesssim \|F\|_{L^q(P_{t,u'})}\|Q_{\leq 1}(N_1)\|_{L^\infty} \lesssim \|F\|_{L^q(P_{t,u'})},$$

where we used the fact that $P_{\leq \frac{j}{2}}$ is bounded on $L^q(P_{t,u'})$ and the fact that $Q_{\leq 1}$ is bounded on L^∞.

Let $f = g(F(\cdot,\omega),Q_{\leq 1}(N_1))$. In order to prove the decomposition (8.65) (8.66) for $g(F(\cdot,\omega''),Q_{\leq 1}(N_1))$, it suffices to show that f satisfies the assumptions of Proposition 8.16. This was already done in the proof of Corollary 8.2, up to the estimate of $\nabla_{bN}f$ which is the only one for which the proof has to be adapted. We have:

$$\begin{aligned}
&\|\Lambda^{-1}(\nabla_{bN}f)\|_{L_u^\infty L^2(\mathcal{H}_u)} \qquad (8.67)\\
\lesssim\ & \|\Lambda^{-1}(\mathbf{g}(Q_{\leq 1}(N_1),\nabla\!\!\!/_{bN}F))\|_{L_u^\infty L^2(\mathcal{H}_u)} + \|\Lambda^{-1}(\mathbf{g}(\nabla\!\!\!/_{bN}Q_{\leq 1}(N_1),F))\|_{L_u^\infty L^2(\mathcal{H}_u)}\\
\lesssim\ & \|\Lambda^{-1}(\mathbf{g}(Q_{\leq 1}(N_1),\nabla\!\!\!/ H_1+H_2))\|_{L_u^\infty L^2(\mathcal{H}_u)} + \|b\|_{L^\infty}\|\nabla\!\!\!/_N Q_{\leq 1}(N_1)\|_{L_t^\infty L^4_{x'}}\|F\|_{L_t^2L^4_{x'}}\\
\lesssim\ & \|\Lambda^{-1}(\mathbf{g}(Q_{\leq 1}(N_1),\nabla\!\!\!/ H_1))\|_{L_u^\infty L^2(\mathcal{H}_u)} + \|\Lambda^{-1}(\mathbf{g}(Q_{\leq 1}(N_1),H_2))\|_{L_u^\infty L^2(\mathcal{H}_u)} + \varepsilon,
\end{aligned}$$

where we used the fact that Λ^{-1} in bounded on $L^2(P_{t,u})$, the assumptions on F and in particular the decomposition for $\nabla\!\!\!/_N F$, the estimate (2.69) for b, and the estimate (8.41) for $\nabla\!\!\!/_N Q_{\leq 1}(N_1)$. We consider the first term in the right-hand side of (8.67). We have

$$\mathbf{g}(Q_{\leq 1}(N_1),\nabla\!\!\!/ H_1) = \nabla\!\!\!/(\mathbf{g}(Q_{\leq 1}(N_1),H_1)) - \mathbf{g}(\nabla\!\!\!/ Q_{\leq 1}(N_1),H_1)$$

and thus

$$\begin{aligned}
&\|\Lambda^{-1}(\mathbf{g}(Q_{\leq 1}(N_1),\nabla\!\!\!/ H_1))\|_{L_u^\infty L^2(\mathcal{H}_u)} \qquad (8.68)\\
\lesssim\ & \|\Lambda^{-1}\nabla\!\!\!/(\mathbf{g}(Q_{\leq 1}(N_1),H_1))\|_{L_u^\infty L^2(\mathcal{H}_u)} + \|\Lambda^{-1}(\mathbf{g}(\nabla\!\!\!/ Q_{\leq 1}(N_1),H_1))\|_{L_u^\infty L^2(\mathcal{H}_u)}\\
\lesssim\ & \|Q_{\leq 1}(N_1)\|_{L^\infty}\|H_1\|_{L_u^\infty L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/ Q_{\leq 1}(N_1)\|_{L_t^\infty L^4_{x'}}\|H_1\|_{L_u^\infty L^2(\mathcal{H}_u)}\\
\lesssim\ & \varepsilon,
\end{aligned}$$

where we used the fact that $\Lambda^{-1}\nabla\!\!\!/$ in bounded on $L^2(P_{t,u})$, the fact that Λ^{-1} is bounded from $L^{\frac{4}{3}}(P_{t,u})$ to $L^2(P_{t,u})$, the assumption on H_1, and the estimate (8.41) for $\nabla\!\!\!/_N Q_{\leq 1}(N_1)$. Next, we estimate the first term in the right-hand side of (8.67). We have

$$\begin{aligned} \|\Lambda^{-1}(\mathbf{g}(Q_{\leq 1}(N_1), H_2))\|_{L^\infty_u L^2(\mathcal{H}_u)} &\lesssim \|Q_{\leq 1}(N_1)\|_{L^\infty}\|H_2\|_{L^2_t L^{\frac{4}{3}}_{x'}} \\ &\lesssim \varepsilon, \end{aligned} \tag{8.69}$$

where we used the fact that Λ^{-1} in bounded from $L^{\frac{4}{3}}(P_{t,u})$ to $L^2(P_{t,u})$ and the assumption on H_2. In view of (8.67), (8.68) and (8.69), we finally obtain

$$\|\Lambda^{-1}(\nabla_{bN} f)\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon.$$

Together with the other estimates for f which may be derived as in Corollary 8.2, we obtain that f satisfies the assumptions of Proposition 8.16, which in turn yields the decomposition (8.65)-(8.66) for $g(F(\cdot, \omega''), Q_{\leq 1}(N_1))$. This concludes the proof of Corollary 8.17.

8.6 Decompositions involving ζ, $\nabla\!\!\!/ b$ and $\partial_\omega b$

The goal of this section is to prove Propositions 2.39 and Proposition 2.40. The proof of Proposition 2.39 is given in section 8.6.1, and the proof of Proposition 2.40 is given in section 8.6.2.

We will need the following two lemmas.

Lemma 8.19 *$\nabla\!\!\!/_{bN}\nabla\!\!\!/ b$ and $\nabla\!\!\!/_{bN}\zeta$ satisfy the following decomposition:*

$$\nabla\!\!\!/_{bN}\nabla\!\!\!/ b,\ \nabla\!\!\!/_{bN}\zeta = \nabla\!\!\!/ h_1 + H_2,$$

where the scalar h_1 and the tensor H_2 satisfy

$$\|h_1\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|H_2\|_{L^2_t L^{\frac{4}{3}}_{x'}} \lesssim \varepsilon.$$

Lemma 8.20 *There holds the following estimate*

$$\|\Lambda^{-1}(\nabla_{bN}\partial_\omega b)\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon.$$

The proof of Lemma 8.19 is postponed to section D.10, and the proof of Lemma 8.20 is postponed to section D.11.

8.6.1 Proof of Corollary 2.39

In view of the estimate (2.72) for ζ, the estimate (2.77) for $\partial_\omega\zeta$ and Lemma 8.19, ζ satisfies the assumption of Corollary 8.17. Also, in view of the estimate (2.69) for b, the estimate (2.77) for $\partial_\omega b$ and Lemma 8.19, $\nabla\!\!\!/ b$ satisfies the assumptions of Corollary 8.17. Thus, the desired decomposition of Corollary 2.39 for ζ and $\nabla\!\!\!/ b$ follows from Corollary 8.17. This concludes the proof of Corollary 2.39.

8.6.2 Proof of Corollary 2.40

We have:

$$b(\cdot,\omega)-b(,.\omega')=\int_{[\omega,\omega']}\partial_\omega b(\cdot,\omega'')d\omega''(\omega-\omega'). \tag{8.70}$$

We denote $\partial_\omega b''=\partial_\omega b(\cdot,\omega'')$. In view of the estimate (2.77) for $\partial_\omega b$, the estimate (2.86) for $\partial_\omega^2 b$ and Lemma 8.20, $\partial_\omega b$ satisfies the assumptions of Proposition 8.16. Thus, we have the following decomposition for $\partial_\omega b''$

$$\partial_\omega b''=f_1^j+f_2^j, \tag{8.71}$$

where the scalar f_1^j only depends on ω' and satisfies:

$$\|f_1^j\|_{L^\infty}\lesssim\|\partial_\omega b\|_{L^\infty}\lesssim\varepsilon \tag{8.72}$$

in view of the estimate (2.77) for $\partial_\omega b$, and where the scalar f_2^j satisfies:

$$\|f_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)}\lesssim 2^{-\frac{j}{4}}\varepsilon. \tag{8.73}$$

Injecting the decomposition (8.71) in (8.70), and in view of (8.72) (8.73), we obtain the desired decomposition for $b(.\omega)-b(,.\omega')$. This concludes the proof of Corollary 2.40.

9 Additional estimates for trχ

The goal of this section is to prove Proposition 2.41 and Proposition 2.42.

9.1 Commutator estimates between P_j and $\nabla\!\!\!/_L, \nabla\!\!\!/_N$

Proposition 9.1 *Let F as tensor on $\mathcal{M}$. Let a real number a such that $0<a<\frac{1}{4}$. Then, we have*

$$\|[\nabla\!\!\!/_{bN},P_j]F\|_{L^\infty_t L^{\frac{4}{3}}_u L^2(P_{t,u})}\lesssim 2^{ja}\|\nabla F\|_{L^\infty_t L^2_{x'}}. \tag{9.1}$$

Proposition 9.2 *Let a scalar function f on $\mathcal{H}_u$. Then, we have for $l\geq 0$*

$$\|[bN,P_l]f\|_{L^\infty_u L^2(\mathcal{H}_u)}+2^{-l}\|\nabla\!\!\!/[bN,P_l]f\|_{L^\infty_u L^2(\mathcal{H}_u)}\lesssim\varepsilon\mathcal{N}_1(f), \tag{9.2}$$

and

$$\|[nL,P_l]f\|_{L^\infty_u L^2(\mathcal{H}_u)}+2^{-l}\|\nabla\!\!\!/[nL,P_l]f\|_{L^\infty_u L^2(\mathcal{H}_u)}\lesssim\varepsilon\mathcal{N}_1(f). \tag{9.3}$$

Proposition 9.3 *Let f a scalar on $\mathcal{M}$. Then, we have*

$$\|[nL,P_q]f\|_{L^\infty_u L^2(\mathcal{H}_u)}+\|[bN,P_q]f\|_{L^\infty_u L^2(\mathcal{H}_u)}\lesssim 2^q\|f\|_{L^\infty_u L^2(\mathcal{H}_u)}. \tag{9.4}$$

The proof of Proposition 9.1 is postponed to section E.1, the proof of Proposition 9.2 is postponed to section E.2, and the proof of Proposition 9.3 is postponed to section E.3.

9.2 Commutator estimates acting on $\mathbf{tr}\chi$

Proposition 9.4 *We have the following commutator estimate*

$$2^j\|[nL,P_j]tr\chi\|_{L^1_tL^2_{x'}}+\|\nabla\!\!\!/[nL,P_j]tr\chi\|_{L^1_tL^2_{x'}}\lesssim\varepsilon. \tag{9.5}$$

Proposition 9.5 *We have*

$$2^{\frac{j}{2}}\|[bN,P_j]tr\chi\|_{L^\infty_uL^2(\mathcal{H}_u)}+2^{-\frac{j}{2}}\|\nabla\!\!\!/[bN,P_j]tr\chi\|_{L^\infty_uL^2(\mathcal{H}_u)}\lesssim\varepsilon. \tag{9.6}$$

and

$$2^{\frac{j}{2}}\|[nL,P_j]tr\chi\|_{L^\infty_uL^2(\mathcal{H}_u)}+2^{-\frac{j}{2}}\|\nabla\!\!\!/[nL,P_j]tr\chi\|_{L^\infty_uL^2(\mathcal{H}_u)}\lesssim\varepsilon. \tag{9.7}$$

Proposition 9.6 *We have the following commutator estimate*

$$\|[\nabla\!\!\!/,P_j]tr\chi\|_{L^2_tL^4_{x'}}\lesssim\varepsilon. \tag{9.8}$$

The proof of Proposition 9.4 is postponed to section E.4, the proof of Proposition 9.5 is postponed to section E.5, and the proof of Proposition 9.6 is postponed to section E.6.

9.3 Additional estimates for $P_j\mathbf{tr}\chi$

The goal of this section is to prove Proposition 2.41 and Proposition 2.42. Note that the finite band property for P_j together with the estimate (2.70) for $\mathrm{tr}\chi$ yields

$$\|P_j\mathrm{tr}\chi\|_{L^\infty_tL^2_{x'}}\lesssim 2^{-j}\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^\infty_tL^2_{x'}}\lesssim 2^{-j}\varepsilon. \tag{9.9}$$

Also, the boundedness on $L^2(P_{t,u})$ of P_j together with the estimate (2.70) for $\mathrm{tr}\chi$ yields

$$\|\nabla\!\!\!/P_{\leq j}\mathrm{tr}\chi\|_{L^\infty_tL^2_{x'}}=\|(-\Delta\!\!\!/)^{\frac{1}{2}}P_{\leq j}\mathrm{tr}\chi\|_{L^\infty_tL^2_{x'}}\lesssim\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^\infty_tL^2_{x'}}\lesssim 2^{-j}\varepsilon. \tag{9.10}$$

In order to prove Proposition 2.41 and Proposition 2.42, we need in particular to obtain (9.9) and (9.10), where the norm $L^\infty_tL^2_{x'}$ is replaced by its stronger version $L^2_{x'}L^\infty_t$. We will need the following lemmas.

Lemma 9.7 *Let h a scalar on $P_{t,u}$, and let F a tensor on $P_{t,u}$. Then, we have*

$$\|[P_{>j},P_{\leq j}(h)]F\|_{L^2(P_{t,u})}\lesssim\|\nabla\!\!\!/h\|_{L^2(P_{t,u})}\|F\|_{L^2(P_{t,u})}. \tag{9.11}$$

Lemma 9.8 *Let h a scalar on $P_{t,u}$, and let F a tensor on $P_{t,u}$. Then, we have*

$$\|\nabla\!\!\!/[P_j,P_{\leq j}(h)]F\|_{L^2(P_{t,u})}\lesssim 2^j(\|\nabla\!\!\!/h\|_{L^2(P_{t,u})}+\|K\|_{L^2(P_{t,u})}\|h\|_{L^2(P_{t,u})})\|F\|_{L^2(P_{t,u})}. \tag{9.12}$$

Lemma 9.9 *Let h a scalar on $P_{t,u}$, and let $a>0$. Then, we have*

$$\|[P_{\leq j},\nabla\!\!\!/]h\|_{L^2(P_{t,u})}\lesssim\|K\|_{L^2(P_{t,u})}(\|K\|_{L^2(P_{t,u})}\|h\|_{L^2(P_{t,u})}+\|\Lambda^a h\|_{L^2(P_{t,u})}). \tag{9.13}$$

Lemma 9.10 *Let h a scalar on $P_{t,u}$, and let $a>0$. Then, we have*

$$\|\nabla\!\!\!/[P_j,\nabla\!\!\!/]h\|_{L^2(P_{t,u})}\lesssim 2^j\|K\|_{L^2(P_{t,u})}(\|K\|_{L^2(P_{t,u})}\|h\|_{L^2(P_{t,u})}+\|\Lambda^a h\|_{L^2(P_{t,u})}). \tag{9.14}$$

Lemma 9.11 *Let f a scalar on $P_{t,u}$. We have*

$$\|\nabla\!\!\!/ f\|_{L^\infty(P_{t,u})} \lesssim \|\Delta\!\!\!/ f\|_{L^2(P_{t,u})} + \|\nabla\!\!\!/\Delta\!\!\!/ f\|^{\frac{1}{2}}_{L^2(P_{t,u})}\|\nabla\!\!\!/ f\|^{\frac{1}{2}}_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})}\|\nabla\!\!\!/ f\|_{L^2(P_{t,u})}. \quad (9.15)$$

In the subsequent sections, we provide a proof of Proposition 2.41 and Proposition 2.42. The proof of Lemma 9.7 is postponed to section E.7, the proof of Lemma 9.8 is postponed to section E.8, the proof of Lemma 9.9 is postponed to section E.9, the proof of Lemma 9.10 is postponed to section E.10, and the proof of Lemma 9.11 is postponed to section E.11.

9.3.1 Proof of Proposition 2.41

Using the estimate (3.66) for transport equations, we have

$$\begin{aligned} \|P_j\mathrm{tr}\chi\|_{L^2_{x'}L^\infty_t} &\lesssim \|P_j\mathrm{tr}\chi\|_{L^2(P_{0,u})} + \|nLP_j\mathrm{tr}\chi\|_{L^2_{x'}L^1_t} \quad (9.16)\\ &\lesssim 2^{-j}\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2(P_{0,u})} + \|P_j(nL\mathrm{tr}\chi)\|_{L^2_{x'}L^1_t} + \|[nL, P_j]\mathrm{tr}\chi\|_{L^2_{x'}L^1_t}\\ &\lesssim \|P_j(nL\mathrm{tr}\chi)\|_{L^2_{x'}L^1_t} + 2^{-j}\varepsilon, \end{aligned}$$

where we used the estimate (9.5) and (2.62) in the last inequality. Now, (2.91) follows from (2.92) and (9.16). Thus, it remains to prove (2.92).

Next, using the Raychaudhuri equation (2.28), we have

$$\begin{aligned} \|P_j(nL\mathrm{tr}\chi)\|_{L^2_{x'}L^1_t} &\lesssim \|P_j(n|\widehat{\chi}|^2)\|_{L^2_{x'}L^1_t} + \left\|P_j\left(n\left(\frac{1}{2}(\mathrm{tr}\chi)^2 + \overline{\delta}\mathrm{tr}\chi\right)\right)\right\|_{L^1_{x'}L^2_t}\\ &\lesssim \|P_j(n|\widehat{\chi}|^2)\|_{L^2_{x'}L^1_t} + 2^{-j}\left\|\nabla\!\!\!/\left(n\left(\frac{1}{2}(\mathrm{tr}\chi)^2 + \overline{\delta}\mathrm{tr}\chi\right)\right)\right\|_{L^2(\mathcal{H}_u)}, \end{aligned}$$

where we used the finite band property for P_j in the last inequality. Together with the estimate (2.70) for $\mathrm{tr}\chi$, the estimate (2.67) for n, and the estimates (2.67) (2.68) for $\overline{\delta}$, we obtain

$$\|P_j(nL\mathrm{tr}\chi)\|_{L^2_{x'}L^1_t} \lesssim \|P_j(n|\widehat{\chi}|^2)\|_{L^2_{x'}L^1_t} + 2^{-j}\varepsilon.$$

Thus, it remains to prove

$$\|P_j(n|\widehat{\chi}|^2)\|_{L^2_{x'}L^1_t} \lesssim 2^{-j}\varepsilon. \quad (9.17)$$

We have

$$P_j(n|\widehat{\chi}|^2) = 2^{-2j}\Delta\!\!\!/ P_j(n|\widehat{\chi}|^2) = 2^{-2j}\mathrm{div}\!\!\!/(\nabla\!\!\!/ P_j(n|\widehat{\chi}|^2)).$$

Thus, we deduce

$$\|P_j(n|\widehat{\chi}|^2)\|_{L^2_{x'}L^1_t} \lesssim 2^{-2j}\|\mathrm{div}\!\!\!/ P_j\nabla\!\!\!/(n|\widehat{\chi}|^2)\|_{L^2_{x'}L^1_t} + 2^{-2j}\|\mathrm{div}\!\!\!/[\nabla\!\!\!/, P_j](n|\widehat{\chi}|^2)\|_{L^1_{x'}L^2_t}. \quad (9.18)$$

Now, in view of (9.14), we have for any $a > 0$

$$\|\mathrm{div}\!\!\!/[\nabla\!\!\!/, P_j](n|\widehat{\chi}|^2)\|_{L^2(P_{t,u})} \lesssim 2^j\|K\|_{L^2(P_{t,u})}(\|K\|_{L^2(P_{t,u})}\|n|\widehat{\chi}|^2\|_{L^2(P_{t,u})} + \|\Lambda^a(n|\widehat{\chi}|^2)\|_{L^2(P_{t,u})}).$$

Taking the L^1_t norm, we obtain

$$\begin{aligned}\|\mathrm{div}\!\!\!/[\nabla\!\!\!/, P_j](n|\widehat{\chi}|^2)\|_{L^1_t L^2_{x'}} &\lesssim 2^j\|K\|_{L^2(\mathcal{H}_u)}(\|K\|_{L^2(\mathcal{H}_u)}\|n\|_{L^\infty}\|\widehat{\chi}\|^2_{L^\infty_t L^4_{x'}} + \|\Lambda^a(n|\widehat{\chi}|^2)\|_{L^2(\mathcal{H}_u)})\\ &\lesssim 2^j\varepsilon(1+\|\Lambda^a(n|\widehat{\chi}|^2)\|_{L^2(\mathcal{H}_u)}),\end{aligned} \tag{9.19}$$

where we used in the last inequality the estimate (4.29) for K, the estimate (2.67) for n, and the estimate (2.71) for $\widehat{\chi}$. Now, choosing $0 < a < \frac{1}{2}$, the non sharp product estimate (5.15) yields

$$\begin{aligned}\|\Lambda^a(n|\widehat{\chi}|^2)\|_{L^2(\mathcal{H}_u)} &\lesssim \|n|\widehat{\chi}|^2\|_{\mathcal{P}^a}\\ &\lesssim \mathcal{N}_1(\widehat{\chi})(\|n\widehat{\chi}\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/(n\widehat{\chi})\|_{L^2(\mathcal{H}_u)})\\ &\lesssim \varepsilon,\end{aligned} \tag{9.20}$$

where we used in the last inequality the estimate (2.67) for n, and the estimate (2.71) for $\widehat{\chi}$. Together with (9.18) and (9.19), we obtain

$$\|P_j(n|\widehat{\chi}|^2)\|_{L^2_{x'}L^1_t} \lesssim 2^{-2j}\|\mathrm{div}\!\!\!/ P_j\nabla\!\!\!/(n|\widehat{\chi}|^2)\|_{L^2_{x'}L^1_t} + 2^{-j}\varepsilon. \tag{9.21}$$

Next, we estimate the right-hand side of (9.21). We have

$$\||\widehat{\chi}|^2\nabla\!\!\!/ n\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \|\nabla\!\!\!/ n\|_{L^\infty}\|\widehat{\chi}\|^2_{L^4(\mathcal{H}_u)} \lesssim \varepsilon, \tag{9.22}$$

where we used the estimate (2.67) for n and the estimate (2.71) for $\widehat{\chi}$. Together with (9.21) and the finite band property for P_j, we obtain

$$\|P_j(n|\widehat{\chi}|^2)\|_{L^2_{x'}L^1_t} \lesssim 2^{-2j}\|\mathrm{div}\!\!\!/ P_j(n\widehat{\chi}\cdot\nabla\!\!\!/\widehat{\chi})\|_{L^2_{x'}L^1_t} + 2^{-j}\varepsilon. \tag{9.23}$$

We define a scalar h and a vectorfield F by

$$h = |\widehat{\chi}| \text{ and } F = n\frac{\widehat{\chi}}{|\widehat{\chi}|}\cdot\nabla\!\!\!/\chi, \tag{9.24}$$

and we decompose

$$n\widehat{\chi}\cdot\nabla\!\!\!/\widehat{\chi} = hF = P_{\leq j}(h)F + P_{>j}(h)F. \tag{9.25}$$

Note in particular in view of the estimate (2.71) for $\widehat{\chi}$ and the estimate (2.67) for n that we have the following estimate for h and F

$$\mathcal{N}_1(h) + \|h\|_{L^\infty_{x'}L^2_t} \lesssim \varepsilon \text{ and } \|F\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{9.26}$$

We have

$$\begin{aligned}\|\mathrm{div}\!\!\!/ P_j(P_{>j}(h)F)\|_{L^2_{x'}L^1_t} &\lesssim \|\mathrm{div}\!\!\!/ P_j(P_{>j}(h)F)\|_{L^1_t L^2_{x'}}\\ &\lesssim 2^j\|P_j(P_{>j}(h)F)\|_{L^1_t L^2_{x'}},\end{aligned}$$

where we used in the last inequality the finite band property for P_j. Together with the dual of the sharp Bernstein inequality for tensors (4.37), we obtain

$$\begin{aligned}\|\mathrm{div}\!\!\!/ P_j(P_{>j}(h)F)\|_{L^2_{x'}L^1_t} &\lesssim 2^j\left\|(2^j + \|K\|_{L^2(P_{t,u})})\|P_{>j}(h)F\|_{L^1(P_{t,u})}\right\|_{L^1_t}\\ &\lesssim 2^{2j}\|P_{>j}(h)F\|_{L^1(\mathcal{H}_u)} + 2^j\|K\|_{L^2(\mathcal{H}_u)}\|P_{>j}(h)F\|_{L^2_t L^1_{x'}}\\ &\lesssim \left(2^{2j}\|P_{>j}(h)\|_{L^2(\mathcal{H}_u)} + 2^j\|K\|_{L^2(\mathcal{H}_u)}\|P_{>j}(h)\|_{L^\infty_t L^2_{x'}}\right)\|F\|_{L^2(\mathcal{H}_u)}.\end{aligned}$$

Together with the finite band property and the boundedness on $L^2(P_{t,u})$ of P_j, we obtain

$$\begin{aligned}\|\not{\mathrm{div}} P_j(P_{>j}(h)F)\|_{L^2_{x'}L^1_t} &\lesssim 2^j\Big(\|\not{\nabla} h\|_{L^2(\mathcal{H}_u)} + \|K\|_{L^2(\mathcal{H}_u)}\|h\|_{L^\infty_t L^2_{x'}}\Big)\|F\|_{L^2(\mathcal{H}_u)}\\ &\lesssim 2^j\varepsilon, \end{aligned} \tag{9.27}$$

where we used in the last inequality the estimate (4.29) for K and the estimate (9.26) for h and F.

Next, we evaluate the first term in the decomposition (9.25). We have

$$\|\not{\mathrm{div}} P_j(P_{\leq j}(h)F)\|_{L^2_{x'}L^1_t} \lesssim \|P_{\leq j}(h)\not{\mathrm{div}} P_j(F)\|_{L^2_{x'}L^1_t} + \|\not{\mathrm{div}}[P_j, P_{\leq j}(h)]F\|_{L^1_t L^2_{x'}}. \tag{9.28}$$

In view of (9.12), we have

$$\begin{aligned}\|\not{\mathrm{div}}[P_j, P_{\leq j}(h)]F\|_{L^1_t L^2_{x'}} &\lesssim 2^j\Big\|(\|\not{\nabla} h\|_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})}\|h\|_{L^2(P_{t,u})})\|F\|_{L^2(P_{t,u})}\Big\|_{L^1_t}\\ &\lesssim 2^j(\|\not{\nabla} h\|_{L^2(\mathcal{H}_u)} + \|K\|_{L^2(\mathcal{H}_u)}\|h\|_{L^\infty_t L^2_{x'}})\|F\|_{L^2(\mathcal{H}_u)}\\ &\lesssim 2^j\varepsilon, \end{aligned} \tag{9.29}$$

where we used in the last inequality the estimate (4.29) for K and the estimate (9.26) for h and F. Next, we consider the first term in the right-hand side of (9.28). We have

$$\|P_{\leq j}(h)\not{\mathrm{div}} P_j(F)\|_{L^2_{x'}L^1_t} \lesssim \|h\not{\mathrm{div}} P_j(F)\|_{L^2_{x'}L^1_t} + \|P_{>j}(h)\not{\mathrm{div}} P_j(F)\|_{L^2_{x'}L^1_t}. \tag{9.30}$$

The first term in the right-hand side of (9.30) is estimated as follows

$$\begin{aligned}\|h\not{\mathrm{div}} P_j(F)\|_{L^2_{x'}L^1_t} &\lesssim \|h\|_{L^\infty_{x'}L^2_t}\|\not{\mathrm{div}} P_j(F)\|_{L^2(\mathcal{H}_u)}\\ &\lesssim 2^j\|h\|_{L^\infty_{x'}L^2_t}\|F\|_{L^2(\mathcal{H}_u)}\\ &\lesssim 2^j\varepsilon, \end{aligned} \tag{9.31}$$

where we used the finite band property for P_j and the estimate (9.26) for h and F. Next, we estimate the second term in (9.30). We have

$$\begin{aligned}\|P_{>j}(h)\not{\mathrm{div}} P_j(F)\|_{L^2(P_{t,u})} &\lesssim \|P_{>j}(h)\|_{L^4(P_{t,u})}\|\not{\mathrm{div}} P_j(F)\|_{L^4(P_{t,u})}\\ &\lesssim \|P_{>j}(h)\|_{L^4(P_{t,u})}\|\not{\nabla}^2 P_j(F)\|^{\frac{1}{2}}_{L^2(P_{t,u})}\|\not{\nabla} P_j(F)\|^{\frac{1}{2}}_{L^2(P_{t,u})}\\ &\lesssim \|P_{>j}(h)\|_{L^4(P_{t,u})}\|\not{\nabla}^2 P_j(F)\|^{\frac{1}{2}}_{L^2(P_{t,u})}2^{\frac{j}{2}}\|P_j(F)\|^{\frac{1}{2}}_{L^2(P_{t,u})}, \end{aligned} \tag{9.32}$$

where we used the Gagliardo-Nirenberg inequality (3.3) and the finite band property for P_j. Using the Bochner identity for tensors (3.7), we have

$$\begin{aligned}&\|\not{\nabla}^2 P_j(F)\|_{L^2(P_{t,u})}\\ \lesssim\ &\|\not{\Delta} P_j(F)\|_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})}\|\not{\nabla} P_j(F)\|_{L^2(P_{t,u})} + \|K\|^2_{L^2(P_{t,u})}\|P_j(F)\|_{L^2(P_{t,u})}\\ \lesssim\ &(2^{2j} + \|K\|^2_{L^2(P_{t,u})})\|F\|_{L^2(P_{t,u})}, \end{aligned}$$

where we used in the last inequality the finite band property for P_j. Together with (9.32), we obtain

$$\begin{aligned}
&\|P_{>j}(h)\mathrm{div}\!\!\!/ P_j(F)\|_{L^2(P_{t,u})}\\
\lesssim\ & \|P_{>j}(h)\|_{L^4(P_{t,u})}2^{\frac{3j}{2}}\|F\|_{L^2(P_{t,u})} + \|P_{>j}(h)\|_{L^4(P_{t,u})}2^{\frac{j}{2}}\|K\|_{L^2(P_{t,u})}\|F\|_{L^2(P_{t,u})}\\
\lesssim\ & \left(\sum_{l>j}2^{\frac{l}{2}}\|P_l h\|_{L^2(P_{t,u})}\right)2^{\frac{3j}{2}}\|F\|_{L^2(P_{t,u})} + \|h\|_{L^4(P_{t,u})}2^{\frac{j}{2}}\|K\|_{L^2(P_{t,u})}\|F\|_{L^2(P_{t,u})}\\
\lesssim\ & \left(\left(\sum_{l>j}2^{-\frac{l}{2}}\right)\|\nabla\!\!\!/ h\|_{L^2(P_{t,u})}2^{\frac{3j}{2}} + 2^{\frac{j}{2}}\|K\|_{L^2(P_{t,u})}\|h\|_{L^4(P_{t,u})}\right)\|F\|_{L^2(P_{t,u})}\\
\lesssim\ & 2^j\left(\|\nabla\!\!\!/ h\|_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})}\|h\|_{L^4(P_{t,u})}\right)\|F\|_{L^2(P_{t,u})},
\end{aligned}$$

where we used Bernstein, the boundedness on $L^4(P_{t,u})$ and the finite band property for P_l. Taking the L^1_t norm, we obtain

$$\begin{aligned}
\|P_{>j}(h)\mathrm{div}\!\!\!/ P_j(F)\|_{L^1_t L^2_{x'}} &\lesssim 2^j\left(\|\nabla\!\!\!/ h\|_{L^2(\mathcal{H}_u)} + \|K\|_{L^2(\mathcal{H}_u)}\|h\|_{L^\infty_t L^4_{x'}}\right)\|F\|_{L^2(\mathcal{H}_u)}\\
&\lesssim 2^j\varepsilon, \qquad (9.33)
\end{aligned}$$

where we used in the last inequality the estimate (4.29) for K and the estimate (9.26) for h and F. Now, (9.30), (9.31) and (9.33) imply

$$\|P_{\leq j}(h)\mathrm{div}\!\!\!/ P_j(F)\|_{L^2_{x'}L^1_t} \lesssim 2^j\varepsilon.$$

Together with (9.28) and (9.29), this yields

$$\|\mathrm{div}\!\!\!/ P_j(P_{\leq j}(h)F)\|_{L^2_{x'}L^1_t} \lesssim 2^j\varepsilon.$$

Together with (9.25) and (9.27), we obtain

$$\|\mathrm{div}\!\!\!/ P_j(n\widehat{\chi}\cdot\nabla\!\!\!/\widehat{\chi})\|_{L^2_{x'}L^1_t} \lesssim 2^j\varepsilon.$$

Together with (9.23), this yields the desired estimate (9.17). This concludes the proof of the proposition.

9.3.2 Proof of Proposition 2.42

Using the estimate (3.66) for transport equations, we have

$$\begin{aligned}
\|\nabla\!\!\!/ P_{\leq j}\mathrm{tr}\chi\|_{L^2_{x'}L^\infty_t} &\lesssim \|\nabla\!\!\!/ P_{\leq j}\mathrm{tr}\chi\|_{L^2(P_{0,u})} + \|\nabla\!\!\!/_{nL}\nabla\!\!\!/ P_{\leq j}\mathrm{tr}\chi\|_{L^2_{x'}L^1_t}\\
&\lesssim \|\nabla\!\!\!/ P_{\leq j}(nL\mathrm{tr}\chi)\|_{L^2_{x'}L^1_t} + \|[\nabla\!\!\!/_{nL},\nabla\!\!\!/]P_{\leq j}\mathrm{tr}\chi\|_{L^2_{x'}L^1_t} + \|\nabla\!\!\!/[nL,P_{\leq j}]\mathrm{tr}\chi\|_{L^2_{x'}L^1_t} + \varepsilon\\
&\lesssim \|\nabla\!\!\!/ P_{\leq j}(nL\mathrm{tr}\chi)\|_{L^2_{x'}L^1_t} + \|n\chi\nabla\!\!\!/ P_{\leq j}\mathrm{tr}\chi\|_{L^2_{x'}L^1_t} + \varepsilon\\
&\lesssim \|\nabla\!\!\!/ P_{\leq j}(nL\mathrm{tr}\chi)\|_{L^2_{x'}L^1_t} + \|n\|_{L^\infty}\|\chi\|_{L^\infty_{x'}L^2_t}\|\nabla\!\!\!/ P_{\leq j}\mathrm{tr}\chi\|_{L^2_{x'}L^1_t} + \varepsilon\\
&\lesssim \|\nabla\!\!\!/ P_{\leq j}(nL\mathrm{tr}\chi)\|_{L^2_{x'}L^1_t} + \varepsilon\|\nabla\!\!\!/ P_{\leq j}\mathrm{tr}\chi\|_{L^2_{x'}L^\infty_t} + \varepsilon,
\end{aligned}$$

where we used (2.62), the commutator formula (2.48), the commutator estimate (9.5), and the estimates (2.67) for n and (2.70) (2.71) for χ. Since $\varepsilon > 0$ is small, we obtain

$$\|\nabla\!\!\!/ P_{\leq j}\mathrm{tr}\chi\|_{L^2_{x'}L^\infty_t} \lesssim \|\nabla\!\!\!/ P_{\leq j}(nL\mathrm{tr}\chi)\|_{L^2_{x'}L^1_t} + 2^{-j}\varepsilon. \tag{9.34}$$

Now, (2.93) follows from (2.94) and (9.34). Thus, it remains to prove (2.94).

Next, using the Raychaudhuri equation (2.28), we have

$$\begin{aligned}\|\nabla\!\!\!/ P_{\leq j}(nL\mathrm{tr}\chi)\|_{L^2_{x'}L^1_t} &\lesssim \|\nabla\!\!\!/ P_{\leq j}(n|\widehat{\chi}|^2)\|_{L^2_{x'}L^1_t} + \left\|\nabla\!\!\!/ P_{\leq j}\left(n\left(\frac{1}{2}(\mathrm{tr}\chi)^2 + \overline{\delta}\mathrm{tr}\chi\right)\right)\right\|_{L^1_{x'}L^2_t} \\ &\lesssim \|\nabla\!\!\!/ P_{\leq j}(n|\widehat{\chi}|^2)\|_{L^2_{x'}L^1_t} + \left\|\nabla\!\!\!/\left(n\left(\frac{1}{2}(\mathrm{tr}\chi)^2 + \overline{\delta}\mathrm{tr}\chi\right)\right)\right\|_{L^2(\mathcal{H}_u)},\end{aligned}$$

where we used the finite band property for P_j in the last inequality. Together with the estimate (2.70) for $\mathrm{tr}\chi$, the estimate (2.67) for n, and the estimates (2.67) (2.68) for $\overline{\delta}$, we obtain

$$\|\nabla\!\!\!/ P_{\leq j}(nL\mathrm{tr}\chi)\|_{L^2_{x'}L^1_t} \lesssim \|\nabla\!\!\!/ P_{\leq j}(n|\widehat{\chi}|^2)\|_{L^2_{x'}L^1_t} + \varepsilon.$$

Thus, it remains to prove

$$\|\nabla\!\!\!/ P_{\leq j}(n|\widehat{\chi}|^2)\|_{L^2_{x'}L^1_t} \lesssim \varepsilon. \tag{9.35}$$

In view of (9.13), we have for any $a > 0$

$$\|[\nabla\!\!\!/, P_{\leq j}](n|\widehat{\chi}|^2)\|_{L^2(P_{t,u})} \lesssim \|K\|_{L^2(P_{t,u})}(\|K\|_{L^2(P_{t,u})}\|n|\widehat{\chi}|^2\|_{L^2(P_{t,u})} + \|\Lambda^a(n|\widehat{\chi}|^2)\|_{L^2(P_{t,u})}).$$

Taking the L^1_t norm, we obtain

$$\begin{aligned}\|[\nabla\!\!\!/, P_{\leq j}](n|\widehat{\chi}|^2)\|_{L^1_tL^2_{x'}} &\lesssim \|K\|_{L^2(\mathcal{H}_u)}(\|K\|_{L^2(\mathcal{H}_u)}\|n\|_{L^\infty}\|\widehat{\chi}\|^2_{L^\infty_tL^4_{x'}} + \|\Lambda^a(n|\widehat{\chi}|^2)\|_{L^2(\mathcal{H}_u)}) \\ &\lesssim \varepsilon,\end{aligned}$$

where we used in the last inequality the estimate (4.29) for K, the estimate (2.67) for n, the estimate (2.71) for $\widehat{\chi}$ and the estimate (9.20) with the choice $0 < a < \frac{1}{2}$. Thus, we obtain

$$\|\nabla\!\!\!/ P_{\leq j}(n|\widehat{\chi}|^2)\|_{L^2_{x'}L^1_t} \lesssim \|P_{\leq j}\nabla\!\!\!/(n|\widehat{\chi}|^2)\|_{L^2_{x'}L^1_t} + \varepsilon. \tag{9.36}$$

Next, we estimate the right-hand side of (9.36). In view of (9.22) and the boundedness on $L^2(P_{t,u})$ of $P_{\leq j}$, we have

$$\|\nabla\!\!\!/ P_{\leq j}(n|\widehat{\chi}|^2)\|_{L^2_{x'}L^1_t} \lesssim \|P_j(n\widehat{\chi}\cdot\nabla\!\!\!/\widehat{\chi})\|_{L^2_{x'}L^1_t} + \varepsilon. \tag{9.37}$$

Now, recall the definition (9.24) of the scalar h and the vectorfield F, the decomposition (9.25) of $n\widehat{\chi}\cdot\nabla\!\!\!/\widehat{\chi}$ and the estimate (9.26) for h and F. Using Bernstein for $P_{\leq j}$, we have

$$\begin{aligned}\|P_{\leq j}(P_{>j}(h)F)\|_{L^1_{x'}L^2_t} &\lesssim 2^{\frac{j}{2}}\|P_{>j}(h)F\|_{L^1_tL^{\frac{4}{3}}_{x'}} \\ &\lesssim 2^{\frac{j}{2}}\|P_{>j}h\|_{L^2_tL^4_{x'}}\|F\|_{L^2(\mathcal{H}_u)} \\ &\lesssim 2^{\frac{j}{2}}\left(\sum_{l>j}2^{\frac{l}{2}}\|P_lh\|_{L^2(\mathcal{H}_u)}\right)\|F\|_{L^2(\mathcal{H}_u)} \\ &\lesssim 2^{\frac{j}{2}}\left(\sum_{l>j}2^{-\frac{l}{2}}\right)\|\nabla\!\!\!/ h\|_{L^2(\mathcal{H}_u)}\|F\|_{L^2(\mathcal{H}_u)} \\ &\lesssim \varepsilon,\end{aligned} \tag{9.38}$$

where we used Bernstein and the finite band property for P_l and the estimate (9.26) for h and F.

Next, we evaluate the first term in the decomposition (9.25) of $n\widehat{\chi}\cdot\nabla\!\!\!/\widehat{\chi}$. We have

$$\|P_{\leq j}(P_{\leq j}(h)F)\|_{L^2_{x'}L^1_t} \lesssim \|P_{\leq j}(h)P_{\leq j}F\|_{L^2_{x'}L^1_t} + \|[P_{\leq j}, P_{\leq j}(h)]F)\|_{L^1_{x'}L^2_t}. \tag{9.39}$$

Since $[P_{\leq j}, P_{\leq j}(h)] = [P_{>j}, P_{\leq j}(h)]$, we have in view of the commutator estimate (9.11)

$$\begin{aligned} \|[P_{\leq j}, P_{\leq j}(h)]F)\|_{L^1_tL^2_{x'}} &= \|[P_{>j}, P_{\leq j}(h)]F)\|_{L^1_tL^2_{x'}} \\ &\lesssim \big\| \|\nabla\!\!\!/ h\|_{L^2(P_{t,u})}\|F\|_{L^2(P_{t,u})}\big\|_{L^1_t} \\ &\lesssim \|\nabla\!\!\!/ h\|_{L^2(\mathcal{H}_u)}\|F\|_{L^2(\mathcal{H}_u)}, \end{aligned} \tag{9.40}$$

where we used in the last inequality the estimate (9.26) for h and F. Next, we consider the first term in the right-hand side of (9.39). We have

$$\|P_{\leq j}(h)P_{\leq j}(F)\|_{L^2_{x'}L^1_t} \lesssim \|hP_{\leq j}(F)\|_{L^2_{x'}L^1_t} + \|P_{>j}(h)P_{\leq j}(F)\|_{L^2_{x'}L^1_t}. \tag{9.41}$$

The first term in the right-hand side of (9.41) is estimated as follows

$$\begin{aligned} \|hP_{\leq j}(F)\|_{L^2_{x'}L^1_t} &\lesssim \|h\|_{L^\infty_{x'}L^2_t}\|P_{\leq j}(F)\|_{L^2(\mathcal{H}_u)} \\ &\lesssim \|h\|_{L^\infty_{x'}L^2_t}\|F\|_{L^2(\mathcal{H}_u)} \\ &\lesssim \varepsilon, \end{aligned} \tag{9.42}$$

where we used the boundedness on $L^2(P_{t,u})$ of $P_{\leq j}$ and the estimate (9.26) for h and F. Next, we estimate the second term in (9.41). We have

$$\begin{aligned} \|P_{>j}(h)P_{\leq j}(F)\|_{L^2(P_{t,u})} &\lesssim \|P_{>j}(h)\|_{L^4(P_{t,u})}\|P_{\leq j}(F)\|_{L^4(P_{t,u})} \\ &\lesssim \left(\sum_{l>j} 2^{\frac{l}{2}}\|P_l h\|_{L^2(P_{t,u})}\right) 2^{\frac{j}{2}}\|F\|_{L^2(P_{t,u})} \\ &\lesssim \left(\sum_{l>j} 2^{-\frac{l}{2}}\right)\|\nabla\!\!\!/ h\|_{L^2(P_{t,u})}2^{\frac{j}{2}}\|F\|_{L^2(P_{t,u})} \\ &\lesssim \|\nabla\!\!\!/ h\|_{L^2(P_{t,u})}\|F\|_{L^2(P_{t,u})}, \end{aligned}$$

where we used Bernstein for P_l and P_j, and the finite band property for P_l. Taking the L^1_t norm, we obtain

$$\begin{aligned} \|P_{>j}(h)P_{\leq j}(F)\|_{L^1_tL^2_{x'}} &\lesssim \|\nabla\!\!\!/ h\|_{L^2(\mathcal{H}_u)}\|F\|_{L^2(\mathcal{H}_u)} \\ &\lesssim \varepsilon, \end{aligned} \tag{9.43}$$

where we used in the last inequality the estimate (9.26) for h and F. Now, (9.41), (9.42) and (9.43) imply

$$\|P_{\leq j}(h)P_{\leq j}(F)\|_{L^2_{x'}L^1_t} \lesssim \varepsilon.$$

Together with (9.39) and (9.40), this yields

$$\|P_{\leq j}(P_{\leq j}(h)F)\|_{L^2_{x'}L^1_t} \lesssim \varepsilon.$$

Together with (9.38) and the decomposition of $n\widehat{\chi}\cdot\nabla\!\!\!/\widehat{\chi}$ given by (9.25), we obtain

$$\|P_{\leq j}(n\widehat{\chi}\cdot\nabla\!\!\!/\widehat{\chi})\|_{L^2_{x'}L^1_t}\lesssim\varepsilon.$$

Together with (9.37), this yields the desired estimate (9.35). This concludes the proof of the proposition.

A Appendix to section 4

A.1 Proof of Proposition 4.11

Recall from the Gauss equation (2.38) that:

$$K=\frac{1}{2}\widehat{\chi}_{AB}\widehat{\underline{\chi}}_{AB}-\frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}-\rho.$$

First, remark from (3.58) that:

$$\left\|\frac{1}{2}\widehat{\chi}_{AB}\widehat{\underline{\chi}}_{AB}-\frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}\right\|_{L^\infty_tL^2_{x'}}\lesssim\|\chi\|^2_{L^\infty_tL^4_{x'}}\lesssim\mathcal{N}_1(\chi)^2\lesssim\varepsilon. \tag{A.1}$$

Furthermore, from the assumptions on the curvature flux (2.58) (2.59), we have:

$$\|\rho\|_{L^2(\mathcal{H}_u)}\leq\varepsilon. \tag{A.2}$$

(A.1) and (A.2) imply (4.29).

We now concentrate on (4.30). We assume:

$$\sum_{j\geq0}2^{-j}\|P_jK\|^2_{L^\infty_tL^2_{x'}}+\|P_{<0}K\|^2_{L^\infty_tL^2_{x'}}\lesssim E^2\varepsilon^2, \tag{A.3}$$

where E is a large enough constant. We will then try to improve (A.3). Note that (3.36), (3.37) and (A.3) yield for any scalar function f on $P_{t,u}$:

$$\|\nabla\!\!\!/^2f\|^2_{L^2(P_{t,u})}\lesssim\|\Delta\!\!\!/f\|^2_{L^2(P_{t,u})}+(E\varepsilon+E^4\varepsilon^4)\|\nabla\!\!\!/f\|^2_{L^2(P_{t,u})}. \tag{A.4}$$

In view of (A.1), we just need to bound $\|\Lambda^{-\frac{1}{2}}\rho\|_{L^\infty_tL^2_{x'}}$. Note from (3.37) that it suffices to bound:

$$\|P_{<0}\rho\|^2_{L^\infty_tL^2_{x'}}+\sum_{j\geq0}2^{-j}\|P_j\rho\|^2_{L^\infty_tL^2_{x'}}.$$

The term $\|P_{<0}\rho\|_{L^\infty_tL^2_{x'}}$ is easier to bound, so we concentrate on estimating the sum $\sum_{j\geq0}2^{-j}\|P_j\rho\|_{L^\infty_tL^2_{x'}}$. We will use the following variant of (3.62) where we do not yet use Cauchy-Schwarz in t for the integral containing $\mathbf{D}_LF$:

$$\|F\|^2_{L^\infty_tL^2_{x'}}\lesssim\int_0^1\|\mathbf{D}_LF\|_{L^2(P_{t,u})}\|F\|_{L^2(P_{t,u})}dt+\|F\|^2_{L^2(\mathcal{H}_u)}. \tag{A.5}$$

Using (A.5), properties (ii) and (iii) of Theorem 3.10 for P_j, the bound on ρ given by (2.59) and the bound on n given by the bootstrap assumption (4.1), we have:

$$\begin{aligned}
&\sum_{j\geq 0} 2^{-j}\|P_j\rho\|^2_{L^\infty_t L^2_{x'}}\\
\lesssim\ &\sum_{j\geq 0} 2^{-j}\left(\int_0^1 \|P_j\rho\|_{L^2(P_{t,u})}\|\mathbf{D}_L P_j\rho\|_{L^2(P_{t,u})}dt + \|P_j\rho\|^2_{L^2(\mathcal{H}_u)}\right)\\
\lesssim\ &\sum_{j\geq 0} 2^{-j}\left(\int_0^1 \|P_j\rho\|_{L^2(P_{t,u})}\|\mathbf{D}_L P_j\rho\|_{L^2(P_{t,u})}dt\right) + \sum_{j\geq 0}2^{-j}\|\rho\|^2_{L^2(\mathcal{H}_u)}\\
\lesssim\ &\sum_{j\geq 0} 2^{-j}\left(\int_0^1 \|P_j\rho\|_{L^2(P_{t,u})}\|nLP_j\rho\|_{L^2(P_{t,u})}dt\right) + \varepsilon^2.
\end{aligned} \tag{A.6}$$

We have:

$$nLU(\tau)\rho = U(\tau)nL\rho + V(\tau) \tag{A.7}$$

where $V(\tau)$ is satisfies:

$$(\partial_\tau - \not{\Delta})V(\tau) = [nL, \not{\Delta}]U(\tau)\rho,\ V(0) = 0. \tag{A.8}$$

Using (3.15) and (A.7), we obtain:

$$nLP_j\rho = P_j nL\rho + \int_0^\infty m_j(\tau)V(\tau)d\tau. \tag{A.9}$$

We now estimate $\|P_j nL\rho\|_{L^2(\mathcal{H}_u)}$. We may assume the existence of $\widetilde{P}_j$ with the same properties than P_j such that $P_j = \widetilde{P}_j{}^2$ (see [10]), and for simplicity we write $P_j = P_j^2$. Also, using the fact that $\Lambda\Lambda^{-1} = I$ and that Λ commutes with P_j, we obtain:

$$P_j = \Lambda P_j P_j \Lambda^{-1}, \tag{A.10}$$

which together with property (iii) of Theorem 3.10 for P_j yields:

$$\|P_j nL\rho\|_{L^2(\mathcal{H}_u)} \lesssim \|\Lambda P_j(P_j\Lambda^{-1}nL\rho)\|_{L^2(\mathcal{H}_u)} \lesssim 2^j\|P_j\Lambda^{-1}nL\rho\|_{L^2(\mathcal{H}_u)}. \tag{A.11}$$

Using the Bianchi identity (2.53), we have:

$$nL\rho = \not{\mathrm{div}}(n\beta) - \not{\nabla}(n)\beta - \frac{3}{2}n\mathrm{tr}\chi\rho - \frac{n}{2}\underline{\widehat{\chi}}\cdot\alpha + n(\epsilon - 2\overline{\epsilon})\cdot\beta. \tag{A.12}$$

Together with properties (3.25) and (3.27) of Λ, this yields:

$$\begin{aligned}
\|\Lambda^{-1}nL\rho\|_{L^2(\mathcal{H}_u)} &\lesssim \|n\beta\|_{L^2(\mathcal{H}_u)} + \left\|-\not{\nabla}(n)\beta - \frac{3}{2}n\mathrm{tr}\chi\rho - \frac{n}{2}\underline{\widehat{\chi}}\cdot\alpha + n(\epsilon-2\overline{\epsilon})\cdot\beta\right\|_{L^2_t L^{\frac{4}{3}}_{x'}}\\
&\lesssim \varepsilon + \|\not{\nabla}(n)\|_{L^\infty_t L^4_{x'}}\|\beta\|_{L^2(\mathcal{H}_u)} + \|\mathrm{tr}\chi\|_{L^\infty_t L^4_{x'}}\|\rho\|_{L^2(\mathcal{H}_u)}\\
&\quad + \|\underline{\widehat{\chi}}\|_{L^\infty_t L^4_{x'}}\|\alpha\|_{L^2(\mathcal{H}_u)} + \|\epsilon - 2\overline{\epsilon}\|_{L^\infty_t L^4_{x'}}\|\beta\|_{L^2(\mathcal{H}_u)}\\
&\lesssim \varepsilon(1 + \mathcal{N}_1(\not{\nabla}(n)) + \mathcal{N}_1(\underline{\widehat{\chi}}) + \mathcal{N}_1(\epsilon) + \mathcal{N}_1(\overline{\epsilon}))\\
&\lesssim \varepsilon,
\end{aligned} \tag{A.13}$$

where we have also used (3.58) to bound the $L^\infty_t L^4_{x'}$ norms, (2.59) to estimate α, β, ρ, and the bootstrap assumptions (4.1)-(4.5). Now, (A.11) and (A.13) yield:

$$\sum_{j\geq 0} 2^{-2j}\|P_j nL\rho\|^2_{L^2(\mathcal{H}_u)} \lesssim \sum_{j\geq 0}\|P_j\Lambda^{-1}nL\rho\|^2_{L^2(\mathcal{H}_u)} \lesssim \|\Lambda^{-1}nL\rho\|^2_{L^2(\mathcal{H}_u)} \lesssim \varepsilon^2. \tag{A.14}$$

Property (ii) of Theorem 3.10, (A.6), (A.9), (A.2), (A.11) and (A.13) imply:

$$\begin{aligned}
&\sum_{j\geq 0} 2^{-j}\|P_j\rho\|^2_{L^\infty_t L^2_{x'}} \lesssim \sum_{j\geq 0} 2^{-j}\left(\int_0^1 \|P_j\rho\|_{L^2(P_{t,u})}\|nLP_j\rho\|_{L^2(P_{t,u})}dt\right) + \varepsilon^2\\
&\lesssim \sum_{j\geq 0} 2^{-j}\|P_j\rho\|_{L^2(\mathcal{H}_u)}\|P_j nL\rho\|_{L^2(\mathcal{H}_u)}\\
&+ \sum_{j\geq 0} 2^{-j}\left(\int_0^1 \|P_j\rho\|_{L^2(P_{t,u})}\left\|\int_0^\infty m_j(\tau)V(\tau)d\tau\right\|_{L^2(P_{t,u})}dt\right) + \varepsilon^2\\
&\lesssim \sum_{j\geq 0}\|P_j\rho\|^2_{L^2(\mathcal{H}_u)} + \sum_{j\geq 0} 2^{-2j}\|P_j nL\rho\|^2_{L^2(\mathcal{H}_u)}\\
&+ \sum_{j\geq 0} 2^{-j}\|P_j\rho\|_{L^\infty_t L^2_{x'}}\left\|\int_0^\infty m_j(\tau)V(\tau)d\tau\right\|_{L^1_t L^2_{x'}} + \varepsilon^2\\
&\lesssim \left(\sum_{j\geq 0} 2^{-j}\|P_j\rho\|^2_{L^\infty_t L^2_{x'}}\right)^{\frac{1}{2}}\\
&\times \left(\sum_{j\geq 0} 2^{-j}\left\|\int_0^\infty m_j(\tau)\|V(\tau)\|_{L^2(P_{t,u})}d\tau\right\|^2_{L^1(0,1)}\right)^{\frac{1}{2}} + \varepsilon^2,
\end{aligned} \tag{A.15}$$

which yields:

$$\sum_{j\geq 0} 2^{-j}\|P_j\rho\|^2_{L^\infty_t L^2_{x'}} \lesssim \sum_{j\geq 0} 2^{-j}\left\|\int_0^\infty m_j(\tau)\|V(\tau)\|_{L^2(P_{t,u})}d\tau\right\|^2_{L^1(0,1)} + \varepsilon^2. \tag{A.16}$$

In view of (A.16), we have to estimate $\|V(\tau)\|_{L^2(P_{t,u})}$. Let a, p real numbers satisfying:

$$0 < a < \frac{1}{2},\ 2 < p < +\infty,\ \text{such that } p < \min\left(\frac{2}{1-a}, \frac{8}{3}\right). \tag{A.17}$$

(3.30) implies:

$$\begin{aligned}
&\|\Lambda^{-a}V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^2_{L^2(P_{t,u})}d\tau'\\
\lesssim\ & \int_0^\tau\int_{P_{t,u}} \Lambda^{-2a}V(\tau')[nL, \Delta\!\!\!/]U(\tau')\rho d\mu_{t,u}d\tau'.
\end{aligned} \tag{A.18}$$

Let p be defined in (A.17), and let p' such that $\frac{1}{p} + \frac{1}{p'} = \frac{1}{2}$. Using the commutator formula

(2.49), (A.7), and integrating by parts the term $\nabla\!\!\!/^2 U(\tau)\rho$ yields:

$$\begin{aligned}
&\int_0^\tau \int_{P_{t,u}} \Lambda^{-2a} V(\tau')[nL, \Delta\!\!\!/] U(\tau')\rho d\mu_{t,u} d\tau' \qquad (A.19)\\
\lesssim\ & (\|\nabla\!\!\!/(n\chi)\|_{L^2(P_{t,u})} + \|n(2\widehat{\chi}\epsilon - \epsilon \mathrm{tr}\chi - \nabla\!\!\!/ \mathrm{tr}\chi)\|_{L^2(P_{t,u})}) \int_0^\tau \|\nabla\!\!\!/ U(\tau')\rho\|_{L^p(P_{t,u})}\\
&\times \|\Lambda^{-2a} V(\tau')\|_{L^{p'}(P_{t,u})} d\tau' + \|n\chi\|_{L^{p'}(P_{t,u})} \int_0^\tau \|\nabla\!\!\!/ U(\tau')\rho\|_{L^p(P_{t,u})} \|\nabla\!\!\!/ \Lambda^{-2a} V(\tau')\|_{L^2(P_{t,u})} d\tau'.
\end{aligned}$$

In view of (3.3) and the fact that $2 < p' < +\infty$, and (3.23) and (3.22), we have

$$\begin{aligned}
&\|\Lambda^{-2a} V(\tau')\|_{L^{p'}(P_{t,u})} + \|\nabla\!\!\!/ \Lambda^{-2a} V(\tau')\|_{L^2(P_{t,u})}\\
\lesssim\ & \|\Lambda^{-2a} V(\tau')\|_{L^2(P_{t,u})}^{\frac{2}{p'}} \|\nabla\!\!\!/ \Lambda^{-2a} V(\tau')\|_{L^2(P_{t,u})}^{1-\frac{2}{p'}} + \|\nabla\!\!\!/ \Lambda^{-2a} V(\tau')\|_{L^2(P_{t,u})}\\
\lesssim\ & \|\Lambda^{-2a} V(\tau')\|_{L^2(P_{t,u})} + \|\nabla\!\!\!/ \Lambda^{-2a} V(\tau')\|_{L^2(P_{t,u})}\\
\lesssim\ & \|\Lambda^{1-2a} V(\tau')\|_{L^2(P_{t,u})}\\
\lesssim\ & \|\Lambda^{-a} V(\tau')\|_{L^{p'}(P_{t,u})}^{a} \|\Lambda^{1-a} V(\tau')\|_{L^{p'}(P_{t,u})}^{1-a}.
\end{aligned}$$

Together with (3.3) and (A.4) we infer:

$$\begin{aligned}
&\int_0^\tau \|\nabla\!\!\!/ U(\tau')\rho\|_{L^p(P_{t,u})} \|\Lambda^{-2a} V(\tau')\|_{L^{p'}(P_{t,u})} d\tau' + \int_0^\tau \|\nabla\!\!\!/ U(\tau')\rho\|_{L^p(P_{t,u})} \|\nabla\!\!\!/ \Lambda^{-2a} V(\tau')\|_{L^2(P_{t,u})} d\tau'\\
\lesssim\ & \int_0^\tau \|\nabla\!\!\!/ U(\tau')\rho\|_{L^2(P_{t,u})}^{\frac{2}{p}} \|\nabla\!\!\!/^2 U(\tau')\rho\|_{L^2(P_{t,u})}^{1-\frac{2}{p}} \|\Lambda^{-a} V(\tau')\|_{L^2(P_{t,u})}^{a} \|\Lambda^{1-a} V(\tau')\|_{L^2(P_{t,u})}^{1-a} d\tau'\\
\lesssim\ & E^{2(1-\frac{2}{p})} \int_0^\tau \|\nabla\!\!\!/ U(\tau')\rho\|_{L^2(P_{t,u})}^{\frac{2}{p}} \|\Delta\!\!\!/ U(\tau')\rho\|_{L^2(P_{t,u})}^{1-\frac{2}{p}} \|\Lambda^{-a} V(\tau')\|_{L^2(P_{t,u})}^{a} \|\Lambda^{1-a} V(\tau')\|_{L^2(P_{t,u})}^{1-a} d\tau'\\
\lesssim\ & E^{2(1-\frac{2}{p})} \int_0^\tau \|\nabla\!\!\!/ U(\tau')\rho\|_{L^2(P_{t,u})}^{\frac{2}{p}} \left(\sqrt{\tau'} \|\Delta\!\!\!/ U(\tau')\rho\|_{L^2(P_{t,u})}\right)^{1-\frac{2}{p}}\\
&\times \left(\tau'^{-\frac{p-2}{2ap}} \|\Lambda^{-a} V(\tau')\|_{L^2(P_{t,u})}\right)^{a} \|\Lambda^{1-a} V(\tau')\|_{L^2(P_{t,u})}^{1-a} d\tau'\\
\lesssim\ & \left(E^{4(1-\frac{2}{p})} \int_0^\tau \|\nabla\!\!\!/ U(\tau')\rho\|_{L^2(P_{t,u})}^2 d\tau' + E^{4(1-\frac{2}{p})} \int_0^\tau \tau' \|\Delta\!\!\!/ U(\tau')\rho\|_{L^2(P_{t,u})}^2 d\tau'\right)^{\frac{1}{2}}\\
&\times \left(\frac{1}{2} \int_0^\tau \|\Lambda^{1-a} V(\tau')\|_{L^2(P_{t,u})}^2 d\tau' + \int_0^\tau \tau'^{-\frac{p-2}{ap}} \|\Lambda^{-a} V(\tau')\|_{L^2(P_{t,u})}^2 d\tau'\right)^{\frac{1}{2}} \qquad (A.20)
\end{aligned}$$

which together with the estimates for the heat flow (3.8), (3.10) implies:

$$\begin{aligned}
&\int_0^\tau \|\nabla\!\!\!/ U(\tau')\rho\|_{L^p(P_{t,u})} \|\Lambda^{-2a} V(\tau')\|_{L^{p'}(P_{t,u})} d\tau' + \int_0^\tau \|\nabla\!\!\!/ U(\tau')\rho\|_{L^p(P_{t,u})} \|\nabla\!\!\!/ \Lambda^{-2a} V(\tau')\|_{L^2(P_{t,u})} d\tau'\\
\lesssim\ & E^{\frac{1}{2}} \|\rho\|_{L^2(P_{t,u})} \left(\int_0^\tau \|\Lambda^{1-a} V(\tau')\|_{L^2(P_{t,u})}^2 d\tau' + \int_0^\tau \tau'^{-\frac{p-2}{ap}} \|\Lambda^{-a} V(\tau')\|_{L^2(P_{t,u})}^2 d\tau'\right)^{\frac{1}{2}}\\
\lesssim\ & E^{\frac{1}{2}} \|\rho\|_{L^2(P_{t,u})} \Bigg(\int_0^\tau (\|\Lambda^{-a} V(\tau')\|_{L^2(P_{t,u})}^2 + \|\nabla\!\!\!/ \Lambda^{-a} V(\tau')\|_{L^2(P_{t,u})}^2) d\tau'\\
&+ \int_0^\tau \tau'^{-\frac{p-2}{ap}} \|\Lambda^{-a} V(\tau')\|_{L^2(P_{t,u})}^2 d\tau'\Bigg)^{\frac{1}{2}} \qquad (A.21)
\end{aligned}$$

where we used (3.23) in the last esimtate.

Finally, the choice of p (A.17), (A.18), (A.19) and (A.21) yield for $0 \leq \tau \leq 1$:

$$\begin{aligned} &\|\Lambda^{-a} V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/ \Lambda^{-a} V(\tau')\|^2_{L^2(P_{t,u})} d\tau' \\ &\lesssim E(\|\nabla\!\!\!/(n\chi)\|_{L^2(P_{t,u})} + \|n(2\widehat{\chi}\overline{\epsilon} - \epsilon \mathrm{tr}\chi - \nabla\!\!\!/ \mathrm{tr}\chi)\|_{L^2(P_{t,u})})^2 \|\rho\|^2_{L^2(P_{t,u})}. \end{aligned} \tag{A.22}$$

Using the interpolation inequality (3.22), we obtain:

$$\begin{aligned} &\int_0^1 \|V(\tau)\|^{\frac{2}{a}}_{L^2(P_{t,u})} d\tau \lesssim \int_0^1 \|\Lambda^{-a} V(\tau')\|^{\frac{2(1-a)}{a}}_{L^2(P_{t,u})} \|\nabla\!\!\!/ \Lambda^{-a} V(\tau')\|^2_{L^2(P_{t,u})} d\tau' \\ &\lesssim E^{\frac{1}{a}} (\|\nabla\!\!\!/(n\chi)\|_{L^2(P_{t,u})} + \|n(2\widehat{\chi}\overline{\epsilon} - \epsilon \mathrm{tr}\chi - \nabla\!\!\!/ \mathrm{tr}\chi)\|_{L^2(P_{t,u})})^{\frac{2}{a}} \|\rho\|^{\frac{2}{a}}_{L^2(P_{t,u})}, \end{aligned} \tag{A.23}$$

which together with the bootstrap assumptions (4.1)-(4.5), the estimate on ρ (A.2), and the fact that m is supported in $(0,1)$ yields for $j \geq 0$:

$$\begin{aligned} &\left\| \int_0^{+\infty} m_j(\tau) \|V(\tau)\|_{L^2(P_{t,u})} d\tau \right\|_{L^1(0,1)} \lesssim 2^{ja} \left\| \left(\int_0^1 \|V(\tau)\|^{\frac{2}{a}}_{L^2(P_{t,u})} d\tau \right)^{\frac{a}{2}} \right\|_{L^1(0,1)} \\ &\lesssim 2^{ja} E^{\frac{1}{2}} (\mathcal{N}_1(\chi)(\mathcal{N}_1(\nabla\!\!\!/ n) + \|n\|_{L^\infty(\mathcal{H}_u)}) + \|n\|_{L^\infty(\mathcal{H}_u)} (\mathcal{N}_1(\chi)\mathcal{N}_1(\epsilon) + \mathcal{N}_1(\chi)) \|\rho\|_{L^2(\mathcal{H}_u)} \\ &\lesssim 2^{ja} E^{\frac{1}{2}} \varepsilon \end{aligned} \tag{A.24}$$

In turn, we obtain together with (A.16) and the fact that $0 < a < \frac{1}{2}$:

$$\sum_{j \geq 0} 2^{-j} \|P_j \rho\|^2_{L^\infty_t L^2_{x'}} \lesssim \sum_{j \geq 0} 2^{-j} 2^{2ja} E\varepsilon^2 + \varepsilon^2 \lesssim E\varepsilon^2. \tag{A.25}$$

Using (A.1), we obtain for K:

$$\sum_{j \geq 0} 2^{-j} \|P_j K\|^2_{L^\infty_t L^2_{x'}} \lesssim E\varepsilon^2, \tag{A.26}$$

which is an improvement of (A.3). Thus, we have proved:

$$\sum_{j \geq 0} 2^{-j} \|P_j K\|^2_{L^\infty_t L^2_{x'}} \lesssim \varepsilon^2, \tag{A.27}$$

which together with (3.37) concludes the proof of (4.30).

A.2 Proof of Lemma 4.14

First, note that

$$\int_{\mathbb{R}} \left| \int_{P_{t,u}} F(u,\cdot) \nabla\!\!\!/ G(u,\cdot) \right| du \lesssim \int_{\Sigma_t} b^{-1} |F \nabla\!\!\!/ G| \lesssim \|F\|_{L^2(\Sigma_t)} \|G\|_{H^1(\Sigma_t)} < +\infty$$

so that we have

$$\liminf_{u \to -\infty} \int_{P_{t,u}} F(u,\cdot) \nabla\!\!\!/ G(u,\cdot) = 0. \tag{A.28}$$

Next, let $u_0 < u_1$. We have

$$\begin{aligned}\int_{P_{t,u_1}} F(u_1,\cdot)\cdot \nabla\!\!\!/ G(u_1,\cdot) &= \int_{P_{t,u_0}} F(u_0,\cdot)\cdot \nabla\!\!\!/ G(u_0,\cdot) + \int_{u_0}^{u_1} \partial_u \left(\int_{P_{t,u}} F\cdot \nabla\!\!\!/ G\right) du \\ &\lesssim \int_{P_{t,u_0}} F(u_0,\cdot)\nabla\!\!\!/ G(u_0,\cdot) + \int_{\mathbb{R}} \left|\partial_u \left(\int_{P_{t,u}} F\cdot \nabla\!\!\!/ G\right)\right| du.\end{aligned}$$

Letting $u_0 \to -\infty$, using (A.28), and taking the supremum in u_1, this yields

$$\sup_u \left(\int_{P_{t,u}} F\cdot \nabla\!\!\!/ G\right) \lesssim \int_{\mathbb{R}} \left|\partial_u \left(\int_{P_{t,u}} F\cdot \nabla\!\!\!/ G\right)\right| du.$$

Together with (3.77), this yields[21]

$$\begin{aligned}&\sup_u \left(\int_{P_{t,u}} F\cdot \nabla\!\!\!/ G\right) \\ \lesssim\ & \int_u \left|\int_{P_{t,u}} b(\nabla\!\!\!/_N F\cdot \nabla\!\!\!/ G + F\cdot \nabla\!\!\!/_N \nabla\!\!\!/ G + \mathrm{tr}\theta F\cdot \nabla\!\!\!/ G) d\mu_{t,u}\right| du \\ \lesssim\ & \int_u \left|\int_{P_{t,u}} b(\nabla\!\!\!/_N F\cdot \nabla\!\!\!/ G + F\cdot \nabla\!\!\!/ \nabla\!\!\!/_N G + F\cdot [\nabla\!\!\!/_N, \nabla\!\!\!/] G + \mathrm{tr}\theta F\cdot \nabla\!\!\!/ G) d\mu_{t,u}\right| du.\end{aligned}$$

Decomposing

$$F\cdot \nabla\!\!\!/ \nabla\!\!\!/_N G = \not{\mathrm{div}}(F\cdot \nabla\!\!\!/_N G) - \nabla\!\!\!/ F\cdot \nabla\!\!\!/_N G$$

and integrating by parts the divergence term on $P_{t,u}$, we deduce

$$\begin{aligned}&\sup_u \left(\int_{P_{t,u}} F\cdot \nabla\!\!\!/ G\right) \\ \lesssim\ & \int_u \int_{P_{t,u}} \left|\nabla\!\!\!/_N F\cdot \nabla\!\!\!/ G - \nabla\!\!\!/ F\cdot \nabla\!\!\!/_N G - b^{-1}\nabla\!\!\!/ b\cdot F\cdot \nabla\!\!\!/_N G + F\cdot [\nabla\!\!\!/_N, \nabla\!\!\!/] G + \mathrm{tr}\theta F\cdot \nabla\!\!\!/ G\right| b d\mu_{t,u} du,\end{aligned}$$

which in view of the coarea formula (3.76) yields

$$\begin{aligned}&\sup_u \left(\int_{P_{t,u}} F\cdot \nabla\!\!\!/ G\right) \qquad\qquad (A.29)\\ \lesssim\ & \int_{\Sigma_t} \left|\nabla\!\!\!/_N F\cdot \nabla\!\!\!/ G - \nabla\!\!\!/ F\cdot \nabla\!\!\!/_N G - b^{-1}\nabla\!\!\!/ b\cdot F\cdot \nabla\!\!\!/_N G + F\cdot [\nabla\!\!\!/_N, \nabla\!\!\!/] G + \mathrm{tr}\theta F\cdot \nabla\!\!\!/ G\right| d\Sigma_t \\ \lesssim\ & \|\nabla F\|_{L^2(\Sigma_t)} \|\nabla G\|_{L^2(\Sigma_t)} + (\|b^{-1}\nabla\!\!\!/ b\|_{L^4(\Sigma_t)} + \|\mathrm{tr}\theta\|_{L^4(\Sigma_t)}) \|F\|_{L^4(\Sigma_t)} \|\nabla G\|_{L^2(\Sigma_t)} \\ & + \|F\|_{L^4(\Sigma_t)} \|[\nabla\!\!\!/_N, \nabla\!\!\!/] G\|_{L^{\frac{4}{3}}(\Sigma_t)}.\end{aligned}$$

[21] Recall that $F\cdot \nabla\!\!\!/ G$ is a scalar by assumption so that we may apply (3.77) with $f = F\cdot \nabla\!\!\!/ G$. Recall also that the definition of ∇_N and of $\nabla\!\!\!/_N$ coincide on scalars so that $\nabla_N f = \nabla\!\!\!/_N F\cdot \nabla\!\!\!/ G + F\cdot \nabla\!\!\!/_N \nabla\!\!\!/ G$.

The commutator formula (2.47) implies[22]

$$\begin{aligned}
&\|[\nabla\!\!\!/_N, \nabla\!\!\!/]G\|_{L^{\frac{4}{3}}(\Sigma_t)} \\
\lesssim\ & (\|b^{-1}\nabla\!\!\!/ b\|_{L^4(\Sigma_t)} + \|\chi\|_{L^4(\Sigma_t)} + \|\eta\|_{L^4(\Sigma_t)})\|\nabla G\|_{L^2(\Sigma_t)} \\
&+(\|\mathbf{R}\|_{L^2(\Sigma_t)} + (\|\chi\|_{L^4(\Sigma_t)} + \|\overline{\epsilon}\|_{L^4(\Sigma_t)} + \|\underline{\xi}\|_{L^4(\Sigma_t)} + \|\underline{\chi}\|_{L^4(\Sigma_t)} + \|\zeta\|_{L^4(\Sigma_t)})^2)\|G\|_{L^4(\Sigma_t)},
\end{aligned}$$

which together with (A.29) and the Sobolev embedding (3.71) on Σ_t implies

$$\begin{aligned}
&\sup_u \left(\int_{P_{t,u}} F \cdot \nabla\!\!\!/ G\right) \qquad\qquad (A.30)\\
\lesssim\ & \Big(1 + \|b^{-1}\nabla\!\!\!/ b\|_{L^4(\Sigma_t)} + \|\chi\|_{L^4(\Sigma_t)} + \|\eta\|_{L^4(\Sigma_t)} + (\|\mathbf{R}\|_{L^2(\Sigma_t)} + (\|\chi\|_{L^4(\Sigma_t)} + \|\overline{\epsilon}\|_{L^4(\Sigma_t)} \\
&+\|\underline{\xi}\|_{L^4(\Sigma_t)} + \|\underline{\chi}\|_{L^4(\Sigma_t)} + \|\zeta\|_{L^4(\Sigma_t)})^2\Big)\|F\|_{H^1(\Sigma_t)}\|G\|_{H^1(\Sigma_t)},
\end{aligned}$$

where we used in the last inequality the definition (4.61) of θ in the last inequality. Now, in view of the embedding (3.58), we have

$$\begin{aligned}
&\|b^{-1}\nabla\!\!\!/ b\|_{L^4(\Sigma_t)} + \|\chi\|_{L^4(\Sigma_t)} + \|\eta\|_{L^4(\Sigma_t)} + \|\overline{\epsilon}\|_{L^4(\Sigma_t)} + \|\underline{\xi}\|_{L^4(\Sigma_t)} + \|\underline{\chi}\|_{L^4(\Sigma_t)} + \|\zeta\|_{L^4(\Sigma_t)} \\
\lesssim\ & \|b^{-1}\nabla\!\!\!/ b\|_{L^\infty_t L^4_{x'}} + \|\chi\|_{L^\infty_t L^4_{x'}} + \|\eta\|_{L^\infty_t L^4_{x'}} + \|\overline{\epsilon}\|_{L^\infty_t L^4_{x'}} + \|\underline{\xi}\|_{L^\infty_t L^4_{x'}} + \|\underline{\chi}\|_{L^\infty_t L^4_{x'}} + \|\zeta\|_{L^\infty_t L^4_{x'}} \\
\lesssim\ & \mathcal{N}_1(b^{-1}\nabla\!\!\!/ b) + \|\mathrm{tr}\chi\|_{L^\infty_t L^4_{x'}} + \mathcal{N}_1(\widehat{\chi}) + \mathcal{N}_1(k) + \|n^{-1}\nabla\!\!\!/ n\|_{L^\infty_t L^4_{x'}} + \mathcal{N}_1(\zeta)
\end{aligned}$$

where we used the identities (2.25) for $\underline{\xi}$ and for $\underline{\chi}$, as well as the definition (2.21) of η and ϵ in terms of k. We deduce

$$\begin{aligned}
&\|b^{-1}\nabla\!\!\!/ b\|_{L^4(\Sigma_t)} + \|\chi\|_{L^4(\Sigma_t)} + \|\eta\|_{L^4(\Sigma_t)} + \|\overline{\epsilon}\|_{L^4(\Sigma_t)} + \|\underline{\xi}\|_{L^4(\Sigma_t)} + \|\underline{\chi}\|_{L^4(\Sigma_t)} + \|\zeta\|_{L^4(\Sigma_t)} \\
\lesssim\ & \mathcal{N}_1(b^{-1}\nabla\!\!\!/ b) + \|\mathrm{tr}\chi\|_{L^\infty_t L^4_{x'}} + \mathcal{N}_1(\widehat{\chi}) + \mathcal{N}_1(k) + \|n^{-1}\nabla\!\!\!/ n\|_{L^\infty_t L^4_{x'}} + \mathcal{N}_1(\zeta) \\
\lesssim\ & \mathcal{N}_2(b-1) + \|\mathrm{tr}\chi\|_{L^\infty_t L^2_{x'}} + \|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^\infty_t L^2_{x'}} + \mathcal{N}_1(\widehat{\chi}) + \mathcal{N}_1(k) \\
&+\|\nabla\!\!\!/ n\|_{L^\infty_t L^2_{x'}} + \|\nabla\!\!\!/^2 n\|_{L^\infty_t L^2_{x'}} + \mathcal{N}_1(\zeta) \\
\lesssim\ & 1, \qquad\qquad (A.31)
\end{aligned}$$

where we used in the last inequality the bootstrap assumptions (4.1)-(4.6) for b, n, $\mathrm{tr}\chi$, $\widehat{\chi}$, k and ζ. Finally, (A.30), (A.31) and the assumption (2.59) on $\mathbf{R}$ yield

$$\sup_u \left(\int_{P_{t,u}} F \cdot \nabla\!\!\!/ G\right) \lesssim \|F\|_{H^1(\Sigma_t)}\|G\|_{H^1(\Sigma_t)},$$

which is the desired estimate. This concludes the proof of Lemma 4.14.

[22]Recall that by definition, $\|\mathbf{R}\|_{L^2(\Sigma_t)}$ controls the $L^2(\Sigma_t)$ norm of any curvature component, and in particular of β and $\underline{\beta}$ appearing in (2.47).

A.3 Proof of Lemma 4.24

Using the formula (3.54) for the commutator $[{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}]$, we have:

$$[{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta}) = -{}^*\mathcal{D}_1^{-1}[{}^*\mathcal{D}_1, \nabla\!\!\!/_{nL}]({}^*\mathcal{D}_1^{-1}(\underline{\beta}))$$

which in view of Lemma 3.17 yields:

$$\begin{aligned} & \|[{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta})\|_{L_t^2 L_{x'}^3} + \|[{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta})\|_{L_t^1 L_{x'}^4} \\ \lesssim \; & \|[{}^*\mathcal{D}_1, \nabla\!\!\!/_{nL}]({}^*\mathcal{D}_1^{-1}(\underline{\beta}))\|_{L_t^2 L_{x'}^{\frac{4}{3}}} + \|[{}^*\mathcal{D}_1, \nabla\!\!\!/_{nL}]({}^*\mathcal{D}_1^{-1}(\underline{\beta}))\|_{L_t^1 L_{x'}^{\frac{3}{2}}}. \end{aligned} \tag{A.32}$$

Now, from the commutator formula (2.48) and the fact that ${}^*\mathcal{D}_1^{-1}(\underline{\beta})$ is a scalar, we have

$$[{}^*\mathcal{D}_1, \nabla\!\!\!/_{nL}]({}^*\mathcal{D}_1^{-1}(\underline{\beta})) = n\chi \cdot \nabla\!\!\!/({}^*\mathcal{D}_1^{-1}(\underline{\beta}))$$

which together with (A.32) implies:

$$\begin{aligned} & \|[{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta})\|_{L_t^2 L_{x'}^3} + \|[{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta})\|_{L_t^1 L_{x'}^4} \\ \lesssim \; & \|n\chi \cdot \nabla\!\!\!/({}^*\mathcal{D}_1^{-1}(\underline{\beta}))\|_{L_t^2 L_{x'}^{\frac{4}{3}}} + \|n\chi \cdot \nabla\!\!\!/({}^*\mathcal{D}_1^{-1}(\underline{\beta}))\|_{L_t^1 L_{x'}^{\frac{3}{2}}} \\ \lesssim \; & \|n\|_{L^\infty}(\|\chi\|_{L_t^\infty L_{x'}^4} + \|\chi\|_{L_t^2 L_{x'}^6})\|\nabla\!\!\!/({}^*\mathcal{D}_1^{-1}(\underline{\beta})\|_{L^2(\mathcal{H}_u)} \\ \lesssim \; & \Big(\mathcal{N}_1(\widehat{\chi}) + \|\mathrm{tr}\chi\|_{L_t^\infty L_{x'}^2} + \|\nabla\!\!\!/\mathrm{tr}\chi\|_{L_t^\infty L_{x'}^2}\Big)\|\underline{\beta}\|_{L^2(\mathcal{H}_u)} \\ \lesssim \; & D\varepsilon^2 \\ \lesssim \; & \varepsilon, \end{aligned}$$

where we used the bootstrap assumption (4.1) for n, the bootstrap assumptions (4.4) (4.5) for $\mathrm{tr}\chi$ and $\widehat{\chi}$, the curvature bound (2.59) for $\underline{\beta}$, and the estimate (3.50) for ${}^*\mathcal{D}_1^{-1}$. This concludes the proof of Lemma 4.24.

B Appendix to section 5

B.1 Proof of Lemma 5.6

We decompose $\|Pj(H \cdot F)\|_{L^2(\mathcal{H}_u)}$ using the property (3.16) of the geometric Littlewood-Paley projections:

$$\|P_j(H \cdot F)\|_{L^2(\mathcal{H}_u)} \lesssim \|P_j(H \cdot P_{<0}F)\|_{L^2(\mathcal{H}_u)} + \sum_{l \geq 0} \|P_j(H \cdot P_l F)\|_{L^2(\mathcal{H}_u)}. \tag{B.1}$$

We focus on the second term in the right-hand side of (B.1), the other being easier to handle. We start with the case $l \leq j$. Using the assumption (5.81) for F, the Sobolev inequality (3.58) and the weak Bernstein inequality iv) of Theorem 3.10, we have:

$$\begin{aligned} \|P_j(H \cdot P_l F)\|_{L^2(\mathcal{H}_u)} & \lesssim \|H \cdot P_l F\|_{L^2(\mathcal{H}_u)} \\ & \lesssim \|H\|_{L_t^\infty L_{x'}^4} \|P_l F\|_{L_t^2 L_{x'}^4} \\ & \lesssim \mathcal{N}_1(H) 2^{\frac{l}{2}} \|P_l F\|_{L^2(\mathcal{H}_u)} \\ & \lesssim \mathcal{N}_1(H)(2^l C_1 + 2^{\frac{l}{2}} C_2), \end{aligned}$$

which yields:

$$\sum_{l\leq j}\|P_j(H\cdot P_lF)\|_{L^2(\mathcal{H}_u)}\lesssim\sum_{l\leq j}\mathcal{N}_1(H)(2^lC_1+2^{\frac{l}{2}}C_2)\lesssim\mathcal{N}_1(H)(2^jC_1+2^{\frac{j}{2}}C_2). \tag{B.2}$$

We now focus on the case $l>j$. We further decompose:

$$\|P_j(H\cdot P_lF)\|_{L^2(\mathcal{H}_u)}\lesssim\|P_j(P_{\leq l}H\cdot P_lF)\|_{L^2(\mathcal{H}_u)}+\|P_j(P_{>l}H\cdot P_lF)\|_{L^2(\mathcal{H}_u)}. \tag{B.3}$$

We evaluate first the second term in the right-hand side of (B.3). Using the dual of the sharp Bernstein inequality (4.32) for scalars, we obtain:

$$\begin{aligned}\|P_j(P_{>l}H\cdot P_lF)\|_{L^2(\mathcal{H}_u)}&\lesssim 2^j\|P_{>l}H\cdot P_lF\|_{L^2_tL^1_{x'}}\\&\lesssim 2^j\|P_{>l}H\|_{L^\infty_tL^2_{x'}}\|P_lF\|_{L^2(\mathcal{H}_u)}\\&\lesssim 2^{j-\frac{l}{2}}\mathcal{N}_1(H)(C_1+2^{-\frac{l}{2}}C_2),\end{aligned} \tag{B.4}$$

where we used the assumption (5.81) for F and the estimate (5.87) for H.

We now consider the first term in the right-hand side of (B.3). We use the existence of a Littlewood-Paley projector $\widetilde{P}_l$ associated to a function $\widetilde{m}$ satisfying the same properties than m, such that

$$P_l=2^{-2l}\not{\Delta}\widetilde{P}_l, \tag{B.5}$$

see Theorem 5.5 in [10]. Using the identity (B.5), (5.88) with $p=\frac{4}{3}$, the dual of the sharp Bernstein inequality (4.32) for scalars and (5.87), we obtain:

$$\begin{aligned}&\|P_j(P_{\leq l}H\cdot P_lF)\|_{L^2(\mathcal{H}_u)}\\=\;&2^{-2l}\|P_j(P_{\leq l}H\cdot\not{\Delta}\widetilde{P}_lF)\|_{L^2(\mathcal{H}_u)}\\\lesssim\;&2^{-2l}\|P_j(\not{\mathrm{div}}(P_{\leq l}H\cdot\not{\nabla}\widetilde{P}_lF))\|_{L^2(\mathcal{H}_u)}+2^{-2l}\|P_j(\not{\nabla}P_{\leq l}H\cdot\not{\nabla}\widetilde{P}_lF)\|_{L^2(\mathcal{H}_u)}\\\lesssim\;&2^{-2l}2^{\frac{3}{2}j}\|P_{\leq l}H\cdot\not{\nabla}\widetilde{P}_lF\|_{L^2_tL^{\frac{4}{3}}_{x'}}+2^{-2l+j}\|\not{\nabla}P_{\leq l}H\cdot\not{\nabla}\widetilde{P}_lF\|_{L^2_tL^1_{x'}}\\\lesssim\;&2^{-2l+\frac{3}{2}j}\|H\|_{L^\infty_tL^4_{x'}}\|\not{\nabla}\widetilde{P}_lF\|_{L^2(\mathcal{H}_u)}+2^{-2l+j}\|\not{\nabla}P_{\leq l}H\|_{L^\infty_tL^2_{x'}}\|\not{\nabla}\widetilde{P}_lF\|_{L^2(\mathcal{H}_u)}\\\lesssim\;&(2^{\frac{3}{2}j-l}+2^{j-\frac{1}{2}l})\mathcal{N}_1(H)(C_1+2^{-\frac{l}{2}}C_2).\end{aligned} \tag{B.6}$$

B.2 Proof of Lemma 5.7

We decompose $\|Pj(hf)\|_{L^p_tL^2_{x'}}$ using the property (3.16) of the geometric Littlewood-Paley projections:

$$\|P_j(hf)\|_{L^p_tL^2_{x'}}\lesssim\|P_j(hP_{<0}f)\|_{L^p_tL^2_{x'}}+\sum_{l\geq 0}\|P_j(hP_lf)\|_{L^p_tL^2_{x'}}. \tag{B.7}$$

We focus on the second term in the right-hand side of (B.7), the other being easier to handle. Using the L^2-boundedness of the Littlewood-Paley projection P_j, we have:

$$\|P_j(hP_lf)\|_{L^p_tL^2_{x'}}\lesssim\|hP_lf\|_{L^p_tL^2_{x'}}\lesssim\|h\|_{L^\infty}(2^lC_1+2^{\frac{l}{2}}C_2) \tag{B.8}$$

which gives the desired estimate for the term

$$\sum_{l\leq j}\|P_j(hP_lf)\|_{L^p_tL^2_{x'}}.$$

We now decompose $\|P_j(hP_lf)\|_{L^p_tL^2_{x'}}$ again using the property (3.16) of the geometric Littlewood-Paley projections:

$$\|P_j(hP_l(f))\|_{L^p_tL^2_{x'}}\lesssim\|P_j(P_{<0}(h)P_l(f))\|_{L^p_tL^2_{x'}}+\sum_{q\geq 0}\|P_j(P_q(h)P_l(f))\|_{L^p_tL^2_{x'}}. \tag{B.9}$$

We focus on the second term in the right-hand side of (B.9), the other being easier to handle. We have:

$$\begin{aligned}\|P_j(P_q(h)P_l(f))\|_{L^p_tL^2_{x'}} &\lesssim 2^j\|P_q(h)P_l(f)\|_{L^p_tL^1_{x'}} \\ &\lesssim 2^j\|P_q(h)\|_{L^\infty_tL^2_{x'}}\|P_l(f)\|_{L^p_tL^2_{x'}} \\ &\lesssim 2^j(2^lC_1+2^{\frac{l}{2}})\|P_q(h)\|_{L^\infty_tL^2_{x'}},\end{aligned} \tag{B.10}$$

where we used in the last inequality the assumption (5.83) for f.

We now derive a second estimate. Using the properties of the Littlewood-Paley projection P_l, we have:

$$\begin{aligned}\|P_j(P_q(h)P_lf)\|_{L^p_tL^2_{x'}} &\lesssim 2^{-2l}\|P_j(P_q(h)\not\Delta P_lf)\|_{L^p_tL^2_{x'}} \\ &\lesssim 2^{-2l}\|P_j(\not\Delta(P_q(h)P_lf))\|_{L^p_tL^2_{x'}}+2^{-2l}\|P_j(\not{\rm div}(\not\nabla P_q(h)P_lf))\|_{L^p_tL^2_{x'}} \\ &\quad+2^{-2l}\|P_j(\not\Delta(P_q(h))P_lf)\|_{L^p_tL^2_{x'}} \\ &\lesssim 2^{2j-2l}\|P_q(h)P_lf\|_{L^p_tL^2_{x'}}+2^{j-2l}\|\not\nabla P_q(h)P_lf\|_{L^p_tL^2_{x'}} \\ &\quad+2^{j-2l}\|\not\Delta P_q(h)P_lf\|_{L^p_tL^1_{x'}} \\ &\lesssim 2^{2j-2l}\|P_q(h)\|_{L^\infty}\|P_lf\|_{L^p_tL^2_{x'}}+2^{j-2l}\|\not\nabla P_q(h)\|_{L^\infty_tL^4_{x'}}\|P_lf\|_{L^p_tL^4_{x'}} \\ &\quad+2^{j+2q-2l}\|P_q(h)\|_{L^\infty_tL^2_{x'}}\|P_lf\|_{L^p_tL^2_{x'}} \\ &\lesssim \Big(2^{2j-2l+q}\|P_q(h)\|_{L^\infty_tL^2_{x'}}+2^{j-\frac{3l}{2}}\|\not\nabla^2P_q(h)\|^{\frac12}_{L^\infty_tL^2_{x'}}\|\not\nabla P_q(h)\|^{\frac12}_{L^\infty_tL^2_{x'}} \\ &\quad+2^{j-\frac{3l}{2}}\|\not\nabla P_q(h)\|_{L^\infty_tL^2_{x'}}+2^{j+2q-2l}\|P_q(h)\|_{L^\infty_tL^2_{x'}}\Big)\|P_lf\|_{L^p_tL^2_{x'}} \\ &\lesssim (2^{2j-2l+q}+2^{j-\frac{3l}{2}+\frac{3q}{2}}+2^{j+2q-2l})\|P_q(h)\|_{L^\infty_tL^2_{x'}}(2^l\varepsilon+2^{\frac{l}{2}}\varepsilon\gamma(u)),\end{aligned} \tag{B.11}$$

where we used the dual of the sharp Bernstein inequality (4.32) and the finite band property for P_j, the weak Bernstein inequality for P_l, the Gagliardo-Nirenberg inequality (3.3), the Bochner inequality (4.34) for scalars, the finite band property and the sharp Bernstein inequality (4.32) for P_q, and the assumption (5.83) for f.

Then, using (B.10) when $q>l$, and (B.11) when $q\leq l$, we obtain:

$$\sum_{q,l>j}\|P_j(P_q(b)P_lf)\|_{L^p_tL^2_{x'}}\lesssim\|h\|_{\mathcal{B}^1}(2^jC_1+2^{\frac{j}{2}}C_2)$$

which together with (B.9) yields:

$$\sum_{l>j}\|P_j(hP_lf)\|_{L^p_tL^2_{x'}} \lesssim \|h\|_{\mathcal{B}^1}(2^jC_1+2^{\frac{j}{2}}C_2). \tag{B.12}$$

Finally, using (B.8) when $l \leq j$ and (B.12) when $l > j$, we obtain:

$$\sum_{l}\|P_j(hP_lf)\|_{L^p_tL^2_{x'}} \lesssim (\|h\|_{L^\infty}+\|h\|_{\mathcal{B}^1})(2^jC_1+2^{\frac{j}{2}}C_2),$$

which together with (B.7) implies:

$$\|P_j(hf)\|_{L^p_tL^2_{x'}} \lesssim (\|h\|_{L^\infty}+\|h\|_{\mathcal{B}^1})(2^jC_1+2^{\frac{j}{2}}C_2). \tag{B.13}$$

Now, the embedding (5.9) applied to h together with (B.13) concludes the proof of Lemma 5.7.

B.3 Proof of Lemma 5.8

Let f the scalar function on $\mathcal{H}_u$ defined by $f = \mathcal{D}_1(F)$. The assumption (5.85) now reads for all $j \geq 0$:

$$\|P_jf\|_{L^2(\mathcal{H}_u)} \lesssim 2^jC_1+2^{\frac{j}{2}}C_2 \tag{B.14}$$

where C_1, C_2 are constants possibly depending on u. From the definition of f, we have $F = \mathcal{D}_1^{-1}(f)$. We decompose the norm $\|P_jF\|_{L^2(\mathcal{H}_u)}$ using the property (3.16) of the geometric Littlewood-Paley projections:

$$\|P_jF\|_{L^2(\mathcal{H}_u)} \lesssim \|P_j\mathcal{D}^{-1}P_{<0}(f)\|_{L^2(\mathcal{H}_u)} + \sum_{q\geq 0}\|P_j\mathcal{D}^{-1}P_q(f)\|_{L^2(\mathcal{H}_u)}. \tag{B.15}$$

The first term in the right-hand side of (B.14) is easier to handle, so we focus on the sum in q. We have:

$$\|P_j\mathcal{D}_1^{-1}P_q(f)\|_{L^2(\mathcal{H}_u)} \lesssim 2^{-j}\|\nabla\!\!\!/\mathcal{D}_1^{-1}\|_{\mathcal{L}(L^2(P_{t,u}))}\|P_q(f)\|_{L^2(\mathcal{H}_u)} \lesssim 2^{-j}(2^qC_1+2^{\frac{q}{2}}C_2), \tag{B.16}$$

where we used the finite band properties of the Littlewood-Paley projection P_j, the estimate (3.50) for $\mathcal{D}_1^{-1}$ and (B.14). We now derive a second estimate. Using the properties of the Littlewood-Paley projection P_q and the identity (3.39) for $\mathcal{D}_1$, we have:

$$\begin{aligned}\|P_j\mathcal{D}_1^{-1}P_q(f)\|_{L^2(\mathcal{H}_u)} &\lesssim 2^{-2q}\|P_j\mathcal{D}_1^{-1}\Delta\!\!\!/P_q(f)\|_{L^2(P_{t,u})}\\ &\lesssim 2^{-2q}\|P_j{}^*\mathcal{D}_1P_q(f)\|_{L^2(P_{t,u})}\\ &\lesssim 2^{j-2q}\|P_q(f)\|_{L^2(P_{t,u})}\\ &\lesssim 2^{j-2q}(2^qC_1+2^{\frac{q}{2}}C_2),\end{aligned} \tag{B.17}$$

where we used the finite band properties of the Littlewood-Paley projection P_j and (B.14).

We now use (B.16) for $q \leq j$ and (B.17) for $q > j$ to obtain:

$$\begin{aligned}\sum_{q\geq 0}\|P_j\mathcal{D}_1^{-1}P_q(f)\|_{L^2(\mathcal{H}_u)} &\lesssim \sum_{q\leq j}2^{-j}(2^qC_1+2^{\frac{q}{2}}C_2)+\sum_{q>j}2^{j-2q}(2^qC_1+2^{\frac{q}{2}}C_2)\\ &\lesssim C_1+2^{-\frac{j}{2}}C_2\end{aligned}$$

which together with (B.15) concludes the proof of Lemma 5.8.

B.4 Proof of Lemma 5.9

First, from the finite band property of the Littlewood-Paley projection P_j, we have:

$$\|\nabla\!\!\!/ P_j F\|_{L^\infty_t L^2_{x'}} \lesssim 2^j \|P_j F\|_{L^\infty_t L^2_{x'}} \tag{B.18}$$

so that we only need to estimate the first term in the left-hand side of (5.87).

Using (A.5), properties (ii) and (iii) of Theorem 3.10 for P_j, and the L^∞ bound on n given by (2.67), we have:

$$\begin{aligned}
&\sum_{j\geq 0} 2^j \|P_j F\|^2_{L^\infty_t L^2_{x'}} \\
\lesssim\ & \sum_{j\geq 0} 2^j \left(\int_0^1 \|P_j F\|_{L^2(P_{t,u})} \|\nabla\!\!\!/_L P_j F\|_{L^2(P_{t,u})} dt + \|P_j F\|^2_{L^2(\mathcal{H}_u)} \right) \\
\lesssim\ & \sum_{j\geq 0} 2^j \left(\int_0^1 \|P_j F\|_{L^2(P_{t,u})} \|\nabla\!\!\!/_{nL} P_j F\|_{L^2(P_{t,u})} dt \right) + \sum_{j\geq 0} 2^j \|P_j F\|^2_{L^2(\mathcal{H}_u)} \\
\lesssim\ & \sum_{j\geq 0} 2^j \|P_j F\|_{L^2(\mathcal{H}_u)} \|P_j \nabla\!\!\!/_{nL} F\|_{L^2(\mathcal{H}_u)} + \sum_{j\geq 0} 2^j \|P_j F\|_{L^\infty_t L^2_{x'}} \|[P_j, \nabla\!\!\!/_{nL}] F\|_{L^1_t L^2_{x'}} + \mathcal{N}_1(F)^2 \\
\lesssim\ & \left(\sum_{j\geq 0} 2^j \|P_j F\|^2_{L^\infty_t L^2_{x'}} \right)^{\frac{1}{2}} \left(\sum_{j\geq 0} 2^j \|[P_j, \nabla\!\!\!/_{nL}] F\|^2_{L^1_t L^2_{x'}} \right)^{\frac{1}{2}} + \mathcal{N}_1(F)^2
\end{aligned}$$

which yields:

$$\sum_{j\geq 0} 2^j \|P_j F\|^2_{L^\infty_t L^2_{x'}} \lesssim \sum_{j\geq 0} 2^j \|[P_j, \nabla\!\!\!/_{nL}] F\|^2_{L^1_t L^2_{x'}} + \mathcal{N}_1(F)^2. \tag{B.19}$$

Now, the commutator estimate (5.24) and (B.19) yield

$$\sum_{j\geq 0} 2^j \|P_j F\|^2_{L^\infty_t L^2_{x'}} \lesssim \sum_{j\geq 0} 2^j 2^{-2j_+} \mathcal{N}_1(F)^2 + \mathcal{N}_1(F)^2 \lesssim \mathcal{N}_1(F)^2$$

which together with (B.18) concludes the proof of Lemma 5.9.

B.5 Proof of Lemma 5.10

By duality, it suffices to prove for any scalar function f on $P_{t,u}$, for any $2 \leq p < +\infty$ and for all $j \geq 0$ the following inequality:

$$\|\nabla\!\!\!/ P_j f\|_{L^p(P_{t,u})} \lesssim 2^{2(1-\frac{1}{p})j} \|f\|_{L^2(P_{t,u})}. \tag{B.20}$$

Now, using the Gagliardo-Nirenberg inequality (3.3), the Bochner inequality for scalar functions (4.34), and the property iii) of Theorem 3.10 for Littlewood-Paley projections, we have:

$$\begin{aligned}
\|\nabla\!\!\!/ P_j f\|_{L^p(P_{t,u})} &\lesssim \|\nabla\!\!\!/^2 P_j f\|^{1-\frac{2}{p}}_{L^2(P_{t,u})} \|\nabla\!\!\!/ P_j f\|^{\frac{2}{p}}_{L^2(P_{t,u})} + \|\nabla\!\!\!/ P_j f\|_{L^2(P_{t,u})} \\
&\lesssim (\|\Delta\!\!\!/ P_j f\|_{L^2(P_{t,u})} + \|\nabla\!\!\!/ P_j f\|_{L^2(P_{t,u})})^{1-\frac{2}{p}} \|\nabla\!\!\!/ P_j f\|^{\frac{2}{p}}_{L^2(P_{t,u})} + 2^j \|f\|_{L^2(P_{t,u})} \\
&\lesssim 2^{2j(1-\frac{1}{p})} \|f\|_{L^2(P_{t,u})},
\end{aligned}$$

which is (B.20). This concludes the proof of Lemma 5.10.

B.6 Proof of Lemma 5.11

We first decompose f using the property (3.16) for the Littlewood-Paley projections P_l. We have:

$$f = \sum_l f_l \tag{B.21}$$

where f_l is the solution of the following transport equation:

$$nL(f_l) = 0,\ f_l = P_l f_0 \text{ on } P_{0,u}. \tag{B.22}$$

Using the L^2 boundedness of P_j, the equation (B.22), and the estimate (3.66) for transport equations applied to f_l, we have:

$$\|P_j f_l\|_{L^\infty_t L^2_{x'}} \lesssim \|f_l\|_{L^\infty_t L^2_{x'}} \lesssim \|f_l\|_{L^2_{x'} L^\infty_t} \lesssim \|P_l f_0\|_{L^2(P_{0,u})} \lesssim C 2^{\frac{l}{2}}. \tag{B.23}$$

Next, we derive a second estimate for $\|P_j f_l\|_{L^\infty_t L^2_{x'}}$. We define v_l as

$$v_l = -2^{-2l} \not\Delta f_l + f_l$$

which in view of (B.22) satisfies the following transport equation:

$$nL(v_l) = -2^{-2l}[nL, \not\Delta] f_l,\ v_l = 0 \text{ on } P_{0,u}. \tag{B.24}$$

The definition of v_l yields:

$$P_j(f_l) = 2^{-2l} P_j(\not\Delta f_l) + P_j(v_l)$$

which together with the finite band property for P_j implies:

$$\begin{aligned} \|P_j(f_l)\|_{L^\infty_t L^2_{x'}} &\lesssim 2^{2j-2l} \|f_l\|_{L^\infty_t L^2_{x'}} + \|P_j(v_l)\|_{L^\infty_t L^2_{x'}} \\ &\lesssim 2^{2j-\frac{3l}{2}} C + 2^{-2l} \left\| P_j \left(\int_0^t ([nL, \not\Delta] f_l) \right) \right\|_{L^\infty_t L^2_{x'}}, \end{aligned} \tag{B.25}$$

where we used the estimate (B.23) for f_l, and the transport equation (B.24) for v_l in the last inequality. Next, we estimate the second term in the right-hand side of (B.25). The commutator formula (2.49), written schematically, implies:

$$\begin{aligned} \left\| P_j \left(\int_0^t ([nL, \not\Delta] f_l) \right) \right\|_{L^\infty_t L^2_{x'}} &\lesssim \left\| P_j \left(\int_0^t \not{\mathrm{div}}(n\chi \cdot \not\nabla f_l) \right) \right\|_{L^\infty_t L^2_{x'}} \\ &\quad + \left\| P_j \left(\int_0^t (n(\not\nabla \chi + \chi \cdot (\overline{\epsilon} + n^{-1} \not\nabla n) \not\nabla f_l) \right) \right\|_{L^\infty_t L^2_{x'}} \end{aligned}$$

which together with Lemma 5.16 and the dual of the sharp Bernstein inequality (4.32)

for P_j yields:

$$\begin{aligned}
&\left\| P_j\left(\int_0^t([nL,\not{\Delta}]f)\right)\right\|_{L_t^\infty L^2_{x'}} \qquad\qquad (B.26)\\
\lesssim\ & 2^j\|n\chi\cdot\not{\nabla} f_l\|_{L^2_{x'}L^1_t}+2^j\left\|\int_0^t(n(\not{\nabla}\chi+\chi\cdot(\overline{\epsilon}+n^{-1}\not{\nabla} n)\not{\nabla} f_l)\right\|_{L_t^\infty L^1_{x'}}\\
\lesssim\ & 2^j\|n\|_{L^\infty}\|\chi\|_{L^\infty_{x'}L^2_t}\|\not{\nabla} f_l\|_{L^2(\mathcal{H}_u)}+2^j\|n(\not{\nabla}\chi+\chi\cdot(\overline{\epsilon}+n^{-1}\not{\nabla} n)\not{\nabla} f_l\|_{L^1(\mathcal{H}_u)}\\
\lesssim\ & 2^j\varepsilon\|\not{\nabla} f_l\|_{L^2(\mathcal{H}_u)}+2^j\|n\|_{L^\infty}(\|\not{\nabla}\chi\|_{L^2(\mathcal{H}_u)}+\mathcal{N}_1(\chi)(\mathcal{N}_1(\overline{\epsilon})+\mathcal{N}_1(n^{-1}\not{\nabla} n))\|\not{\nabla} f_l\|_{L^2(\mathcal{H}_u)}\\
\lesssim\ & 2^j\varepsilon\|\not{\nabla} f_l\|_{L^2(\mathcal{H}_u)},
\end{aligned}$$

where we used the estimate (3.66) for transport equations, and the estimates[23] (2.67)-(2.71) for n,χ, and $\overline{\epsilon}$.

In view of (B.26), we need to estimate $\|\not{\nabla} f_l\|_{L^2(\mathcal{H}_u)}$. In view of the transport equation (B.22) satisfied by f_l, we have:

$$nL(\not{\nabla} f_l)=[nL,\not{\nabla}]f_l,\ \not{\nabla} f_l=\not{\nabla} P_l f_0 \text{ on } P_{0,u}.$$

Together with the estimate (3.66) and the commutator formula (2.48), this yields:

$$\begin{aligned}
\|\not{\nabla} f_l\|_{L^2(\mathcal{H}_u)} &\lesssim \|\not{\nabla} P_l f_0\|_{L^2(P_{0,u})}+\|[nL,\not{\nabla}]f_l\|_{L^2_{x'}L^1_t}\\
&\lesssim 2^l\|P_l f_0\|_{L^2(P_{0,u})}+\|n\chi\cdot\not{\nabla} f_l\|_{L^2_{x'}L^1_t}\\
&\lesssim 2^{\frac{3l}{2}}C+\|n\|_{L^\infty}\|\chi\|_{L^\infty_{x'}L^2_t}\|\not{\nabla} f_l\|_{L^2(\mathcal{H}_u)}\\
&\lesssim 2^{\frac{3l}{2}}C+\varepsilon\|\not{\nabla} f_l\|_{L^2(\mathcal{H}_u)},
\end{aligned}$$

where we used the finite band property for P_l, the assumption on f_0, and the estimates (2.67)-(2.71) for n and χ. Since ε is small, we obtain:

$$\|\not{\nabla} f_l\|_{L^2(\mathcal{H}_u)}\lesssim 2^{\frac{3l}{2}}C. \qquad (B.27)$$

Finally, (B.26) and (B.27) imply:

$$\left\| P_j\left(\int_0^t([nL,\not{\Delta}]f)\right)\right\|_{L_t^\infty L^2_{x'}}\lesssim 2^{j+\frac{3l}{2}}C\varepsilon,$$

which together with (B.25) yields:

$$\|P_j(f_l)\|_{L_t^\infty L^2_{x'}}\lesssim 2^{2j-\frac{3l}{2}}C+2^{j-\frac{l}{2}}C\varepsilon. \qquad (B.28)$$

[23]For n, note

$$\begin{aligned}
\mathcal{N}_1(n^{-1}\not{\nabla} n) &\lesssim \|n^{-1}\not{\nabla} n\|_{L_t^\infty L^2_{x'}}+\|\not{\nabla}(n^{-1}\not{\nabla} n)\|_{L_t^\infty L^2_{x'}}+\|\not{\nabla}_L(n^{-1}\not{\nabla} n)\|_{L_t^\infty L^2_{x'}}\\
&\lesssim \|\nabla n\|_{L_t^\infty L^2_{x'}}+\|\nabla^2 n\|_{L_t^\infty L^2_{x'}}+\|\nabla D_T n\|_{L_t^\infty L^2_{x'}}\\
&\lesssim \varepsilon.
\end{aligned}$$

Now, using (B.21), and summing (B.23) for $l \leq j$ and (B.28) for $l > j$, we obtain:

$$\|P_j f\|_{L^\infty_t L^2_{x'}} \lesssim \sum_l \|P_j(f_l)\|_{L^\infty_t L^2_{x'}} \lesssim C 2^{\frac{j}{2}}$$

which yields the conclusion of Lemma 5.11.

B.7 Proof of Lemma 5.12

We decompose $\|P_j \left(\int_0^t (f\mu_1) d\tau\right)\|_{L^2(\mathcal{H}_u)}$ using the property (3.16) of the geometric Littlewood-Paley projections:

$$\begin{aligned} & \left\| P_j \left(\int_0^t (f\mu_1) d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \\ \lesssim & \left\| P_j \left(\int_0^t (f P_{<0}\mu_1) d\tau \right) \right\|_{L^2(\mathcal{H}_u)} + \sum_{l \geq 0} \left\| P_j \left(\int_0^t (f P_l \mu_1) d\tau \right) \right\|_{L^2(\mathcal{H}_u)}. \end{aligned} \tag{B.29}$$

We focus on the second term in the right-hand side of (B.29), the other being easier to handle. Using the L^2 boundedness of the Littlewood-Paley projection P_j and the estimate for transport equations (3.66), we have:

$$\begin{aligned} \left\| P_j \left(\int_0^t (f P_l \mu_1) d\tau \right) \right\|_{L^2(\mathcal{H}_u)} & \lesssim \left\| \int_0^t (f P_l \mu_1) d\tau \right\|_{L^2(\mathcal{H}_u)} \\ & \lesssim \|f P_l \mu_1\|_{L^2_{x'} L^1_t} \\ & \lesssim \|f\|_{L^\infty_{x'} L^2_t} \|P_l \mu_1\|_{L^2(\mathcal{H}_u)} \\ & \lesssim D\varepsilon^2 2^l + D\varepsilon^2 2^{\frac{l}{2}} \gamma(u), \end{aligned} \tag{B.30}$$

where we used the estimate (5.93) for μ_1 and the assumption of Lemma 5.12 for f.

We now make another decomposition using the property (3.16) of the geometric Littlewood-Paley projections:

$$\begin{aligned} & \left\| P_j \left(\int_0^t (f P_l \mu_1) d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \\ \lesssim & \left\| P_j \left(\int_0^t (P_{<0}(f) P_l(\mu_1)) d\tau \right) \right\|_{L^2(\mathcal{H}_u)} + \sum_{q \geq 0} \left\| P_j \left(\int_0^t (P_q(f) P_l(\mu_1)) d\tau \right) \right\|_{L^2(\mathcal{H}_u)}. \end{aligned} \tag{B.31}$$

We focus on the second term in the right-hand side of (B.31), the other being easier to handle. Using the dual of the sharp Bernstein inequality (4.32) and the estimate for transport equations (3.66), we have:

$$\begin{aligned} \left\| P_j \left(\int_0^t (P_q(f) P_l(\mu_1)) d\tau \right) \right\|_{L^2(\mathcal{H}_u)} & \lesssim 2^j \left\| \int_0^t (P_q(f) P_l(\mu_1)) d\tau \right\|_{L^2_t L^1_{x'}} \\ & \lesssim 2^j \|P_q(f) P_l(\mu_1)\|_{L^1(\mathcal{H}_u)} \\ & \lesssim 2^j \|P_q f\|_{L^2(\mathcal{H}_u)} \|P_l f\|_{L^2(\mathcal{H}_u)} \\ & \lesssim 2^j (\varepsilon 2^l + \varepsilon 2^{\frac{l}{2}} \gamma(u)) \|P_q f\|_{L^2(\mathcal{H}_u)} \end{aligned} \tag{B.32}$$

where we used the estimate (5.93) for μ_1 in the last inequality.

We now derive a second estimate. Using the property of the Littlewood-Paley projection P_l, we have:

$$\begin{aligned}
&\left\|P_j\left(\int_0^t (P_q(f)P_l(\mu_1))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}\\
\lesssim\ & 2^{-2l}\left\|P_j\left(\int_0^t (P_q(f)\not{\Delta}P_l(\mu_1))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}\\
\lesssim\ & 2^{-2l}\left\|P_j\left(\int_0^t \not{\Delta}(P_q(f)P_l(\mu_1))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} + 2^{-2l}\left\|P_j\left(\int_0^t \not{\mathrm{div}}(\not{\nabla}(P_q(f))P_l(\mu_1))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}\\
&+2^{-2l}\left\|P_j\left(\int_0^t (\not{\Delta}(P_q(f))P_l(\mu_1))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}
\end{aligned}$$

which together with Lemma 5.15, Lemma 5.16, the dual of the sharp Bernstein inequality (4.32) for P_j, and the estimate for transport equations (3.66) implies:

$$\begin{aligned}
&\left\|P_j\left(\int_0^t (P_q(f)P_l(\mu_1))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \qquad\qquad (B.33)\\
\lesssim\ & 2^{2j-2l}\|P_q(f)P_l\mu_1\|_{L^2_{x'}L^1_t} + 2^{\frac{3j}{2}-2l}\|\not{\nabla}(P_q(f))P_l\mu_1\|_{L^{\frac{4}{3}}_{x'}L^1_t} + 2^{j-2l}\|\not{\Delta}(P_q(f))P_l\mu_1\|_{L^1(\mathcal{H}_u)}\\
\lesssim\ & 2^{2j-2l}\|P_q(f)\|_{L^2_tL^\infty_{x'}}\|P_l\mu_1\|_{L^2(\mathcal{H}_u)} + 2^{\frac{3j}{2}-2l}\|\not{\nabla}(P_q(f))\|_{L^2(\mathcal{H}_u)}\|P_l\mu_1\|_{L^2_tL^4_{x'}}\\
&+2^{j-2l}\|\not{\Delta}(P_q(f))\|_{L^2(\mathcal{H}_u)}\|P_l\mu_1\|_{L^2(\mathcal{H}_u)}\\
\lesssim\ & 2^{2j+q-2l}\|P_q(f)\|_{L^2(\mathcal{H}_u)}\|P_l\mu_1\|_{L^2(\mathcal{H}_u)} + 2^{\frac{3j}{2}+q-\frac{3l}{2}}\|P_q(f)\|_{L^2(\mathcal{H}_u)}\|P_l\mu_1\|_{L^2(\mathcal{H}_u)}\\
&+2^{j+2q-2l}\|P_q(f)\|_{L^2(\mathcal{H}_u)}\|P_l\mu_1\|_{L^2(\mathcal{H}_u)}\\
\lesssim\ & (2^{2j+q-2l} + 2^{\frac{3j}{2}+q-\frac{3l}{2}} + 2^{j+2q-2l})(\varepsilon 2^l + \varepsilon 2^{\frac{l}{2}}\gamma(u))\|P_q(f)\|_{L^2(\mathcal{H}_u)},
\end{aligned}$$

where we have used the weak Bernstein inequality (see Theorem 3.10) for P_q and P_l, the sharp Bernstein inequality (4.32) for P_q, and the estimate (5.93) for μ_1.

Then, using (B.32) for $q > l$ and (B.33) for $q \le l$, we obtain:

$$\begin{aligned}
&\sum_{q\ge 0}\left\|P_j\left(\int_0^t (P_q(f)P_l(\mu_1))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}\\
\lesssim\ & (2^{2j-2l} + 2^{\frac{3j}{2}-\frac{3l}{2}})(\varepsilon 2^l + \varepsilon 2^{\frac{l}{2}}\gamma(u))\left(\sum_{q\ge 0} 2^q\|P_q(f)\|_{L^2(\mathcal{H}_u)}\right)\\
&+2^j(D\varepsilon + D\varepsilon 2^{-\frac{l}{2}}\gamma(u))\left(\sum_{q\ge 0} 2^{-|q-l|}2^q\|P_q(f)\|_{L^2(\mathcal{H}_u)}\right)\\
\lesssim\ & (2^{2j-2l} + 2^{\frac{3j}{2}-\frac{3l}{2}})(D\varepsilon 2^l + D\varepsilon 2^{\frac{l}{2}}\gamma(u))\varepsilon\\
&+2^j(D\varepsilon + D\varepsilon 2^{-\frac{l}{2}}\gamma(u))\left(\sum_{q\ge 0} 2^{-|q-l|}2^q\|P_q(f)\|_{L^2(\mathcal{H}_u)}\right),
\end{aligned}$$

where we used the bound on $\|\nabla\!\!\!/ f\|_{\mathcal{P}^0}$ given by the assumptions of Lemma 5.12 in the last inequality. Together with (B.31), we obtain:

$$\begin{aligned}\left\|P_j\left(\int_0^t (fP_l\mu_1)d\tau\right)\right\|_{L^2(\mathcal{H}_u)} &\lesssim (2^{2j-2l}+2^{\frac{3j}{2}-\frac{3l}{2}})(D\varepsilon 2^l + D\varepsilon 2^{\frac{l}{2}}\gamma(u))\varepsilon \\ &\quad +2^j(D\varepsilon + D\varepsilon 2^{-\frac{l}{2}}\gamma(u))\left(\sum_{q\geq 0} 2^{-|q-l|}2^q\|P_q(f)\|_{L^2(\mathcal{H}_u)}\right).\end{aligned} \tag{B.34}$$

Finally, using (B.29), (B.30) for $l \leq j$ and (B.34) for $l > j$, we get:

$$\begin{aligned}\left\|P_j\left(\int_0^t (f\mu_1)d\tau\right)\right\|_{L^2(\mathcal{H}_u)} &\lesssim (D\varepsilon 2^j + D\varepsilon 2^{\frac{j}{2}}\gamma(u))\varepsilon \\ &\quad +(D\varepsilon 2^j + D\varepsilon 2^{\frac{j}{2}}\gamma(u))\left(\sum_{l,q\geq 0} 2^{-|q-l|}2^q\|P_q(f)\|_{L^2(\mathcal{H}_u)}\right) \\ &\lesssim (D\varepsilon 2^j + D\varepsilon 2^{\frac{j}{2}}\gamma(u))(\varepsilon + \|\nabla\!\!\!/ f\|_{\mathcal{B}^0}) \\ &\lesssim (D\varepsilon^2 2^j + D\varepsilon^2 2^{\frac{j}{2}}\gamma(u)),\end{aligned} \tag{B.35}$$

where we used the bound on $\|\nabla\!\!\!/ f\|_{\mathcal{P}^0}$ given by the assumptions of Lemma 5.12 in the last inequality. This concludes the proof of Lemma 5.12.

B.8 Proof of Lemma 5.13

We have:

$$\begin{aligned}&\left\|P_j\left(\int_0^t (F\cdot\nabla\!\!\!/\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \\ \lesssim\ &\left\|P_j\left(\int_0^t \mathrm{di}\!\!\!/\mathrm{v}(F\cdot\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} + \left\|P_j\left(\int_0^t (\nabla\!\!\!/ F\cdot\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \\ \lesssim\ &\left\|P_j\left(\int_0^t \mathrm{di}\!\!\!/\mathrm{v}(F\cdot\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} + \left\|P_j\left(\int_0^t (\nabla\!\!\!/_{nL}(P)\cdot\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \\ &+\left\|P_j\left(\int_0^t (E\cdot\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \\ \lesssim\ &\left\|P_j\left(\int_0^t \mathrm{di}\!\!\!/\mathrm{v}(F\cdot\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} + \|P_j(P\cdot\nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)} \\ &+\left\|P_j\left(\int_0^t (P\cdot\nabla\!\!\!/_{nL}\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} + \left\|P_j\left(\int_0^t (E\cdot\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)},\end{aligned} \tag{B.36}$$

where we used the assumption of Lemma 5.13 $\nabla\!\!\!/ F = \nabla\!\!\!/_{nL}(P) + E$, and then where we integrated by part in t. Since $\|E\|_{\mathcal{P}^0} \lesssim \varepsilon$ and $\nabla\!\!\!/_{\underline{L}}(\zeta)$ satisfies (5.92), the fourth term in the right-hand side of (B.36) is estimated using Lemma 5.14:

$$\left\|P_j\left(\int_0^t (E\cdot\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \lesssim D\varepsilon^2 2^j + D\varepsilon^2 2^{\frac{j}{2}}\gamma(u). \tag{B.37}$$

Next, we estimate the first, the second and the third term in the right-hand side of (B.36).

B.8.1 Estimate of the first term in the right-hand side of (B.36)

We decompose
$\|P_j\left(\int_0^t \mathrm{div}\!\!\!/(F\cdot \nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\|_{L^2(\mathcal{H}_u)}$ using the property (3.16) of the geometric Littlewood-Paley projections:

$$\begin{aligned}\left\|P_j\left(\int_0^t \mathrm{div}\!\!\!/(F\cdot \nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \lesssim & \left\|P_j\left(\int_0^t \mathrm{div}\!\!\!/(F\cdot P_{<0}\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \\ & +\sum_{l\geq 0}\left\|P_j\left(\int_0^t \mathrm{div}\!\!\!/(F\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}.\end{aligned} \tag{B.38}$$

We focus on the second term in the right-hand side of (B.38), the other being easier to handle[24]. Using the L^2 boundedness of the Littlewood-Paley projection P_j, the weak Bernstein inequality for P_j, and the estimate for transport equations (3.66), we have:

$$\begin{aligned}& \left\|P_j\left(\int_0^t \mathrm{div}\!\!\!/(F\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \\ \lesssim & \left\|P_j\left(\int_0^t (\nabla\!\!\!/F\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}+\left\|P_j\left(\int_0^t (F\cdot \nabla\!\!\!/P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \\ \lesssim & 2^{\frac{j}{3}}\left\|\int_0^t (\nabla\!\!\!/F\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right\|_{L^2_tL^{\frac{3}{2}}_{x'}}+\left\|\int_0^t (F\cdot \nabla\!\!\!/P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right\|_{L^2(\mathcal{H}_u)} \\ \lesssim & 2^{\frac{j}{3}}\|\nabla\!\!\!/F\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^{\frac{3}{2}}_{x'}L^1_t}+\|F\cdot \nabla\!\!\!/P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2_{x'}L^1_t} \\ \lesssim & 2^{\frac{j}{3}}\|\nabla\!\!\!/F\|_{L^2(\mathcal{H}_u)}\|P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2_tL^6_{x'}}+\|F\|_{L^\infty_{x'}L^2_t}\|\nabla\!\!\!/P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(\mathcal{H}_u)} \\ \lesssim & 2^{\frac{j}{3}+\frac{2l}{3}}(D\varepsilon^2+D\varepsilon^2 2^{-\frac{l}{2}}\gamma(u))+D\varepsilon^2 2^l+D\varepsilon^2 2^{\frac{l}{2}}\gamma(u),\end{aligned} \tag{B.39}$$

where we used the finite band property for P_l, the weak Bernstein inequality for P_l, the estimate (5.92) for $\nabla\!\!\!/_{\underline{L}}(\zeta)$ and the assumption of Lemma 5.13 for F.

We will need another estimate for $\|P_j\left(\int_0^t \mathrm{div}\!\!\!/(F\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\|_{L^2(\mathcal{H}_u)}$. We decompose $\|P_j\left(\int_0^t \mathrm{div}\!\!\!/(F\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\|_{L^2(\mathcal{H}_u)}$ using the property (3.16) of the geometric Littlewood-

[24]The terms involving the projection on low frequencies $P_{<0}$ are always significantly easier to estimate. In the present case, the first term in the right-hand side of (B.38) can be estimated exactly as (B.39) with $l=0$, i.e. we obtain

$$\left\|P_j\left(\int_0^t \mathrm{div}\!\!\!/(F\cdot P_{<0}\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \lesssim 2^{\frac{j}{3}}(D\varepsilon^2+D\varepsilon^2\gamma(u)),$$

which immediately yields the desired conclusion (B.55) for this term (with $2^{\frac{2j}{3}}$ room) without having to go through (B.40)-(B.54). In the sequel, we will consistently leave to the reader this type of terms.

Paley projections:

$$\begin{aligned}\left\|P_j\left(\int_0^t \mathrm{d}\!\!\!/\mathrm{iv}(F\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \lesssim\ & \left\|P_j\left(\int_0^t \mathrm{d}\!\!\!/\mathrm{iv}(P_{\leq l}(F)\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \\ &+\sum_{q>l}\left\|P_j\left(\int_0^t \mathrm{d}\!\!\!/\mathrm{iv}(P_q(F)\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}.\end{aligned} \tag{B.40}$$

We first estimate the second term in the right-hand side of (B.40). Using Lemma 5.16 with $p=\frac{4}{3}$, we have:

$$\begin{aligned}\left\|P_j\left(\int_0^t \mathrm{d}\!\!\!/\mathrm{iv}(P_q(F)\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \lesssim\ & 2^{\frac{3j}{2}}\|P_q(F)\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^{\frac{4}{3}}_{x'}L^1_t} \\ \lesssim\ & 2^{\frac{3j}{2}}\|P_q(F)\|_{L^2(\mathcal{H}_u)}\|P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2_tL^4_{x'}} \\ \lesssim\ & 2^{\frac{3j}{2}}2^{-q}\|\nabla\!\!\!/ F\|_{L^2(\mathcal{H}_u)}2^{\frac{l}{2}}\|P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(\mathcal{H}_u)} \\ \lesssim\ & 2^{\frac{3j}{2}-q+\frac{l}{2}}\varepsilon(D\varepsilon+D\varepsilon 2^{-\frac{l}{2}}\gamma(u)),\end{aligned}$$

where we used the finite band property for P_q, the weak Bernstein inequality for P_l, the assumption on F and the estimate (5.92) for $\nabla\!\!\!/_{\underline{L}}(\zeta)$. This yields the following estimate for the the second term in the right-hand side of (B.40):

$$\sum_{q>l}\left\|P_j\left(\int_0^t \mathrm{d}\!\!\!/\mathrm{iv}(P_q(F)\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \lesssim 2^{\frac{3j}{2}-\frac{l}{2}}\varepsilon(D\varepsilon+D\varepsilon 2^{-\frac{l}{2}}\gamma(u)). \tag{B.41}$$

We now estimate the first term in the right-hand side of (B.40). Using the property of the Littlewood-Paley projection P_l, we have:

$$\begin{aligned}&\left\|P_j\left(\int_0^t \mathrm{d}\!\!\!/\mathrm{iv}(P_{\leq l}(F)\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \\ \lesssim\ & 2^{-2l}\left\|P_j\left(\int_0^t \mathrm{d}\!\!\!/\mathrm{iv}(P_{\leq l}(F)\cdot \Delta\!\!\!/ P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \\ \lesssim\ & 2^{-2l}\left\|P_j\left(\int_0^t \mathrm{d}\!\!\!/\mathrm{iv}\,\mathrm{d}\!\!\!/\mathrm{iv}(P_{\leq l}(F)\cdot \nabla\!\!\!/ P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \\ &+2^{-2l}\left\|P_j\left(\int_0^t \mathrm{d}\!\!\!/\mathrm{iv}(\nabla\!\!\!/ P_{\leq l}(F)\cdot \nabla\!\!\!/ P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}\end{aligned}$$

which together with Lemma 5.15 and Lemma 5.16 with $p=\frac{4}{3}$ yields:

$$\begin{aligned}&\left\|P_j\left(\int_0^t \mathrm{d}\!\!\!/\mathrm{iv}(P_{\leq l}(F)\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \\ \lesssim\ & 2^{2j-2l}\|P_{\leq l}(F)\cdot \nabla\!\!\!/ P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2_{x'}L^1_t}+2^{\frac{3j}{2}-2l}\|\nabla\!\!\!/ P_{\leq l}(F)\cdot \nabla\!\!\!/ P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^{\frac{4}{3}}_{x'}L^1_t}.\end{aligned} \tag{B.42}$$

Using the fact that $P_{\leq l}(F) = F - P_{>l}(F)$, we estimate the first term in the right-hand side of (B.42) as follows:

$$\begin{aligned} & \|P_{\leq l}(F) \cdot \nabla\!\!\!/ P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2_{x'}L^1_t} \qquad \text{(B.43)} \\ \lesssim\ & \|F \cdot \nabla\!\!\!/ P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2_{x'}L^1_t} + \sum_{q>l} \|P_q(F) \cdot \nabla\!\!\!/ P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2_{x'}L^1_t} \\ \lesssim\ & \|F\|_{L^\infty_{x'}L^2_t} \|\nabla\!\!\!/ P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(\mathcal{H}_u)} + \sum_{q>l} \Big\| \|P_q(F)\|_{L^4(P_{t,u})} \|\nabla\!\!\!/ P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^4(P_{t,u})} \Big\|_{L^1_t} \\ \lesssim\ & \varepsilon 2^l (D\varepsilon + 2^{\frac{l}{2}} D\varepsilon\gamma(u)) + \sum_{q>l} \Big\| \|P_q(F)\|_{L^4(P_{t,u})} \|\nabla\!\!\!/ P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^4(P_{t,u})} \Big\|_{L^1_t}, \end{aligned}$$

where we used the finite band property for P_l, the assumption on F and the estimate (5.92) for $\nabla\!\!\!/_{\underline{L}}(\zeta)$. We consider the second term in the right-hand side of (B.43). The Gagliardo-Nirenberg inequality (3.3), the Bochner inequality for tensors (3.7), and the weak Bernstein inequality for P_l yield:

$$\begin{aligned} & \|\nabla\!\!\!/ P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^4(P_{t,u})} \qquad \text{(B.44)} \\ \lesssim\ & \|\nabla\!\!\!/^2 P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|^{\frac{1}{2}}_{L^2(P_{t,u})} \|\nabla\!\!\!/ P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|^{\frac{1}{2}}_{L^2(P_{t,u})} + \|\nabla\!\!\!/ P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(P_{t,u})} \\ \lesssim\ & \Big(\|\Delta\!\!\!/ P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(P_{t,u})} \\ & + \|K\|^{\frac{3}{2}}_{L^2(P_{t,u})} \|\nabla\!\!\!/ P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|^{\frac{1}{2}}_{L^2(P_{t,u})} \|P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|^{\frac{1}{2}}_{L^2(P_{t,u})} \\ & + \|P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(P_{t,u})} \Big)^{\frac{1}{2}} 2^{\frac{l}{2}} \|P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|^{\frac{1}{2}}_{L^2(P_{t,u})} + 2^l \|P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(P_{t,u})} \\ \lesssim\ & (2^{2l} + 2^{\frac{l}{2}} \|K\|^{\frac{3}{2}}_{L^2(P_{t,u})})^{\frac{1}{2}} 2^{\frac{l}{2}} \|P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(P_{t,u})}. \end{aligned}$$

Now, (B.44), the weak Bernstein inequality for P_q, the finite band property for P_q, and Lemma 5.9 imply:

$$\begin{aligned} & \Big\| \|P_q(F)\|_{L^4(P_{t,u})} \|\nabla\!\!\!/ P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^4(P_{t,u})} \Big\|_{L^1_t} \qquad \text{(B.45)} \\ \lesssim\ & \Big\| \|P_q(F)\|_{L^4(P_{t,u})} (2^{2l} + 2^{\frac{l}{2}} \|K\|^{\frac{3}{2}}_{L^2(P_{t,u})})^{\frac{1}{2}} 2^{\frac{l}{2}} \|P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(P_{t,u})} \Big\|_{L^1_t} \\ \lesssim\ & \|P_q(F)\|_{L^2_t L^4_{x'}} 2^{\frac{3l}{2}} \|P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(\mathcal{H}_u)} + \|P_q(F)\|_{L^8_t L^4_{x'}} 2^{\frac{3l}{4}} \|K\|^{\frac{3}{4}}_{L^2(\mathcal{H}_u)} \|P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(\mathcal{H}_u)} \\ \lesssim\ & 2^{-\frac{q}{2}+\frac{3l}{2}} \|\nabla\!\!\!/ F\|_{L^2(\mathcal{H}_u)} (D\varepsilon + D\varepsilon 2^{-\frac{l}{2}}\gamma(u)) + 2^{-\frac{q}{8}+\frac{3l}{4}} \mathcal{N}_1(F) \varepsilon^{\frac{3}{4}} (D\varepsilon + D\varepsilon 2^{-\frac{l}{2}}\gamma(u)) \\ \lesssim\ & (2^{-\frac{q}{2}+\frac{3l}{2}} + 2^{\frac{3l}{4}-\frac{q}{8}})(D\varepsilon^2 + D\varepsilon^2 2^{-\frac{l}{2}}\gamma(u)) \end{aligned}$$

where we used the bound (4.29) for K, the assumptions on F and the estimate (5.92) for $\nabla\!\!\!/_{\underline{L}}(\zeta)$. (B.43) and (B.45) yield:

$$\begin{aligned} \|P_{\leq l}(F) \cdot \nabla\!\!\!/ P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2_{x'}L^1_t} &\lesssim \left(2^l + \sum_{q>l} (2^{-\frac{q}{2}+\frac{3l}{2}} + 2^{\frac{3l}{4}-\frac{q}{8}}) \right) (D\varepsilon^2 + D\varepsilon^2 2^{-\frac{l}{2}}\gamma(u)) \\ &\lesssim D\varepsilon^2 2^l + D\varepsilon^2 2^{\frac{l}{2}}\gamma(u). \qquad \text{(B.46)} \end{aligned}$$

Next, we estimate the second term in the right-hand side of (B.42):

$$\begin{aligned}\|\nabla\!\!\!/ P_{\le l}(F)\cdot\nabla\!\!\!/ P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^{\frac{4}{3}}_{x'}L^1_t} &\lesssim \|\nabla\!\!\!/ P_{\le l}(F)\|_{L^2_tL^4_{x'}}\|\nabla\!\!\!/ P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(\mathcal{H}_u)} \\ &\lesssim \left(\sum_{q\le l}\|\nabla\!\!\!/ P_q(F)\|_{L^2_tL^4_{x'}}\right)2^l(D\varepsilon+D\varepsilon 2^{-\frac{l}{2}}\gamma(u)),\end{aligned} \tag{B.47}$$

where we used the finite band property of P_l and the estimate (5.92) for $\nabla\!\!\!/_{\underline{L}}(\zeta)$ in the last inequality. We estimate $\|\nabla\!\!\!/ P_q(F)\|_{L^4(P_{t,u})}$ using the Gagliardo-Nirenberg inequality (3.3), the Bochner inequality for tensors (3.7), and the weak Bernstein inequality for P_q:

$$\begin{aligned}\|\nabla\!\!\!/ P_q(F)\|_{L^4(P_{t,u})} &\lesssim \|\nabla\!\!\!/^2P_q(F)\|^{\frac{1}{2}}_{L^2(P_{t,u})}\|\nabla\!\!\!/ P_q(F)\|^{\frac{1}{2}}_{L^2(P_{t,u})}+\|\nabla\!\!\!/ P_q(F)\|_{L^2(P_{t,u})} \\ &\lesssim (\|\Delta\!\!\!/ P_q(F)\|_{L^2(P_{t,u})}+\|K\|_{L^2(P_{t,u})}\|\nabla\!\!\!/ P_q(F)\|_{L^2(P_{t,u})} \\ &\quad +\|K\|^2_{L^2(P_{t,u})}\|P_q(F)\|_{L^2(P_{t,u})})^{\frac{1}{2}}2^{\frac{q}{2}}\|P_q(F)\|^{\frac{1}{2}}_{L^2(P_{t,u})}+2^q\|P_q(F)\|_{L^2(P_{t,u})} \\ &\lesssim (2^{2q}+\|K\|^2_{L^2(P_{t,u})})^{\frac{1}{2}}2^{\frac{q}{2}}\|P_q(F)\|_{L^2(P_{t,u})}\end{aligned}$$

which together with the finite band property for P_q, and Lemma 5.9 implies:

$$\begin{aligned}\|\nabla\!\!\!/ P_q(F)\|_{L^2_tL^4_{x'}} &\lesssim 2^{\frac{3q}{2}}\|P_q(F)\|_{L^2(\mathcal{H}_u)}+\|K\|_{L^2(P_{t,u})}2^{\frac{q}{2}}\|P_q(F)\|_{L^\infty_tL^2_{x'}} \\ &\lesssim 2^{\frac{q}{2}}\|\nabla\!\!\!/ F\|_{L^2(\mathcal{H}_u)}+\varepsilon\mathcal{N}_1(F) \\ &\lesssim 2^{\frac{q}{2}}\varepsilon\end{aligned} \tag{B.48}$$

where we used the bound (4.29) for K and the assumptions on F. (B.47) and (B.48) yield:

$$\begin{aligned}\|\nabla\!\!\!/ P_{\le l}(F)\cdot\nabla\!\!\!/ P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^{\frac{4}{3}}_{x'}L^1_t} &\lesssim \left(\sum_{q\le l}2^{\frac{q}{2}}\varepsilon\right)2^l(D\varepsilon+D\varepsilon 2^{-\frac{l}{2}}\gamma(u)) \\ &\lesssim 2^{\frac{3l}{2}}(D\varepsilon^2+D\varepsilon^2 2^{-\frac{l}{2}}\gamma(u)).\end{aligned} \tag{B.49}$$

Finally, (B.42), (B.46) and (B.49) imply:

$$\left\|P_j\left(\int_0^t \mathrm{d}\!\!\!/\mathrm{iv}(P_{\le l}(F)\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}\lesssim(2^{2j-l}+2^{\frac{3j}{2}-\frac{l}{2}})(D\varepsilon^2+D\varepsilon^2 2^{-\frac{l}{2}}\gamma(u)). \tag{B.50}$$

Now, (B.40), (B.41) and (B.50) yield:

$$\left\|P_j\left(\int_0^t \mathrm{d}\!\!\!/\mathrm{iv}(F\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}\lesssim(2^{2j-l}+2^{\frac{3j}{2}-\frac{l}{2}})(D\varepsilon^2+D\varepsilon^2 2^{-\frac{l}{2}}\gamma(u)). \tag{B.51}$$

Using (B.38), (B.39) for $l\le j$ and (B.51) for $l>j$, we obtain:

$$\left\|P_j\left(\int_0^t \mathrm{d}\!\!\!/\mathrm{iv}(F\cdot\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}\lesssim D\varepsilon^2 2^j+D\varepsilon^2 2^{\frac{j}{2}}\gamma(u), \tag{B.52}$$

which is the desired estimate of the first term in the right-hand side of (B.36).

B.8.2 Estimate of the second term in the right-hand side of (B.36)

We decompose $\|P_j(P\cdot\nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)}$ using the property (3.16) of the geometric Littlewood-Paley projections:

$$\|P_j(P\cdot\nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)}\lesssim\|P_j(P\cdot P_{<0}\nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)}+\sum_{l\geq 0}\|P_j(P\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)}. \quad \text{(B.53)}$$

We focus on the second term in the right-hand side of (B.53), the other being easier to handle. Using the weak Bernstein inequality for P_j, we have:

$$\begin{aligned}\|P_j(P\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)} &\lesssim 2^{\frac{j}{2}}\|P\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2_tL^{\frac{4}{3}}_{x'}} \quad \text{(B.54)}\\ &\lesssim 2^{\frac{j}{2}}\|P\|_{L^\infty_tL^4_{x'}}\|P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(\mathcal{H}_u)}\\ &\lesssim 2^{\frac{j}{2}}\mathcal{N}_1(P)(D\varepsilon+D\varepsilon 2^{-\frac{l}{2}}\gamma(u))\\ &\lesssim 2^{\frac{j}{2}}(D\varepsilon^2+D\varepsilon^2 2^{-\frac{l}{2}}\gamma(u)),\end{aligned}$$

where we used the assumption of Lemma 5.13 for P and the estimate (5.92) for $\nabla\!\!\!/_{\underline{L}}(\zeta)$.

We will need another estimate for $\|P_j(P\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)}$. We decompose $\|P_j(P\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)}$ using the property (3.16) of the geometric Littlewood-Paley projections:

$$\|P_j(P\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)}\lesssim\|P_j(P_{\leq l}P\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)}+\sum_{q>l}\|P_j(P_q(P)\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)}. \quad \text{(B.55)}$$

We first estimate the second term in the right-hand side of (B.55). Using the dual of the sharp Bernstein inequality (4.32), we have:

$$\begin{aligned}\|P_j(P_q(P)\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)} &\lesssim 2^j\|P_q(P)\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2_tL^1_{x'}}\\ &\lesssim 2^j\|P_q(P)\|_{L^\infty_tL^2_{x'}}\|P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(\mathcal{H}_u)}\\ &\lesssim 2^{j-\frac{q}{2}}\mathcal{N}_1(P)(D\varepsilon+D\varepsilon 2^{-\frac{l}{2}}\gamma(u))\\ &\lesssim 2^{j-\frac{q}{2}}(D\varepsilon^2+D\varepsilon^2 2^{-\frac{l}{2}}\gamma(u)),\end{aligned}$$

where we used Lemma 5.9, the assumption of Lemma 5.13 on P and the estimate (5.92) for $\nabla\!\!\!/_{\underline{L}}(\zeta)$. This yields the following estimate for the second term in the right-hand side of (B.55):

$$\sum_{q>l}\|P_j(P_q(P)\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)}\lesssim 2^{j-\frac{l}{2}}(D\varepsilon^2+D\varepsilon^2 2^{-\frac{l}{2}}\gamma(u)). \quad \text{(B.56)}$$

We now estimate the first term in the right-hand side of (B.55). Using the property of the Littlewood-Paley projection P_l, we have:

$$\begin{aligned}&\|P_j(P_{\leq l}P\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)}\\ \lesssim\ & 2^{-2l}\|P_j(P_{\leq l}P\cdot\Delta\!\!\!/P_l\nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)}\\ \lesssim\ & 2^{-2l}\|P_j\mathrm{div}\!\!\!/(P_{\leq l}P\cdot\nabla\!\!\!/P_l\nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)}+2^{-2l}\|P_j(\nabla\!\!\!/P_{\leq l}P\cdot\nabla\!\!\!/P_l\nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)}\end{aligned}$$

which together with the property (4.36) of P_j with $p = \frac{4}{3}$ and the dual of the sharp Bernstein inequality (4.32) yields:

$$\begin{aligned}
&\|P_j(P_{\leq l}P \cdot P_l \nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)} \qquad \text{(B.57)}\\
\lesssim\ & 2^{\frac{3j}{2}-2l}\|P_{\leq l}P \cdot \nabla\!\!\!/ P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2_t L^{\frac{4}{3}}_{x'}} + 2^{j-2l}\|\nabla\!\!\!/ P_{\leq l}P \cdot \nabla\!\!\!/ P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2_t L^1_{x'}}\\
\lesssim\ & 2^{\frac{3j}{2}-2l}\|P_{\leq l}P\|_{L^\infty_t L^4_{x'}}\|\nabla\!\!\!/ P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(\mathcal{H}_u)} + 2^{j-2l}\|\nabla\!\!\!/ P_{\leq l}P\|_{L^\infty_t L^2_{x'}}\|\nabla\!\!\!/ P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(\mathcal{H}_u)}\\
\lesssim\ & 2^{\frac{3j}{2}-2l}\|P\|_{L^\infty_t L^4_{x'}} 2^l \|P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(\mathcal{H}_u)} + 2^{j-2l}\left(\sum_{q\leq l} 2^q \|P_q P\|_{L^\infty_t L^2_{x'}}\right) 2^l \|P_l \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(\mathcal{H}_u)}\\
\lesssim\ & 2^{\frac{3j}{2}-l}\mathcal{N}_1(P)(D\varepsilon + D\varepsilon 2^{-\frac{l}{2}}\gamma(u)) + 2^{j-l}\left(\sum_{q\leq l} 2^{\frac{q}{2}}\mathcal{N}_1(P)\right)(D\varepsilon + D\varepsilon 2^{-\frac{l}{2}}\gamma(u))\\
\lesssim\ & (2^{\frac{3j}{2}-l} + 2^{j-\frac{l}{2}})(D\varepsilon^2 + D\varepsilon^2 2^{-\frac{l}{2}}\gamma(u))
\end{aligned}$$

where we used the finite band property of P_l and P_q, the embedding (3.58), Lemma 5.9, the estimate (5.92) for $\nabla\!\!\!/_{\underline{L}}(\zeta)$, and the assumption of Lemma 5.13 for P.

Finally, (B.55), (B.56) and (B.57) imply:

$$\|P_j(P \cdot P_l \nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)} \lesssim (2^{\frac{3j}{2}-l} + 2^{j-\frac{l}{2}})(D\varepsilon^2 + D\varepsilon^2 2^{-\frac{l}{2}}\gamma(u)). \qquad \text{(B.58)}$$

Using (B.53), (B.54) for $l \geq j$, and (B.58) for $l > j$, we obtain:

$$\|P_j(P \cdot \nabla\!\!\!/_{\underline{L}}(\zeta))\|_{L^2(\mathcal{H}_u)} \lesssim D\varepsilon^2 2^{\frac{j}{2}} + D\varepsilon^2 2^{\frac{j}{2}}\gamma(u)), \qquad \text{(B.59)}$$

which is the desired estimate of the second term in the right-hand side of (B.36).

B.8.3 Estimate of the third term in the right-hand side of (B.36)

We start by deriving an equation for $\nabla\!\!\!/_{nL}\nabla\!\!\!/_{\underline{L}}(\zeta)$. Differentiating the transport equation (2.30) satisfied by ζ with respect to $\underline{L}$, we obtain:

$$\nabla\!\!\!/_{\underline{L}}\nabla\!\!\!/_L \zeta_A = -(\overline{\epsilon}_B + \zeta_B)\nabla\!\!\!/_{\underline{L}}(\chi)_{AB} - (\nabla\!\!\!/_{\underline{L}}(\overline{\epsilon})_B + \nabla\!\!\!/_{\underline{L}}(\zeta)_B)\chi_{AB} - \nabla\!\!\!/_{\underline{L}}(\beta)_A$$

which together with the commutator formula (2.46) and the Bianchi identity (2.52) yields after multiplication by n:

$$\nabla\!\!\!/_{nL}\nabla\!\!\!/_{\underline{L}}\zeta = n(\overline{\delta} - \chi\cdot)\nabla\!\!\!/_{\underline{L}}(\zeta) + B - \nabla\!\!\!/(n\rho) - (\nabla\!\!\!/(n\sigma))^* \qquad \text{(B.60)}$$

where the 1-form B is given by:

$$\begin{aligned}
B \ =\ & -n(\delta + n^{-1}N(n))\nabla\!\!\!/_L(\zeta) - 2n(\zeta - \underline{\zeta})\cdot\nabla\!\!\!/\zeta + 2n(\underline{\zeta}\wedge\zeta + \in {}^*\sigma)\cdot\zeta\\
& -n(\zeta + \overline{\epsilon})\cdot\nabla\!\!\!/_{\underline{L}}(\chi) - n\nabla\!\!\!/_{\underline{L}}(\overline{\epsilon})\cdot\chi + n\mathrm{tr}\underline{\chi}\beta - 2n\widehat{\chi}\cdot\underline{\beta} - 2n(\delta + n^{-1}N(n))\beta - n\underline{\xi}\cdot\alpha\\
& -3n(\zeta\rho + {}^*\zeta\sigma) + \nabla\!\!\!/(n)\rho + \nabla\!\!\!/(n)\sigma.
\end{aligned}$$

We estimate the $L^2_t L^{\frac{4}{3}}_{x'}$ norm of B. We have:

$$\begin{aligned}
\|B\|_{L^2_t L^{\frac{4}{3}}_{x'}} \lesssim\ & \|n\|_{L^\infty}\Big(\|\delta + n^{-1}N(n)\|_{L^\infty_t L^4_{x'}}\|\nabla\!\!\!/_L(\zeta)\|_{L^2(\mathcal{H}_u)} + \|\zeta - \underline{\zeta}\|_{L^\infty_t L^4_{x'}}\|\nabla\!\!\!/\zeta\|_{L^2(\mathcal{H}_u)} \\
& + \|\underline{\zeta}\wedge\zeta + \in {}^*\sigma\|_{L^2(\mathcal{H}_u)}\|\zeta\|_{L^\infty_t L^4_{x'}} + \|\zeta + \overline{\epsilon}\|_{L^\infty_t L^4_{x'}}\|\nabla\!\!\!/_{\underline{L}}(\chi)\|_{L^2(\mathcal{H}_u)} \\
& + \|\nabla\!\!\!/_{\underline{L}}(\overline{\epsilon})\|_{L^2(\mathcal{H}_u)}\|\chi\|_{L^\infty_t L^4_{x'}} + \|\mathrm{tr}\underline{\chi}\|_{L^\infty_t L^4_{x'}}\|\beta\|_{L^2(\mathcal{H}_u)} + \|\widehat{\chi}\|_{L^\infty_t L^4_{x'}}\|\underline{\beta}\|_{L^2(\mathcal{H}_u)} \\
& + \|\delta + n^{-1}N(n)\|_{L^\infty_t L^4_{x'}}\|\beta\|_{L^2(\mathcal{H}_u)} + \|\underline{\xi}\|_{L^\infty_t L^4_{x'}}\|\alpha\|_{L^2(\mathcal{H}_u)} \\
& + \|\zeta\|_{L^\infty_t L^4_{x'}}(\|\rho\|_{L^2(\mathcal{H}_u)} + \|\sigma\|_{L^2(\mathcal{H}_u)}) + \|n^{-1}\nabla\!\!\!/ n\|_{L^\infty_t L^4_{x'}}(\|\rho\|_{L^2(\mathcal{H}_u)} + \|\sigma\|_{L^2(\mathcal{H}_u)})\Big) \qquad \text{(B.61)} \\
\lesssim\ & \|n\|_{L^\infty}\Big(\mathcal{N}_1(\zeta)^2 + \mathcal{N}_1(\delta)^2 + \mathcal{N}_1(\nabla n)^2 + \mathcal{N}_1(\underline{\zeta})^2 + \mathcal{N}_1(\overline{\epsilon})^2 + \mathcal{N}_1(\chi)^2 + \mathcal{N}_1(\underline{\xi})^2 \\
& + \|\nabla\!\!\!/_{\underline{L}}(\chi)\|^2_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/_{\underline{L}}(\overline{\epsilon})\|^2_{L^2(\mathcal{H}_u)} + \|\sigma\|^2_{L^2(\mathcal{H}_u)} + \|\rho\|^2_{L^2(\mathcal{H}_u)} + \|\beta\|^2_{L^2(\mathcal{H}_u)} \\
& + \|\alpha\|^2_{L^2(\mathcal{H}_u)} + \|\underline{\beta}\|^2_{L^2(\mathcal{H}_u)}\Big) \\
\lesssim\ & \varepsilon^2,
\end{aligned}$$

where we used the curvature bound (2.59) for $\alpha, \beta, \rho, \sigma$ and $\underline{\beta}$, and the estimates (2.67)-(2.72) for $n, \delta, \overline{\epsilon}, \underline{\zeta}, \chi, \underline{\xi}$ and ζ.

We have the following estimate for the third term in the right-hand side of (B.36):

$$\begin{aligned}
& \left\| P_j\left(\int_0^t (P\cdot\nabla\!\!\!/_{nL}\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \qquad \text{(B.62)} \\
\lesssim\ & \left\| P_j\left(\int_0^t (nP\cdot(\overline{\delta} - \chi\cdot)\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} + \left\| P_j\left(\int_0^t (P\cdot B)d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \\
& + \left\| P_j\left(\int_0^t (P\cdot(\nabla\!\!\!/(n\rho) + (\nabla\!\!\!/(n\sigma))^*)d\tau\right)\right\|_{L^2(\mathcal{H}_u)}
\end{aligned}$$

We estimate the three terms in the right-hand side of (B.62) starting with the first one. The non sharp product estimates (5.14) and (5.15) imply:

$$\|nP(\overline{\delta} - \cdot\chi)\|_{\mathcal{P}^0} \lesssim \mathcal{N}_2(n-1)\|P(\overline{\delta} - \cdot\chi)\|_{\mathcal{P}^0} \lesssim \mathcal{N}_2(n-1)\mathcal{N}_1(P)(\mathcal{N}_1(\overline{\delta}) + \mathcal{N}_1(\chi)) \lesssim \varepsilon^2$$

which together with Lemma 5.14 yields the following estimate for the first term in the right-hand side of (B.62):

$$\left\| P_j\left(\int_0^t (nP\cdot(\overline{\delta} - \chi\cdot)\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \lesssim D\varepsilon^2 2^j + D\varepsilon^2 2^{\frac{j}{2}}\gamma(u). \qquad \text{(B.63)}$$

To estimate the second term in the right-hand side of (B.62), we use the dual of the sharp

Bernstein inequality (4.32) and the estimate for transport equations (3.66). We have:

$$\begin{aligned}\left\|P_j\left(\int_0^t (P\cdot B)d\tau\right)\right\|_{L^2(\mathcal{H}_u)} &\lesssim 2^j\left\|\int_0^t (P\cdot B)d\tau\right\|_{L^2_tL^1_{x'}} \\ &\lesssim 2^j\|P\cdot B\|_{L^1(\mathcal{H}_u)} \\ &\lesssim 2^j\|P\|_{L^2_tL^4_{x'}}\|B\|_{L^2_tL^{\frac{4}{3}}_{x'}} \\ &\lesssim 2^j\mathcal{N}_1(P)\varepsilon^2 \\ &\lesssim 2^j\varepsilon,\end{aligned} \tag{B.64}$$

where we used the assumption on P in Lemma 5.13, and the estimate (B.61).

We now focus on estimating the third term in the right-hand side of (B.62). Using the decomposition of $\nabla\!\!\!/(n\rho)+(\nabla\!\!\!/(n\sigma))^*$ given by Lemma 5.17, we estimate the third term in the right-hand side of (B.62) as follows:

$$\begin{aligned}&\left\|P_j\left(\int_0^t (P\cdot(\nabla\!\!\!/(n\rho)+(\nabla\!\!\!/(n\sigma))^*)d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \\ \lesssim\ &\left\|P_j\left(\int_0^t (P\cdot{}^*\mathcal{D}_1\cdot J\cdot{}^*\mathcal{D}_1^{-1}(\nabla_{nL}(\underline{\beta}))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}+\left\|P_j\left(\int_0^t (P\cdot{}^*\mathcal{D}_1(H))d\tau\right)\right\|_{L^2(\mathcal{H}_u)}.\end{aligned} \tag{B.65}$$

Next, we estimate both terms in the right-hand side of (B.65) starting with the second one. We have:

$$\begin{aligned}\left\|P_j\left(\int_0^t (P\cdot{}^*\mathcal{D}_1(H))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} &\lesssim \left\|P_j\left(\int_0^t \mathrm{div}\!\!\!/(P\cdot H)d\tau\right)\right\|_{L^2(\mathcal{H}_u)} \\ &\quad+\left\|P_j\left(\int_0^t (\nabla\!\!\!/P\cdot H)d\tau\right)\right\|_{L^2(\mathcal{H}_u)}\end{aligned}$$

which together with the finite band property for P_j, the sharp Bernstein inequality (4.32), and the estimate for transport equations (3.66) yields:

$$\begin{aligned}\left\|P_j\left(\int_0^t (P\cdot{}^*\mathcal{D}_1(H))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} &\lesssim 2^j\left\|\int_0^t (P\cdot H)d\tau\right\|_{L^2(\mathcal{H}_u)}+2^j\left\|\int_0^t (\nabla\!\!\!/P\cdot H)d\tau\right\|_{L^2_tL^1_{x'}} \\ &\lesssim 2^j\|P\cdot H\|_{L^1_tL^2_{x'}}+2^j\|\nabla\!\!\!/P\cdot H\|_{L^1(\mathcal{H}_u)} \\ &\lesssim 2^j\|P\|_{L^2_tL^6_{x'}}\|H\|_{L^2_tL^3_{x'}}+2^j\|\nabla\!\!\!/P\|_{L^2(\mathcal{H}_u)}\|H\|_{L^2(\mathcal{H}_u)} \\ &\lesssim 2^j\varepsilon,\end{aligned} \tag{B.66}$$

where we used the estimate (B.97) for H, and the assumption of Lemma 5.13 on P.

We turn to the first term in the right-hand side of (B.65). We have:

$$\begin{aligned}
&\left\| P_j \left(\int_0^t (P \cdot {}^*\mathcal{D}_1 \cdot J \cdot {}^*\mathcal{D}_1^{-1}(\nabla\!\!\!/_{nL}(\underline{\beta}))d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \qquad \text{(B.67)}\\
\lesssim\ & \left\| P_j \left(\int_0^t (P \cdot \nabla\!\!\!/_{nL} {}^*\mathcal{D}_1 \cdot J \cdot {}^*\mathcal{D}_1^{-1}(\underline{\beta}))d\tau \right) \right\|_{L^2(\mathcal{H}_u)}\\
&+ \left\| P_j \left(\int_0^t (P \cdot [{}^*\mathcal{D}_1, \nabla\!\!\!/_{nL}] \cdot J \cdot {}^*\mathcal{D}_1^{-1}(\underline{\beta}))d\tau \right) \right\|_{L^2(\mathcal{H}_u)}\\
&+ \left\| P_j \left(\int_0^t (P \cdot {}^*\mathcal{D}_1 \cdot J \cdot [{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta}))d\tau \right) \right\|_{L^2(\mathcal{H}_u)}\\
\lesssim\ & \|P_j(P \cdot {}^*\mathcal{D}_1 \cdot J \cdot {}^*\mathcal{D}_1^{-1}(\underline{\beta}))\|_{L^2(\mathcal{H}_u)} + \left\| P_j \left(\int_0^t (\nabla\!\!\!/_{nL}(P) \cdot {}^*\mathcal{D}_1 \cdot J \cdot {}^*\mathcal{D}_1^{-1}(\underline{\beta}))d\tau \right) \right\|_{L^2(\mathcal{H}_u)}\\
&+ \left\| P_j \left(\int_0^t (P \cdot [{}^*\mathcal{D}_1, \nabla\!\!\!/_{nL}] \cdot J \cdot {}^*\mathcal{D}_1^{-1}(\underline{\beta}))d\tau \right) \right\|_{L^2(\mathcal{H}_u)}\\
&+ \left\| P_j \left(\int_0^t (P \cdot {}^*\mathcal{D}_1 \cdot J \cdot [{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta}))d\tau \right) \right\|_{L^2(\mathcal{H}_u)}.
\end{aligned}$$

Next, we estimate the four terms in the right-hand side of (B.67) starting with the first one.

Using the dual of the sharp Bernstein inequality (4.32), we have:

$$\begin{aligned}
\|P_j(P \cdot {}^*\mathcal{D}_1 \cdot J \cdot {}^*\mathcal{D}_1^{-1}(\underline{\beta}))\|_{L^2(\mathcal{H}_u)} &\lesssim 2^j \|P \cdot {}^*\mathcal{D}_1 \cdot J \cdot {}^*\mathcal{D}_1^{-1}(\underline{\beta})\|_{L^2_t L^1_{x'}} \qquad \text{(B.68)}\\
&\lesssim 2^j \|P\|_{L^\infty_t L^2_{x'}} \|{}^*\mathcal{D}_1 \cdot J \cdot {}^*\mathcal{D}_1^{-1}(\underline{\beta})\|_{L^2(\mathcal{H}_u)}\\
&\lesssim 2^j \mathcal{N}_1(P) \|\underline{\beta}\|_{L^2(\mathcal{H}_u)}\\
&\lesssim 2^j \varepsilon^2,
\end{aligned}$$

where we used the estimate (3.50) for ${}^*\mathcal{D}_1^{-1}$, the assumption of Lemma 5.13 for P and the curvature bound (2.59) for $\underline{\beta}$.

We now consider the second term in the right-hand side of (B.67). Using the dual of the sharp Bernstein inequality (4.32) and the estimate for transport equations (3.66), we have:

$$\begin{aligned}
&\left\| P_j \left(\int_0^t (\nabla\!\!\!/_{nL}(P) \cdot {}^*\mathcal{D}_1 \cdot J \cdot {}^*\mathcal{D}_1^{-1}(\underline{\beta}))d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \qquad \text{(B.69)}\\
\lesssim\ & 2^j \left\| \int_0^t (\nabla\!\!\!/_{nL}(P) \cdot {}^*\mathcal{D}_1 \cdot J \cdot {}^*\mathcal{D}_1^{-1}(\underline{\beta}))d\tau \right\|_{L^2_t L^1_{x'}}\\
\lesssim\ & 2^j \|\nabla\!\!\!/_{nL}(P) \cdot {}^*\mathcal{D}_1 \cdot J \cdot {}^*\mathcal{D}_1^{-1}(\underline{\beta})\|_{L^1(\mathcal{H}_u)}\\
\lesssim\ & 2^j \|\nabla\!\!\!/_{nL}(P)\|_{L^2(\mathcal{H}_u)} \|{}^*\mathcal{D}_1 \cdot J \cdot {}^*\mathcal{D}_1^{-1}(\underline{\beta})\|_{L^2(\mathcal{H}_u)}\\
\lesssim\ & 2^j \mathcal{N}_1(P) \|\underline{\beta}\|_{L^2(\mathcal{H}_u)}\\
\lesssim\ & 2^j \varepsilon^2,
\end{aligned}$$

where we used the estimate (3.50) for ${}^*\mathcal{D}_1^{-1}$, the assumption of Lemma 5.13 for P and the curvature bound (2.59) for $\underline{\beta}$.

We consider the third term in the right-hand side of (B.67). From the commutator formula (2.48) and the fact that $\mathcal{D}_1^{-1}(\underline{\beta})$ is a scalar, we have

$$[{}^*\mathcal{D}_1, \nabla\!\!\!/_{nL}] \cdot J \cdot ({}^*\mathcal{D}_1^{-1}(\underline{\beta})) = n\chi \cdot \nabla\!\!\!/({}^*\mathcal{D}_1^{-1}(\underline{\beta}))$$

which together with the dual of the sharp Bernstein inequality (4.32) and the estimate for transport equations (3.66) yields:

$$\begin{aligned}
& \left\| P_j \left(\int_0^t (P \cdot [{}^*\mathcal{D}_1, \nabla\!\!\!/_{nL}] \cdot J \cdot {}^*\mathcal{D}_1^{-1}(\underline{\beta}))d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \\
\lesssim & \left\| P_j \left(\int_0^t (P \cdot n\chi \cdot \nabla\!\!\!/({}^*\mathcal{D}_1^{-1}(\underline{\beta})))d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \\
\lesssim & 2^j \left\| \int_0^t (P \cdot n\chi \cdot \nabla\!\!\!/({}^*\mathcal{D}_1^{-1}(\underline{\beta})))d\tau \right\|_{L^2_t L^1_{x'}} \\
\lesssim & 2^j \|P \cdot n\chi \cdot \nabla\!\!\!/({}^*\mathcal{D}_1^{-1}(\underline{\beta}))\|_{L^1(\mathcal{H}_u)} \\
\lesssim & 2^j \|P\|_{L^4(\mathcal{H}_u)} \|n\|_{L^\infty} \|\chi\|_{L^4(\mathcal{H}_u)} \|\nabla\!\!\!/({}^*\mathcal{D}_1^{-1}(\underline{\beta}))\|_{L^2(\mathcal{H}_u)} \\
\lesssim & 2^j \mathcal{N}_1(P)\mathcal{N}_1(\chi)\|\underline{\beta}\|_{L^2(\mathcal{H}_u)} \\
\lesssim & 2^j \varepsilon^2,
\end{aligned} \tag{B.70}$$

where we used the estimate (3.50) for ${}^*\mathcal{D}_1^{-1}$, the assumption of Lemma 5.13 for P, the curvature bound (2.59) for $\underline{\beta}$, and the L^∞ bound for n provided by (2.67).

We now consider the last term in the right-hand side of (B.67). We have:

$$\begin{aligned}
& \left\| P_j \left(\int_0^t (P \cdot {}^*\mathcal{D}_1 \cdot J \cdot [{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta}))d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \\
\lesssim & \left\| P_j \left(\int_0^t \mathrm{div}\!\!\!/(P \cdot J \cdot [{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta}))d\tau \right) \right\|_{L^2(\mathcal{H}_u)} + \left\| P_j \left(\int_0^t (\nabla\!\!\!/ P \cdot J \cdot [{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta}))d\tau \right) \right\|_{L^2(\mathcal{H}_u)}
\end{aligned}$$

which together with Lemma 3.17, the dual of the sharp Bernstein inequality (4.32) and the estimate for transport equations (3.66) yields:

$$\begin{aligned}
& \left\| P_j \left(\int_0^t (P \cdot {}^*\mathcal{D}_1 \cdot J \cdot [{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta}))d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \\
\lesssim & 2^j \|P \cdot J \cdot [{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta})\|_{L^1_t L^2_{x'}} + 2^j \|\nabla\!\!\!/ P \cdot J \cdot [{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta}))\|_{L^1(\mathcal{H}_u)} \\
\lesssim & 2^j \|P\|_{L^2_t L^6_{x'}} \|[{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta})\|_{L^2_t L^3_{x'}} + 2^j \|\nabla\!\!\!/ P\|_{L^2(\mathcal{H}_u)} \|[{}^*\mathcal{D}_1^{-1}, \nabla\!\!\!/_{nL}](\underline{\beta}))\|_{L^2(\mathcal{H}_u)} \\
\lesssim & 2^j \mathcal{N}_1(P)\varepsilon \\
\lesssim & 2^j \varepsilon^2,
\end{aligned} \tag{B.71}$$

where we used Lemma 4.24 for the commutator term, and the assumption of Lemma 5.13 for P.

Finally, (B.67), (B.68), (B.69), (B.70) and (B.71) imply:

$$\left\| P_j \left(\int_0^t (P \cdot {}^*\mathcal{D}_1 \cdot J \cdot {}^*\mathcal{D}_1^{-1}(\nabla\!\!\!/_{nL}(\underline{\beta}))d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \lesssim 2^j \varepsilon.$$

Together with (B.65) and (B.66), we obtain:

$$\left\| P_j \left(\int_0^t (P \cdot (\nabla\!\!\!/(n\rho) + (\nabla\!\!\!/(n\sigma))^*)d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \lesssim 2^j \varepsilon. \tag{B.72}$$

Now, (B.62), (B.63), (B.64) and (B.72) yield:

$$\left\| P_j \left(\int_0^t (P \cdot \nabla\!\!\!/_{nL} \nabla\!\!\!/_{\underline{L}}(\zeta))d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \lesssim 2^j \varepsilon + D\varepsilon^2 2^j + D\varepsilon^2 2^{\frac{j}{2}} \gamma(u), \tag{B.73}$$

which is the desired estimate for the third term in the right-hand side of (B.36).

B.8.4 End of the proof of Lemma 5.13

(B.36), (B.37), (B.52), (B.59) and (B.73) imply:

$$\left\| P_j \left(\int_0^t (F \cdot \nabla\!\!\!/ \nabla\!\!\!/_{\underline{L}}(\zeta))d\tau \right) \right\|_{L^2(\mathcal{H}_u)} \lesssim 2^j \varepsilon + D\varepsilon^2 2^j + D\varepsilon^2 2^{\frac{j}{2}} \gamma(u), \tag{B.74}$$

which concludes the proof of Lemma 5.13.

B.9 Proof of Lemma 5.14

We decompose $\|P_j \left(\int_0^t (F \cdot \nabla\!\!\!/_{\underline{L}}(\zeta))d\tau \right)\|_{L^2(\mathcal{H}_u)}$ using the property (3.16) of the geometric Littlewood-Paley projections:

$$\begin{aligned} \left\| P_j \left(\int_0^t (F \cdot \nabla\!\!\!/_{\underline{L}}(\zeta))d\tau \right) \right\|_{L_t^\infty L^2_{x'}} \lesssim & \left\| P_j \left(\int_0^t (F \cdot P_{<0} \nabla\!\!\!/_{\underline{L}}(\zeta))d\tau \right) \right\|_{L_t^\infty L^2_{x'}} \\ & + \sum_{l \geq 0} \left\| P_j \left(\int_0^t (F \cdot P_l \nabla\!\!\!/_{\underline{L}}(\zeta))d\tau \right) \right\|_{L_t^\infty L^2_{x'}}. \end{aligned} \tag{B.75}$$

We focus on the second term in the right-hand side of (B.75), the other being easier to handle. Using the weak Bernstein inequality for P_j and the estimate for transport equations (3.66), we have:

$$\begin{aligned} \left\| P_j \left(\int_0^t (F \cdot P_l \nabla\!\!\!/_{\underline{L}}(\zeta))d\tau \right) \right\|_{L_t^\infty L^2_{x'}} &\lesssim 2^{\frac{j}{3}} \left\| \int_0^t (F \cdot P_l \nabla\!\!\!/_{\underline{L}}(\zeta))d\tau \right\|_{L_t^\infty L^{\frac{3}{2}}_{x'}} \\ &\lesssim 2^{\frac{j}{3}} \| F \cdot P_l \nabla\!\!\!/_{\underline{L}}(\zeta) \|_{L^1_{x'} L^{\frac{3}{2}}_t} \\ &\lesssim 2^{\frac{j}{3}} \| F \|_{L^2(\mathcal{H}_u)} \| P_l \nabla\!\!\!/_{\underline{L}}(\zeta) \|_{L^2_t L^6_{x'}} \\ &\lesssim 2^{\frac{j}{3}} 2^{\frac{2l}{3}} \varepsilon \| P_l \nabla\!\!\!/_{\underline{L}}(\zeta) \|_{L^2(\mathcal{H}_u)} \\ &\lesssim 2^{\frac{j}{3} + \frac{2l}{3}} (D\varepsilon^2 + D\varepsilon^2 2^{-\frac{l}{2}} \gamma(u), \end{aligned} \tag{B.76}$$

where we used the weak Bernstein inequality for P_l, the estimate (5.92) for $\nabla\!\!\!/_{\underline{L}}(\zeta)$ and the assumption of Lemma 5.14 for F.

We now make another decomposition using the property (3.16) of the geometric Littlewood-Paley projections:

$$\begin{aligned}\left\|P_j\left(\int_0^t (F\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L_t^\infty L_{x'}^2} \lesssim\ & \left\|P_j\left(\int_0^t (P_{<0}(F)\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L_t^\infty L_{x'}^2} \\ &+\sum_{q\geq 0}\left\|P_j\left(\int_0^t (P_q(F)\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L_t^\infty L_{x'}^2}.\end{aligned} \tag{B.77}$$

We focus on the second term in the right-hand side of (B.77), the other being easier to handle. Using the property of the Littlewood-Paley projection P_l, we have:

$$\begin{aligned}&\left\|P_j\left(\int_0^t (P_q(F)\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L_t^\infty L_{x'}^2}\\ \lesssim\ & 2^{-2l}\left\|P_j\left(\int_0^t (P_q(F)\cdot \Delta\!\!\!/ P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L_t^\infty L_{x'}^2}\\ \lesssim\ & 2^{-2l}\left\|P_j\left(\int_0^t \mathrm{div}\!\!\!/(P_q(F)\cdot \nabla\!\!\!/ P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L_t^\infty L_{x'}^2} + 2^{-2l}\left\|P_j\left(\int_0^t (\nabla\!\!\!/ P_q(F)\cdot \nabla\!\!\!/ P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L_t^\infty L_{x'}^2}\end{aligned}$$

which together with Lemma 5.16 with $p=\frac{4}{3}$, the strong Bernstein inequality (4.32) and the estimate for transport equations (3.66) yields:

$$\begin{aligned}&\left\|P_j\left(\int_0^t (P_q(F)\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L_t^\infty L_{x'}^2} \\ \lesssim\ & 2^{\frac{3j}{2}-2l}\|P_q(F)\cdot \nabla\!\!\!/ P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L_t^1 L_{x'}^{\frac{4}{3}}} + 2^{j-2l}\|\nabla\!\!\!/ P_q(F)\cdot \nabla\!\!\!/ P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^1(\mathcal{H}_u)}\\ \lesssim\ & 2^{\frac{3j}{2}-2l}\|P_q(F)\|_{L_t^2 L_{x'}^4}\|\nabla\!\!\!/ P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(\mathcal{H}_u)} + 2^{j-2l}\|\nabla\!\!\!/ P_q(F)\|_{L^2(\mathcal{H}_u)}\|\nabla\!\!\!/ P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(\mathcal{H}_u)}\\ \lesssim\ & (2^{\frac{3j}{2}+\frac{q}{2}-l}+2^{j+q-l})\|P_q(F)\|_{L^2(\mathcal{H}_u)}\|P_l\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(\mathcal{H}_u)}\\ \lesssim\ & (2^{\frac{3j}{2}+\frac{q}{2}-l}+2^{j+q-l})\|P_q(F)\|_{L^2(\mathcal{H}_u)}(D\varepsilon + D\varepsilon 2^{-\frac{l}{2}}\gamma(u)),\end{aligned} \tag{B.78}$$

where we used the weak Bernstein inequality for P_q, the finite band property for P_q and P_l, and the estimate (5.92) for $\nabla\!\!\!/_{\underline{L}}(\zeta)$. Similarly, we may exchange the role of l and q and obtain:

$$\begin{aligned}&\left\|P_j\left(\int_0^t (P_q(F)\cdot P_l\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L_t^\infty L_{x'}^2}\\ \lesssim\ & (2^{\frac{3j}{2}+\frac{l}{2}-q}+2^{j+l-q})\|P_q(F)\|_{L^2(\mathcal{H}_u)}(D\varepsilon + D\varepsilon 2^{-\frac{l}{2}}\gamma(u)).\end{aligned} \tag{B.79}$$

Now, using (B.78) for $q \leq l$ and (B.79) for $q > l$ and assuming $l > j$ yields:

$$\begin{aligned}
&\sum_{l>j,q\geq 0} \left\| P_j \left(\int_0^t (P_q(F) \cdot P_l \nabla\!\!\!/_{\underline{L}}(\zeta)) d\tau \right) \right\|_{L^\infty_t L^2_{x'}} \\
\lesssim & \sum_{l>j,q\geq 0} (2^{j-\frac{|l-q|}{2}} + 2^{j-|l-q|}) \|P_q(F)\|_{L^2(\mathcal{H}_u)} (D\varepsilon + D\varepsilon 2^{-\frac{j}{2}}\gamma(u)) \\
\lesssim & \left(\sum_{q\geq 0} \|P_q(F)\|_{L^2(\mathcal{H}_u)} \right) 2^j (D\varepsilon + D\varepsilon 2^{-\frac{j}{2}}\gamma(u)) \\
\lesssim & \|F\|_{\mathcal{P}^0} 2^j (D\varepsilon + D\varepsilon 2^{-\frac{j}{2}}\gamma(u)) \\
\lesssim & D\varepsilon^2 2^j + D\varepsilon^2 2^{\frac{j}{2}}\gamma(u),
\end{aligned}$$

where we used the definition of $\mathcal{P}^0$ and the assumption of Lemma 5.14 on F. Together with (B.77), this yields:

$$\sum_{l>j} \left\| P_j \left(\int_0^t (F \cdot P_l \nabla\!\!\!/_{\underline{L}}(\zeta)) d\tau \right) \right\|_{L^\infty_t L^2_{x'}} \lesssim D\varepsilon^2 2^j + D\varepsilon^2 2^{\frac{j}{2}}\gamma(u). \tag{B.80}$$

Finally, using (B.75), (B.76) for $l \leq j$, and (B.80) for $l > j$, we obtain:

$$\left\| P_j \left(\int_0^t (F \cdot \nabla\!\!\!/_{\underline{L}}(\zeta)) d\tau \right) \right\|_{L^\infty_t L^2_{x'}} \lesssim D\varepsilon^2 2^j + D\varepsilon^2 2^{\frac{j}{2}}\gamma(u),$$

which concludes the proof of Lemma 5.14.

B.10 Proof of Lemma 5.15

Since P_j is selfadjoint, since the dual of $\mathrm{div}\!\!\!/\mathrm{div}\!\!\!/$ acting on $P_{t,u}$ tangent 2-tensors is given by $\nabla\!\!\!/^2$ acting on scalars, and since the dual of $\int_0^t$ is $\int_1^t$, we infer by duality that the second estimate of Lemma 5.15 is equivalent to the estimate:

$$\left\| \nabla\!\!\!/^2 \left(\int_1^t P_j f d\tau \right) \right\|_{L^2_{x'} L^\infty_t} \lesssim 2^{2j} \|f\|_{L^2(\mathcal{H}_u)} \tag{B.81}$$

for any scalar function f on $\mathcal{H}_u$ and any $j \geq 0$. Similarly, we infer by duality that the first estimate of Lemma 5.15 is equivalent to the estimate:

$$\left\| \Delta\!\!\!/ \left(\int_1^t P_j f d\tau \right) \right\|_{L^2_{x'} L^\infty_t} \lesssim 2^{2j} \|f\|_{L^2(\mathcal{H}_u)}$$

for any scalar function f on $\mathcal{H}_u$ and any $j \geq 0$. This estimate is actually implied by (B.81), and hence we focus on the proof of (B.81) from now on.

Let w the solution of the following transport equation:

$$nL(w) = P_j f,\ w = 0 \text{ on } P_{1,u}. \tag{B.82}$$

Then, (B.81) may be rewritten as:

$$\|\nabla\!\!\!/^2 w\|_{L^2_{x'}L^\infty_t} \lesssim 2^{2j}\|f\|_{L^2(\mathcal{H}_u)}. \tag{B.83}$$

From now on, we focus on obtaining (B.83). We first derive an estimate for $\|\nabla\!\!\!/ w\|_{L^\infty}$. Differentiating (B.82) with respect to $\nabla\!\!\!/$ and using the commutator formula (2.48), which in the particular case of a scalar function w is given by

$$[\nabla\!\!\!/, \nabla\!\!\!/_{nL}]w \;\; = \;\; n\chi\nabla\!\!\!/ w,$$

we obtain:

$$\nabla\!\!\!/_{nL}(\nabla\!\!\!/ w) = n\chi\nabla\!\!\!/ w + \nabla\!\!\!/ P_j f,\ \nabla\!\!\!/ w = 0 \text{ on } P_{1,u}$$

which together with the estimate for transport equations (3.66) implies:

$$\|\nabla\!\!\!/ w\|_{L^\infty} \lesssim \|n\chi\|_{L^\infty_{x'}L^2_t}\|\nabla\!\!\!/ w\|_{L^\infty} + \|\nabla\!\!\!/ P_j f\|_{L^1_t L^\infty_{x'}}.$$

Using the L^∞ bound for n given by (2.67) and the trace bound for χ given by (2.70) (2.71), we get:

$$\|\nabla\!\!\!/ w\|_{L^\infty} \lesssim \|\nabla\!\!\!/ P_j f\|_{L^1_t L^\infty_{x'}}. \tag{B.84}$$

In view of (B.84), we need to estimate $\|\nabla\!\!\!/ P_j f\|_{L^1_t L^\infty_{x'}}$. Using the L^∞ bound (3.38) for tensors on $P_{t,u}$ with the choice $p = 2$, we have:

$$\begin{aligned}
\|\nabla\!\!\!/ P_j f\|_{L^\infty(P_{t,u})} &\lesssim \|\nabla\!\!\!/^3 P_j f\|^{\frac{1}{2}}_{L^2(P_{t,u})}\|\nabla\!\!\!/ P_j f\|^{\frac{1}{2}}_{L^2(P_{t,u})} + \|\nabla\!\!\!/^2 P_j f\|_{L^2(P_{t,u})} \\
&\lesssim \|\nabla\!\!\!/^3 P_j f\|^{\frac{1}{2}}_{L^2(P_{t,u})} 2^{\frac{j}{2}}\|P_j f\|^{\frac{1}{2}}_{L^2(P_{t,u})} + \|\Delta\!\!\!/ P_j f\|_{L^2(P_{t,u})} + \|\nabla\!\!\!/ P_j f\|_{L^2(P_{t,u})} \\
&\lesssim 2^{\frac{j}{2}}\|\nabla\!\!\!/^3 P_j f\|^{\frac{1}{2}}_{L^2(P_{t,u})}\|f\|^{\frac{1}{2}}_{L^2(P_{t,u})} + 2^{2j}\|f\|_{L^2(P_{t,u})}
\end{aligned} \tag{B.85}$$

where we used the Bochner inequality (4.34), and the L^2 boundedness and the finite band property of P_j. In view of (B.85), we need to estimate $\|\nabla\!\!\!/^3 P_j f\|_{L^2(P_{t,u})}$. Using the Bochner inequality for tensors (3.7), we have:

$$\begin{aligned}
&\|\nabla\!\!\!/^3 P_j f\|_{L^2(P_{t,u})} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (B.86)\\
\lesssim\ & \|\Delta\!\!\!/\nabla\!\!\!/ P_j f\|_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})}\|\nabla\!\!\!/^2 P_j f\|_{L^2(P_{t,u})} + \|K\|^2_{L^2(P_{t,u})}\|\nabla\!\!\!/ P_j f\|_{L^2(P_{t,u})} \\
\lesssim\ & \|[\Delta\!\!\!/, \nabla\!\!\!/]P_j f\|_{L^2(P_{t,u})} + \|\nabla\!\!\!/\Delta\!\!\!/ P_j f\|_{L^2(P_{t,u})} \\
& + \|K\|_{L^2(P_{t,u})}(\|\Delta\!\!\!/ P_j f\|_{L^2(P_{t,u})} + \|\nabla\!\!\!/ P_j f\|_{L^2(P_{t,u})}) + \|K\|^2_{L^2(P_{t,u})}2^j\|P_j f\|_{L^2(P_{t,u})} \\
\lesssim\ & \|[\Delta\!\!\!/, \nabla\!\!\!/]P_j f\|_{L^2(P_{t,u})} + 2^{3j}\|f\|_{L^2(P_{t,u})} + 2^{2j}\|K\|_{L^2(P_{t,u})}\|f\|_{L^2(P_{t,u})} \\
& + 2^j\|K\|^2_{L^2(P_{t,u})}\|f\|_{L^2(P_{t,u})}
\end{aligned}$$

where we used the Bochner inequality (4.34), and the L^2 boundedness and the finite band property of P_j. Now, for any scalar function f on $P_{t,u}$, there holds the following commutator formula:

$$[\nabla\!\!\!/, \Delta\!\!\!/]f = K\nabla\!\!\!/ f \tag{B.87}$$

which together with (B.86) yields:

$$\begin{aligned}\|\nabla\!\!\!/^3 P_j f\|_{L^2(P_{t,u})} &\lesssim \|K\nabla\!\!\!/ P_j f\|_{L^2(P_{t,u})} + 2^{3j}\|f\|_{L^2(P_{t,u})} + 2^{2j}\|K\|_{L^2(P_{t,u})}\|f\|_{L^2(P_{t,u})} \\ &\quad +2^j\|K\|^2_{L^2(P_{t,u})}\|f\|_{L^2(P_{t,u})} \\ &\lesssim \|K\|_{L^2(P_{t,u})}\|\nabla\!\!\!/ P_j f\|_{L^\infty(P_{t,u})} + 2^{3j}\|f\|_{L^2(P_{t,u})} + 2^{2j}\|K\|_{L^2(P_{t,u})}\|f\|_{L^2(P_{t,u})} \\ &\quad +2^j\|K\|^2_{L^2(P_{t,u})}\|f\|_{L^2(P_{t,u})}. \end{aligned} \tag{B.88}$$

Now, (B.85) and (B.88) imply:

$$\begin{aligned}\|\nabla\!\!\!/ P_j f\|_{L^\infty(P_{t,u})} &\lesssim 2^{\frac{j}{2}}\|K\|^{\frac{1}{2}}_{L^2(P_{t,u})}\|\nabla\!\!\!/ P_j f\|^{\frac{1}{2}}_{L^\infty(P_{t,u})}\|f\|^{\frac{1}{2}}_{L^2(P_{t,u})} \\ &\quad +2^{2j}(1+\|K\|_{L^2(P_{t,u})})\|f\|_{L^2(P_{t,u})}\end{aligned}$$

which yields:

$$\|\nabla\!\!\!/ P_j f\|_{L^\infty(P_{t,u})} \lesssim 2^{2j}(1+\|K\|_{L^2(P_{t,u})})\|f\|_{L^2(P_{t,u})}. \tag{B.89}$$

Integrating (B.89) and using the bound (4.29) for K, we obtain:

$$\|\nabla\!\!\!/ P_j f\|_{L^1_t L^\infty_{x'}} \lesssim 2^{2j}(1+\|K\|_{L^2(\mathcal{H}_u)})\|f\|_{L^2(\mathcal{H}_u)} \lesssim 2^{2j}\|f\|_{L^2(\mathcal{H}_u)}. \tag{B.90}$$

Next, we come back to w. (B.84) and (B.89) yield:

$$\|\nabla\!\!\!/ w\|_{L^\infty} \lesssim 2^{2j}\|f\|_{L^2(\mathcal{H}_u)}. \tag{B.91}$$

Using twice the commutator formula (2.48), we have schematically

$$\begin{aligned}[\nabla\!\!\!/_{nL}, \nabla\!\!\!/^2]w &= [\nabla\!\!\!/_{nL}, \nabla\!\!\!/]\nabla\!\!\!/ w + \nabla\!\!\!/[\nabla\!\!\!/_{nL}, \nabla\!\!\!/]w \\ &= (n\chi\nabla\!\!\!/ + n\chi\overline{\epsilon} + n\beta)\nabla\!\!\!/ w + \nabla\!\!\!/(n\chi\nabla\!\!\!/ w) \\ &= n\chi\nabla\!\!\!/^2 w + (\nabla\!\!\!/(n\chi) + n\beta + n\chi\overline{\epsilon})w.\end{aligned}$$

Differentiating (B.82) with respect to $\nabla\!\!\!/^2$, we infer:

$$\nabla\!\!\!/_{nL}(\nabla\!\!\!/^2 w) = n\chi\nabla\!\!\!/^2 w + (n\chi\overline{\epsilon} + \nabla\!\!\!/(n\chi) + n\beta)\nabla\!\!\!/ w + \nabla\!\!\!/^2 P_j f, \quad \nabla\!\!\!/^2 w = 0 \text{ on } P_{0,u}$$

which together with the estimate for transport equations (3.66) implies:

$$\begin{aligned}\|\nabla\!\!\!/^2 w\|_{L^2_{x'}L^\infty_t} &\lesssim \|n\chi\nabla\!\!\!/^2 w\|_{L^2_{x'}L^1_t} + \|(n\chi\overline{\epsilon} + \nabla\!\!\!/(n\chi) + n\beta)\nabla\!\!\!/ w\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/^2 P_j f\|_{L^2(\mathcal{H}_u)} \\ &\lesssim \|n\|_{L^\infty}\|\chi\|_{L^\infty_{x'}L^2_t}\|\nabla\!\!\!/^2 w\|_{L^2(\mathcal{H}_u)} + \|n\|_{L^\infty}\|\nabla\!\!\!/ w\|_{L^\infty}(\mathcal{N}_1(\chi)^2 + \mathcal{N}_1(\overline{\epsilon})^2 \\ &\quad +\mathcal{N}_1(\nabla\!\!\!/ n)^2 + \|\beta\|_{L^2(\mathcal{H}_u)}) + \|\Delta\!\!\!/ P_j f\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/ P_j f\|_{L^2(\mathcal{H}_u)},\end{aligned}$$

where we used the Bochner inequality (4.34) in the last inequality. Now, using (B.91), the L^2 boundedness and the finite band property of P_j, the bound (2.59) for β, and the estimates (2.67)-(2.71) for n, χ and $\overline{\epsilon}$, we obtain:

$$\|\nabla\!\!\!/^2 w\|_{L^2_{x'}L^\infty_t} \lesssim \varepsilon\|\nabla\!\!\!/^2 w\|_{L^2(\mathcal{H}_u)} + 2^{2j}\|f\|_{L^2(\mathcal{H}_u)}.$$

This yields (B.83) which concludes the proof of Lemma 5.15.

B.11 Proof of Lemma 5.16

Since P_j is selfadjoint, since the dual of $\nabla\!\!\!/$ acting on $P_{t,u}$ tangent 1-forms is given by $\nabla\!\!\!/$ acting on scalars, and since the dual of $\int_0^t$ is $\int_1^t$, we infer by duality that the conclusion of Lemma 5.16 is equivalent to the estimate:

$$\left\| \nabla\!\!\!/ \left(\int_1^t P_j f d\tau \right) \right\|_{L^p_{x'} L^\infty_t} \lesssim 2^{2j(1-\frac{1}{p})} \|f\|_{L^1_t L^2_{x'}} \tag{B.92}$$

for any scalar function f on $\mathcal{H}_u$, any $2 \leq p < +\infty$ and any $j \geq 0$. Consider again w the solution of the transport equation (B.82). Then, (B.92) may be rewritten as:

$$\|\nabla\!\!\!/ w\|_{L^p_{x'} L^\infty_t} \lesssim 2^{2j(1-\frac{1}{p})} \|f\|_{L^1_t L^2_{x'}}. \tag{B.93}$$

From now on, we focus on obtaining (B.93). Differentiating (B.82) with respect to $\nabla\!\!\!/$ and using the commutator formula (2.48), we obtain:

$$\nabla\!\!\!/_{nL}(\nabla\!\!\!/ w) = n\chi \nabla\!\!\!/ w + \nabla\!\!\!/ P_j f, \; \nabla\!\!\!/ w = 0 \text{ on } P_{1,u}$$

which together with the estimate for transport equations (3.66) implies:

$$\|\nabla\!\!\!/ w\|_{L^p_{x'} L^\infty_t} \lesssim \|n\chi\|_{L^\infty_{x'} L^2_t} \|\nabla\!\!\!/ w\|_{L^p_{x'} L^2_t} + \|\nabla\!\!\!/ P_j f\|_{L^1_t L^p_{x'}}.$$

Using the L^∞ bound for n given by (2.67) and the trace bound for χ given by (2.70) (2.71), we get:

$$\|\nabla\!\!\!/ w\|_{L^p_{x'} L^\infty_t} \lesssim \|\nabla\!\!\!/ P_j f\|_{L^1_t L^p_{x'}}. \tag{B.94}$$

In view of (B.94), we need to estimate $\|\nabla\!\!\!/ P_j f\|_{L^1_t L^p_{x'}}$. Recall (4.35):

$$\|\nabla\!\!\!/ P_j f\|_{L^p(P_{t,u})} \lesssim 2^{2j(1-\frac{1}{p})} \|f\|_{L^2(P_{t,u})}. \tag{B.95}$$

Integrating (B.95), we obtain:

$$\|\nabla\!\!\!/ P_j f\|_{L^1_t L^p_{x'}} \lesssim 2^{2j(1-\frac{1}{p})} \|f\|_{L^1_t L^2_{x'}}$$

which together with (B.94) yields (B.93). This concludes the proof of Lemma 5.16.

B.12 Proof of Lemma 5.17

Recall that J denotes the involution $(\rho, \sigma) \to (-\rho, \sigma)$. Then, $\nabla\!\!\!/(n\rho) + (\nabla\!\!\!/(n\sigma))^*$ may be rewritten as:

$$\nabla\!\!\!/(n\rho) + (\nabla\!\!\!/(n\sigma))^* = {}^*\mathcal{D}_1 \cdot J(n\rho, n\sigma).$$

Now, in view of the Bianchi identity (2.57), we have:

$$(n\rho, n\sigma) = {}^*\mathcal{D}_1^{-1} \left(\nabla\!\!\!/_{nL}(\underline{\beta}) - \nabla\!\!\!/(n)\rho + \nabla\!\!\!/(n)\sigma + n\mathrm{tr}\chi \underline{\beta} - 2n\underline{\widehat{\chi}} \cdot \beta - n\overline{\delta}\underline{\beta} + 3n(\underline{\zeta}\rho - {}^*\underline{\zeta}\sigma) \right)$$

which yields:

$$\nabla\!\!\!/(n\rho) + (\nabla\!\!\!/(n\sigma))^* = {}^*\mathcal{D}_1 \cdot J \cdot {}^*\mathcal{D}_1^{-1}(\nabla\!\!\!/_{nL}(\underline{\beta})) + {}^*\mathcal{D}_1(H) \tag{B.96}$$

where H is given by:

$$H = J \cdot {}^*\mathcal{D}_1^{-1}\bigg(- \nabla\!\!\!/(n)\rho + \nabla\!\!\!/(n)\sigma + n\mathrm{tr}\chi\underline{\beta} - 2n\underline{\widehat{\chi}} \cdot \beta - n\overline{\delta}\underline{\beta} + 3n(\underline{\zeta}\rho - {}^*\underline{\zeta}\sigma)\bigg).$$

Now, using Lemma 3.17 with $p = \frac{4}{3}$, $q = 3$, we obtain for H the following estimate:

$$\begin{aligned}
&\|H\|_{L^2_t L^3_{x'}} \qquad\qquad (B.97)\\
\lesssim\ & \bigg\| J \cdot {}^*\mathcal{D}_1^{-1}\bigg(- \nabla\!\!\!/(n)\rho + \nabla\!\!\!/(n)\sigma + n\mathrm{tr}\chi\underline{\beta} - 2n\underline{\widehat{\chi}} \cdot \beta - n\overline{\delta}\underline{\beta} + 3n(\underline{\zeta}\rho - {}^*\underline{\zeta}\sigma)\bigg)\bigg\|_{L^2_t L^3_{x'}}\\
\lesssim\ & \big\| - \nabla\!\!\!/(n)\rho + \nabla\!\!\!/(n)\sigma + n\mathrm{tr}\chi\underline{\beta} - 2n\underline{\widehat{\chi}} \cdot \beta - n\overline{\delta}\underline{\beta} + 3n(\underline{\zeta}\rho - {}^*\underline{\zeta}\sigma)\big\|_{L^2_t L^{\frac{4}{3}}_{x'}}\\
\lesssim\ & \|\nabla\!\!\!/ n\|_{L^\infty_t L^4_{x'}}\|\rho\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/ n\|_{L^\infty_t L^4_{x'}}\|\sigma\|_{L^2(\mathcal{H}_u)} + \|n\mathrm{tr}\chi\|_{L^\infty_t L^4_{x'}}\|\underline{\beta}\|_{L^2(\mathcal{H}_u)}\\
& + \|n\underline{\widehat{\chi}}\|_{L^\infty_t L^4_{x'}}\|\beta\|_{L^2(\mathcal{H}_u)} + \|n\overline{\delta}\|_{L^\infty_t L^4_{x'}}\|\underline{\beta}\|_{L^2(\mathcal{H}_u)} + \|n\underline{\zeta}\|_{L^\infty_t L^4_{x'}}\|\rho\|_{L^2(\mathcal{H}_u)} + \|n\underline{\zeta}\|_{L^\infty_t L^4_{x'}}\|\sigma\|_{L^2(\mathcal{H}_u)}\\
\lesssim\ & \varepsilon^2
\end{aligned}$$

where we used the curvature bound (2.59) for β, ρ, σ and $\underline{\beta}$, and the estimates (2.67)-(2.72) for $n, \overline{\delta}, \underline{\widehat{\chi}}$ and ζ. (B.96) and (B.97) give the conclusion of the proof of Lemma 5.17.

C Appendix to section 6

C.1 Proof of Lemma 6.9

Recall the transport equation (6.27) satisfied by $\Pi(\partial_\omega\chi)$. We have

$$\begin{aligned}
\nabla\!\!\!/_L(\Pi(\partial_\omega\chi))_{AB} \ =\ & -\nabla\!\!\!/_{\partial_\omega N}\chi_{AB} - (\partial_\omega\chi)_{AC}\chi_{CB} - \chi_{AC}(\partial_\omega\chi)_{CB} - \overline{\delta}\partial_\omega\chi_{AB}\\
& + \overline{\epsilon}_A\chi_{\partial_\omega N B} + \overline{\epsilon}_B\chi_{A\partial_\omega N} + (\partial_\omega N)_A\chi_{CB}\epsilon_C + (\partial_\omega N)_B\chi_{AC}\overline{\epsilon}_C\\
& - (2\epsilon_{\partial_\omega N} - n^{-1}\nabla_{\partial_\omega N}n)\chi_{AB} + (\partial_\omega N)_C(\in_{AC} {}^*\beta_B + \in_{BC} {}^*\beta_A).
\end{aligned}$$

Differentiating with respect to $\nabla\!\!\!/_{\underline{L}}$, this yields, schematically

$$\begin{aligned}
& \nabla\!\!\!/_L(\nabla\!\!\!/_{\underline{L}}(\Pi(\partial_\omega\chi))) + [\nabla\!\!\!/_{\underline{L}}, \nabla\!\!\!/_L](\Pi(\partial_\omega\chi)) \qquad\qquad (C.1)\\
=\ & -\nabla\!\!\!/_{\underline{L}}\Pi(\partial_\omega\chi)\cdot\chi - \chi\cdot\nabla\!\!\!/_{\underline{L}}(\Pi(\partial_\omega\chi)) - \overline{\delta}\nabla\!\!\!/_{\underline{L}}(\Pi(\partial_\omega\chi)) + \partial_\omega N\nabla\!\!\!/_{\underline{L}}\beta + \nabla\!\!\!/(F_3) + F_4,
\end{aligned}$$

where the tensors F_3 and F_4 are given schematically by

$$F_3 = \partial_\omega N\nabla\!\!\!/_{\underline{L}}\chi,$$

and

$$\begin{aligned}
F_4 \ =\ & \nabla\!\!\!/(\partial_\omega N)\nabla\!\!\!/_{\underline{L}}\chi - \nabla\!\!\!/_{\nabla\!\!\!/_{\underline{L}}\partial_\omega N}\chi - (\partial_\omega N)[\nabla\!\!\!/_{\underline{L}}, \nabla\!\!\!/]\chi - \partial_\omega\chi\cdot\nabla\!\!\!/_{\underline{L}}\chi - \nabla\!\!\!/_{\underline{L}}\chi\cdot\partial_\omega\chi - \underline{L}(\overline{\delta})\partial_\omega\chi\\
& + \nabla\!\!\!/_{\underline{L}}(\overline{\epsilon})\chi\partial_\omega N + \overline{\epsilon}\nabla\!\!\!/_{\underline{L}}(\chi)\partial_\omega N + \overline{\epsilon}\chi\nabla\!\!\!/_{\underline{L}}\partial_\omega N - (2\epsilon_{\partial_\omega N} - n^{-1}\nabla_{\partial_\omega N}n)\nabla\!\!\!/_{\underline{L}}\chi\\
& - (2\nabla\!\!\!/_{\underline{L}}(\epsilon)\partial_\omega N + 2\epsilon\nabla\!\!\!/_{\underline{L}}\partial_\omega N - \nabla\!\!\!/_{\underline{L}}(n^{-1}\nabla_{\partial_\omega N}n))\chi + \nabla\!\!\!/_{\underline{L}}\partial_\omega N\beta.
\end{aligned}$$

F_3 satisfies the following estimate

$$\|F_3\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \|\partial_\omega N\|_{L^\infty} \|\nabla\!\!\!/_{\underline{L}} \chi\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon, \tag{C.2}$$

where we used in the last inequality the estimate (2.76) for $\partial_\omega N$ and the estimates (2.70) (2.71) for χ. Also, F_4 satisfies the following estimate

$$\begin{aligned}
&\|F_4\|_{L^2_t L^1_{x'}} \qquad\qquad (C.3)\\
\lesssim\ & \|\mathbf{D}\partial_\omega N\|_{L^\infty_t L^2_{x'}} \|\mathbf{D}\chi\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\partial_\omega N\|_{L^\infty} \|[\nabla\!\!\!/_{\underline{L}}, \nabla\!\!\!/]\chi\|_{L^2_t L^1_{x'}} + \|\partial_\omega \chi\|_{L^\infty_t L^2_{x'}} \|\nabla\!\!\!/_{\underline{L}} \chi\|_{L^\infty_u L^2(\mathcal{H}_u)}\\
&+\|\underline{L}(\overline{\delta})\|_{L^\infty_u L^2(\mathcal{H}_u)} \|\partial_\omega \chi\|_{L^\infty_t L^2_{x'}} + \|\partial_\omega N\|_{L^\infty} \Big(\|\nabla\!\!\!/_{\underline{L}}(\overline{\epsilon})\|_{L^\infty_u L^2(\mathcal{H}_u)} \|\chi\|_{L^\infty_t L^2_{x'}}\\
&+\|\overline{\epsilon}\|_{L^\infty_t L^2_{x'}} \|\nabla\!\!\!/_{\underline{L}}(\chi)\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|n^{-1}\nabla n\|_{L^\infty_t L^2_{x'}} \|\nabla\!\!\!/_{\underline{L}} \chi\|_{L^\infty_u L^2(\mathcal{H}_u)} \Big)\\
&+\|\overline{\epsilon}\|_{L^\infty_t L^4_{x'}} \|\chi\|_{L^\infty_t L^4_{x'}} \|\nabla\!\!\!/_{\underline{L}} \partial_\omega N\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/_{\underline{L}}(n^{-1}\nabla_{\partial_\omega N} n)\|_{L^\infty_u L^2(\mathcal{H}_u)} \|\chi\|_{L^\infty_t L^2_{x'}}\\
&+\|\epsilon\|_{L^\infty_t L^4_{x'}} \|\chi\|_{L^\infty_t L^4_{x'}} \|\nabla\!\!\!/_{\underline{L}} \partial_\omega N\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/_{\underline{L}} \partial_\omega N\|_{L^\infty_t L^2_{x'}} \|\beta\|_{L^\infty_u L^2(\mathcal{H}_u)}\\
\lesssim\ & \varepsilon + \|[\nabla\!\!\!/_{\underline{L}}, \nabla\!\!\!/]\chi\|_{L^2_t L^1_{x'}},
\end{aligned}$$

where we used in the last inequality the estimates (2.67)-(2.71) for χ, $\overline{\delta}$, ϵ, and n, the assumption (2.59) for β, the estimate (2.76) for $\partial_\omega N$, and the estimate (2.77) for $\partial_\omega N$ and $\partial_\omega \chi$. Now, in view of the commutator formula (2.45), we have

$$\begin{aligned}
&\|[\nabla\!\!\!/_{\underline{L}}, \nabla\!\!\!/]\chi\|_{L^2_t L^1_{x'}}\\
\lesssim\ & (\|\underline{\chi}\|_{L^\infty_t L^2_{x'}} + \|\underline{\xi}\|_{L^\infty_t L^2_{x'}} + \|b^{-1}\nabla\!\!\!/ b\|_{L^\infty_t L^2_{x'}}) \|\mathbf{D}\chi\|_{L^\infty_u L^2(\mathcal{H}_u)}\\
&+(\|\chi\|_{L^\infty_t L^4_{x'}} (\|\overline{\epsilon}\|_{L^\infty_t L^4_{x'}} + \|\underline{\xi}\|_{L^\infty_t L^4_{x'}}) + \|\underline{\chi}\|_{L^\infty_t L^4_{x'}} \|\zeta\|_{L^\infty_t L^4_{x'}}) (\|\beta\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\underline{\beta}\|_{L^\infty_u L^2(\mathcal{H}_u)})\\
\lesssim\ & \varepsilon,
\end{aligned}$$

where we used in the last inequality the estimates (2.67)-(2.71) for χ, $\underline{\chi}$, $\underline{\xi}$, b, and ζ, and the assumption (2.59) for β and $\underline{\beta}$. Injecting in (C.3), we obtain

$$\|F_4\|_{L^2_t L^1_{x'}} \lesssim \varepsilon. \tag{C.4}$$

Next, we estimate the commutator term in the right-hand side of (C.1). In view of the commutator formula (2.46), we have

$$[\nabla\!\!\!/_{\underline{L}}, \nabla\!\!\!/_L](\Pi(\partial_\omega \chi)) = -\overline{\delta} \nabla\!\!\!/_{\underline{L}}(\Pi(\partial_\omega \chi)) + \nabla\!\!\!/(F_5) + F_6, \tag{C.5}$$

where the tensors F_5 and F_6 are given schematically by

$$F_5 = (\zeta - \underline{\zeta})\Pi(\partial_\omega \chi),$$

and

$$F_6 = (\delta + n^{-1}N(n))\nabla\!\!\!/_L(\Pi(\partial_\omega \chi)) + (\nabla\!\!\!/\zeta - \nabla\!\!\!/\underline{\zeta})\Pi(\partial_\omega \chi) + (\underline{\zeta}\zeta + \sigma)(\Pi(\partial_\omega \chi)).$$

F_5 satisfies the following estimate

$$\|F_5\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim (\|\zeta\|_{L^\infty_{x'} L^2_t} + \|\underline{\zeta}\|_{L^\infty_{x'} L^2_t}) \|\partial_\omega \chi\|_{L^2_{x'} L^\infty_t} \lesssim \varepsilon, \tag{C.6}$$

where we used in the last inequality the estimate (2.68) for $\underline{\zeta}$, the estimate (2.72) for ζ, and the estimate (2.77) for $\partial_\omega\chi$. Next, we estimate F_6. We have

$$\begin{aligned} & \|F_6\|_{L^2_t L^1_{x'}} \\ \lesssim\ & (\|\delta\|_{L^\infty_t L^2_{x'}} + \|n^{-1}N(n)\|_{L^\infty_t L^2_{x'}})\|\nabla\!\!\!/_L(\Pi(\partial_\omega\chi))\|_{L^\infty_u L^2(\mathcal{H}_u)} + (\|\nabla\!\!\!/\zeta\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ & + \|\nabla\!\!\!/\underline{\zeta}\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\zeta\|_{L^\infty_u L^4(\mathcal{H}_u)}\|\underline{\zeta}\|_{L^\infty_u L^4(\mathcal{H}_u)} + \|\sigma\|_{L^\infty_u L^2(\mathcal{H}_u)})\|\partial_\omega\chi\|_{L^\infty_t L^2_{x'}} \\ \lesssim\ & \varepsilon + \|\nabla\!\!\!/_L(\Pi(\partial_\omega\chi))\|_{L^\infty_u L^2(\mathcal{H}_u)}, \end{aligned} \tag{C.7}$$

where we used in the last inequality the estimate (2.67) for n, the estimate (2.68) for δ and $\underline{\zeta}$, the estimate (2.72) for ζ, the assumption (2.59) for σ and the estimate (2.77) for $\partial_\omega\chi$. Now, the estimate (6.34) for $\nabla\!\!\!/_L(\Pi(\partial_\omega\chi))$ together with the estimate (2.77) for $\partial_\omega\chi$ implies

$$\|\nabla\!\!\!/_L(\Pi(\partial_\omega\chi))\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon.$$

Injecting in (C.7), we obtain

$$\|F_6\|_{L^2_t L^1_{x'}} \lesssim \varepsilon. \tag{C.8}$$

Next, we evaluate the term involving $\nabla\!\!\!/_{\underline{L}}\beta$ in the right-hand side of (C.1). In view of the bianchi identity (2.52), we have

$$\partial_\omega N\nabla\!\!\!/_{\underline{L}}\beta = \nabla\!\!\!/(F_7) + F_8, \tag{C.9}$$

where the tensors F_7 and F_8 are given schematically by

$$F_7 = \partial_\omega N\rho + \partial_\omega N\sigma,$$

and

$$F_8 = \nabla\!\!\!/\partial_\omega N(\rho+\sigma) + \partial_\omega N\Big(\mathrm{tr}\underline{\chi}\beta + \widehat{\chi}\underline{\beta} + (\delta + n^{-1}N(n))\beta + \underline{\xi}\cdot\alpha + \zeta\rho + {}^*\zeta\sigma\Big).$$

F_7 satisfies the following estimate

$$\|F_7\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \|\partial_\omega N\|_{L^\infty}(\|\rho\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\sigma\|_{L^\infty_u L^2(\mathcal{H}_u)}) \lesssim \varepsilon. \tag{C.10}$$

Next, we estimate F_8. We have

$$\begin{aligned} \|F_8\|_{L^2_t L^1_{x'}} \lesssim\ & \Big(\|\nabla\!\!\!/\partial_\omega N\|_{L^\infty_t L^2_{x'}} + \|\partial_\omega N\|_{L^\infty}(\|\mathrm{tr}\underline{\chi}\|_{L^\infty_t L^2_{x'}} + \|\widehat{\chi}\|_{L^\infty_t L^2_{x'}} + \|\delta\|_{L^\infty_t L^2_{x'}} \\ & + \|n^{-1}N(n)\|_{L^\infty_t L^2_{x'}} + \|\underline{\xi}\|_{L^\infty_t L^2_{x'}} + \|\zeta\|_{L^\infty_t L^2_{x'}})\Big)\|(\alpha,\beta,\rho,\sigma,\underline{\beta})\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ \lesssim\ & \varepsilon, \end{aligned} \tag{C.11}$$

where we used in the last inequality the estimates (2.67)-(2.72) for $\mathrm{tr}\underline{\chi}$, $\widehat{\chi}$, δ, n, $\underline{\xi}$ and ζ, the assumption (2.59) on $(\alpha,\beta,\rho,\sigma,\underline{\beta})$, and the estimates (2.76) (2.77) for $\partial_\omega N$.

Finally, in view of (C.1), (C.5) and (C.9), we obtain

$$\nabla\!\!\!/_L(\nabla\!\!\!/_{\underline{L}}(\Pi(\partial_\omega\chi))) = -\nabla\!\!\!/_{\underline{L}}\Pi(\partial_\omega\chi)\cdot\chi - \chi\cdot\nabla\!\!\!/_{\underline{L}}(\Pi(\partial_\omega\chi)) + \nabla\!\!\!/(F_1) + F_2,$$

where the tensors F_1 and F_2 are given by

$$F_1 = F_3 + F_5 + F_7 \text{ and } F_2 = F_4 + F_6 + F_8.$$

In view of (C.2), (C.6) and (C.10), we have

$$\|F_1\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon.$$

Also, in view of (C.4), (C.8) and (C.11), we have

$$\|F_2\|_{L^2_t L^1_{x'}} \lesssim \varepsilon.$$

This concludes the proof of the lemma.

C.2 Proof of Lemma 6.10

Applying the estimate (3.66) for transport equations to the transport equation (6.48) for M, we obtain:

$$\begin{aligned} \|M\|_{L^\infty} &\lesssim \|\gamma\|_{L^\infty(P_{0,u})} + \|M \cdot \chi\|_{L^\infty_{x'} L^1_t} \\ &\lesssim 1 + \|M\|_{L^\infty} \|\chi\|_{L^\infty_{x'} L^2_t} \\ &\lesssim 1 + \varepsilon \|M\|_{L^\infty} \end{aligned}$$

where we used the estimates (2.70) (2.71) for χ in the last inequality. This yields:

$$\|M\|_{L^\infty} \lesssim 1. \tag{C.12}$$

Now, since $\nabla\!\!\!/_L \gamma = 0$, we may rewrite the transport equation (6.48) for M as:

$$\nabla\!\!\!/_L (M - \gamma)_{AB} = M_{AC} \chi_{CB},\ (M - \gamma)_{AB} = 0 \text{ on } P_{0,u},$$

Together with the estimate (3.66) for transport equations, the estimates (2.70) (2.71) for χ, and the estimate (C.12), this implies:

$$\begin{aligned} \|M - \gamma\|_{L^\infty} &\lesssim \|M \cdot \chi\|_{L^\infty_{x'} L^1_t} \\ &\lesssim \|M\|_{L^\infty} \|\chi\|_{L^\infty_{x'} L^2_t} \\ &\lesssim \varepsilon. \end{aligned} \tag{C.13}$$

Next, we estimate $\nabla\!\!\!/ M$. We rewrite the transport equation (6.48) for M as:

$$\nabla\!\!\!/_{nL} M = n\chi \cdot M,\ M_{AB} = \gamma_{AB} \text{ on } P_{0,u}.$$

Differentiating with respect to $\nabla\!\!\!/$ and using the commutator formula (2.48), we obtain schematically:

$$\begin{aligned} \nabla\!\!\!/_{nL}(\nabla\!\!\!/ M) &= [\nabla\!\!\!/_{nL}, \nabla\!\!\!/] M + \nabla\!\!\!/ \nabla\!\!\!/_{nL} M \\ &= n\chi \cdot \nabla\!\!\!/ M + (n\chi \cdot \overline{\epsilon} + n\beta + \nabla\!\!\!/(n\chi)) \cdot M \end{aligned}$$

Together with the decomposition (5.31) for $n\beta$ and the decomposition (5.55) for $\nabla\!\!\!/(n\chi)$, we obtain:

$$\nabla\!\!\!/_{nL}(\nabla\!\!\!/ M) = n\chi \cdot \nabla\!\!\!/ M + (n\chi \cdot \overline{\epsilon} + \nabla\!\!\!/_{nL} P + E) \cdot M, \tag{C.14}$$

where P and E satisfy:

$$\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} \lesssim \varepsilon. \tag{C.15}$$

(C.14) and the sharp trace theorem for transport equations (5.22) imply:

$$\begin{aligned} \|\nabla\!\!\!/ M\|_{\mathcal{B}^0} &\lesssim \big(\mathcal{N}_1(n\chi) + \|n\chi\|_{L^\infty_{x'} L^2_t}\big)\|\nabla\!\!\!/ M\|_{\mathcal{P}^0} \\ &\quad + \big(\mathcal{N}_1(M) + \|M\|_{L^\infty_{x'} L^2_t}\big) \cdot (\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} + \|n\chi \cdot \overline{\epsilon}\|_{\mathcal{P}^0}) \\ &\lesssim \varepsilon + \varepsilon \|\nabla\!\!\!/ M\|_{\mathcal{P}^0}, \end{aligned} \tag{C.16}$$

where we used the estimates (2.67) (2.70) (2.71) for n and χ, the estimate (C.12) for M, the estimate (C.15) for P and E, and the estimate:

$$\|n\chi \cdot \overline{\epsilon}\|_{\mathcal{P}^0} \lesssim \mathcal{N}_2(n-1)\mathcal{N}_1(\chi)\mathcal{N}_1(\overline{\epsilon}) \lesssim \varepsilon,$$

which follows form the non sharp product estimates (5.14) (5.15) and the estimates (2.67)-(2.71) for n, $\overline{\epsilon}$ and χ.

Finally, (C.16) yields:

$$\|\nabla\!\!\!/ M\|_{\mathcal{B}^0} \lesssim \varepsilon$$

which together with (C.13) yields the conclusion of Lemma 6.10.

C.3 Proof of Lemma 6.12

(6.52) follows immediately from the sharp Bernstein inequality for tensors (4.37). Then, (6.53) follows immediately from (6.52) by taking the dual. This concludes the proof of Lemma 6.12.

C.4 Proof of Lemma 6.13

It suffices to prove the dual inequality. Let H the solution of the following transport equation:

$$\nabla\!\!\!/_{nL}(H) = P_j F,\ H = 0 \text{ on } P_{1,u}. \tag{C.17}$$

Then, since P_j is selfadjoint, since the dual of $\nabla\!\!\!/$ acting on $P_{t,u}$ tangent tensors is given by $-\nabla\!\!\!/$, and since the dual of $\int_0^t$ is $\int_1^t$, we infer by duality that the conclusion of Lemma 6.13 is equivalent to the estimate:

$$\|\nabla\!\!\!/ H\|_{L^\infty_t L^2_{x'}} \lesssim 2^j \|F\|_{L^p_t L^2_{x'}}, \tag{C.18}$$

for any $1 < p \leq 2$.

From now on, we focus on proving (C.18). Note first from the estimate on transport equations (3.66) and the transport equation (C.17) satisfied by H:

$$\|H\|_{L^\infty} \lesssim \|P_j F\|_{L^1_t L^\infty_{x'}} \lesssim 2^j \|F\|_{L^p_t L^2_{x'}}, \tag{C.19}$$

where we used in the last inequality the sharp Bernstein inequality for tensors (6.52).

Next, we differentiate the transport equation (C.17) for H with respect to $\nabla\!\!\!/$:

$$\begin{aligned}\nabla\!\!\!/_{nL}(\nabla\!\!\!/ H) &= [\nabla\!\!\!/_{nL}, \nabla\!\!\!/]H + \nabla\!\!\!/\nabla\!\!\!/_{nL}(H)\\ &= n\chi\cdot\nabla\!\!\!/ H + (n\chi\cdot\overline{\epsilon}+n\beta)\cdot H + \nabla\!\!\!/ P_j F,\end{aligned}$$

where we used in the last equality the commutator formula (2.48). Together with the estimate for transport equations (3.66), this yields:

$$\begin{aligned}&\|\nabla\!\!\!/ H\|_{L^\infty_t L^2_{x'}} \qquad\qquad (C.20)\\ \lesssim\ & \|n\chi\cdot\nabla\!\!\!/ H + (n\chi\cdot\overline{\epsilon}+n\beta)\cdot H + \nabla\!\!\!/ P_j F\|_{L^2_{x'}L^1_t}\\ \lesssim\ & \|n\|_{L^\infty}\big(\|\chi\|_{L^\infty_{x'}L^2_t}\|\nabla\!\!\!/ H\|_{L^2(\mathcal{H}_u)} + (\mathcal{N}_1(\chi)\mathcal{N}_1(\overline{\epsilon}) + \|\beta\|_{L^2(\mathcal{H}_u)})\|H\|_{L^\infty}\big) + \|\nabla\!\!\!/ P_j F\|_{L^1_t L^2_{x'}}\\ \lesssim\ & \varepsilon\|\nabla\!\!\!/ H\|_{L^2(\mathcal{H}_u)} + 2^j\|F\|_{L^p_t L^2_{x'}},\end{aligned}$$

where we used in the last inequality the estimates (2.67)-(2.71) for $n, \overline{\epsilon}$ and χ, the estimate (C.19) for H, and the finite band property for P_j.

Finally, (C.20) yields (C.18) which concludes the proof of Lemma 6.13

C.5 Proof of Lemma 6.14

Using the product estimate (5.10), we have:

$$\begin{aligned}\|\nabla\!\!\!/(M^{-1})\|_{\mathcal{B}^0} &= \|M^{-1}(\nabla\!\!\!/ M)M^{-1}\|_{\mathcal{B}^0} \qquad (C.21)\\ &\lesssim (\|\nabla\!\!\!/(M^{-1})\|_{L^\infty_t L^2_{x'}} + \|M^{-1}\|_{L^\infty})^2\|\nabla\!\!\!/ M\|_{\mathcal{B}^0}\\ &\lesssim (\|\nabla\!\!\!/ M\|_{L^\infty_t L^2_{x'}}\|M^{-1}\|^2_{L^\infty} + \|M^{-1}\|_{L^\infty})^2\|\nabla\!\!\!/ M\|_{\mathcal{B}^0}\\ &\lesssim \varepsilon\end{aligned}$$

where we used in the last inequality the fact that $\|M-\gamma\|_{L^\infty} + \|\nabla\!\!\!/ M\|_{\mathcal{B}^0} \lesssim \varepsilon$ from the assumptions of Lemma 6.14. Then, in view of (C.21), Lemma 6.14 is an immediate consequence of the following slightly more general lemma.

Lemma C.1 *Let F a $P_{t,u}$-tangent tensor and $2 < p \leq +\infty$ such that for all $j \geq 0$:*

$$\|P_j F\|_{L^p_t L^2_{x'}} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u).$$

Also, let H a $P_{t,u}$-tangent tensor such that for any $2 \leq r < +\infty$, we have

$$\|H\|_{L^r_t L^\infty_{x'}} + \|\nabla\!\!\!/ H\|_{L^r_t B^0_{2,1}(P_{t,u})} \lesssim 1,$$

where $B^0_{2,1}(P_{t,u})$ has been defined in (5.4). Then, we have for any $2 \leq q < p$ and all $j \geq 0$:

$$\|P_j(HF)\|_{L^q_t L^2_{x'}} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u).$$

We conclude this section with the proof of Lemma C.1. Using the property (3.16) of the Littlewood-Paley projections, we have:

$$\|P_j(HF)\|_{L_t^q L_{x'}^2} \lesssim \sum_l \|P_j(HP_lF)\|_{L_t^q L_{x'}^2}. \tag{C.22}$$

We estimate the right-hand side of (C.22). Using the L^2 boundedness of P_j, the assumption $\|H\|_{L_t^r L_{x'}^\infty} \lesssim 1$ on H with r large enough, and the assumption for F:

$$\|P_j(HP_lF)\|_{L_t^q L_{x'}^2} \lesssim \|H\|_{L_t^r L_{x'}^\infty} \|P_lF\|_{L_t^p L_{x'}^2} \lesssim 2^l\varepsilon + 2^{\frac{l}{2}}\varepsilon\gamma(u). \tag{C.23}$$

We will need a second estimate for the right-hand side of (C.22). Recall from (B.5) the existence of a Littlewood-Paley projector $\widetilde{P}_l$ associated to a function $\widetilde{m}$ satisfying the same properties than m, such that

$$P_l = 2^{-2l}\not{\Delta}\widetilde{P}_l. \tag{C.24}$$

We infer:

$$\begin{aligned} & \|P_j(HP_lF)\|_{L_t^q L_{x'}^2} \\ = \;& 2^{-2l}\|P_j(H\not{\Delta}\widetilde{P}_lF)\|_{L_t^q L_{x'}^2} \\ \lesssim \;& 2^{-2l}\|P_j \not{\mathrm{div}}(H\not{\nabla}\widetilde{P}_lF)\|_{L_t^q L_{x'}^2} + 2^{-2l}\|P_j(\not{\nabla}H\not{\nabla}\widetilde{P}_lF)\|_{L_t^q L_{x'}^2} \\ \lesssim \;& 2^{-2l}\|P_j\not{\Delta}(H\widetilde{P}_lF)\|_{L_t^q L_{x'}^2} + 2^{-2l}\|P_j \not{\mathrm{div}}(\not{\nabla}H\widetilde{P}_lF)\|_{L_t^q L_{x'}^2} + 2^{-2l}\|P_j(\not{\nabla}H\not{\nabla}\widetilde{P}_lF)\|_{L_t^q L_{x'}^2}. \end{aligned} \tag{C.25}$$

Next, we bound the three terms in the right-hand side of (C.25) starting with the first one. Using the finite band property for P_j, we have:

$$\begin{aligned} \|P_j\not{\Delta}(H\widetilde{P}_lF)\|_{L_t^q L_{x'}^2} &\lesssim 2^{2j}\|H\widetilde{P}_lF\|_{L_t^q L_{x'}^2} \\ &\lesssim 2^{2j}\|H\|_{L_t^r L_{x'}^\infty}\|\widetilde{P}_lF\|_{L_t^p L_{x'}^2} \\ &\lesssim 2^{2j}(2^l\varepsilon + 2^{\frac{l}{2}}\varepsilon\gamma(u)), \end{aligned} \tag{C.26}$$

where we used the assumption $\|H\|_{L_t^r L_{x'}^\infty} \lesssim 1$ on H, the fact that we may choose r large enough, and the assumption for F.

Next, we estimate the second term in the right-hand side of (C.25). Using the property (3.16) of the Littlewood-Paley projections, we have:

$$\|P_j \not{\mathrm{div}}(\not{\nabla}(H)\widetilde{P}_lF)\|_{L_t^q L_{x'}^2} \lesssim \sum_m \|P_j \not{\mathrm{div}}(P_m(\not{\nabla}H)\widetilde{P}_lF)\|_{L_t^q L_{x'}^2}. \tag{C.27}$$

Using the finite band property for P_j, and the weak Bernstein inequality for $\widetilde{P}_l$ and P_m, we have:

$$\begin{aligned} \|P_j \not{\mathrm{div}}(P_m(\not{\nabla}H)\widetilde{P}_lF)\|_{L_t^q L_{x'}^2} &\lesssim 2^j\|P_m(\not{\nabla}H)\widetilde{P}_lF\|_{L_t^q L_{x'}^2} \\ &\lesssim 2^j\|P_m(\not{\nabla}H)\|_{L_t^r L_{x'}^4}\|\widetilde{P}_lF\|_{L_t^p L_{x'}^4} \\ &\lesssim 2^{j+\frac{m}{2}+\frac{l}{2}}\|P_m(\not{\nabla}H)\|_{L_t^r L_{x'}^2}\|\widetilde{P}_lF\|_{L_t^p L_{x'}^2} \\ &\lesssim 2^{j+\frac{m}{2}+\frac{l}{2}}\|P_m(\not{\nabla}H)\|_{L_t^r L_{x'}^2}(2^l\varepsilon + 2^{\frac{l}{2}}\varepsilon\gamma(u)), \end{aligned} \tag{C.28}$$

where we used the assumption for F.

We will need another estimate for the right-hand side of (C.27). First, let q_+ such that $q < q_+ < p_- < p$. Using the finite band property for P_j, we have for r large enough:

$$\begin{aligned}\|P_j \mathrm{div}\!\!\!/(P_m(\nabla\!\!\!/ H)\widetilde{P}_l F)\|_{L_t^{q_+}L_{x'}^2} &\lesssim 2^j \|P_m(\nabla\!\!\!/ H)\widetilde{P}_l F\|_{L_t^{q_+}L_{x'}^2} \\ &\lesssim 2^j \|P_m(\nabla\!\!\!/ H)\|_{L_t^r L_{x'}^2}\|\widetilde{P}_l F\|_{L_t^{p_-}L_{x'}^\infty} \\ &\lesssim 2^{j+l}\|P_m(\nabla\!\!\!/ H)\|_{L_t^r L_{x'}^2}\|\widetilde{P}_l F\|_{L_t^p L_{x'}^2},\end{aligned} \tag{C.29}$$

where we used in the last inequality the sharp Bernstein inequality for tensors (6.52). Also, using again (C.24), this time for P_m, we have:

$$\begin{aligned}&\|P_j \mathrm{div}\!\!\!/(P_m(\nabla\!\!\!/ H)\widetilde{P}_l F)\|_{L^2(P_{t,u})} \\ =\ & 2^{-2m}\|P_j \mathrm{div}\!\!\!/(\Delta\!\!\!/(\widetilde{P}_m(\nabla\!\!\!/ H))\widetilde{P}_l F)\|_{L^2(P_{t,u})} \\ \lesssim\ & 2^{-2m}\|P_j \mathrm{div}\!\!\!/\mathrm{div}\!\!\!/(\nabla\!\!\!/(\widetilde{P}_m(\nabla\!\!\!/ H))\widetilde{P}_l F)\|_{L^2(P_{t,u})} + 2^{-2m}\|P_j \mathrm{div}\!\!\!/(\nabla\!\!\!/(\widetilde{P}_m(\nabla\!\!\!/ H))\nabla\!\!\!/\widetilde{P}_l F)\|_{L^2(P_{t,u})}.\end{aligned}$$

Together with the finite band property for P_j, this yields:

$$\begin{aligned}&\|P_j \mathrm{div}\!\!\!/(P_m(\nabla\!\!\!/ H)\widetilde{P}_l F)\|_{L^2(P_{t,u})} \\ \lesssim\ & 2^{-2m}\|P_j \mathrm{div}\!\!\!/\mathrm{div}\!\!\!/\|_{\mathcal{L}(L^2(P_{t,u}))}\|\nabla\!\!\!/(\widetilde{P}_m(\nabla\!\!\!/ H))\widetilde{P}_l F\|_{L^2(P_{t,u})} + 2^{j-2m}\|\nabla\!\!\!/(\widetilde{P}_m(\nabla\!\!\!/ H))\nabla\!\!\!/\widetilde{P}_l F\|_{L^2(P_{t,u})} \\ \lesssim\ & 2^{-2m}\|\nabla\!\!\!/^2 P_j\|_{\mathcal{L}(L^2(P_{t,u}))}\|\nabla\!\!\!/(\widetilde{P}_m(\nabla\!\!\!/ H))\|_{L^2(P_{t,u})}\|\widetilde{P}_l F\|_{L^\infty(P_{t,u})} \\ & + 2^{j-2m}\|\nabla\!\!\!/(\widetilde{P}_m(\nabla\!\!\!/ H))\|_{L^4(P_{t,u})}\|\nabla\!\!\!/\widetilde{P}_l F\|_{L^4(P_{t,u})} \\ \lesssim\ & 2^{-m}\|\nabla\!\!\!/^2 P_j\|_{\mathcal{L}(L^2(P_{t,u}))}\|\widetilde{P}_m(\nabla\!\!\!/ H)\|_{L^2(P_{t,u})}\|\widetilde{P}_l F\|_{L^\infty(P_{t,u})} \\ & + 2^{j-2m}\Big(\|\nabla\!\!\!/^2(\widetilde{P}_m(\nabla\!\!\!/ H))\|_{L^2(P_{t,u})}^{\frac{1}{2}}\|\nabla\!\!\!/(\widetilde{P}_m(\nabla\!\!\!/ H))\|_{L^2(P_{t,u})}^{\frac{1}{2}} + \|\nabla\!\!\!/(\widetilde{P}_m(\nabla\!\!\!/ H))\|_{L^2(P_{t,u})}\Big) \\ & \times\Big(\|\nabla\!\!\!/^2\widetilde{P}_l F\|_{L^2(P_{t,u})}^{\frac{1}{2}}\|\nabla\!\!\!/\widetilde{P}_l F\|_{L^2(P_{t,u})}^{\frac{1}{2}} + \|\nabla\!\!\!/\widetilde{P}_l F\|_{L^2(P_{t,u})}\Big),\end{aligned}$$

where we used in the last inequality the Gagliardo-Nirenberg inequality (3.3) and the finite band property for $\widetilde{P}_l$. Finally, using the Bochner inequality for tensors (3.7), the sharp Bernstein inequality (4.37) for tensors, and the fact that $p > 2$, we obtain for r large enough:

$$\|P_j \mathrm{div}\!\!\!/(\widetilde{P}_m(\nabla\!\!\!/ H)\widetilde{P}_l F)\|_{L_t^1 L_{x'}^2} \lesssim (2^{2j+l-m} + 2^{j-\frac{m}{2}+\frac{3l}{2}})\|\widetilde{P}_m(\nabla\!\!\!/ H)\|_{L_t^r L_{x'}^2}\|\widetilde{P}_l F\|_{L_t^p L_{x'}^2},$$

which in the case $j < l < m$ implies:

$$\|P_j \mathrm{div}\!\!\!/(P_m(\nabla\!\!\!/ H)\widetilde{P}_l F)\|_{L_t^1 L_{x'}^2} \lesssim 2^{j-\frac{m}{2}+\frac{3l}{2}}\|\widetilde{P}_m(\nabla\!\!\!/ H)\|_{L_t^r L_{x'}^2}\|\widetilde{P}_l F\|_{L_t^p L_{x'}^2}. \tag{C.30}$$

Since $1 < q < q_+$, we may interpolate (C.29) and (C.30). We obtain for $j < l < m$:

$$\begin{aligned}\|P_j \mathrm{div}\!\!\!/(P_m(\nabla\!\!\!/ H)\widetilde{P}_l F)\|_{L_t^q L_{x'}^2} &\lesssim 2^{j+l-|m-l|_+}\|P_m(\nabla\!\!\!/ H)\|_{L_t^r L_{x'}^2}\|\widetilde{P}_l F\|_{L_t^p L_{x'}^2} \\ &\lesssim 2^{j+l-(m-l)_+}\|P_m(\nabla\!\!\!/ H)\|_{L_t^r L_{x'}^2}(2^l\varepsilon + 2^{\frac{l}{2}}\varepsilon\gamma(u)),\end{aligned} \tag{C.31}$$

where we used the assumption for F. Now, using (C.27), (C.28) for $m \leq l$ and (C.31) for $j < l < m$ yield for any $j < l$:

$$\|P_j \mathrm{div}((\nabla\!\!\!/ H)\widetilde{P}_l F)\|_{L_t^q L_{x'}^2} \lesssim \sum_m 2^{j+l-|m-l|_+} \|P_m(\nabla\!\!\!/ H)\|_{L_t^r L_{x'}^2}(2^l\varepsilon + 2^{\frac{l}{2}}\varepsilon\gamma(u)). \tag{C.32}$$

The the third term in the right-hand side of (C.25) satisfies for r large enough the following estimate:

$$\|P_j(\nabla\!\!\!/ H \nabla\!\!\!/ \widetilde{P}_l F)\|_{L_t^q L_{x'}^2} \lesssim \sum_m 2^{j+l-|m-l|_+} \|P_m(\nabla\!\!\!/ H)\|_{L_t^r L_{x'}^2}(2^l\varepsilon + 2^{\frac{l}{2}}\varepsilon\gamma(u)). \tag{C.33}$$

The proof of (C.33) is similar to the proof of (C.32), so we skip it.

(C.25), (C.26), (C.32) and (C.33) yield for any $j < l$:

$$\begin{aligned} \|P_j(HP_lF)\|_{L_t^q L_{x'}^2} \lesssim\ & 2^{2j-2l}(2^l\varepsilon + 2^{\frac{l}{2}}\varepsilon\gamma(u)) \\ & + \sum_m 2^{j-l-|m-l|_+} \|P_m(\nabla\!\!\!/ H)\|_{L_t^r L_{x'}^2}(2^l\varepsilon + 2^{\frac{l}{2}}\varepsilon\gamma(u)), \end{aligned} \tag{C.34}$$

where r is large enough. Finally, summing (C.22) for $l \leq j$ and (C.34) for $l > j$ implies for r large enough:

$$\sum_l \|P_j(HP_lF)\|_{L_t^q L_{x'}^2} \lesssim (1 + \|\nabla\!\!\!/ H\|_{L_t^r B^0_{2,1}(P_{t,u})})(2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u)),$$

which together with the bound (C.21) for H and the inequality (C.22) yields:

$$\|P_j(HF)\|_{L_t^q L_{x'}^2} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\varepsilon\gamma(u).$$

This concludes the proof of Lemma C.1.

C.6 Proof of Lemma 6.16

Using the property (3.16) of the Littlewood-Paley projections, we have:

$$\|F \cdot H\|_{L_t^r B^0_{2,1}(P_{t,u})} \lesssim \sum_{j,q,l} \|P_j(P_q(F) \cdot P_l(H))\|_{L_t^r L_{x'}^2}. \tag{C.35}$$

Note first that (6.85) is symmetric with respect to F and H. Thus, we may assume for instance $l \leq q$ in (C.35). We will estimate the right-hand side (C.35) in the two cases $q \leq j$ and $q > j$ starting with the first one. We have:

$$\begin{aligned} & \|P_j(P_q(F) \cdot P_l(H))\|_{L^2(P_{t,u})} \\ \lesssim\ & 2^{-2j}\|P_j \not{\Delta}(P_q(F) \cdot P_l(H))\|_{L^2(P_{t,u})} \\ \lesssim\ & 2^{-2j}\|P_j \mathrm{div}(\nabla\!\!\!/ P_q(F) \cdot P_l(H))\|_{L^2(P_{t,u})} + 2^{-2j}\|P_j \mathrm{div}(P_q(F) \cdot \nabla\!\!\!/ P_l(H))\|_{L^2(P_{t,u})}, \end{aligned}$$

which yields:

$$
\begin{aligned}
&\|P_j(P_q(F)\cdot P_l(H))\|_{L^2(P_{t,u})} \qquad\qquad (C.36)\\
\lesssim\ & 2^{-2j}\|P_j \not{\mathrm{div}}\|_{\mathcal{L}(L^p(P_{t,u}),L^2(P_{t,u}))}\Big(\|\not{\nabla}P_q(F)\cdot P_l(H))\|_{L^p(P_{t,u})}+\|P_q(F)\cdot \not{\nabla}P_l(H))\|_{L^p(P_{t,u})}\Big)\\
\lesssim\ & 2^{-2j}\|P_j \not{\mathrm{div}}\|_{\mathcal{L}(L^p(P_{t,u}),L^2(P_{t,u}))}\Big(\|\not{\nabla}P_q(F)\|_{L^2(P_{t,u})}\|P_l(H))\|_{L^{\frac{2p}{2-p}}(P_{t,u})}\\
&+\|P_q(F)\|_{L^{\frac{2p}{2-p}}(P_{t,u})}\|\not{\nabla}P_l(H))\|_{L^2(P_{t,u})}\Big)\\
\lesssim\ & 2^{-2j}(2^{(2-\frac{2}{p})l+q}+2^{(2-\frac{2}{p})q+l})\|P_j \not{\mathrm{div}}\|_{\mathcal{L}(L^p(P_{t,u}),L^2(P_{t,u}))}\|P_q(F)\|_{L^2(P_{t,u})}\|P_l(H))\|_{L^2(P_{t,u})},
\end{aligned}
$$

where $1<p<2$ will be chosen later, and where we used the finite band property for P_l and P_q, and the weak Bernstein inequality for P_l and P_q. In view of (C.36) we need to evaluate $\|P_j \not{\mathrm{div}}\|_{\mathcal{L}(L^p(P_{t,u}),L^2(P_{t,u}))}$. Let p' the conjugate exponent of p, i.e. $\frac{1}{p}+\frac{1}{p'}=1$. Using the Gagliardo-Nirenberg inequality (3.3), we have:

$$
\begin{aligned}
&\|\not{\nabla}P_jF\|_{L^{p'}(P_{t,u})}\\
\lesssim\ & \|\not{\nabla}P_jF\|^{\frac{2}{p'}}_{L^2(P_{t,u})}\|\not{\nabla}^2P_jF\|^{1-\frac{2}{p'}}_{L^2(P_{t,u})}+\|\not{\nabla}P_jF\|_{L^2(P_{t,u})}\\
\lesssim\ & 2^{\frac{2j}{p'}}\|P_jF\|^{\frac{2}{p'}}_{L^2(P_{t,u})}\big(\|\not{\Delta}P_jF\|_{L^2(P_{t,u})}+(\|K\|_{L^2(P_{t,u})}+\|K\|^{\frac{1}{2}}_{L^2(P_{t,u})})\|\not{\nabla}P_jF\|_{L^2(P_{t,u})}\\
&+\|K\|^2_{L^2(P_{t,u})}\|P_jF\|_{L^2(P_{t,u})}\big)^{1-\frac{2}{p'}}+2^j\|P_jF\|_{L^2(P_{t,u})}\\
\lesssim\ & (1+\|K\|^{2-\frac{4}{p'}}_{L^2(P_{t,u})})2^{\frac{2j}{p}}\|P_jF\|_{L^2(P_{t,u})}, \qquad\qquad (C.37)
\end{aligned}
$$

where we used the weak Bernstein inequality and the Bochner inequality for tensors (3.7). In view of (C.37), we have:

$$
\|P_j \not{\mathrm{div}}\|_{\mathcal{L}(L^p(P_{t,u}),L^2(P_{t,u}))}=\|\not{\nabla}P_j\|_{\mathcal{L}(L^2(P_{t,u}),L^{p'}(P_{t,u}))}\lesssim(1+\|K\|^{2-\frac{4}{p'}}_{L^2(P_{t,u})})2^{\frac{2j}{p}},
$$

which together with (C.36) implies:

$$
\begin{aligned}
&\|P_j(P_q(F)\cdot P_l(H))\|_{L^2(P_{t,u})}\\
\lesssim\ & 2^{-(2-\frac{2}{p})j}(2^{(2-\frac{2}{p})l+q}+2^{(2-\frac{2}{p})q+l})(1+\|K\|^{2-\frac{4}{p'}}_{L^2(P_{t,u})})\|P_q(F)\|_{L^2(P_{t,u})}\|P_l(H))\|_{L^2(P_{t,u})}.
\end{aligned}
$$

We fix $p=\frac{2r}{r+1}$ which satisfies $1<p<2$. Using the estimate (4.29) for K, and the fact that $l\le q\le j$, we obtain:

$$
\|P_jP_q(F)\cdot P_l(H)\|_{L^r_tL^2_{x'}}\lesssim 2^{-(\frac{1}{2}-\frac{1}{r})|j-q|-(\frac{1}{2}-\frac{1}{r})|j-l|}2^{\frac{q}{2}}\|P_q(F)\|_{L^\infty_tL^2_{x'}}2^{\frac{l}{2}}\|P_l(H))\|_{L^\infty_tL^2_{x'}}. \quad (C.38)
$$

Next, we consider the case $q>j$. Using the weak Bernstein inequality for P_j and P_l, we have:

$$
\begin{aligned}
\|P_j(P_q(F)\cdot P_l(H))\|_{L^2(P_{t,u})} &\lesssim 2^{\frac{j}{3}}\|P_q(F)\cdot P_l(H)\|_{L^{\frac{3}{2}}(P_{t,u})}\\
&\lesssim 2^{\frac{j}{3}}\|P_q(F)\|_{L^2(P_{t,u})}\|P_l(H)\|_{L^6(P_{t,u})}\\
&\lesssim 2^{\frac{j}{3}+\frac{2l}{3}}\|P_q(F)\|_{L^2(P_{t,u})}\|P_l(H)\|_{L^2(P_{t,u})}\\
&\lesssim 2^{-\frac{|j-q|}{6}-\frac{|l-j|}{6}}2^{\frac{q}{2}}\|P_q(F)\|_{L^2(P_{t,u})}2^{\frac{l}{2}}\|P_l(H)\|_{L^2(P_{t,u})}
\end{aligned}
$$

where we used the fact that $q > j$ and $q \geq l$. This yields:

$$\|P_j(P_q(F) \cdot P_l(H))\|_{L_t^\infty L_{x'}^2} \lesssim 2^{-\frac{|j-q|}{6}-\frac{|l-j|}{6}} 2^{\frac{q}{2}} \|P_q(F)\|_{L_t^\infty L_{x'}^2} 2^{\frac{l}{2}} \|P_l(H)\|_{L_t^\infty L_{x'}^2}. \tag{C.39}$$

Recall from (5.87) that:

$$\sum_q 2^q \|P_q(F)\|^2_{L_t^\infty L_{x'}^2} \lesssim \mathcal{N}_1(F)^2 \text{ and } \sum_l 2^l \|P_l(H)\|^2_{L_t^\infty L_{x'}^2} \lesssim \mathcal{N}_1(H)^2. \tag{C.40}$$

(C.35), (C.38), (C.39) and (C.40) imply (6.85) which concludes the proof of Lemma 6.16.

C.7 Proof of Lemma 6.17

Since $h = (\rho, \sigma)$, Lemma 6.21 yields:

$$\|P_l h\|_{L_t^\infty L_{x'}^2} \lesssim \varepsilon 2^{\frac{l}{2}}. \tag{C.41}$$

We estimate the quantity $\|P_j \mathcal{D}_2^{-1}(F P_l h)\|_{L_t^\infty L_{x'}^{4-}}$. Using the weak Bernstein inequality and the finite band property for P_j, we have:

$$\begin{aligned} \|P_j \mathcal{D}_2^{-1}(F P_l h)\|_{L_t^\infty L_{x'}^{4-}} &\lesssim 2^{-j(\frac{1}{2})+} \|\nabla\!\!\!/ \mathcal{D}_2^{-1}(F P_l h)\|_{L_t^\infty L_{x'}^2} \\ &\lesssim 2^{-j(\frac{1}{2})+} \|F P_l h\|_{L_t^\infty L_{x'}^2} \\ &\lesssim 2^{-j(\frac{1}{2})+} \|F\|_{L^\infty} \|P_l h\|_{L_t^\infty L_{x'}^2} \\ &\lesssim 2^{-j(\frac{1}{2})+ + \frac{l}{2}} \|F\|_{L^\infty} \varepsilon, \end{aligned} \tag{C.42}$$

where we used the estimate (3.50) for $\mathcal{D}_2^{-1}$ and the estimate (C.41) for h.

We derive a second estimate for $\|P_j \mathcal{D}_2^{-1}(F \cdot P_l h)\|_{L_t^\infty L_{x'}^{4-}}$. We have:

$$\begin{aligned} & \|P_j \mathcal{D}_2^{-1}(F P_l h)\|_{L_t^\infty L_{x'}^{4-}} \\ = \; & 2^{-2l} \|P_j \mathcal{D}_2^{-1}(F \Delta\!\!\!/ P_l h)\|_{L_t^\infty L_{x'}^{4-}} \\ \lesssim \; & 2^{-2l} \|P_j \mathcal{D}_2^{-1} \mathrm{div}\!\!\!/(F \nabla\!\!\!/ P_l h)\|_{L_t^\infty L_{x'}^{4-}} + 2^{-2l} \|P_j \mathcal{D}_2^{-1}(\nabla\!\!\!/ F \nabla\!\!\!/ P_l h)\|_{L_t^\infty L_{x'}^{4-}} \end{aligned}$$

which together with the weak Bernstein inequality for P_j yields:

$$\begin{aligned} & \|P_j \mathcal{D}_2^{-1}(F P_l h)\|_{L_t^\infty L_{x'}^{4-}} \\ \lesssim \; & 2^{j(\frac{1}{2})- - 2l} \|\mathcal{D}_2^{-1} \mathrm{div}\!\!\!/(F \nabla\!\!\!/ P_l h)\|_{L_t^\infty L_{x'}^2} + 2^{j(\frac{1}{2})- - 2l} \|\mathcal{D}_2^{-1}(\nabla\!\!\!/ F \nabla\!\!\!/ P_l h)\|_{L_t^\infty L_{x'}^2} \\ \lesssim \; & 2^{j(\frac{1}{2})- - 2l} \|F \nabla\!\!\!/ P_l h\|_{L_t^\infty L_{x'}^2} + 2^{j(\frac{1}{2})- - 2l} \|\nabla\!\!\!/ F \nabla\!\!\!/ P_l h\|_{L_t^\infty L_{x'}^{1+}} \\ \lesssim \; & 2^{j(\frac{1}{2})- - 2l} \|F\|_{L^\infty} \|\nabla\!\!\!/ P_l h\|_{L_t^\infty L_{x'}^2} + 2^{j(\frac{1}{2})- - 2l} \|\nabla\!\!\!/ F\|_{L_t^\infty L_{x'}^2} \|\nabla\!\!\!/ P_l h\|_{L_t^\infty L_{x'}^{2+}} \\ \lesssim \; & 2^{j(\frac{1}{2})- - l(\frac{1}{2})-} (\|F\|_{L^\infty} + \|\nabla\!\!\!/ F\|_{L_t^\infty L_{x'}^2}) \varepsilon, \end{aligned} \tag{C.43}$$

where we used the estimate (3.50) and Remark 3.15 for $\mathcal{D}_2^{-1}$, the Gagliardo-Nirenberg inequality (3.3), the Bochner identity for scalars (4.34), and the estimate (C.41) for h. In the last inequality, note that as soon as 4_- is fixed in the $L_t^\infty L_{x'}^{4_-}$ norm, then $(\frac{1}{2})_-$ is fixed in $j(\frac{1}{2})_-$. Let us fix $j(\frac{1}{2})_- = j(\frac{1}{2}-a)$ for some $a>0$, then we may choose $l(\frac{1}{2})_- = l(\frac{1}{2}-\frac{a}{2})$ in order to obtain:

$$2^{j(\frac{1}{2})_- - l(\frac{1}{2})_-} = 2^{j(\frac{1}{2}-a)-l(\frac{1}{2}-\frac{a}{2})},$$

which together with (C.43) yields:

$$\|P_j \mathcal{D}_2^{-1}(FP_l h)\|_{L_t^\infty L_{x'}^{4_-}} \lesssim 2^{j(\frac{1}{2}-a)-l(\frac{1}{2}-\frac{a}{2})}(\|F\|_{L^\infty} + \|\nabla\!\!\!/ F\|_{L_t^\infty L_{x'}^2})\varepsilon. \tag{C.44}$$

Summing on j and l and taking (C.42) for $l \leq j$ and (C.44) for $l > j$, we obtain:

$$\begin{aligned}
\|\mathcal{D}_2^{-1}(Fh)\|_{L_t^\infty L_{x'}^{4_-}} &\lesssim \sum_{j,l} \|P_j \mathcal{D}_2^{-1}(FP_l h)\|_{L_t^\infty L_{x'}^{4_-}} \\
&\lesssim \left(\sum_{l \leq j} 2^{-j(\frac{1}{2})_+ + \frac{l}{2}} + \sum_{l>j} 2^{j(\frac{1}{2}-a)-l(\frac{1}{2}-\frac{a}{2})} \right) (\|F\|_{L^\infty} + \|\nabla\!\!\!/ F\|_{L_t^\infty L_{x'}^2})\varepsilon \\
&\lesssim (\|F\|_{L^\infty} + \|\nabla\!\!\!/ F\|_{L_t^\infty L_{x'}^2})\varepsilon
\end{aligned}$$

which yields the conclusion of the Lemma.

C.8 Proof of Lemma 6.18

Since $h = (\rho, \sigma)$, Lemma 6.21 yields:

$$\|P_l h\|_{L_t^\infty L_{x'}^2} \lesssim \varepsilon 2^{\frac{l}{2}}. \tag{C.45}$$

We have:

$$\|\mathcal{D}_2^{-1} b^{-1} \mathcal{D}_2^{-1}(b\nabla\!\!\!/ h)\|_{L_t^p L_{x'}^{4_-}} \lesssim \|\mathcal{D}_2^{-1} b^{-1} \mathcal{D}_2^{-1}((\nabla\!\!\!/ b)h)\|_{L_t^p L_{x'}^{4_-}} + \|\mathcal{D}_2^{-1} b^{-1} \mathcal{D}_2^{-1}(\nabla\!\!\!/(bh)\|_{L_t^p L_{x'}^{4_-}}. \tag{C.46}$$

Lemma 6.19 applied to the first term in the right-hand side of (C.46) with $F = \nabla\!\!\!/ b$ and $H = h = (\rho, \sigma)$ yields:

$$\|\mathcal{D}_2^{-1} b^{-1} \mathcal{D}_2^{-1}((\nabla\!\!\!/ b)h)\|_{L_t^p L_{x'}^{4_-}} \lesssim \mathcal{N}_1(\nabla\!\!\!/ b)\varepsilon \lesssim \varepsilon, \tag{C.47}$$

where we used the estimate (2.69) for b.

Next, we evaluate $\|P_l(bP_j h)\|_{L_t^\infty L_{x'}^2}$. Using the L^2 boundedness of P_l, we have:

$$\begin{aligned}
\|P_l(bP_j h)\|_{L_t^\infty L_{x'}^2} &\lesssim \|bP_j h\|_{L_t^\infty L_{x'}^2} \\
&\lesssim \|b\|_{L^\infty} \|P_j h\|_{L_t^\infty L_{x'}^2} \\
&\lesssim 2^{\frac{j}{2}} \varepsilon,
\end{aligned} \tag{C.48}$$

where we used the estimate (2.69) for b and the estimate (C.45) for h.

We derive a second estimate for $\|P_l(bP_jh)\|_{L_t^\infty L_{x'}^2}$. We have:

$$\begin{aligned}\|P_l(bP_jh)\|_{L_t^\infty L_{x'}^2} &\lesssim 2^{-2j}\|P_l(b\not\Delta P_jh)\|_{L_t^\infty L_{x'}^2}\\ &\lesssim 2^{-2j}\|P_l\not{\rm div}(b\not\nabla P_jh)\|_{L_t^\infty L_{x'}^2}+2^{-2j}\|P_l(\not\nabla b\not\nabla P_jh)\|_{L_t^\infty L_{x'}^2}\end{aligned}$$

which together with the finite band property and the weak Bernstein inequality for P_l yields:

$$\begin{aligned}\|P_l(bP_jh)\|_{L_t^\infty L_{x'}^2} &\lesssim 2^{l-2j}\|b\not\nabla P_jh\|_{L_t^\infty L_{x'}^2}+2^{\frac{l}{2}-2j}\|\not\nabla b\not\nabla P_jh\|_{L_t^\infty L_{x'}^{\frac{4}{3}}} \qquad \text{(C.49)}\\ &\lesssim 2^{l-2j}\|b\|_{L^\infty}\|\not\nabla P_jh\|_{L_t^\infty L_{x'}^2}+2^{\frac{l}{2}-2j}\|\not\nabla b\|_{L_t^\infty L_{x'}^4}\|\not\nabla P_jh\|_{L_t^\infty L_{x'}^2}\\ &\lesssim 2^{l-j}\|P_jh\|_{L_t^\infty L_{x'}^2}+2^{\frac{l}{2}-j}\mathcal{N}_1(\not\nabla b)\|P_jh\|_{L_t^\infty L_{x'}^2}\\ &\lesssim 2^{l-\frac{j}{2}}\varepsilon,\end{aligned}$$

where we used the finite band property for P_l, the estimate (2.69) for b and the estimate (C.45) for h. Finally, (C.48) for $j\le l$ and (C.49) for $j>l$ yield:

$$\|P_l(bh)\|_{L_t^\infty L_{x'}^2}\lesssim\sum_j\|P_l(bP_jh)\|_{L_t^\infty L_{x'}^2}\lesssim 2^{\frac{l}{2}}\varepsilon. \qquad \text{(C.50)}$$

In view of (C.46), we need to evaluate $\|\mathcal{D}_2^{-1}b^{-1}\mathcal{D}_2^{-1}(\not\nabla(bh)\|_{L_t^pL_{x'}^{4-}}$. Note first that we have the following commutator formula:

$$\mathcal{D}_2^{-1}\not\nabla-\not\nabla\mathcal{D}_1^{-1}=\mathcal{D}_2^{-1}K\mathcal{D}_1^{-1}$$

which yields:

$$\|\mathcal{D}_2^{-1}b^{-1}\mathcal{D}_2^{-1}(\not\nabla(bh)\|_{L_t^pL_{x'}^{4-}}\lesssim\|\mathcal{D}_2^{-1}b^{-1}\not\nabla\mathcal{D}_1^{-1}(bh)\|_{L_t^pL_{x'}^{4-}}+\|\mathcal{D}_2^{-1}b^{-1}\mathcal{D}_2^{-1}K\mathcal{D}_1^{-1}(bh)\|_{L_t^pL_{x'}^{4-}}. \qquad \text{(C.51)}$$

We first evaluate the first term in the right-hand side of (C.51). Using the weak Bernstein inequality for P_j, we have:

$$\begin{aligned}&\|P_j\mathcal{D}_2^{-1}b^{-1}\not\nabla\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^\infty L_{x'}^{4-}} \qquad \text{(C.52)}\\ \lesssim\ & 2^{j(\frac12)-}\|\mathcal{D}_2^{-1}b^{-1}\not\nabla\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^\infty L_{x'}^2}\\ \lesssim\ & 2^{j(\frac12)-}\|\mathcal{D}_2^{-1}\not\nabla(b^{-1}\mathcal{D}_1^{-1}P_l(bh))\|_{L_t^\infty L_{x'}^2}+2^{j(\frac12)-}\|\mathcal{D}_2^{-1}(\not\nabla(b^{-1})\mathcal{D}_1^{-1}P_l(bh))\|_{L_t^\infty L_{x'}^2}\\ \lesssim\ & 2^{j(\frac12)-}\|b^{-1}\|_{L^\infty}\|\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^\infty L_{x'}^2}+2^{j(\frac12)-}\|\not\nabla(b^{-1})\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^\infty L_{x'}^{\frac43}}\\ \lesssim\ & 2^{j(\frac12)-}(\|b^{-1}\|_{L^\infty}+\|\not\nabla(b^{-1})\|_{L_t^\infty L_{x'}^4})\|\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^\infty L_{x'}^2}\\ \lesssim\ & 2^{j(\frac12)--\frac{l}{2}}\varepsilon,\end{aligned}$$

where we used the estimate (3.50) and the Remark 3.15 for $\mathcal{D}_2^{-1}$, the estimate (3.50) for $\mathcal{D}_1^{-1}$, the finite band property for P_l, the estimate (2.69) for b, the estimate (C.50) for bh, and the estimate (3.51).

We derive a second estimate for $\|P_j\mathcal{D}_2^{-1}b^{-1}\nabla\!\!\!/\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^\infty L_{x'}^{4-}}$. Using the weak Bernstein inequality and the finite band property for P_j, we have:

$$\begin{aligned}
&\|P_j\mathcal{D}_2^{-1}b^{-1}\nabla\!\!\!/\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^\infty L_{x'}^{4-}} \qquad \text{(C.53)}\\
\lesssim\ & 2^{j(\frac{1}{2})-}\|P_j\mathcal{D}_2^{-1}b^{-1}\nabla\!\!\!/\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^\infty L_{x'}^2}\\
\lesssim\ & 2^{-j(\frac{1}{2})+}\|\nabla\!\!\!/\mathcal{D}_2^{-1}b^{-1}\nabla\!\!\!/\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^\infty L_{x'}^2}\\
\lesssim\ & 2^{-j(\frac{1}{2})+}\|b^{-1}\|_{L^\infty}\|\nabla\!\!\!/\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^\infty L_{x'}^2}\\
\lesssim\ & 2^{-j(\frac{1}{2})+}\|P_l(bh)\|_{L_t^\infty L_{x'}^2}\\
\lesssim\ & 2^{-j(\frac{1}{2})++\frac{l}{2}}\varepsilon,
\end{aligned}$$

where we used the estimate (3.50) for $\mathcal{D}_2^{-1}$, the estimate (3.50) for $\mathcal{D}_1^{-1}$, the estimate (2.69) for b, and the estimate (C.50) for bh. Summing (C.52) for $j \le l$ and (C.53) for $j > l$ yields:

$$\|\mathcal{D}_2^{-1}b^{-1}\nabla\!\!\!/\mathcal{D}_1^{-1}(bh)\|_{L_t^pL_{x'}^{4-}} \lesssim \sum_{j\le l}2^{j(\frac{1}{2})--\frac{l}{2}}\varepsilon + \sum_{j>l}2^{-j(\frac{1}{2})++\frac{l}{2}}\varepsilon \lesssim \varepsilon. \qquad \text{(C.54)}$$

Next, we evaluate the second term in the right-hand side of (C.51). Using Remark 3.15 for $\mathcal{D}_2^{-1}$, we have:

$$\begin{aligned}
\|\mathcal{D}_2^{-1}b^{-1}\mathcal{D}_2^{-1}K\mathcal{D}_1^{-1}(bh)\|_{L_t^pL_{x'}^{4-}} &\lesssim \|b^{-1}\|_{L^\infty}\|\mathcal{D}_2^{-1}K\mathcal{D}_1^{-1}(bh)\|_{L_t^pL_{x'}^2} \qquad \text{(C.55)}\\
&\lesssim \|\mathcal{D}_2^{-1}K\mathcal{D}_1^{-1}(bh)\|_{L_t^pL_{x'}^2}.
\end{aligned}$$

Using Remark 3.15 for $\mathcal{D}_2^{-1}$, we have:

$$\begin{aligned}
\|\mathcal{D}_2^{-1}P_j(K)\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^pL_{x'}^2} &\lesssim \|P_j(K)\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^pL_{x'}^{1+}} \qquad \text{(C.56)}\\
&\lesssim \|P_j(K)\|_{L_t^pL_{x'}^{2+}}\|\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^\infty L_{x'}^2}\\
&\lesssim 2^{j(0)+}\|P_j(K)\|_{L^2(\mathcal{H}_u)}^{\frac{2}{p}}\|P_j(K)\|_{L_t^\infty L_{x'}^2}^{1-\frac{2}{p}}2^{-l}\|P_l(bh)\|_{L_t^\infty L_{x'}^2}\\
&\lesssim 2^{j(\frac{1}{2})-}\|K\|_{L^2(\mathcal{H}_u)}^{\frac{2}{p}}\|\Lambda^{-\frac{1}{2}}K\|_{L_t^\infty L_{x'}^2}^{1-\frac{2}{p}}2^{-\frac{l}{2}}\varepsilon\\
&\lesssim 2^{j(\frac{1}{2})--\frac{l}{2}}\varepsilon,
\end{aligned}$$

where we used the weak Bernstein inequality for P_j, (3.51) for P_l, the estimates (4.29) and (4.30) for K, and the estimate (C.50) for bh.

We derive another estimate for $\|\mathcal{D}_2^{-1}P_j(K)\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^pL_{x'}^2}$. We have:

$$\begin{aligned}
&\|\mathcal{D}_2^{-1}P_j(K)\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^pL_{x'}^2}\\
=\ & 2^{-2j}\|\mathcal{D}_2^{-1}\Delta\!\!\!/(P_j(K)\mathcal{D}_1^{-1}P_l(bh))\|_{L_t^pL_{x'}^2}\\
\lesssim\ & 2^{-2j}\|\mathcal{D}_2^{-1}\mathrm{div}\!\!\!/(\nabla\!\!\!/P_j(K)\mathcal{D}_1^{-1}P_l(bh))\|_{L_t^pL_{x'}^2} + 2^{-2j}\|\mathcal{D}_2^{-1}\nabla\!\!\!/P_j(K)\nabla\!\!\!/\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^pL_{x'}^2}
\end{aligned}$$

which together with the estimate (3.50) and the Remark 3.15 for $\mathcal{D}_2^{-1}$ yields:

$$\begin{aligned}
&\|\mathcal{D}_2^{-1}P_j(K)\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^pL_{x'}^2} \qquad (C.57)\\
\lesssim\ & 2^{-2j}\|\nabla\!\!\!/ P_j(K)\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^pL_{x'}^2}+2^{-2j}\|\nabla\!\!\!/ P_j(K)\nabla\!\!\!/\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^pL_{x'}^{1+}}\\
\lesssim\ & 2^{-2j}\|\nabla\!\!\!/ P_j(K)\|_{L_t^pL_{x'}^{2+}}\left(\|\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^\infty L_{x'}^{\frac{4+}{2+-2}}}+\|\nabla\!\!\!/\mathcal{D}_1^{-1}P_l(bh)\|_{L_t^\infty L_{x'}^2}\right)\\
\lesssim\ & 2^{-2j}\left(\|\nabla\!\!\!/^2P_j(K)\|_{L_t^pL_{x'}^2}^{1-\frac{2}{2+}}\|\nabla\!\!\!/ P_j(K)\|_{L_t^pL_{x'}^2}^{\frac{2}{2+}}+\|\nabla\!\!\!/ P_j(K)\|_{L_t^pL_{x'}^2}\right)\|P_l(bh)\|_{L_t^\infty L_{x'}^2}\\
\lesssim\ & 2^{-j_-+\frac{l}{2}}\|P_j(K)\|_{L_t^pL_{x'}^2}\varepsilon\\
\lesssim\ & 2^{-j(\frac12)_++\frac{l}{2}}\|K\|_{L^2(\mathcal{H}_u)}^{\frac{2}{p}}\|\Lambda^{-\frac12}K\|_{L_t^\infty L_{x'}^2}^{1-\frac{2}{p}}\varepsilon\\
\lesssim\ & 2^{-j(\frac12)_++\frac{l}{2}}\varepsilon,
\end{aligned}$$

where we used the finite band property for P_j, the estimate (3.50) and the Remark 3.15 for $\mathcal{D}_1^{-1}$, the Bochner inequality for scalars (4.34), the estimates (4.29) and (4.30) for K, and the estimate (C.50) for bh, as well as the estimate

$$\begin{aligned}
\|P_jK\|_{L^2(P_{t,u})} &= \|\Lambda^{\frac12}P_j\Lambda^{-\frac12}K\|_{L^2(P_{t,u})}\\
&\lesssim \|\Lambda P_j\Lambda^{-\frac12}K\|_{L^2(P_{t,u})}^{\frac12}\|P_j\Lambda^{-\frac12}K\|_{L^2(P_{t,u})}^{\frac12}\\
&\lesssim (\|P_j\Lambda^{-\frac12}K\|_{L^2(P_{t,u})}+\|\nabla\!\!\!/ P_j\Lambda^{-\frac12}K\|_{L^2(P_{t,u})})^{\frac12}\|P_j\Lambda^{-\frac12}K\|_{L^2(P_{t,u})}^{\frac12}\\
&\lesssim 2^{\frac{j}{2}}\|\Lambda^{-\frac12}K\|_{L^2(P_{t,u})} \qquad (C.58)
\end{aligned}$$

which uses the finite band property for P_j, the fact that P_j commutes with any power of Λ, and the properties (3.22) (3.23) of Λ.

Using (C.55), and summing (C.56) for $j\le l$ and (C.57) for $j>l$ yields:

$$\|\mathcal{D}_2^{-1}b^{-1}\mathcal{D}_2^{-1}K\mathcal{D}_1^{-1}(bh)\|_{L_t^pL_{x'}^{4-}}\lesssim\sum_{j\le l}2^{j(\frac12)_--\frac{l}{2}}\varepsilon+\sum_{j>l}2^{-j(\frac12)_++\frac{l}{2}}\varepsilon\lesssim\varepsilon. \qquad (C.59)$$

Finally, (C.46), (C.47), (C.51), (C.54) and (C.59) yield the conclusion of Lemma 6.18.

C.9 Proof of Lemma 6.19

Since $H=(\rho,\sigma,\beta,\underline{\beta})$ and $\mathcal{N}_1(G)\lesssim\varepsilon$, the curvature estimate (2.59) and the finite band property for P_l yield:

$$\|P_lH\|_{L^2(\mathcal{H}_u)}\lesssim\varepsilon \text{ and } \|P_lG\|_{L^2(\mathcal{H}_u)}\lesssim 2^{-l}\varepsilon. \qquad (C.60)$$

while Lemma 6.21 and Lemma 5.9 yield:

$$\|P_lH\|_{L_t^\infty L_{x'}^2}\lesssim 2^{\frac{l}{2}}\varepsilon \text{ and } \|P_lG\|_{L_t^\infty L_{x'}^2}\lesssim 2^{-\frac{l}{2}}\varepsilon. \qquad (C.61)$$

Using Remark 3.15 for $\mathcal{D}_2^{-1}$, we have:

$$\begin{aligned} & \|\mathcal{D}_2^{-1}b^{-1}(\mathcal{D}_2^{-1}(F\cdot H))\|_{L_t^pL_{x'}^{4-}} + \|\mathcal{D}_2^{-1}b^{-1}(\mathcal{D}_2^{-1}(F\cdot \nabla\!\!\!/ G))\|_{L_t^pL_{x'}^{4-}} \qquad \text{(C.62)}\\ \lesssim\ & \|b^{-1}(\mathcal{D}_2^{-1}(F\cdot H))\|_{L_t^pL_{x'}^2} + \|b^{-1}(\mathcal{D}_2^{-1}(F\cdot \nabla\!\!\!/ G))\|_{L_t^pL_{x'}^2}\\ \lesssim\ & \|\mathcal{D}_2^{-1}(F\cdot H)\|_{L_t^pL_{x'}^2} + \|\mathcal{D}_2^{-1}(F\cdot \nabla\!\!\!/ G)\|_{L_t^pL_{x'}^2}, \end{aligned}$$

where we used the estimate (2.69) for b in the last inequality.

Next, we estimate the two terms in the right-hand side of (C.62). Using Remark 3.15 for $\mathcal{D}_2^{-1}$, we have:

$$\begin{aligned} & \|\mathcal{D}_2^{-1}(P_q(F)\cdot P_lH)\|_{L_t^pL_{x'}^2} + \|\mathcal{D}_2^{-1}(P_q(F)\cdot \nabla\!\!\!/ P_lG)\|_{L_t^pL_{x'}^2} \qquad \text{(C.63)}\\ \lesssim\ & \|P_q(F)\cdot P_lH\|_{L_t^pL_{x'}^{1+}} + \|P_q(F)\cdot \nabla\!\!\!/ P_lG\|_{L_t^pL_{x'}^{1+}}\\ \lesssim\ & \|P_q(F)\|_{L_t^\infty L_{x'}^2}\|P_lH\|_{L_t^pL_{x'}^{2+}} + \|P_q(F)\|_{L_t^\infty L_{x'}^2}\|\nabla\!\!\!/ P_lG\|_{L_t^pL_{x'}^{2+}}\\ \lesssim\ & 2^{-\frac{q}{2}}\mathcal{N}_1(F)\bigg(2^{l(0)+}\|P_lH\|_{L_t^pL_{x'}^2} + \bigg\|\|\nabla\!\!\!/^2P_lG\|_{L^2(P_{t,u})}^{1-\frac{2}{2_+}}\|\nabla\!\!\!/ P_lG\|_{L^2(P_{t,u})}^{\frac{2}{2_+}} + \|\nabla\!\!\!/ P_lG\|_{L^2(P_{t,u})}\bigg\|_{L^p(0,1)}\bigg)\\ \lesssim\ & 2^{-\frac{q}{2}}\mathcal{N}_1(F)\bigg(2^{l(0)+}\|P_lH\|_{L_t^\infty L_{x'}^2}^{1-\frac{2}{p}}\|P_lH\|_{L^2(\mathcal{H}_u)}^{\frac{2}{p}} + \bigg\|(\|\Delta\!\!\!/ P_lG\|_{L^2(P_{t,u})}\\ & +\|K\|_{L^2(P_{t,u})}\|\nabla\!\!\!/ P_lG\|_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})}^2\|P_lG\|_{L_t^pL_{x'}^2})^{1-\frac{2}{2_+}}2^l\|P_lG\|_{L^2(P_{t,u})}^{\frac{2}{2_+}}\\ & +2^l\|P_lG\|_{L^2(P_{t,u})}\bigg\|_{L^p(0,1)}\bigg)\\ \lesssim\ & 2^{l(\frac{3}{2}-\frac{2}{2_+}-\frac{1}{p})-\frac{q}{2}}\mathcal{N}_1(F)(1+\|K\|_{L^2(\mathcal{H}_u)}^{2(1-\frac{2}{2_+})})\varepsilon\\ \lesssim\ & 2^{l(\frac{1}{2})--\frac{q}{2}}\mathcal{N}_1(F)\varepsilon, \end{aligned}$$

where we used Lemma 5.9 for $\|P_q(F)\|_{L_t^\infty L_{x'}^2}$, the Bochner inequality for tensors (3.7), (C.60) and (C.61) for G and H, and the estimate (4.29) for K. We also used the fact that once $p<+\infty$ is fixed, we may choose $2_+>2$ such that $1-\frac{2}{2_+}-\frac{1}{p}<0$.

We derive a second estimate for $\|\mathcal{D}_2^{-1}(P_q(F)\cdot P_lH)\|_{L_t^pL_{x'}^2}$ and $\|\mathcal{D}_2^{-1}(P_q(F)\cdot \nabla\!\!\!/ P_lG)\|_{L_t^pL_{x'}^2}$. We have:

$$\begin{aligned} & \|\mathcal{D}_2^{-1}(P_q(F)\cdot P_lH)\|_{L_t^pL_{x'}^2} + \|\mathcal{D}_2^{-1}(P_q(F)\cdot \nabla\!\!\!/ P_lG)\|_{L_t^pL_{x'}^2}\\ \lesssim\ & 2^{-2l}\|\mathcal{D}_2^{-1}(P_q(F)\cdot \Delta\!\!\!/ P_lH)\|_{L_t^pL_{x'}^2} + \|\mathcal{D}_2^{-1}\mathrm{div}\!\!\!/(P_q(F)\cdot P_lG)\|_{L_t^pL_{x'}^2}\\ & +\|\mathcal{D}_2^{-1}(\nabla\!\!\!/ P_q(F)\cdot P_lG)\|_{L_t^pL_{x'}^2}\\ \lesssim\ & 2^{-2l}\|\mathcal{D}_2^{-1}\mathrm{div}\!\!\!/(P_q(F)\cdot \nabla\!\!\!/ P_lH)\|_{L_t^pL_{x'}^2} + 2^{-2l}\|\mathcal{D}_2^{-1}(\nabla\!\!\!/ P_q(F)\cdot \nabla\!\!\!/ P_lH)\|_{L_t^pL_{x'}^2}\\ & +\|\mathcal{D}_2^{-1}\mathrm{div}\!\!\!/(P_q(F)\cdot P_lG)\|_{L_t^pL_{x'}^2} + \|\mathcal{D}_2^{-1}(\nabla\!\!\!/ P_q(F)\cdot P_lG)\|_{L_t^pL_{x'}^2} \end{aligned}$$

which together with the estimate (3.50) and Remark 3.15 for $\mathcal{D}_2^{-1}$ implies:

$$\begin{aligned}
&\|\mathcal{D}_2^{-1}(P_q(F)\cdot P_lH)\|_{L_t^pL_{x'}^2}+\|\mathcal{D}_2^{-1}(P_q(F)\cdot \nabla\!\!\!/ P_lG)\|_{L_t^pL_{x'}^2} \qquad \text{(C.64)}\\
\lesssim\ & 2^{-2l}\|P_q(F)\cdot \nabla\!\!\!/ P_lH\|_{L_t^pL_{x'}^2}+2^{-2l}\|\nabla\!\!\!/ P_q(F)\cdot \nabla\!\!\!/ P_lH\|_{L_t^pL_{x'}^{1+}}\\
&+\|P_q(F)\cdot P_lG\|_{L_t^pL_{x'}^2}+\|\nabla\!\!\!/ P_q(F)\cdot P_lG)\|_{L_t^pL_{x'}^{1+}}\\
\lesssim\ & 2^{-2l}\|P_q(F)\|_{L_t^pL_{x'}^\infty}\|\nabla\!\!\!/ P_lH\|_{L_t^\infty L_{x'}^2}+2^{-2l}\|\nabla\!\!\!/ P_q(F)\|_{L_t^pL_{x'}^{2+}}\|\nabla\!\!\!/ P_lH\|_{L_t^\infty L_{x'}^2}\\
&+\|P_q(F)\|_{L_t^pL_{x'}^\infty}\|P_lG\|_{L_t^\infty L_{x'}^2}+\|\nabla\!\!\!/ P_q(F)\|_{L_t^pL_{x'}^{2+}}\|P_lG\|_{L_t^\infty L_{x'}^2}\\
\lesssim\ & 2^{-\frac{l}{2}}\Big\|\|\nabla\!\!\!/^2P_q(F)\|_{L^2(P_{t,u})}^{1-\frac{2}{2+}}\|\nabla\!\!\!/ P_q(F)\|_{L^2(P_{t,u})}^{\frac{2}{2+}}\Big\|_{L^p(0,1)}\varepsilon\\
\lesssim\ & 2^{-\frac{l}{2}}\Big\|(\|\Delta\!\!\!/P_qF\|_{L^2(P_{t,u})}+\|K\|_{L^2(P_{t,u})}\|\nabla\!\!\!/ P_qF\|_{L^2(P_{t,u})}+\|K\|^2_{L^2(P_{t,u})}\|P_qF\|_{L_t^pL_{x'}^2})^{1-\frac{2}{2+}}\\
&\times 2^q\|P_qF\|_{L^2(P_{t,u})}^{\frac{2}{2+}}\Big\|_{L^p(0,1)}\varepsilon\\
\lesssim\ & 2^{q(\frac{1}{2})_--\frac{l}{2}}\mathcal{N}_1(F)\varepsilon,
\end{aligned}$$

where we used Lemma 5.9 for $\|P_q(F)\|_{L_t^\infty L_{x'}^2}$, the finite band property for P_q and P_l, the Bochner inequality for tensors (3.7), (3.38) to bound the $L^\infty(P_{t,u})$ norm of $P_q(F)$, (C.60) and (C.61) for G and H, and the estimate (4.29) for K.

Finally, summing (C.63) for $l\le q$ and (C.64) for $l>q$ implies:

$$\|\mathcal{D}_2^{-1}(F\cdot H)\|_{L_t^pL_{x'}^2}+\|\mathcal{D}_2^{-1}(F\cdot \nabla\!\!\!/ G)\|_{L_t^pL_{x'}^2}\lesssim\mathcal{N}_1(F)\varepsilon$$

which together with (C.62) yields the conclusion of Lemma 6.19.

C.10 Proof of Lemma 6.20

Lemma 3.17 implies:

$$\begin{aligned}
\|\mathcal{D}_2^{-1}(FGH)\|_{L_t^\infty L_{x'}^{4-}}&\lesssim\|FGH\|_{L_t^\infty L_{x'}^{\frac{4}{3}}}\\
&\lesssim\|F\|_{L_t^\infty L_{x'}^4}\|G\|_{L_t^\infty L_{x'}^4}\|H\|_{L_t^\infty L_{x'}^4}\\
&\lesssim\mathcal{N}_1(F)\mathcal{N}_1(G)\mathcal{N}_1(H),
\end{aligned}$$

which concludes the proof of Lemma 6.20.

C.11 Proof of Lemma 6.21

Note first from the curvature bound (2.59) for $\beta,\underline{\beta},\rho,\sigma$ that H satisfies the following estimate:

$$\|H\|_{L^2(\mathcal{H}_u)}\le\varepsilon. \qquad \text{(C.65)}$$

The proof follows the same strategy as the one of Proposition 4.11. However, one has to be more careful since β and $\underline{\beta}$ are tensors unlike K. In particular, using the estimate (A.5), the L^2 boundedness of P_j, and the estimate (C.65) for H, we obtain:

$$\begin{aligned}
&\|P_jH\|^2_{L^\infty_t L^2_{x'}} \qquad (C.66)\\
\lesssim\ & \left(\int_0^1 \|P_jH\|_{L^2(P_{t,u})}\|\nabla\!\!\!/_{nL}P_jH\|_{L^2(P_{t,u})}dt\right) + \|P_jH\|^2_{L^2(\mathcal{H}_u)}\\
\lesssim\ & \|P_jH\|_{L^2(\mathcal{H}_u)}\|P_j\nabla\!\!\!/_{nL}H\|_{L^2(\mathcal{H}_u)} + \left(\int_0^1 \|P_jH\|_{L^2(P_{t,u})}\|[\nabla\!\!\!/_{nL},P_j]H\|_{L^2(P_{t,u})}dt\right) + \varepsilon^2\\
\lesssim\ & \varepsilon\|P_j\nabla\!\!\!/_{nL}H\|_{L^2(\mathcal{H}_u)} + \left(\int_0^1 \|P_jH\|_{L^2(P_{t,u})}\|[\nabla\!\!\!/_{nL},P_j]H\|_{L^2(P_{t,u})}dt\right) + \varepsilon^2.
\end{aligned}$$

Now, the Bianchi identities (2.51), (2.53), (2.55) and (2.57) for $\nabla\!\!\!/_L(\beta), L(\rho), L(\sigma)$ and $\nabla\!\!\!/_L(\underline{\beta})$ have the following structure:

$$\nabla\!\!\!/_L H = (\mathrm{di}\!\!\!/\mathrm{v}(\alpha), \mathrm{di}\!\!\!/\mathrm{v}(\beta), \mathrm{cu}\!\!\!/\mathrm{l}(\beta), \nabla\!\!\!/\rho, \nabla\!\!\!/\sigma) + F\cdot(\alpha,\beta,\rho,\sigma,\underline{\beta})$$

where in view of the estimates (2.67)-(2.72), F satisfies $\mathcal{N}_1(F)\lesssim\varepsilon$. Thus, using the finite band property and the weak Bernstein inequality for P_j, we obtain for $\nabla\!\!\!/_L H$ the following estimate:

$$\begin{aligned}
\|P_j\nabla\!\!\!/_L H\|_{L^2(\mathcal{H}_u)} &\lesssim 2^j\|(\alpha,\beta,\rho,\sigma,\underline{\beta})\|_{L^2(\mathcal{H}_u)} + 2^{\frac{j}{2}}\|F\cdot(\alpha,\beta,\rho,\sigma,\underline{\beta})\|_{L^2_t L^{\frac{4}{3}}_{x'}} \qquad (C.67)\\
&\lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\mathcal{N}_1(F)\varepsilon\\
&\lesssim 2^j\varepsilon,
\end{aligned}$$

where we used the curvature bound (2.59) for α,β,ρ,σ and $\underline{\beta}$. (C.66) and (C.67) imply:

$$\|P_jH\|^2_{L^\infty_t L^2_{x'}} \lesssim \left(\int_0^1 \|P_jH\|_{L^2(P_{t,u})}\|[\nabla\!\!\!/_{nL},P_j]H\|_{L^2(P_{t,u})}dt\right) + 2^j\varepsilon^2,$$

which yields:

$$\|P_jH\|_{L^\infty_t L^2_{x'}} \lesssim \|[\nabla\!\!\!/_{nL},P_j]H\|_{L^1_t L^2_{x'}} + 2^{\frac{j}{2}}\varepsilon. \qquad (C.68)$$

We now evaluate the right-hand side of (C.68). Again, let us say that the difference with the proof of Proposition 4.11 is the fact that H is a tensor unlike K. Using the definition (3.15) of P_j, we have:

$$[\nabla\!\!\!/_{nL},P_j]H = \int_0^\infty m_j(\tau)V(\tau)d\tau, \qquad (C.69)$$

where $V(\tau)$ satisfies:

$$(\partial_\tau - \Delta\!\!\!/)V(\tau) = [\nabla\!\!\!/_{nL},\Delta\!\!\!/]U(\tau)H,\ V(0)=0. \qquad (C.70)$$

(C.69) yields:

$$\|[\nabla\!\!\!/_{nL},P_j]H\|_{L^1_t L^2_{x'}} \lesssim \left\|\int_0^\infty m_j(\tau)\|V(\tau)\|_{L^2(P_{t,u})}d\tau\right\|_{L^1(0,1)}. \qquad (C.71)$$

In view of (C.68) and (C.71), we have to estimate $\|V(\tau)\|_{L^2(P_{t,u})}$. Let a, p real numbers satisfying:

$$0 < a < \frac{1}{2},\ 2 < p < +\infty,\ \text{such that } p < \frac{4}{2-a}. \tag{C.72}$$

The energy estimate (3.30) implies:

$$\begin{aligned} &\|\Lambda^{-a} V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/ \Lambda^{-a} V(\tau')\|^2_{L^2(P_{t,u})} d\tau' \\ \lesssim\ & \int_0^\tau \int_{P_{t,u}} \Lambda^{-2a} V(\tau') [\nabla\!\!\!/_{nL}, \Delta\!\!\!/] U(\tau') H d\mu_{t,u} d\tau'. \end{aligned} \tag{C.73}$$

We need to estimate the commutator term $[\nabla\!\!\!/_{nL}, \Delta\!\!\!/]U$. Using twice the commutator formula (2.48), we have:

$$[\nabla\!\!\!/_{nL}, \Delta\!\!\!/] U = F \nabla\!\!\!/^2 U + G \nabla\!\!\!/ U + \nabla\!\!\!/(GU) \tag{C.74}$$

where the tensors F and G are given by $F = n\chi$ and $G = n\chi\epsilon + n^*\beta$. Using the curvature bound (2.59) for β and the bound (2.67)-(2.71) for n, ϵ and χ, we obtain the following bound for F and G:

$$\|F\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/ F\|_{L^2(\mathcal{H}_u)} + \|G\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon \tag{C.75}$$

Let p defined in (C.72), and let p' such that $\frac{1}{p} + \frac{1}{p'} = \frac{1}{2}$. Using the commutator formula (C.74), and integrating by parts the terms $\nabla\!\!\!/^2 U(\tau) H$ and $\nabla\!\!\!/(GU)$ yields:

$$\begin{aligned} &\int_0^\tau \int_{P_{t,u}} \Lambda^{-2a} V(\tau') [\nabla\!\!\!/_{nL}, \Delta\!\!\!/] U(\tau') H d\mu_{t,u} d\tau' \\ &\lesssim (\|\nabla\!\!\!/ F\|_{L^2(P_{t,u})} + \|G\|_{L^2(P_{t,u})}) \int_0^\tau \|\nabla\!\!\!/ U(\tau') H\|_{L^p(P_{t,u})} \|\Lambda^{-2a} V(\tau')\|_{L^{p'}(P_{t,u})} d\tau' \\ &+ \|F\|_{L^2(P_{t,u})} \int_0^\tau \|\nabla\!\!\!/ U(\tau') H\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ \Lambda^{-2a} V(\tau')\|_{L^2(P_{t,u})} d\tau' \\ &+ \|G\|_{L^2(P_{t,u})} \int_0^\tau \|U(\tau') H\|_{L^\infty(P_{t,u})} \|\nabla\!\!\!/ \Lambda^{-2a} V(\tau')\|_{L^2(P_{t,u})} d\tau' \\ &\lesssim (\|F\|_{L^2(P_{t,u})} + \|\nabla\!\!\!/ F\|_{L^2(P_{t,u})} + \|G\|_{L^2(P_{t,u})}) \\ &\times \int_0^\tau (\|\nabla\!\!\!/ U(\tau') H\|_{L^p(P_{t,u})} + \|\nabla\!\!\!/ U(\tau') H\|_{L^2(P_{t,u})} + \|U(\tau') H\|_{L^p(P_{t,u})}) \|\nabla\!\!\!/ \Lambda^{-2a} V(\tau')\|_{L^2(P_{t,u})} d\tau' \end{aligned} \tag{C.76}$$

where we used the Sobolev embeddings (3.3) and (3.4) in the last inequality.

Next, we concentrate on the first term of the right-hand side of (C.76), the other one being lower order and left to the reader. The Gagliardo-Nirenberg inequality (3.3), the

properties (3.23) and (3.22) of Λ, and the Bochner inequality (3.7) for tensors yield:

$$\begin{aligned}
&\int_0^\tau \|\nabla\!\!\!/ U(\tau')H\|_{L^p(P_{t,u})}\|\nabla\!\!\!/\Lambda^{-2a}V(\tau')\|_{L^2(P_{t,u})}d\tau' \qquad (C.77)\\
\lesssim\ &\int_0^\tau \|\nabla\!\!\!/ U(\tau')H\|^{\frac{2}{p}}_{L^2(P_{t,u})}\|\nabla\!\!\!/^2 U(\tau')H\|^{1-\frac{2}{p}}_{L^2(P_{t,u})}\|\Lambda^{-a}V(\tau')\|^a_{L^2(P_{t,u})}\|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^{1-a}_{L^2(P_{t,u})}d\tau'\\
\lesssim\ &\int_0^\tau \|\nabla\!\!\!/ U(\tau')H\|^{\frac{2}{p}}_{L^2(P_{t,u})}\big(\|\Delta\!\!\!/ U(\tau')H\|_{L^2(P_{t,u})}+\|K\|_{L^2(P_{t,u})}\|\nabla\!\!\!/ U(\tau')H\|_{L^2(P_{t,u})}\\
&+\|K\|^2_{L^2(P_{t,u})}\|U(\tau')H\|_{L^2(P_{t,u})}\big)^{1-\frac{2}{p}}\|\Lambda^{-a}V(\tau')\|^a_{L^2(P_{t,u})}\|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^{1-a}_{L^2(P_{t,u})}d\tau'\\
\lesssim\ &\bigg(\int_0^\tau \|\nabla\!\!\!/ U(\tau')H\|^2_{L^2(P_{t,u})}d\tau'+\int_0^\tau \tau'\|\Delta\!\!\!/ U(\tau')H\|^2_{L^2(P_{t,u})}d\tau'\\
&+\|K\|^2_{L^2(P_{t,u})}\int_0^\tau \tau'\|\nabla\!\!\!/ U(\tau')H\|^2_{L^2(P_{t,u})}d\tau'\bigg)^{\frac{1}{2}}\\
&\times\bigg(\frac{1}{2}\int_0^\tau \|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^2_{L^2(P_{t,u})}d\tau'+\int_0^\tau \tau'^{-\frac{2(p-2)}{ap}}\|\Lambda^{-a}V(\tau')\|^2_{L^2(P_{t,u})}d\tau'\bigg)^{\frac{1}{2}}
\end{aligned}$$

which together with the estimates for the heat flow (3.8), (3.10) and (3.29), implies:

$$\begin{aligned}
&\int_0^\tau \|\nabla\!\!\!/ U(\tau')H\|_{L^p(P_{t,u})}\|\nabla\!\!\!/\Lambda^{-2a}V(\tau')\|_{L^2(P_{t,u})}d\tau'\\
&\lesssim (\|H\|_{L^2(P_{t,u})}+\|K\|_{L^2(P_{t,u})}\|\Lambda^{-1}H\|_{L^2(P_{t,u})}(1+\tau)) \qquad (C.78)\\
&\bigg(\int_0^\tau \|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^2_{L^2(P_{t,u})}d\tau'+\int_0^\tau \tau'^{-\frac{2(p-2)}{ap}}\|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^2_{L^2(P_{t,u})}d\tau'\bigg)^{\frac{1}{2}}.
\end{aligned}$$

Finally, the choice of p (C.72), (C.73), (C.76) and (C.78) implies:

$$\begin{aligned}
&\|\Lambda^{-a}V(\tau)\|^2_{L^2(P_{t,u})}+\int_0^\tau \|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^2_{L^2(P_{t,u})}d\tau'\\
&\lesssim (1+\tau)(\|\nabla\!\!\!/ F\|_{L^2(P_{t,u})}+\|G\|_{L^2(P_{t,u})})(\|H\|_{L^2(P_{t,u})}+\|K\|_{L^2(P_{t,u})}\|\Lambda^{-1}H\|_{L^2(P_{t,u})}). \qquad (C.79)
\end{aligned}$$

Using the interpolation inequality (3.22), we obtain:

$$\begin{aligned}
&\int_0^1 \|V(\tau)\|^{\frac{2}{a}}_{L^2(P_{t,u})}d\tau\lesssim\int_0^1 \|\Lambda^{-a}V(\tau')\|^{\frac{2(1-a)}{a}}_{L^2(P_{t,u})}\|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^2_{L^2(P_{t,u})}d\tau' \qquad (C.80)\\
&\lesssim (\|\nabla\!\!\!/ F\|_{L^2(P_{t,u})}+\|G\|_{L^2(P_{t,u})})(\|H\|_{L^2(P_{t,u})}+\|K\|_{L^2(P_{t,u})}\|\Lambda^{-1}H\|_{L^2(P_{t,u})}),
\end{aligned}$$

which together with the estimate (C.65) for H and the estimate (C.75) for F and G yields for $j\geq 0$:

$$\begin{aligned}
&\bigg\|\int_0^{+\infty} m_j(\tau)\|V(\tau)\|_{L^2(P_{t,u})}d\tau\bigg\|_{L^1(0,1)}\lesssim 2^{ja}\bigg\|\bigg(\int_0^1 \|V(\tau)\|^{\frac{2}{a}}_{L^2(P_{t,u})}d\tau\bigg)^{\frac{a}{2}}\bigg\|_{L^1(0,1)} \qquad (C.81)\\
&\lesssim 2^{ja}(\|\nabla\!\!\!/ F\|_{L^2(\mathcal{H}_u)}+\|G\|_{L^2(\mathcal{H}_u)})(\|H\|_{L^2(\mathcal{H}_u)}+\|K\|_{L^2(\mathcal{H}_u)}\|\Lambda^{-1}H\|_{L^\infty_t L^2_{x'}})\\
&\lesssim 2^{ja}\varepsilon(1+\|\Lambda^{-1}H\|_{L^\infty_t L^2_{x'}}),
\end{aligned}$$

where we used the fact that m is supported in $(0,1)$, see Lemma 3.8. Now, (C.68), (C.71) and (C.81) imply:

$$\begin{aligned} \|P_j H\|_{L_t^\infty L_{x'}^2} &\lesssim 2^{\frac{j}{2}}\varepsilon + 2^{ja}\varepsilon(1+\|\Lambda^{-1}H\|_{L_t^\infty L_{x'}^2}) \\ &\lesssim 2^{\frac{j}{2}}\varepsilon + 2^{\frac{j}{2}}\varepsilon\|\Lambda^{-1}H\|_{L_t^\infty L_{x'}^2}, \end{aligned} \tag{C.82}$$

where we used the choice of a (C.72) in the last inequality. Finally, from the properties of Λ and P_j, we have:

$$\begin{aligned} \|\Lambda^{-1}H\|_{L_t^\infty L_{x'}^2} &\lesssim \|P_{<0}\Lambda^{-1}H\|_{L_t^\infty L_{x'}^2} + \sum_{j\geq 0}\|P_j\Lambda^{-1}H\|_{L_t^\infty L_{x'}^2} \\ &\lesssim \|P_{<0}H\|_{L_t^\infty L_{x'}^2} + \sum_{j\geq 0} 2^{-j}\|P_j H\|_{L_t^\infty L_{x'}^2} \\ &\lesssim \|P_{<0}H\|_{L_t^\infty L_{x'}^2} + \sup_{j\geq 0} 2^{-\frac{j}{2}}\|P_j H\|_{L_t^\infty L_{x'}^2} \end{aligned}$$

which together with (C.82), and an easy estimate for $\|P_{<0}H\|_{L_t^\infty L_{x'}^2}$ left to the reader, implies for $j \geq 0$:

$$\|P_j H\|_{L_t^\infty L_{x'}^2} \lesssim 2^{\frac{j}{2}}\varepsilon.$$

This concludes the proof of Lemma 6.21.

C.12 Proof of Lemma 6.23

Using the L^∞ estimate (3.83), we have:

$$\begin{aligned} \|Q_{>1}N\|_{L^\infty} &\lesssim \|Q_{>1}N\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla Q_{>1}N\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla\!\!\!/\nabla Q_{>1}N\|_{L_t^\infty L^2(\Sigma_t)} \\ &\lesssim \|\nabla N\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla\!\!\!/\nabla N\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla\!\!\!/\nabla Q_{\leq 1}N\|_{L_t^\infty L^2(\Sigma_t)}, \end{aligned} \tag{C.83}$$

where we used in the last inequality the finite band property for $Q_{>1}$ of Theorem 3.35, and the decomposition $N = Q_{\leq 1}(N) + Q_{>1}(N)$.

We now evaluate the various terms in the right-hand side of (C.83). Since $N = \frac{1}{2}(L - \underline{L})$, the Ricci equation (2.23) imply:

$$\nabla_A N = \theta_{AB}e_B,\ \nabla_N N = -b^{-1}\nabla\!\!\!/ b. \tag{C.84}$$

(C.84) implies:

$$\begin{aligned} &\|\nabla N\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla\!\!\!/\nabla N\|_{L_t^\infty L^2(\Sigma_t)} \\ \lesssim\ &\|\theta\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla\!\!\!/\theta\|_{L_t^\infty L^2(\Sigma_t)} + \|b^1\nabla\!\!\!/ b\|_{L_t^\infty L^2(\Sigma_t)} + \|b^{-1}\nabla\!\!\!/ b\|^2_{L_t^\infty L^4(\Sigma_t)} + \|b^{-1}\nabla\!\!\!/^2 b\|_{L_t^\infty L^2(\Sigma_t)}. \end{aligned} \tag{C.85}$$

Furthermore, the Bochner inequality (3.81) and the finite band property for $Q_{\leq 1}$ imply:

$$\begin{aligned} \|\nabla\!\!\!/\nabla Q_{\leq 1}N\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \|\nabla^2 Q_{\leq 1}N\|_{L_t^\infty L^2(\Sigma_t)} \\ &\lesssim \|\Delta Q_{\leq 1}N\|_{L_t^\infty L^2(\Sigma_t)} \\ &\lesssim \|\nabla N\|_{L_t^\infty L^2(\Sigma_t)}. \end{aligned} \tag{C.86}$$

Now, (C.83), (C.85) and (C.86) yield:

$$\begin{aligned}\|Q_{>1}N\|_{L^\infty} &\lesssim \|\theta\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla\!\!\!/\theta\|_{L^\infty_t L^2(\Sigma_t)} + \|b^1\nabla\!\!\!/ b\|_{L^\infty_t L^2(\Sigma_t)} \\ &\quad + \|b^{-1}\nabla\!\!\!/ b\|^2_{L^\infty_t L^4(\Sigma_t)} + \|b^{-1}\nabla\!\!\!/^2 b\|_{L^\infty_t L^2(\Sigma_t)}.\end{aligned} \tag{C.87}$$

In view of (C.87), we need to estimate b and θ on Σ_t. So far, we only proved regularity estimates on $\mathcal{H}_u$. In order to transfer them to Σ_t, we consider the structure equation for the foliation generated by u on Σ_t (see [4] p. 56):

$$\begin{cases} b^{-1}\Delta\!\!\!/ b = -\nabla_N \mathrm{tr}\theta - |\theta|^2 + R_{NN}, \\ \nabla\!\!\!/^B \widehat{\theta}_{AB} = \frac{1}{2}\nabla\!\!\!/_A \mathrm{tr}\theta + R_{NA}. \end{cases} \tag{C.88}$$

Recall from the definition of θ (4.61) that $\mathrm{tr}\theta$ is given by:

$$\mathrm{tr}\theta = \mathrm{tr}\chi - \delta$$

where we used the fact that the time foliation is maximal (2.2). In view of the estimate (2.68) for δ and the estimate (2.70) for $\mathrm{tr}\chi$, we obtain:

$$\|\mathrm{tr}\theta\|_{L^\infty_u L^2(P_{t,u})} + \|\nabla\!\!\!/\mathrm{tr}\theta\|_{L^\infty_u L^2(P_{t,u})} + \|\nabla_N \mathrm{tr}\theta\|_{L^\infty_u L^2(P_{t,u})} \lesssim \varepsilon. \tag{C.89}$$

Furthermore, using the definition of θ (4.61) and the Sobolev embedding (3.58), we have:

$$\||\theta|^2\|_{L^\infty_u L^2(P_{t,u})} \lesssim \|\theta\|^2_{L^\infty_u L^4(P_{t,u})} \lesssim \mathcal{N}_1(\theta)^2 \lesssim \mathcal{N}_1(\chi)^2 + \mathcal{N}_1(\eta)^2 \lesssim \varepsilon^2, \tag{C.90}$$

where we used the estimate (2.68) for η and the estimates (2.71) (2.70) for χ. Also, using the Sobolev embedding (3.62), we have:

$$\|\nabla\!\!\!/ b\|_{L^\infty_u L^2(P_{t,u})} + \|\nabla\!\!\!/ b\|_{L^\infty_u L^4(P_{t,u})} \lesssim \mathcal{N}_1(\nabla\!\!\!/ b) \lesssim \varepsilon, \tag{C.91}$$

where we used the estimate (2.69) for b.

Next, we estimate $\|\nabla\!\!\!/^2 b\|_{L^\infty_t L^2(\Sigma_t)}$. In view of the Bochner inequality (4.34), we have:

$$\begin{aligned}\|\nabla\!\!\!/^2 b\|_{L^\infty_t L^2(\Sigma_t)} &\lesssim \|\Delta\!\!\!/ b\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla\!\!\!/ b\|_{L^\infty_t L^2(\Sigma_t)} \\ &\lesssim \|b\|_{L^\infty}(\|\nabla_N \mathrm{tr}\theta\|_{L^\infty_t L^2(\Sigma_t)} + \||\theta|^2\|_{L^\infty_t L^2(\Sigma_t)} + \|R_{NN}\|_{L^\infty_t L^2(\Sigma_t)}) \\ &\quad + \|\nabla\!\!\!/ b\|_{L^\infty_t L^2(\Sigma_t)} \\ &\lesssim \varepsilon,\end{aligned} \tag{C.92}$$

where we used in the last inequality the curvature bound (2.59) for R_{NN}, the estimate (2.69) for b, and the estimates (C.89)-(C.91) for θ and b.

Next, we estimate $\|\widehat{\theta}\|_{L^\infty_u L^2(P_{t,u})}$. in view of the Hodge estimate (3.50), we have:

$$\begin{aligned}\|\nabla\!\!\!/\widehat{\theta}\|_{L^\infty_t L^2(\Sigma_t)} &\lesssim \|\nabla\!\!\!/\mathrm{tr}\theta\|_{L^\infty_t L^2(\Sigma_t)} + \|R_{AN}\|_{L^\infty_t L^2(\Sigma_t)} \\ &\lesssim \varepsilon,\end{aligned} \tag{C.93}$$

where we used in the last inequality the curvature bound (2.59) for R_{AN} and the estimates (C.89) and (C.91) for θ and b.

Finally, (C.87), (C.89), (C.91), (C.92) and (C.93) yield (6.117). This concludes the proof of Lemma 6.23.

C.13 Proof of Lemma 6.24

We estimate the following quantity:

$$\begin{aligned} & \|\nabla\!\!\!/ Q_{\leq 1}(N')\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla\!\!\!/ \nabla\!\!\!/ Q_{\leq 1}(N')\|_{L^\infty_t L^2(\Sigma_t)} \qquad \text{(C.94)} \\ \lesssim & \ \|\nabla Q_{\leq 1}(N')\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla\!\!\!/ \nabla Q_{\leq 1}(N')\|_{L^\infty_t L^2(\Sigma_t)} + \|[\nabla\!\!\!/, \nabla\!\!\!/_N] Q_{\leq 1}(N')\|_{L^\infty_t L^2(\Sigma_t)} \\ \lesssim & \ \|\nabla N'\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla^2 Q_{\leq 1}(N')\|_{L^\infty_t L^2(\Sigma_t)} + \|[\nabla\!\!\!/, \nabla\!\!\!/_N] Q_{\leq 1}(N')\|_{L^\infty_t L^2(\Sigma_t)} \\ \lesssim & \ \|\nabla N'\|_{L^\infty_t L^2(\Sigma_t)} + \|\Delta Q_{\leq 1}(N')\|_{L^\infty_t L^2(\Sigma_t)} + \|[\nabla\!\!\!/, \nabla\!\!\!/_N] Q_{\leq 1}(N')\|_{L^\infty_t L^2(\Sigma_t)} \\ \lesssim & \ \|\nabla N'\|_{L^\infty_t L^2(\Sigma_t)} + \|[\nabla\!\!\!/, \nabla\!\!\!/_N] Q_{\leq 1}(N')\|_{L^\infty_t L^2(\Sigma_t)}, \end{aligned}$$

where we used several times the finite band property for $Q_{\leq 1}$ and the Bochner inequality (3.81). Now, for any tensor F, the following estimate is a immediate consequence of the proof of (3.83) (see (3.86)):

$$\|[\nabla\!\!\!/, \nabla_N] F\|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon \|\nabla\!\!\!/ \nabla F\|_{L^\infty_t L^2(\Sigma_t)} + \varepsilon \|F\|_{L^\infty} \qquad \text{(C.95)}$$

Using (C.95) with $F = Q_{\leq 1}(N')$ yields:

$$\begin{aligned} \|[\nabla\!\!\!/, \nabla_N] Q_{\leq 1}(N')\|_{L^\infty_t L^2(\Sigma_t)} \lesssim & \ \varepsilon \|\nabla\!\!\!/ \nabla Q_{\leq 1}(N')\|_{L^\infty_t L^2(\Sigma_t)} + \varepsilon \|Q_{\leq 1}(N')\|_{L^\infty} \\ \lesssim & \ \varepsilon \|\nabla^2 Q_{\leq 1}(N')\|_{L^\infty_t L^2(\Sigma_t)} + \varepsilon \|N'\|_{L^\infty} \\ \lesssim & \ \varepsilon \|\Delta Q_{\leq 1}(N')\|_{L^\infty_t L^2(\Sigma_t)} + \varepsilon \|N'\|_{L^\infty} \\ \lesssim & \ \varepsilon \|\nabla N'\|_{L^\infty_t L^2(\Sigma_t)} + \varepsilon \|N'\|_{L^\infty}, \end{aligned}$$

where we used the L^∞ boundedness of $Q_{\leq 1}$, the Bochner inequality (3.81), and the finite band property for $Q_{\leq 1}$.

Note from the proof of Lemma 6.23 (see (C.85) the following estimate:

$$\|\nabla N'\|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon.$$

Together with (C.94) and (C.95), this implies:

$$\begin{aligned} \|\nabla\!\!\!/ Q_{\leq 1}(N')\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla\!\!\!/ \nabla\!\!\!/ Q_{\leq 1}(N')\|_{L^\infty_t L^2(\Sigma_t)} \lesssim & \ \|\nabla N'\|_{L^\infty_t L^2(\Sigma_t)} + \varepsilon \|N'\|_{L^\infty} \\ \lesssim & \ \varepsilon. \qquad \text{(C.96)} \end{aligned}$$

We will prove for any tensor vectorfield F the following non sharp estimate:

$$\|F\|_{\mathcal{B}^0} \lesssim \|F\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla F\|_{L^\infty_t L^2(\Sigma_t)}. \qquad \text{(C.97)}$$

(C.96) and (C.97) immediately yield (6.118).

In order to conclude the proof of Lemma 6.24, it remains to prove (C.97). We estimate

$\|P_jF\|^2_{L^\infty_t L^2_{x'}}$. In view of (3.79), we have:

$$\begin{aligned}
&\|P_jF\|^2_{L^\infty_{t,u}L^2(P_{t,u})} \qquad\qquad (C.98)\\
\lesssim\ & \sup_t\left(\int_u \|P_jF\|_{L^2(P_{t,u})}\|\nabla\!\!\!/_{bN}P_jF\|_{L^2(P_{t,u})}du\right)\\
&+\sup_t\left(\|\mathrm{tr}\theta\|_{L^\infty_u L^4(P_{t,u})}\|P_jF\|_{L^2_uL^4(P_{t,u})}\right)\|P_jF\|_{L^\infty_tL^2(\Sigma_t)}\\
\lesssim\ & \|P_jF\|_{L^\infty_tL^2(\Sigma_t)}\|P_j\nabla\!\!\!/_{bN}F\|_{L^\infty_tL^2(\Sigma_t)}+\left(\int_u\|P_jF\|_{L^2(P_{t,u})}\|[\nabla\!\!\!/_{bN},P_j]F\|_{L^2(P_{t,u})}du\right)\\
&+2^{\frac{j}{2}}\varepsilon\|P_jF\|^2_{L^\infty_tL^2(\Sigma_t)}\\
\lesssim\ & 2^{-j}\|\nabla F\|^2_{L^\infty_tL^2(\Sigma_t)}+\|P_jF\|_{L^4_uL^2(P_{t,u})}\|[\nabla\!\!\!/_{bN},P_j]F\|_{L^{\frac{4}{3}}_uL^2(P_{t,u})}\\
\lesssim\ & 2^{-j}\|\nabla F\|^2_{L^\infty_tL^2(\Sigma_t)}+\|P_jF\|^{\frac12}_{L^\infty_uL^2(P_{t,u})}\|P_jF\|^{\frac12}_{L^\infty_tL^2(\Sigma_t)}\|[\nabla\!\!\!/_{bN},P_j]F\|_{L^{\frac{4}{3}}_uL^2(P_{t,u})}\\
\lesssim\ & 2^{-j}\|\nabla F\|^2_{L^\infty_tL^2(\Sigma_t)}+2^{-\frac{j}{2}}\|P_jF\|^{\frac12}_{L^\infty_uL^2(P_{t,u})}\|\nabla\!\!\!/F\|^{\frac12}_{L^\infty_tL^2(\Sigma_t)}\|[\nabla\!\!\!/_{bN},P_j]F\|_{L^{\frac{4}{3}}_uL^2(P_{t,u})},
\end{aligned}$$

where we used the estimate (C.89) for $\mathrm{tr}\theta$, the Bernstein inequality for P_j, and the finite band property and the L^2 boundedness of P_j. (C.98) implies:

$$\|P_jF\|_{L^\infty_tL^2_{x'}}\lesssim 2^{-\frac{j}{2}}\|\nabla F\|_{L^\infty_tL^2(\Sigma_t)}+2^{-\frac{j}{4}}\|[\nabla\!\!\!/_{bN},P_j]F\|_{L^{\frac{4}{3}}_uL^2(P_{t,u})}. \qquad (C.99)$$

Now, (C.99) and the commutator estimate (9.1) imply:

$$\|P_jF\|_{L^\infty_tL^2_{x'}}\lesssim(2^{-\frac{j}{2}}+2^{-\frac{j}{4}+a})\|\nabla F\|_{L^\infty_tL^2_{x'}},$$

where $0<a<\frac14$. In view of the definition of $\mathcal{B}^0$, this yields (C.97). This concludes the proof of Lemma 6.24.

C.14 Proof of Lemma 6.25

In view of the Ricci equations (2.23), we have:

$$\begin{aligned}
\|\mathbf{D}_L(N')\|_{L^2(\mathcal{H}_u)}\ \lesssim\ & \|\chi'\|_{L^2(\mathcal{H}_u)}+\|\underline{\chi}'\|_{L^2(\mathcal{H}_u)}+\|\epsilon'\|_{L^2(\mathcal{H}_u)}+\|\delta'\|_{L^2(\mathcal{H}_u)} \qquad (C.100)\\
&+\|\zeta'\|_{L^2(\mathcal{H}_u)}+\|n^{-1}\nabla_{N'}n\|_{L^2(\mathcal{H}_u)}+\|\underline{\xi}'\|_{L^2(\mathcal{H}_u)},
\end{aligned}$$

where $\chi',\underline{\chi}',\delta',\zeta',\underline{\xi}'$ are the Ricci coefficients associated to $u(\cdot,\omega')$. We only estimate ζ' since it is the worst term in (C.100). In view of the computations (3.56) and (3.77), we have for any scalar function f:

$$\underline{L}\left(\int_{P_{t,u}}fd\mu_{t,u}\right)=\int_{P_{t,u}}(\underline{L}(f)+\mathrm{tr}\underline{\chi}f)d\mu_{t,u}.$$

Together with the coarea formula (3.55) and the fact that ζ' vanishes at infinity, we obtain:

$$\begin{aligned}\|\zeta'\|^2_{L^2(\mathcal{H}_u)} &\lesssim \int_u\int_{\mathcal{H}_u}(\mathbf{D}_{\underline{L}}(\zeta')\cdot\zeta'+|\zeta'|^2(\mathrm{tr}\underline{\chi}+b^{-1}\mathbf{D}_{\underline{L}}(b)))d\mathcal{H}_u du \qquad (C.101)\\ &\lesssim \left|\int_u\int_{\mathcal{H}_u}\mathbf{D}_{\underline{L}}(\zeta')\cdot\zeta' d\mathcal{H}_u du\right| + \|\zeta'\|^2_{L^4(\mathcal{M})}(\|\mathrm{tr}\underline{\chi}\|_{L^2(\mathcal{M})}+\|b^{-1}\mathbf{D}_{\underline{L}}(b)\|_{L^2(\mathcal{M})})\\ &\lesssim \left|\int_u\int_{\mathcal{H}_u}\mathbf{D}_{\underline{L}}(\zeta')\cdot\zeta' d\mathcal{H}_u du\right| + \varepsilon,\end{aligned}$$

where we used the estimates (2.69)-(2.72) for ζ', b and $\mathrm{tr}\underline{\chi}$.

Next, we estimate the right-hand side of (C.101). Decomposing $\underline{L}$ on the frame $L', \underline{L}', e_1', e_2'$, we have:

$$\begin{aligned}&\left|\int_u\int_{\mathcal{H}_u}\mathbf{D}_{\underline{L}}(\zeta')\cdot\zeta' d\mathcal{H}_u du\right| \qquad (C.102)\\ \lesssim\ &\left|\int_u\int_{\mathcal{H}_u} g(\underline{L},\underline{L}')\mathbf{D}_{L'}(\zeta')\cdot\zeta' d\mathcal{H}_u du\right| + \left|\int_u\int_{\mathcal{H}_u} g(\underline{L},e_A')\mathbf{D}_{e_A'}(\zeta')\cdot\zeta' d\mathcal{H}_u du\right|\\ &+\left|\int_u\int_{\mathcal{H}_u} g(\underline{L},L')\mathbf{D}_{\underline{L}'}(\zeta')\cdot\zeta' d\mathcal{H}_u du\right|\\ \lesssim\ &(\|\nabla\!\!\!/_{L'}(\zeta')\|_{L^2(\mathcal{M})}+\|\nabla\!\!\!/\zeta'\|_{L^2(\mathcal{M})})\|\zeta'\|_{L^2(\mathcal{M})} + \left|\int_u\int_{\mathcal{H}_u} g(\underline{L},L')\mathbf{D}_{\underline{L}'}(\zeta')\cdot\zeta' d\mathcal{H}_u du\right|\\ \lesssim\ &\varepsilon^2+\left|\int_u\int_{\mathcal{H}_u} g(\underline{L},L')\mathbf{D}_{\underline{L}'}(\zeta')\cdot\zeta' d\mathcal{H}_u du\right|,\end{aligned}$$

where we used the estimate (2.72) for ζ' in the last inequality.

Now, we estimate the right-hand side of (C.102). Using the Littlewood-Paley decomposition, we have:

$$\begin{aligned}&\left|\int_u\int_{\mathcal{H}_u}(P_j'\mathbf{D}_{\underline{L}'}(\zeta')\cdot P_j'(g(\underline{L},L')\zeta')d\mathcal{H}_u du\right| \qquad (C.103)\\ \lesssim\ &\|P_j'\mathbf{D}_{\underline{L}'}(\zeta')\|_{L^2(\mathcal{M})}\|P_j'(g(\underline{L},L')\zeta')\|_{L^2(\mathcal{M})}\\ \lesssim\ &\varepsilon 2^{-j}\|\nabla\!\!\!/'(g(\underline{L},L')\zeta')\|_{L^2(\mathcal{M})}\\ \lesssim\ &\varepsilon 2^{-j}(\|\nabla\!\!\!/'\zeta'\|_{L^2(\mathcal{M})}+\|\mathbf{D}_{e_A'}(\underline{L})\zeta')\|_{L^2(\mathcal{M})}+\|\mathbf{D}_{e_A'}(L')\zeta')\|_{L^2(\mathcal{M})})\\ \lesssim\ &\varepsilon 2^{-j}(\|\nabla\!\!\!/'\zeta'\|_{L^2(\mathcal{M})}+\|\zeta'\|_{L^4(\mathcal{M})}(\|\mathbf{D}\underline{L}\|_{L^4(\mathcal{M})}+\|\mathbf{D}L'\|_{L^4(\mathcal{M})}))\\ \lesssim\ &\varepsilon^2 2^{-j}\end{aligned}$$

where we used the estimate (2.74) for $\mathbf{D}_{\underline{L}'}(\zeta')$, the finite band property for P_j', the estimate (2.72) for ζ', and the Ricci equations (2.23) together with the estimates (2.67)-(2.72) of the Ricci coefficients to estimate $\mathbf{D}\underline{L}$ and $\mathbf{D}L'$.

Finally, summing with respect to j in (C.103), together with (C.101) and (C.102) yields:

$$\|\zeta'\|_{L^2(\mathcal{H}_u)}\lesssim\varepsilon.$$

The estimates of the other Ricci coefficients in the right-hand side of (C.100) are easier, and we obtain in the end:

$$\|\mathbf{D}_L(N')\|_{L^2(\mathcal{H}_u)}\lesssim\varepsilon,$$

which concludes the proof of Lemma 6.25.

D Appendix to section 8

D.1 Proof of Lemma 8.3

We have constructed a global coordinate system on Σ_t in section 4.2.2. We will need another global coordinate system. Let $\omega \in \mathbb{S}^2$. Let $\Phi_{t,\omega} : \Sigma_t \to \mathbb{R}^3$ defined by:

$$\Phi_{t,\omega}(t,x) := u(t,x,\omega)\omega + \partial_\omega u(t,x,\omega). \tag{D.1}$$

Then we claim that $\Phi_{t,\omega}$ is a global C^1 diffeomorphism from Σ_t to $\mathbb{R}^3$ and therefore provides a global coordinate system on Σ_t. The proof has been done in [21] for the particular case $t = 0$ of a global coordinate system on Σ_0. The proof for Σ_t is completely analogous an we refer the interested reader to Proposition 2.9 in [21]. The proof also provides the following bound for $d\Phi^{-1}_{t,\omega}$ as well as the determinant of its Jacobian:

$$\|d\Phi^{-1}_{t,\omega}\|_{L^\infty} \lesssim 1, \quad \||\det(\mathrm{Jac}\Phi_{t,\omega})| - 1\|_{L^\infty} \lesssim \varepsilon. \tag{D.2}$$

Recall from (6.31) that we have $\|\mathbf{D}_L(\partial_\omega N)\|_{L^\infty_{x'}L^2_t} \lesssim \varepsilon$. This yields:

$$\|\mathbf{D}_L\left(g(\partial_\omega N, \partial_\omega N) - I\right)\|_{L^\infty_{x'}L^2_t} \lesssim \varepsilon$$

which together with the estimate for transport equation (3.66) and the corresponding estimate at initial time (see [21]) yields:

$$\|g(\partial_\omega N, \partial_\omega N) - I\|_{L^\infty} \lesssim \varepsilon. \tag{D.3}$$

Consider the global coordinate system on Σ_t provided by $\Phi^{-1}_{t,\omega}(u, y')$. Then, for any scalar function f on Σ_t, one easily derives the following formulas:

$$\frac{\partial f}{\partial u} = g\left(N + O(\varepsilon), \nabla f\right) \text{ and } \frac{\partial f}{\partial y'} = g\left(\partial_\omega N + O(\varepsilon), \nabla f\right), \tag{D.4}$$

where we used the fact that $g(N, \partial_\omega N) = 0$, $\nabla u(t,x,\omega) = b^{-1}N$, $\nabla\partial_\omega u(t,x,\omega) = -b^{-2}\partial_\omega b N + b^{-1}\partial_\omega N$, $\|b-1\|_{L^\infty} \lesssim \varepsilon$, $\|\partial_\omega b\|_{L^\infty} \lesssim \varepsilon$ and (D.3).

Finally, u being fixed, $\Phi^{-1}_{t,\omega}(u, y')$ provided a coordinate system on $P_{t,u}$ such that the following estimate holds for the induced metric γ in the coordinate system:

$$|\gamma_{AB}(p)\xi^A\xi^B - |\xi|^2| \lesssim \varepsilon|\xi|^2, \qquad \text{uniformly for all } \ p \in \mathbb{R}^2. \tag{D.5}$$

We evaluate $\|F\|_{L^2(\mathcal{H}_u)}$. Using the global coordinate system on $P_{t,u}$ provided by $\Phi^{-1}_{t,\omega}(u, y')$, we have:

$$\begin{aligned} \|F\|^2_{L^2(\mathcal{H}_u)} &= \int_0^1 \int |F(\Phi^{-1}_{t,\omega'}(u,y'))|^2\sqrt{\gamma}dy'dt \\ &\lesssim \int_0^1 \int |F(\Phi^{-1}_{t,u,\omega'}(y'))|^2 dy'dt, \end{aligned} \tag{D.6}$$

where we used (D.5) in the last inequality. Let (t, x_t) a point on Σ_t. Let $0 \leq \sigma \leq 1$ parametrize the arc on $\mathbb{S}^2$ joining ω and ω', and let $\omega_\sigma \in \mathbb{S}^2$ corresponding to σ. Let $u_\sigma = u(t, x_t, \omega_\sigma)$ and $\partial_\omega u_\sigma = \partial_\omega u(t, x_t, \omega_\sigma)$. Let ρ a positive smooth bounded function on $\mathbb{R}$ vanishing in the neighborhood of 0. We consider the following integral:

$$I(\sigma) = \int_0^1 \int |F(\Phi^{-1}_{t,,\omega_\sigma}(u = u_\sigma, y'))|^2 \rho(\partial_\omega u - \partial_\omega u_\sigma) dy' dt. \tag{D.7}$$

We have:

$$I(0) = \int_0^1 \int |F(\Phi^{-1}_{t,\omega'}(u = u_0, y'))|^2 dy' dt \gtrsim \|F\sqrt{\rho(\partial_\omega u - \partial_\omega u_\sigma)}\|^2_{L^2(\mathcal{H}_{u'=u_0})} \tag{D.8}$$

where we used (D.5) and (D.6), and where $u' = u(\cdot, ., \omega')$. We also have:

$$\begin{aligned} I(1) &= \int_0^1 \int |F(\Phi^{-1}_{t,\omega}(u = u_1, y'))|^2 \rho(\partial_\omega u - \partial_\omega u_\sigma) dy' dt \\ &\lesssim \int_0^1 \int |F(\Phi^{-1}_{t,\omega}(u = u_1, y'))|^2 \sqrt{\gamma} dy' dt \\ &\lesssim \|F\|^2_{L^\infty_u L^2(\mathcal{H}_u)}, \end{aligned} \tag{D.9}$$

using again (D.5) and (D.6).

Next, we evaluate $\frac{dI}{d\sigma}$:

$$\begin{aligned} \frac{dI}{d\sigma} &= 2\int_0^1 \int DF(\Phi^{-1}_{t,\omega_\sigma}(u_\sigma, y')) \frac{d}{d\sigma}\left[\Phi^{-1}_{t,\omega_\sigma}(u_\sigma, y')\right] F(\Phi^{-1}_{t,\omega_\sigma}(u_\sigma, y')) \rho(\partial_\omega u - \partial_\omega u_\sigma) dy' dt \\ &\quad - \int_0^1 \int |F(\Phi^{-1}_{t,,\omega_\sigma}(u = u_\sigma, y'))|^2 \partial^2_\omega u_\sigma \rho'(\partial_\omega u - \partial_\omega u_\sigma) dy' dt. \end{aligned} \tag{D.10}$$

Now, we have

$$\frac{d}{d\sigma}\left[\Phi^{-1}_{t,\omega_\sigma}(u_\sigma, y')\right] = d\Phi^{-1}\left(\frac{d}{d\sigma}\left[\Phi_{t,\omega_\sigma}\right] \circ \Phi^{-1}\right)(u_\sigma, y) + d\Phi^{-1}_{t,\omega_\sigma}\left(\frac{du_\sigma}{d\sigma}, y'\right),$$

which yields:

$$\begin{aligned} \left\|\frac{d}{d\sigma}\left[\Phi^{-1}_{t,\omega_\sigma}(u_\sigma, .)\right]\right\|_{L^\infty} &\leq \|d\Phi^{-1}\|_{L^\infty}\left(\left\|\frac{d}{d\sigma}\left[\Phi_{t,\omega_\sigma}\right]\right\|_{L^\infty} + \left\|\frac{du_\sigma}{d\sigma}\right\|_{L^\infty}\right) \\ &\lesssim \|\partial^2_\omega u\|_{L^\infty} + \|\partial_\omega u\|_{L^\infty} \\ &\lesssim 1 \end{aligned} \tag{D.11}$$

where we used (D.2). Also, differentiating twice the Eikonal equation with respect to ω, we obtain:

$$L(\partial^2_\omega u) = -b^{-1} g(\partial_\omega N, \partial_\omega N).$$

Since $\|\partial_\omega N\|_{L^\infty} \lesssim 1$, the use of the estimate for transport equations (3.66) together with a corresponding estimate at initial time (see [21]) yields:

$$\|\partial^2_\omega u\|_{L^\infty} \lesssim 1.$$

Together with (D.10) and (D.11), we obtain:

$$\left|\frac{dI}{d\sigma}\right| \lesssim \int_0^1 \int (|DF(\Phi^{-1}_{t,\omega_\sigma}(u_\sigma,y'))||F(\Phi^{-1}_{t,\omega_\sigma}(u_\sigma,y'))|\rho(\partial_\omega u-\partial_\omega u_\sigma) \quad (D.12)$$
$$+|F(\Phi^{-1}_{t,\omega_\sigma}(u_\sigma,y'))|^2)\rho'(\partial_\omega u-\partial_\omega u_\sigma))dy'dt.$$

In view of (D.8), (D.9) and (D.12), we obtain:

$$\begin{aligned}&\|F\sqrt{\rho(\partial_\omega u-\partial_\omega u_0)}\|^2_{L^2(\mathcal{H}_{u'=u_0})}\\ \lesssim\ &\|F\|^2_{L^\infty_u L^2(\mathcal{H}_u)}+\int_0^1\int_0^1\int(|DF(\Phi^{-1}_{t,\omega_\sigma}(u_\sigma,y'))||F(\Phi^{-1}_{t,\omega_\sigma}(u_\sigma,y'))|\rho(\partial_\omega u-\partial_\omega u_\sigma)\\ &+|F(\Phi^{-1}_{t,\omega_\sigma}(u_\sigma,y'))|^2\rho'(\partial_\omega u-\partial_\omega u_\sigma))dy'dtd\sigma. \quad (D.13)\end{aligned}$$

Next, we consider the change of variables $(\sigma,y')\to(u,z')$ where:

$$u=u(t,x,\omega),\ y'=\partial_\omega u(t,x,\omega_\sigma) \text{ and } z'=\partial_\omega u(t,x,\omega).$$

Given $(t,x)\in\Sigma_t$, there is only one $\sigma(t,x)$ such that $u(t,x,\omega_\sigma)=u_\sigma$. $\sigma(t,x)$ is given implicitly by the following equation:

$$u(t,x,\omega_\sigma)=u(t,x_0,\omega)$$

which after differentiation provides the formula:

$$\nabla\sigma(t,x)=\frac{\nabla u(t,x)}{\partial_\omega u(t,x,\omega_\sigma)-\partial_\omega u(t,x,\omega_\sigma)}. \quad (D.14)$$

Also, we have:

$$\nabla u(t,x,\omega_\sigma)=b_\sigma^{-1}N_\sigma \text{ and } \nabla\partial_\omega u(t,x,\omega_\sigma)=-b_\sigma^{-2}\partial_\omega b_\sigma N_\sigma+b_\sigma^{-1}\partial_\omega N_\sigma, \quad (D.15)$$

with the notation $N=N(t,x,\omega)$, $b=b(t,x,\omega)$, $N_\sigma=N(t,x,\omega_\sigma)$, and $b_\sigma=b(t,x,\omega_\sigma)$. In view of (D.4), the Jacobian J of the change of variable $(\sigma,y')\to(u,z')$ in Σ_t is the 3×3 matrix given by:

$$J=\begin{pmatrix} g(N+O(\varepsilon),\nabla\sigma(t,x)) & g(N+O(\varepsilon),\nabla\partial_\omega u(t,x,\omega_\sigma))\\ g(\partial_\omega N+O(\varepsilon),\nabla\sigma(t,x)) & g(\partial_\omega N+O(\varepsilon),\nabla\partial_\omega u(t,x,\omega_\sigma))\end{pmatrix}.$$

Together with (D.14) and (D.15), this yields for the determinant $|J|$:

$$\begin{aligned}|J| &= \frac{b_\sigma^{-3}}{\partial_\omega u(t,x,\omega_\sigma)-\partial_\omega u(t,x,\omega_\sigma)} \quad (D.16)\\ &\times\begin{vmatrix} g(N+O(\varepsilon),N_\sigma) & g(N+O(\varepsilon),-b_\sigma^{-1}\partial_\omega b_\sigma N_\sigma+\partial_\omega N_\sigma)\\ g(\partial_\omega N+O(\varepsilon),N_\sigma) & g(\partial_\omega N,-b_\sigma^{-1}\partial_\omega b_\sigma N_\sigma+\partial_\omega N_\sigma)\end{vmatrix}.\end{aligned}$$

Now, recall that:

$$\|b-1\|_{L^\infty}\lesssim 1,\ \|\partial_\omega b\|_{L^\infty}\lesssim\varepsilon \text{ and } \|\partial_\omega N\|_{L^\infty}\lesssim 1$$

which together with (D.16) yields:

$$|J| \lesssim \frac{1}{\partial_\omega u(t,x,\omega_\sigma) - \partial_\omega u(0,x_0,\omega_\sigma)}. \tag{D.17}$$

Now, recall that ρ vanishes in the neighborhood of 0, which together with (D.17) implies:

$$|\rho(\partial_\omega u - \partial_\omega u_\sigma)|J|| + |\rho'(\partial_\omega u - \partial_\omega u_\sigma)|J|| \lesssim 1. \tag{D.18}$$

Next, we consider the range of $u(t,x,\omega)$ in the domain of the integral in the right-hand side of (D.13). We have:

$$\begin{aligned} |u(t,x,\omega) - u_1| &\lesssim |u(t,x,\omega) - u(t,x,\omega_\sigma)| + |u(t,x,\omega_\sigma) - u_\sigma| + |u_\sigma - u_1| \\ &\lesssim |u(t,x,\omega) - u(t,x,\omega_\sigma)| + |u(t,x_t,\omega_\sigma - u(t,x_t,\omega)| \\ &\lesssim \|\partial_\omega u\|_{L^\infty} |\omega_\sigma - \omega| \\ &\lesssim |\omega - \omega'|, \end{aligned} \tag{D.19}$$

where we used the fact that $u(t,x,\omega_\sigma) = u_\sigma$, $u_\sigma = u(t,x_t,\sigma)$, $u_1 = u(t,x_t,\omega)$ and $\|\partial_\omega u\|_{L^\infty} \lesssim 1$.

In view of (D.18) and (D.19), the change of variables $(\sigma, y') \to (u, z')$ in (D.13) yields:

$$\begin{aligned} &\|F\sqrt{\rho(\partial_\omega u - \partial_\omega u_0)}\|^2_{L^2(\mathcal{H}_{u'=u_0})} \\ \lesssim\ & \|F\|^2_{L^\infty_u L^2(\mathcal{H}_u)} + \int_0^1 \int_{u_1}^{u_1+|\omega-\omega'|} \int |DF(\Phi^{-1}_{t,\omega}(u,y'))||F(\Phi^{-1}_{t,\omega}(u,y'))| dy'dtdu \\ \lesssim\ & \|F\|^2_{L^\infty_u L^2(\mathcal{H}_u)} + \int_0^1 \int_{u_1}^{u_1+|\omega-\omega'|} \int |DF(\Phi^{-1}_{t,\omega}(u,y'))||F(\Phi^{-1}_{t,\omega}(u,y'))| dy'dtdu \\ \lesssim\ & \|F\|^2_{L^\infty_u L^2(\mathcal{H}_u)} + \sup_u \left(\int_u^{u+|\omega-\omega'|} \int_0^1 \int |DF(\Phi^{-1}_{t,\omega}(u,y'))||F(\Phi^{-1}_{t,\omega}(u,y'))| dy'dtdu \right) \\ \lesssim\ & \|F\|^2_{L^\infty_u L^2(\mathcal{H}_u)} + |\omega-\omega'|^{\frac{1}{2}} \sup_u \left(\int_0^1 \int |F(\Phi^{-1}_{t,\omega}(u,y'))|^2 dy'dt \right)^{\frac{1}{2}} \\ & \times \sup_u \left(\int_u^{u+|\omega-\omega'|} \int_0^1 \int |DF(\Phi^{-1}_{t,\omega}(u,y'))|^2 dy'dtdu \right)^{\frac{1}{2}}. \end{aligned} \tag{D.20}$$

Now, we have:

$$\|F\|^2_{L^2(\mathcal{H}_u)} = \int_0^1 \int |F(\Phi^{-1}_{t,\omega}(u,y'))|^2 \sqrt{\gamma} dy'dt \gtrsim \int_0^1 \int |F(\Phi^{-1}_{t,\omega}(u,y'))|^2 dy'dt$$

where we used (D.5). Together with (D.20), this yields:

$$\begin{aligned} &\|F\sqrt{\rho(\partial_\omega u - \partial_\omega u_0)}\|^2_{L^2(\mathcal{H}_{u'=u_0})} \\ \lesssim\ & \|F\|^2_{L^\infty_u L^2(\mathcal{H}_u)} + |\omega-\omega'|^{\frac{1}{2}} \|F\|_{L^\infty_u L^2(\mathcal{H}_u)} \left(\sup_u \left(\int_u^{u+|\omega-\omega'|} \|\mathbf{D}F\|^2_{L^2(\mathcal{H}_u)} d\tau \right)^{\frac{1}{2}} \right). \end{aligned} \tag{D.21}$$

Now, (D.21) holds regardless of the choice (t, x_t) on P_{t,u_0}. Also, $\partial_\omega u$ as a map from $P_{t,u}$ to the tangent space $T_\omega \mathbb{S}^2$ is a C^1 diffeomorphism (see [21] Proposition 2.8 for a completely analogous proof in the case $t = 0$ of $P_{0,u}$). Thus, we may choose ρ, and two points (t, x_t^1) and (t, x_t^2) on P_{t,u_0} sufficiently far from each other such that for all $(t, x) \in P_{t,u}$, we have:

$$\rho(\partial_\omega u(t, x, \omega') - \partial_\omega u(t, x_t^1, \omega')) + \rho(\partial_\omega u(t, x, \omega') - \partial_\omega u(t, x_t^2, \omega')) \geq 1.$$

Together with (D.21), we obtain:

$$\|F\|^2_{L^2(\mathcal{H}_{u'=u_0})} \lesssim \|F\|^2_{L^\infty_u L^2(\mathcal{H}_u)} + |\omega-\omega'|^{\frac{1}{2}} \|F\|_{L^\infty_u L^2(\mathcal{H}_u)} \left(\sup_u \left(\int_u^{u+|\omega-\omega'|} \|\mathbf{D}F\|^2_{L^2(\mathcal{H}_u)} d\tau \right)^{\frac{1}{2}} \right).$$

Taking the supremum over u_0 implies:

$$\|F\|^2_{L^\infty_{u'} L^2(\mathcal{H}_{u'})} \lesssim \|F\|^2_{L^\infty_u L^2(\mathcal{H}_u)} + |\omega-\omega'|^{\frac{1}{2}} \|F\|_{L^\infty_u L^2(\mathcal{H}_u)} \left(\sup_u \left(\int_u^{u+|\omega-\omega'|} \|\mathbf{D}F\|^2_{L^2(\mathcal{H}_u)} d\tau \right)^{\frac{1}{2}} \right),$$

which concludes the proof of Lemma 8.3.

Remark D.1 *The change of variables $(\sigma, y') \to (u, z')$ in* (D.13) *is singular at $(t, x) = (t, x_t)$ in view of the determinant of the corresponding Jacobian* (D.17)*. This is also the case in the flat case where $u(t, x, \omega) = t + x \cdot \omega$ and where $P_{t,u}$ are parallel planes in $\mathbb{R}^3$ orthogonal to ω. In this case, the corresponding change of variables corresponds to a change of variable in the plane of $\mathbb{R}^3$ spanned by ω and ω' passing through x_t from polar coordinates with origin at x_t to cartesian coordinates. This explains why the singularity at (t, x_t) in the change of variables $(\sigma, y') \to (u, z')$ in* (D.13) *is natural. Fortunately, one has the freedom to chose the point (t, x_t) around which we rotate the surfaces $P_{t,u}$ which allows us to tackle this issue by considering successively two point in Σ_t (t, x_t^1) and (t, x_t^2) as preformed in the end of the above proof.*

D.2 Proof of Lemma 8.4

Let us apply Lemma 8.3 with $F = P_l f$ where f is a scalar function. Then:

$$\begin{aligned} & \|P_l f\|_{L^\infty_{u'} L^2(\mathcal{H}_{u'})} \qquad (D.22) \\ \lesssim\ & \|P_l f\|_{L^\infty_u L^2(\mathcal{H}_u)} + |\omega-\omega'|^{\frac{1}{4}} \|P_l f\|^{\frac{1}{2}}_{L^\infty_u L^2(\mathcal{H}_u)} \left(\sup_u \left(\int_u^{u+|\omega-\omega'|} \|\mathbf{D}P_l f\|^2_{L^2(\mathcal{H}_\tau)} d\tau \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\ \lesssim\ & 2^{-l} \|\nabla\!\!\!/ f\|_{L^\infty_u L^2(\mathcal{H}_u)} + 2^{-\frac{l}{2}} |\omega-\omega'|^{\frac{1}{2}} \|\nabla\!\!\!/ f\|^{\frac{1}{2}}_{L^\infty_u L^2(\mathcal{H}_u)} \|\mathbf{D}P_l f\|^{\frac{1}{2}}_{L^\infty_u L^2(\mathcal{H}_u)} \end{aligned}$$

where we used the finite band property for P_l in the last inequality.

In order to prove Lemma 8.4, it is enough in view of (D.22) to prove:

$$\|\mathbf{D}P_l f\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \|\mathbf{D}f\|_{L^\infty_u L^2(\mathcal{H}_u)}. \qquad (D.23)$$

Furthermore, (D.23) for $\mathbf{D} = \nabla\!\!\!/$ follows from the properties of P_l, so we may focus on the case of L and $\underline{L}$, or even L and N. Also, the case of L being easier, we focus on the case of N. Thus, the proof of Lemma 8.4 reduces to:

$$\|N(P_l f)\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \|\mathbf{D} f\|_{L^\infty_u L^2(\mathcal{H}_u)}. \tag{D.24}$$

Since $\|b-1\|_{L^\infty} \lesssim \varepsilon$, we have:

$$\begin{aligned}\|N(P_l f)\|_{L^\infty_u L^2(\mathcal{H}_u)} &\lesssim \|bN(P_l f)\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ &\lesssim \|P_l(bN(f))\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|[bN, P_l]f\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ &\lesssim \|\mathbf{D} f\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|[bN, P_l]f\|_{L^\infty_u L^2(\mathcal{H}_u)},\end{aligned}$$

where we used the $L^2(P_{t,u})$ boundedness of P_l in the last inequality. Together with the commutator estimate (9.2), we obtain the desired estimate (D.24). This concludes the proof of Lemma 8.4.

D.3 Proof of Lemma 8.5

Let us apply Lemma 8.3 with $F = P_{\le l} f$ where f is a scalar function. Then:

$$\begin{aligned}&\|P_{\le l} f\|_{L^\infty_{u'} L^2(\mathcal{H}_{u'})} \qquad\qquad (\text{D.25})\\ \lesssim\ & \|P_{\le l} f\|_{L^\infty_u L^2(\mathcal{H}_u)} + |\omega-\omega'|^{\frac14} \|P_{\le l} f\|^{\frac12}_{L^\infty_u L^2(\mathcal{H}_u)} \left(\sup_u \left(\int_u^{u+|\omega-\omega'|} \|\mathbf{D} P_{\le l} f\|^2_{L^2(\mathcal{H}_\tau)} d\tau \right)^{\frac12} \right)^{\frac12} \\ \lesssim\ & \|f\|_{L^\infty_u L^2(\mathcal{H}_u)} + |\omega-\omega'|^{\frac14} \|f\|^{\frac12}_{L^\infty_u L^2(\mathcal{H}_u)} \left(\sup_u \left(\int_u^{u+|\omega-\omega'|} \|\mathbf{D} P_{\le l} f\|^2_{L^2(\mathcal{H}_\tau)} d\tau \right)^{\frac12} \right)^{\frac12}\end{aligned}$$

where we used the finite band property for $P_{\le l}$ in the last inequality. Now, we have:

$$\begin{aligned}\|\mathbf{D} P_{\le l} f\|_{L^2(\mathcal{H}_u)} &\lesssim \|\nabla\!\!\!/ P_{\le l} f\|_{L^2(\mathcal{H}_u)} + \|L(P_{\le l} f)\|_{L^2(\mathcal{H}_u)} + \|N(P_{\le l} f)\|_{L^2(\mathcal{H}_u)} \qquad (\text{D.26})\\ &\lesssim 2^l \|f\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|nL(P_{\le l} f)\|_{L^2(\mathcal{H}_u)} + \|bN(P_{\le l} f)\|_{L^2(\mathcal{H}_u)},\end{aligned}$$

where we used the finite band property for P_l in the last inequality. Also:

$$\begin{aligned}&\|nL(P_{\le l} f)\|_{L^2(\mathcal{H}_u)} + \|bN(P_{\le l} f)\|_{L^2(\mathcal{H}_u)} \\ \lesssim\ & \|P_{\le l}(nL(f))\|_{L^2(\mathcal{H}_u)} + \|P_{\le l}(bN(f))\|_{L^2(\mathcal{H}_u)} + \|[nL, P_{\le l}]f\|_{L^2(\mathcal{H}_u)} + \|[bN, P_{\le l}]f\|_{L^2(\mathcal{H}_u)} \\ \lesssim\ & \sum_{q \le l} \left(\|P_q(nL(f))\|_{L^2(\mathcal{H}_u)} + \|P_q(bN(f))\|_{L^2(\mathcal{H}_u)} + \|[nL, P_q]f\|_{L^2(\mathcal{H}_u)} + \|[bN, P_q]f\|_{L^2(\mathcal{H}_u)} \right),\end{aligned}$$

which together with the commutator estimate (9.4) implies

$$\begin{aligned}&\|nL(P_{\le l} f)\|_{L^2(\mathcal{H}_u)} + \|bN(P_{\le l} f)\|_{L^2(\mathcal{H}_u)} \qquad (\text{D.27})\\ \lesssim\ & \sum_{q \le l} \left(\|P_q(nL(f))\|_{L^2(\mathcal{H}_u)} + \|P_q(bN(f))\|_{L^2(\mathcal{H}_u)} + 2^q \|f\|_{L^\infty_u L^2(\mathcal{H}_u)} \right) \\ \lesssim\ & \sum_{q \le l} \left(\|P_q(nL(f))\|_{L^2(\mathcal{H}_u)} + \|P_q(bN(f))\|_{L^2(\mathcal{H}_u)} \right) + 2^l \|f\|_{L^\infty_u L^2(\mathcal{H}_u)}.\end{aligned}$$

Finally, in view of (D.25)-(D.27), we obtain the conclusion of Lemma 8.5.

D.4 Proof of Lemma 8.6

Lemma 8.9 with the choice $p = 2$ yields the following estimate for any $a > \frac{1}{2}$:

$$\begin{aligned} \|f\|_{L^\infty_{u'}L^2(\mathcal{H}_{u'})} &\leq \sum_j \|P_j f\|_{L^\infty_{u'}L^2(\mathcal{H}_{u'})} \\ &\leq \sum_j \|P_j f\|^{\frac{1}{2}}_{L^\infty_u L^2(\mathcal{H}_u)} \|\nabla\!\!\!/ P_j f\|^{\frac{1}{2}}_{L^\infty_u L^2(\mathcal{H}_u)} \\ &\leq \sum_j 2^{\frac{j}{2}} \|P_j f\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ &\leq \left(\sum_j 2^{j(\frac{1}{2}-a)}\right) \|\Lambda^a f\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ &\leq \|\Lambda^a f\|_{L^\infty_u L^2(\mathcal{H}_u)} \end{aligned} \tag{D.28}$$

where we used the finite band property for P_j and the analog of (C.58) which yields for any $0 \leq a \leq 1$

$$\|P_j f\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-ja} \|\Lambda^a f\|_{L^\infty_u L^2(\mathcal{H}_u)}. \tag{D.29}$$

Next, we evaluate $[\partial_\omega, P_{\leq l}]f$. We have:

$$\partial_\omega U(\tau) f = U(\tau)\partial_\omega f + W(\tau)$$

where $W(\tau)$ satisfies:

$$(\partial_\tau - \Delta\!\!\!/)W(\tau) = [\partial_\omega, \Delta\!\!\!/]U(\tau)f,\ W(0) = 0. \tag{D.30}$$

Using the definition of P_q (3.15) and (D.30), we obtain:

$$[\partial_\omega, P_q]f = \int_0^\infty m_q(\tau) W(\tau) d\tau.$$

Together with (D.28), this yields for any $a > \frac{1}{2}$:

$$\begin{aligned} \|[\partial_\omega, P_{\leq l}]f\|_{L^\infty_{u'}L^2(\mathcal{H}_{u'})} &\lesssim \left\| \sum_{q \leq l} [\partial_\omega, P_q] f \right\|_{L^\infty_{u'}L^2(\mathcal{H}_{u'})} \\ &\lesssim \left\| \int_0^\infty \left| \left(\sum_{q \leq l} m_q(\tau) \right) \right| \|\Lambda^a W(\tau)\|_{L^2(\mathcal{H}_u)} d\tau \right\|_{L^\infty_u} \\ &\lesssim \left(\int_0^\infty \left| \left(\sum_{q \leq l} m_q(\tau) \right) \right| d\tau \right) \left\| \sup_\tau \|\Lambda^a W(\tau)\|_{L^2(\mathcal{H}_u)} \right\|_{L^\infty_u} \\ &\lesssim \left\| \sup_\tau \|\Lambda^a W(\tau)\|_{L^2(\mathcal{H}_u)} \right\|_{L^\infty_u}. \end{aligned} \tag{D.31}$$

Let

$$\frac{1}{2} < a < 1.$$

The energy estimate (3.30) implies:

$$\begin{aligned}
&\|\Lambda^a W(\tau)\|^2_{L^2(\mathcal{H}_u)} + \int_0^\tau \|\nabla\!\!\!/\Lambda^a W(\tau')\|^2_{L^2(\mathcal{H}_u)} d\tau' \\
\lesssim\ & \int_0^\tau \int_0^1 \int_{P_{t,u}} \Lambda^{2a} W(\tau')[\partial_\omega, \Delta\!\!\!/] U(\tau') f d\mu_{t,u} dt d\tau' \\
\lesssim\ & \int_0^\tau \|\Lambda^{1+a} W(\tau')\|_{L^2(\mathcal{H}_u)} \|\Lambda^{-1+a}[\partial_\omega, \Delta\!\!\!/] U(\tau') f\|_{L^2(\mathcal{H}_u)} d\tau' \\
\lesssim\ & \int_0^\tau \|\nabla\!\!\!/\Lambda^a W(\tau')\|_{L^2(\mathcal{H}_u)} \|\Lambda^{-1+a}[\partial_\omega, \Delta\!\!\!/] U(\tau') f\|_{L^2(\mathcal{H}_u)} d\tau',
\end{aligned}$$

where we used the property (3.23) for Λ. Thus, we obtain:

$$\|\Lambda^a W(\tau)\|^2_{L^2(\mathcal{H}_u)} + \int_0^\tau \|\nabla\!\!\!/\Lambda^a W(\tau')\|^2_{L^2(\mathcal{H}_u)} d\tau' \lesssim \int_0^\tau \|\Lambda^{-1+a}[\partial_\omega, \Delta\!\!\!/] U(\tau') f\|^2_{L^2(\mathcal{H}_u)} d\tau'. \tag{D.32}$$

The following formula has been established in [21]:

$$\begin{aligned}
[\partial_\omega, \Delta\!\!\!/] U(\tau) f = \ & -2\nabla\!\!\!/_{\partial_\omega N} \nabla_N U(\tau) f + 2\theta(\partial_\omega N, \nabla\!\!\!/ U(\tau) f) - \mathrm{tr}\theta \nabla\!\!\!/_{\partial_\omega N} U(\tau) f \\
& - \partial_\omega \mathrm{tr}\theta \nabla_N U(\tau) f,
\end{aligned} \tag{D.33}$$

where $\theta = \chi + \eta$ has been defined in (4.61). The estimates (2.68)-(2.71) for θ, (2.76) for $\partial_\omega N$ and (2.77) for $\partial_\omega \mathrm{tr}\theta$ together with the Gagliardo-Nirenberg inequality (3.3), the fact that $a < 1$, the estimate (3.28) for Λ^{-1+a} and (D.33) imply:

$$\begin{aligned}
\|\Lambda^{-1+a}[\partial_\omega, \Delta\!\!\!/] U(\tau) f\|_{L^2(\mathcal{H}_u)} \ &\lesssim\ \|[\partial_\omega, \Delta\!\!\!/] U(\tau) f\|_{L^2_t L^{2-}_{x'}} \\
&\lesssim\ \|\nabla\!\!\!/\nabla_N U(\tau) f\|_{L^2(\mathcal{H}_u)} \|\partial_\omega N\|_{L^\infty} \\
&\quad + \|\theta\|_{L^\infty_t L^4_{x'}} \|\partial_\omega N\|_{L^\infty} \|\nabla\!\!\!/ U(\tau) f\|_{L^2_t L^4_{x'}} \\
&\quad + \|\partial_\omega \mathrm{tr}\theta\|_{L^\infty_t L^2_{x'}} \|\nabla\!\!\!/\nabla_N U(\tau) f\|_{L^2(\mathcal{H}_u)} \\
&\lesssim\ \|\nabla\!\!\!/^2 U(\tau) f\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/\nabla_N U(\tau) f\|_{L^2(\mathcal{H}_u)}.
\end{aligned}$$

Together with the Bochner inequality for scalars (4.34) and the estimates for b (2.69), this yields

$$\|\Lambda^{-1+a}[\partial_\omega, \Delta\!\!\!/] U(\tau) f\|_{L^2(\mathcal{H}_u)} \lesssim \|\Delta\!\!\!/ U(\tau) f\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/\nabla_{bN} U(\tau) f\|_{L^2(\mathcal{H}_u)}. \tag{D.34}$$

Together with the definition of V:

$$(\partial_\tau - \Delta\!\!\!/) V(\tau) = [bN, \Delta\!\!\!/] U(\tau) f,\ V(0) = 0, \tag{D.35}$$

which yields

$$\nabla_{bN} U(\tau) f \ = \ U(\tau) \nabla_{bN} f + V(\tau),$$

we infer

$$\|\Lambda^{-1+a}[\partial_\omega, \not\Delta]U(\tau)f\|_{L^2(\mathcal{H}_u)} \lesssim \|\not\Delta U(\tau)f\|_{L^2(\mathcal{H}_u)} + \|\not\nabla U(\tau)\nabla_{bN}f\|_{L^2(\mathcal{H}_u)} + \|\not\nabla V(\tau)\|_{L^2(\mathcal{H}_u)}. \tag{D.36}$$

Using the Heat flow estimate (3.9), we have:

$$\|\not\nabla U(\tau)f\|^2_{L^2(\mathcal{H}_u)} + \int_0^\tau \|\not\Delta U(\tau)f\|^2_{L^2(\mathcal{H}_u)}d\tau \lesssim \|\not\nabla f\|^2_{L^2(\mathcal{H}_u)}. \tag{D.37}$$

Using the Heat flow estimate (3.8), we have:

$$\|U(\tau)\nabla_N f\|^2_{L^2(\mathcal{H}_u)} + \int_0^\tau \|\not\nabla U(\tau)\nabla_N f\|^2_{L^2(\mathcal{H}_u)}d\tau \lesssim \|\nabla_N f\|^2_{L^2(\mathcal{H}_u)}. \tag{D.38}$$

The estimate (E.17) for $\not\nabla V$ (note that V in (E.17) is defined as in (D.35), so that (E.17) applies), (D.36), (D.37) and (D.38) imply:

$$\int_0^\tau \|\Lambda^{-1+a}[\partial_\omega, \not\Delta]U(\tau')f\|^2_{L^2(\mathcal{H}_u)}d\tau' \lesssim \|\mathbf{D}f\|^2_{L^\infty_u L^2(\mathcal{H}_u)},$$

which together with (D.32) yields:

$$\|\Lambda^a W(\tau)\|^2_{L^2(\mathcal{H}_u)} + \int_0^\tau \|\not\nabla\Lambda^a W(\tau')\|^2_{L^2(\mathcal{H}_u)}d\tau' \lesssim \|\mathbf{D}f\|^2_{L^\infty_u L^2(\mathcal{H}_u)}. \tag{D.39}$$

Since (D.39) holds for any $\frac{1}{2} < a < 1$, we obtain together with (D.31)

$$\|[\partial_\omega, P_{\leq l}]f\|_{L^\infty_{u'}L^2(\mathcal{H}_{u'})} \lesssim \|\mathbf{D}f\|_{L^\infty_u L^2(\mathcal{H}_u)}.$$

This concludes the proof of Lemma 8.6.

D.5 Proof of Lemma 8.7

We have:

$$\begin{aligned}
&\|\mathbf{D}_T Q_{\leq 1}(N)\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla\mathbf{D}_T Q_{\leq 1}(N)\|_{L^\infty_t L^2(\Sigma_t)}\\
\lesssim\ &\|\mathbf{D}_{nT} Q_{\leq 1}(N)\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla\mathbf{D}_{nT} Q_{\leq 1}(N)\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla n\|_{L^\infty_t L^3(\Sigma_t)}\|\mathbf{D}_T Q_{\leq 1}(N)\|_{L^\infty_t L^6(\Sigma_t)}\\
\lesssim\ &\|\mathbf{D}_{nT} Q_{\leq 1}(N)\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla\mathbf{D}_{nT} Q_{\leq 1}(N)\|_{L^\infty_t L^2(\Sigma_t)},
\end{aligned}$$

where we used in the last inequality the estimates (2.67) and (4.47) for n, and the Sobolev embedding (3.71) on Σ_t. This yields:

$$\begin{aligned}
&\|\mathbf{D}_T Q_{\leq 1}(N)\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla\mathbf{D}_T Q_{\leq 1}(N)\|_{L^\infty_t L^2(\Sigma_t)} \qquad (D.40)\\
\lesssim\ &\|Q_{\leq 1}(\mathbf{D}_{nT}N)\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla Q_{\leq 1}(\mathbf{D}_{nT}N)\|_{L^\infty_t L^2(\Sigma_t)} + \|[\mathbf{D}_{nT}, Q_{\leq 1}](N)\|_{L^\infty_t L^2(\Sigma_t)}\\
&+\|\nabla[\mathbf{D}_{nT}, Q_{\leq 1}](N)\|_{L^\infty_t L^2(\Sigma_t)}\\
\lesssim\ &\|\mathbf{D}_{nT}N\|_{L^\infty_t L^2(\Sigma_t)} + \|[\mathbf{D}_{nT}, Q_{\leq 1}](N)\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla[\mathbf{D}_{nT}, Q_{\leq 1}](N)\|_{L^\infty_t L^2(\Sigma_t)}.
\end{aligned}$$

Now, we have in view of the Ricci equations (2.23), we have:

$$\mathbf{D}_T N = n^{-1}N(n)T + (\zeta_A - n^{-1}\not\nabla_A n)e_A$$

which together with the estimates (2.67) for n and (2.72) for ζ yields:

$$\|\mathbf{D}_{nT}N\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \|\nabla n\|_{L_t^\infty L^2(\Sigma_t)} + \|\zeta\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \varepsilon.$$

Together with (D.40), this yields:

$$\begin{aligned} & \|\mathbf{D}_T Q_{\leq 1}(N)\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla \mathbf{D}_T Q_{\leq 1}(N)\|_{L_t^\infty L^2(\Sigma_t)} \\ \lesssim\ & \|[\mathbf{D}_{nT}, Q_{\leq 1}](N)\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla [\mathbf{D}_{nT}, Q_{\leq 1}](N)\|_{L_t^\infty L^2(\Sigma_t)} + \varepsilon. \end{aligned} \tag{D.41}$$

Next, we estimate the commutator terms in the right-hand side of (D.41). Using the definition of Q_j, we have:

$$[\mathbf{D}_{nT}, Q_j]N = \int_0^\infty m_j(\tau) Z(\tau) d\tau, \tag{D.42}$$

where $Z(\tau)$ satisfies:

$$(\partial_\tau - \Delta) Z(\tau) = [\mathbf{D}_{nT}, \Delta] Y(\tau) N,\ Z(0) = 0, \tag{D.43}$$

with $Y(\tau)N$ the solution of:

$$(\partial_\tau - \Delta) Y(\tau) N = 0,\ Y(0)N = N.$$

In view of (D.42), we have:

$$\begin{aligned} & \|[\mathbf{D}_{nT}, Q_{\leq 1}](N)\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla [\mathbf{D}_{nT}, Q_{\leq 1}](N)\|_{L_t^\infty L^2(\Sigma_t)} \\ \lesssim\ & \left\| \sup_\tau \|Z(\tau)\|_{L^2(\Sigma_t)} \right\|_{L_t^\infty} + \left\| \int_0^\infty \|\nabla Z(\tau)\|^2_{L^2(\Sigma_t)} d\tau \right\|^{\frac{1}{2}}_{L_t^\infty}. \end{aligned} \tag{D.44}$$

Our next goal is to evaluate the right-hand side of (D.44). Multiplying (D.43) with $Z(\tau)$ and integrating on Σ_t and with respect to τ yields:

$$\|Z(\tau)\|^2_{L^2(\Sigma_t)} + \int_0^\tau \|\nabla Z(\tau')\|^2_{L^2(\Sigma_t)} d\tau' \lesssim \int_0^\tau \int_{\Sigma_t} Z(\tau') [\mathbf{D}_{nT}, \Delta] Y(\tau') N d\Sigma_t d\tau'. \tag{D.45}$$

In view of the commutator formula (3.95), we have, schematically:

$$\begin{aligned} [\mathbf{D}_{nT}, \Delta] Y(\tau) N \ =\ & nk\nabla^2 Y(\tau) N + (n\mathbf{R} + k\nabla n + n\nabla k)\nabla Y(\tau) N \\ & + (\nabla k \nabla n + k\Delta n) Y(\tau) N + \nabla(n\mathbf{R} Y(\tau) N). \end{aligned}$$

Integrating by parts, this yields:

$$\begin{aligned}
&\int_0^\tau \int_{\Sigma_t} Z(\tau')[\mathbf{D}_{nT}, \Delta]Y(\tau')Nd\Sigma_t d\tau' \\
= &\int_0^\tau \int_{\Sigma_t} nk\nabla Z(\tau')\nabla Y(\tau')Nd\Sigma_t d\tau' + \int_0^\tau \int_{\Sigma_t} Z(\tau')(n\mathbf{R} + k\nabla n + n\nabla k)\nabla Y(\tau')Nd\Sigma_t d\tau' \\
&+ \int_0^\tau \int_{\Sigma_t} Z(\tau')(\nabla k\nabla n + k\Delta n)Y(\tau')Nd\Sigma_t d\tau' + \int_0^\tau \int_{\Sigma_t} n\mathbf{R}\nabla Z(\tau')Y(\tau')Nd\Sigma_t d\tau' \\
\lesssim &\int_0^\tau \|\nabla Z(\tau')\|_{L^2(\Sigma_t)}(\|\nabla Y(\tau')\|_{L^3(\Sigma_t)}\|nk\|_{L^6(\Sigma_t)} + \|Y(\tau')\|_{L^\infty(\Sigma_t)}\|n\mathbf{R}\|_{L^2(\Sigma_t)}) \\
&+\|Z(\tau')\|_{L^6(\Sigma_t)}\Big(\|\nabla Y(\tau')\|_{L^3(\Sigma_t)}(\|n\mathbf{R}\|_{L^2(\Sigma_t)} + \|n\nabla k\|_{L^2(\Sigma_t)} + \|k\nabla n\|_{L^2(\Sigma_t)}) \\
&+\|Y(\tau')\|_{L^6(\Sigma_t)}(\|\nabla k\nabla n\|_{L^{\frac{3}{2}}(\Sigma_t)} + \|k\Delta n\|_{L^{\frac{3}{2}}(\Sigma_t)})\Big)d\tau' \\
\lesssim &\varepsilon\int_0^\tau \|\nabla Z(\tau')\|_{L^2(\Sigma_t)}(\|\nabla Y(\tau')\|_{L^3(\Sigma_t)} + \|Y(\tau')\|_{L^\infty(\Sigma_t)}) \\
&+\|Z(\tau')\|_{L^6(\Sigma_t)}(\|\nabla Y(\tau')\|_{L^3(\Sigma_t)} + \|Y(\tau')\|_{L^6(\Sigma_t)})d\tau'
\end{aligned}$$

where we used in the last inequality the estimates (2.67) and (4.47) for n, the estimate (4.42) for k, the curvature bound (2.59) for $\mathbf{R}$, and the Sobolev embedding (3.71) on Σ_t. Together with the Sobolev embedding (3.71), the L^∞ estimate (3.73), and the Bochner inequality (3.81) on Σ_t, we obtain:

$$\int_0^\tau \int_{\Sigma_t} Z(\tau')[\mathbf{D}_{nT}, \Delta]Y(\tau')Nd\Sigma_t d\tau' \lesssim \varepsilon\int_0^\tau \|\nabla Z(\tau')\|_{L^2(\Sigma_t)}(\|\Delta Y(\tau')\|_{L^2(\Sigma_t)} + \|\nabla Y(\tau')\|_{L^2(\Sigma_t)})d\tau',$$

which together with (D.45) yields:

$$\|Z(\tau)\|^2_{L^2(\Sigma_t)} + \int_0^\tau \|\nabla Z(\tau')\|^2_{L^2(\Sigma_t)}d\tau' \lesssim \int_0^\tau (\|\Delta Y(\tau')\|^2_{L^2(\Sigma_t)} + \|\nabla Y(\tau')\|^2_{L^2(\Sigma_t)})d\tau'. \quad \text{(D.46)}$$

Now, usual Heat flow estimates for $Y(\tau)N$ yield:

$$\int_0^\tau (\|\Delta Y(\tau')\|^2_{L^2(\Sigma_t)} + \|\nabla Y(\tau')\|^2_{L^2(\Sigma_t)})d\tau' \lesssim \|\mathbf{D}N\|_{L^2(\Sigma_t)} \lesssim \varepsilon, \quad \text{(D.47)}$$

where we used in the last inequality the Ricci equations (2.23) to compute $\mathbf{D}N$ in function of the Ricci coefficients, and the estimates (2.67)-(2.72) to estimate the ricci coefficient in $L^\infty_t L^2_{x'}$ which embeds in $L^\infty_t L^2(\Sigma_t)$. Finally, (D.46) and (D.47) yield:

$$\|Z(\tau)\|^2_{L^2(\Sigma_t)} + \int_0^\tau \|\nabla Z(\tau')\|^2_{L^2(\Sigma_t)}d\tau' \lesssim \varepsilon,$$

which together with (D.44) implies:

$$\|[\mathbf{D}_{nT}, Q_{\leq 1}](N)\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla[\mathbf{D}_{nT}, Q_{\leq 1}](N)\|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon.$$

In view of (D.41), this yields

$$\|\mathbf{D}_T Q_{\leq 1}(N)\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla\mathbf{D}_T Q_{\leq 1}(N)\|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon,$$

which concludes the proof of Lemma 8.7.

D.6 Proof of Lemma 8.8

Let $\delta_l^j = 1$ if $j = l$ and 0 otherwise. Our goal is evaluate the L^∞ norm of

$$g(Q_{\leq 1}(N_j), Q_{\leq 1}(N_l)) - \delta_l^j.$$

The L^∞ estimate (3.73) on Σ_t together with the Bochner inequality (3.81) on Σ_t yields:

$$\begin{aligned}
& \|g(Q_{\leq 1}(N_j), Q_{\leq 1}(N_l)) - \delta_l^j\|_{L^\infty} \qquad \text{(D.48)}\\
\lesssim\ & \|g(Q_{\leq 1}(N_j), Q_{\leq 1}(N_l)) - \delta_l^j\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla^2(g(Q_{\leq 1}(N_j), Q_{\leq 1}(N_l)))\|_{L_t^\infty L^2(\Sigma_t)}\\
\lesssim\ & \|g(Q_{\leq 1}(N_j), Q_{\leq 1}(N_l)) - \delta_l^j\|_{L_t^\infty L^2(\Sigma_t)} + \|g(\Delta Q_{\leq 1}(N_j), Q_{\leq 1}(N_l)))\|_{L_t^\infty L^2(\Sigma_t)}\\
\lesssim\ & \|g(Q_{\leq 1}(N_j), Q_{\leq 1}(N_l)) - \delta_l^j\|_{L_t^\infty L^2(\Sigma_t)} + \|\mathbf{D}N_j\|_{L_t^\infty L^2(\Sigma_t)}\|\mathbf{D}N_l\|_{L_t^\infty L^2(\Sigma_t)}\\
& + \|N_j\|_{L^\infty}\|\mathbf{D}N_l\|_{L_t^\infty L^2(\Sigma_t)} + \|\mathbf{D}N_j\|_{L_t^\infty L^2(\Sigma_t)}\|N_l\|_{L^\infty}\\
\lesssim\ & \|g(Q_{\leq 1}(N_j), Q_{\leq 1}(N_l)) - \delta_l^j\|_{L_t^\infty L^2(\Sigma_t)} + \varepsilon,
\end{aligned}$$

where we have used the finite band property for $Q_{\leq 1}$, the boundedness of $Q_{\leq 1}$ on $L^\infty(\Sigma_t)$, the Ricci equations (2.23) to compute $\mathbf{D}N$ in function of the Ricci coefficients, and the estimates (2.67)-(2.72) to estimate the Ricci coefficient in $L_t^\infty L_{x'}^2$ which embeds in $L_t^\infty L^2(\Sigma_t)$. Now, we have:

$$\begin{aligned}
& \|g(Q_{\leq 1}(N_j), Q_{\leq 1}(N_l)) - \delta_l^j\|_{L_t^\infty L^2(\Sigma_t)}\\
\lesssim\ & \|g(N_j, N_l) - \delta_l^j\|_{L_t^\infty L^2(\Sigma_t)} + \|g(Q_{\leq 1}(N_j), Q_{>1}(N_l))\|_{L_t^\infty L^2(\Sigma_t)}\\
& + \|g(Q_{>1}(N_j), Q_{\leq 1}(N_l))\|_{L_t^\infty L^2(\Sigma_t)} + \|g(Q_{>1}(N_j), Q_{>1}(N_l))\|_{L_t^\infty L^2(\Sigma_t)}\\
\lesssim\ & \|g(N_j, N_l) - \delta_l^j\|_{L_t^\infty L^2(\Sigma_t)} + \|N_j\|_{L^\infty}\|\mathbf{D}N_l\|_{L_t^\infty L^2(\Sigma_t)} + \|\mathbf{D}N_j\|_{L_t^\infty L^2(\Sigma_t)}\|N_l\|_{L^\infty}\\
\lesssim\ & \|g(N_j, N_l) - \delta_l^j\|_{L_t^\infty L^2(\Sigma_t)} + \varepsilon,
\end{aligned}$$

where we have used the finite band property for $Q_{>1}$, the boundedness of $Q_{\leq 1}$ on $L^\infty(\Sigma_t)$, the Ricci equations (2.23) to compute $\mathbf{D}N$ in function of the Ricci coefficients, and the estimates (2.67)-(2.72) to estimate the Ricci coefficient in $L_t^\infty L_{x'}^2$ which embeds in $L_t^\infty L^2(\Sigma_t)$. Together with (D.48), this yields:

$$\|g(Q_{\leq 1}(N_j), Q_{\leq 1}(N_l)) - \delta_l^j\|_{L^\infty} \lesssim \|g(N_j, N_l) - \delta_l^j\|_{L_t^\infty L^2(\Sigma_t)} + \varepsilon. \qquad \text{(D.49)}$$

Next, we have:

$$\begin{aligned}
& \|g(N_j, N_l) - \delta_l^j\|_{L_t^\infty L^2(\Sigma_t)}\\
\lesssim\ & \|g(N_j, N_l) - \delta_l^j\|_{L^2(\Sigma_0)} + \|\mathbf{D}_T g(N_j, N_l)\|_{L^2(\mathcal{M})}\\
& \|g(N_j, N_l) - \delta_l^j\|_{L^2(\Sigma_0)} + \|N_j\|_{L^\infty}\|\mathbf{D}N_l\|_{L_t^\infty L^2(\Sigma_t)} + \|\mathbf{D}N_j\|_{L_t^\infty L^2(\Sigma_t)}\|N_l\|_{L^\infty}\\
\lesssim\ & \varepsilon,
\end{aligned}$$

where we have used in the last inequality the estimate on $g(N_j, N_l)$ on Σ_0 derived in [21], the Ricci equations (2.23) to compute $\mathbf{D}N$ in function of the Ricci coefficients, and the estimates (2.67)-(2.72) to estimate the Ricci coefficient in $L_t^\infty L_{x'}^2$ which embeds in $L_t^\infty L^2(\Sigma_t)$. Together with (D.49), we obtain:

$$\|g(Q_{\leq 1}(N_j), Q_{\leq 1}(N_l)) - \delta_l^j\|_{L^\infty} \lesssim \varepsilon.$$

This proves that $Q_{\leq 1}(N_1), Q_{\leq 1}(N_2)$ and $Q_{\leq 1}(N_3)$ form a basis of the tangent space of Σ_t. This concludes the proof of Lemma 8.8.

D.7 Proof of Lemma 8.9

Let (φ, ψ) the spherical coordinates on $\mathbb{S}^2$ such that ψ measures the angle in the plane spanned by ω, ω', and φ measures the angle with the axis $\omega \wedge \omega'$. Then, we have in particular:

$$\partial_\varphi \omega \cdot (\omega - \omega') = 0. \tag{D.50}$$

Now, we claim that we have the analog estimate:

$$|g(\partial_\varphi N, N - N')| \lesssim |\omega - \omega'|(\varepsilon + |\omega - \omega'|). \tag{D.51}$$

Indeed, we have:

$$g(\partial_\varphi N, N - N') = \int_{[\omega,\omega']} g(\partial_\varphi N, \partial_\psi N'') d\omega''(\omega' - \omega),$$

where $\partial_\psi N'' = \partial_\psi N(\cdot, \omega'')$. This yields:

$$|g(\partial_\varphi N, N - N')| \lesssim |\omega - \omega'| \sup_{\omega'' \in [\omega,\omega']} |g(\partial_\varphi N, \partial_\psi N'')|,$$

and (D.51) now follows from:

$$\sup_{\omega'' \in [\omega,\omega']} |g(\partial_\varphi N, \partial_\psi N'')| \lesssim \varepsilon + |\omega - \omega'|. \tag{D.52}$$

Now, let $\omega_1 \in \mathbb{S}^2$ defined as:

$$\omega_1 = \frac{\omega - \omega'}{|\omega - \omega'|}.$$

Arguing as in the proof of (2.83), we have:

$$\|g(\partial_\psi N'', N_1) - 1\|_{L^\infty} \lesssim \varepsilon + |\omega - \omega'|. \tag{D.53}$$

The choice of ω_1 and the fact that φ measures the angle with the axis $\omega \wedge \omega'$ implies

$$\partial_\varphi \omega \cdot \omega_1 = 0.$$

Arguing again as in the proof of (2.83), we obtain:

$$\|g(\partial_\varphi N, N_1)\|_{L^\infty} \lesssim \varepsilon + |\omega - \omega'|,$$

which together with (D.53) yields (D.52). This concludes the proof of (D.51).

Now, we consider the coordinate system on $\mathcal{H}_{u'}$ consisting of the functions t, u and $\partial_\varphi u$, where $u = u(t, x, \omega)$ and $\partial_\varphi u = \partial_\varphi u(t, x, \omega)$. The fact that it is indeed a coordinate system on $\mathcal{H}_{u'}$ follows from the fact that $(u, \partial_\varphi u)$ is a coordinate system on $P_{t,u'}$. The later claim follows from the invertibility of the corresponding matrix of the metric coefficients in the coordinate system $(u, \partial_\varphi u)$ which we check now. Using the fact that $g(N, \partial_\varphi N) = 0$, we easily compute the following identities for the coordinate system $(u, \partial_\varphi u)$ on $P_{t,u'}$:

$$\frac{\partial}{\partial u} = \frac{b}{1 - g(N, N')^2}(N - g(N, N')N') + \frac{1}{g(\partial_\varphi N, \partial_\varphi N)} \left(\partial_\varphi b + \frac{b g(N, N') g(\partial_\varphi N, N')}{1 - g(N, N')^2} \right) \partial_\varphi N.$$

and:

$$\frac{\partial}{\partial\partial_\varphi u} = \frac{b}{g(\partial_\varphi N, \partial_\varphi N)}\partial_\varphi N.$$

Let γ' denote the induced metric on $P_{t,u'}$. The previous identities yield the corresponding coefficients for γ' in the coordinate system $(u, \partial_\varphi u)$:

$$\gamma'\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = \frac{b^2}{1-g(N,N')^2}\left(1 - \frac{2g(N,N')g(N',\partial_\varphi N)\partial_\varphi b}{g(\partial_\varphi N, \partial_\varphi N)b} - \frac{2g(N,N')^2 g(\partial_\varphi N, N')^2}{1-g(N,N')^2}\right), \tag{D.54}$$

$$\gamma'\left(\frac{\partial}{\partial\partial_\varphi u}, \frac{\partial}{\partial u}\right) = \frac{b\partial_\varphi b}{g(\partial_\varphi N, \partial_\varphi N)}, \tag{D.55}$$

and

$$\gamma'\left(\frac{\partial}{\partial\partial_\varphi u}, \frac{\partial}{\partial\partial_\varphi u}\right) = b^2. \tag{D.56}$$

Note that we have:

$$1 - g(N,N')^2 = (1-g(N,N'))(1+g(N,N')) = \frac{g(N-N', N-N')}{2}\frac{g(N+N', N+N')}{2}$$

which together with (2.83) and the fact that $\|\partial_\omega N\|_{L^\infty} \lesssim 1$ yields:

$$1 - g(N,N')^2 \sim |\omega - \omega'|^2. \tag{D.57}$$

Now, since $\|g(\partial_\varphi N, \partial_\varphi N) - 1\|_{L^\infty} \lesssim \varepsilon$, $\|b-1\|_{L^\infty} \lesssim \varepsilon$, $\|\partial_\omega b\|_{L^\infty} \lesssim \varepsilon$, and in view of (D.51) and (D.54)-(D.57), we have:

$$\gamma'\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) \sim \frac{1}{|\omega-\omega'|^2},\ \gamma'\left(\frac{\partial}{\partial\partial_\varphi u}, \frac{\partial}{\partial u}\right) = O(\varepsilon),\ \gamma'\left(\frac{\partial}{\partial\partial_\varphi u}, \frac{\partial}{\partial\partial_\varphi u}\right) = 1 + O(\varepsilon).$$

This yields the following estimate for the determinant $|\gamma'|$:

$$|\gamma'| \sim \frac{1}{|\omega-\omega'|^2}. \tag{D.58}$$

Since $|\gamma'| \neq 0$ in view of (D.58), $(u, \partial_\varphi u)$ is a coordinate system on $P_{t,u'}$. Note also that this coordinate system is global. Indeed, t, u and $\partial_\varphi u$ are defined everywhere on $\mathcal{M}$, and thus everywhere on $\mathcal{H}_{u'}$, so we only need to show that $(t, u, \partial_\varphi u)$ is one-to-one on $\mathcal{H}_{u'}$. t and u being fixed, this is equivalent to check the injectivity of $\partial_\varphi u$ on $P_{t,u} \cap P_{t,u'}$. Next, we check the injectivity of $\partial_\varphi u$ on $P_{t,u} \cap P_{t,u'}$. Let ℓ a curve in $P_{t,u} \cap P_{t,u'}$ parametrized by arc length. We have:

$$\begin{aligned} \frac{d}{d\sigma}[\partial_\varphi u(\ell(\sigma), \omega)] &= g(\nabla\partial_\varphi u, \dot\ell) \\ &= g(-b^{-2}\partial_\varphi b N + b^{-1}\partial_\varphi N, \dot\ell) \\ &= b^{-1} g(\partial_\varphi N, \dot\ell), \end{aligned} \tag{D.59}$$

where we used in the last equality the fact that ℓ is a curve in $P_{t,u} \cap P_{t,u'}$ which yields:

$$g(\ell, N) = g(\ell, N') = 0.$$

Note that this implies the fact that:

$$N,\ \frac{N' - g(N,N')N}{\sqrt{1-g(N,N')^2}},\ \dot{\ell}$$

forms an orthonormal basis of Σ_t. Now, we have:

$$g(\partial_\varphi N, N) = 0$$

and:

$$\begin{aligned}\left| g\left(\partial_\varphi N, \frac{N' - g(N,N')N}{\sqrt{1-g(N,N')^2}}\right)\right| &= \left|\frac{g(N,N')g(\partial_\varphi N, N'-N)}{\sqrt{1-g(N,N')^2}}\right| \\ &\lesssim \varepsilon + |\omega - \omega'|,\end{aligned}$$

where we used (D.51) and (D.57) in the last inequality. Since $g(\partial_\varphi N, \partial_\varphi N) = 1 + O(\varepsilon)$, and since

$$N,\ \frac{N' - g(N,N')N}{\sqrt{1-g(N,N')^2}},\ \dot{\ell}$$

forms an orthonormal basis of Σ_t, we deduce:

$$g(\partial_\varphi N, \dot{\ell}) \neq 0$$

which together with (D.59) and the fact that $b \sim 1$ yields:

$$\frac{d}{d\sigma}[\partial_\varphi u(\ell(\sigma), \omega)] \neq 0 \text{ for all } \sigma.$$

In particular, $\partial_\varphi u$ is one-to-one along ℓ which implies that $\partial_\varphi u$ is one-to-one on any connex component of $P_{t,u} \cap P_{t,u'}$.

Thus, to conclude that $\partial_\varphi u$ is one-to-one on $P_{t,u} \cap P_{t,u'}$, is suffices to show that $P_{t,u} \cap P_{t,u'}$ is connex. Assume for some $0 \leq t_0 \leq 1$ that $P_{t_0,u} \cap P_{t_0,u'}$ is connex. Note that on $\mathcal{H}_{u'}$, we have:

$$\left|\frac{\partial u}{\partial t}\right| = |g(L, L')| = 1 - g(N, N') = \frac{1}{2} g(N - N', N - N') \gtrsim |\omega - \omega'|^2$$

where we used (2.83) in the last inequality. Thus, we have:

$$\frac{\partial u}{\partial t} \neq 0$$

and the implicit function theorem implies that in a neighborhood of $t = t_0$ of size depending only on $|\omega - \omega'|^2$ (but not on t_0), $P_{t,u} \cap P_{t,u'}$ is the image of $P_{t_0,u} \cap P_{t_0,u'}$ by a smooth map. Thus $P_{t,u} \cap P_{t,u'}$ is connex for t in a neighborhood of $t = t_0$ of size depending only on $|\omega - \omega'|^2$. Therefore, if $P_{0,u} \cap P_{0,u'}$ is connex, applying the implicit function theorem successively $O(|\omega - \omega'|^{-2})$, we obtain that $P_{t,u} \cap P_{t,u'}$ is connex for all $0 \leq t \leq 1$. Now, $P_{0,u} \cap P_{0,u'}$ is connex as an easy consequence of the construction in [21] on the initial slice $t = 0$. Therefore, $P_{t,u} \cap P_{t,u'}$ is connex for all $0 \leq t \leq 1$.

Finally, we have obtained the fact that $(t, u, \partial_\varphi u)$ is a global coordinate system on $\mathcal{H}_{u'}$. Now, we use it to estimate the norm of a scalar f in $L^p(\mathcal{H}_{u'})$ for $2 \leq p < +\infty$. Let u_0 a real number. We have:

$$\begin{aligned} \|f\|^p_{L^p(\mathcal{H}_{u'=u_0})} &= \int_0^1 \int |f|^p \sqrt{|\gamma'|} d\partial_\varphi u du dt \\ &\lesssim \frac{1}{|\omega - \omega'|} \left(\int_0^1 \int |f|^p d\partial_\varphi u du dt \right), \end{aligned} \tag{D.60}$$

where we used (D.58) in the last inequality. Note that we have on $u' = u_0$ the estimate:

$$|u - u_0| = |u - u'| \leq \|\partial_\omega u\|_{L^\infty} |\omega - \omega'| \lesssim |\omega - \omega'|$$

which together with (D.60) yields:

$$\|f\|^p_{L^p(\mathcal{H}_{u'=u_0})} \lesssim \frac{1}{|\omega - \omega'|} \left(\int_0^1 \int_{u_0 - |\omega - \omega'|}^{u_0 + |\omega - \omega'|} \int |f|^p d\partial_\varphi u du dt \right). \tag{D.61}$$

Next, recall the global coordinate system $\Phi_{t,\omega}$ on Σ_t introduced in (D.1). Since $\partial_\omega u = (\partial_\varphi u, \partial_\psi u)$, we have in view of (D.61):

$$\|f\|^p_{L^p(\mathcal{H}_{u'=u_0})} \lesssim \frac{1}{|\omega - \omega'|} \left(\int_0^1 \int_{u_0 - |\omega - \omega'|}^{u_0 + |\omega - \omega'|} \int_{y_1} \sup_{y_2} |f(\Phi^{-1}_{t,\omega}(u, y_1, y_2))|^p dy_1 du dt \right). \tag{D.62}$$

From a standard estimate in $\mathbb{R}^2$, we have:

$$\begin{aligned} &\int_{y_1} \sup_{y_2} |f(\Phi^{-1}_{t,\omega}(u, y_1, y_2))|^p dy_1 \\ &\lesssim \left(\int_y |f(\Phi^{-1}_{t,\omega}(u, y_1, y_2))|^{2(p-1)} dy_1 dy_2 \right)^{\frac{1}{2}} \left(\int_y |\partial_{y_2} f(\Phi^{-1}_{t,\omega}(u, y_1, y_2))|^2 dy_1 dy_2 \right)^{\frac{1}{2}} \\ &\lesssim \left(\int_{P_{t,u}} |f|^{2(p-1)} d\mu_{t,u} \right)^{\frac{1}{2}} \left(\int_{P_{t,u}} |\nabla\!\!\!/ f|^2 d\mu_{t,u} \right)^{\frac{1}{2}} \end{aligned} \tag{D.63}$$

where we used the estimate (D.5) for the coefficients of the induced metric γ on $P_{t,u}$ in the global coordinate system $\Phi^{-1}_{t,\omega}(u, y_1, y_2)$. Together with (D.62), this yields:

$$\begin{aligned} \|f\|^p_{L^p(\mathcal{H}_{u'=u_0})} &\lesssim \frac{1}{|\omega - \omega'|} \left(\int_0^1 \int_{u_0 - |\omega - \omega'|}^{u_0 + |\omega - \omega'|} \left(\int_{P_{t,u}} |f|^{2(p-1)} d\mu_{t,u} \right)^{\frac{1}{2}} \left(\int_{P_{t,u}} |\nabla\!\!\!/ f|^2 d\mu_{t,u} \right)^{\frac{1}{2}} du dt \right) \\ &\lesssim \frac{1}{|\omega - \omega'|} \left(\int_{u_0 - |\omega - \omega'|}^{u_0 + |\omega - \omega'|} \|f\|^{p-1}_{L^{2(p-1)}(\mathcal{H}_u)} \|\nabla\!\!\!/ f\|_{L^2(\mathcal{H}_u)} du \right) \\ &\lesssim \|f\|^{p-1}_{L^\infty_u L^{2(p-1)}(\mathcal{H}_u)} \|\nabla\!\!\!/ f\|_{L^\infty_u L^2(\mathcal{H}_u)}. \end{aligned}$$

Since this holds for any real number u_0, we take the supremum which yields:

$$\|f\|^p_{L^\infty_{u'} L^p(\mathcal{H}_{u'})} \lesssim \|f\|^{p-1}_{L^\infty_u L^{p-1}(\mathcal{H}_u)} \|\nabla\!\!\!/ f\|_{L^\infty_u L^2(\mathcal{H}_u)}.$$

Finally, let F a tensor. Applying the previous inequality to $f = |F|$, we obtain

$$\|F\|^p_{L^\infty_{u'}L^p(\mathcal{H}_{u'})} \lesssim \|F\|^{p-1}_{L^\infty_u L^{p-1}(\mathcal{H}_u)} \|\nabla\!\!\!/ F\|_{L^\infty_u L^2(\mathcal{H}_u)}.$$

This concludes the proof of Lemma 8.9.

D.8 Proof of Lemma 8.12

Let $2 \leq r < +\infty$. Then, we have

$$\|FH\|_{L^r_t L^\infty_{x'}} \lesssim \|F\|_{L^{2r}_t L^\infty_{x'}} \|H\|_{L^{2r}_t L^\infty_{x'}} \lesssim \varepsilon.$$

Thus, it suffices to bound $F\nabla\!\!\!/ H$ and $H\nabla\!\!\!/ F$ in $L^r_t B^0_{2,1}(P_{t,u})$. These terms are treated exactly in the same way, so we focus on $F\nabla\!\!\!/ H$. We have

$$\|P_j(F\nabla\!\!\!/ H)\|_{L^r_t L^2_{x'}} \lesssim \sum_l \|P_j(FP_l(\nabla\!\!\!/ H)\|_{L^r_t L^2_{x'}}. \tag{D.64}$$

Next, we estimate the right-hand side of (D.64). Using the finite band property for P_j, we have

$$\begin{aligned} & \|P_j(FP_l(\nabla\!\!\!/ H)\|_{L^r_t L^2_{x'}} \qquad (D.65)\\ \lesssim\ & 2^{-j}\|\nabla\!\!\!/(FP_l(\nabla\!\!\!/ H))\|_{L^r_t L^2_{x'}} \\ \lesssim\ & 2^{-j}\|\nabla\!\!\!/(F)P_l(\nabla\!\!\!/ H)\|_{L^r_t L^2_{x'}} + 2^{-j}\|F\nabla\!\!\!/ P_l(\nabla\!\!\!/ H))\|_{L^r_t L^2_{x'}} \\ \lesssim\ & 2^{-j}\|\nabla\!\!\!/(F)\|_{L^{2r}_t L^2_{x'}}\|P_l(\nabla\!\!\!/ H)\|_{L^{2r}_t L^\infty_{x'}} + 2^{-j}\|F\|_{L^{2r}_t L^\infty_{x'}}\|\nabla\!\!\!/ P_l(\nabla\!\!\!/ H))\|_{L^{2r}_t L^2_{x'}} \\ \lesssim\ & \varepsilon 2^{-j+l}\|P_l(\nabla\!\!\!/ H)\|_{L^{3r}_t L^2_{x'}}, \end{aligned}$$

where we used in the last inequality the finite band property, the sharp Bernstein inequality for tensors (4.37) for P_l, the bound $\|K\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon$ for K, and the assumptions on F. Also, we have

$$\begin{aligned} \|P_j(FP_l(\nabla\!\!\!/ H)\|_{L^r_t L^2_{x'}} \lesssim\ & 2^{-2l}\|P_j(F\Delta\!\!\!/ P_l(\nabla\!\!\!/ H)\|_{L^r_t L^2_{x'}} \\ \lesssim\ & 2^{-2l}\|P_j \mathrm{div}\!\!\!/(F\nabla\!\!\!/ P_l(\nabla\!\!\!/ H)\|_{L^r_t L^2_{x'}} + 2^{-2l}\|P_j(\nabla\!\!\!/ F\nabla\!\!\!/ P_l(\nabla\!\!\!/ H)\|_{L^r_t L^2_{x'}} \\ \lesssim\ & 2^{j-2l}\|F\nabla\!\!\!/ P_l(\nabla\!\!\!/ H)\|_{L^r_t L^2_{x'}} + 2^{j-2l}\|F\nabla\!\!\!/ P_l(\nabla\!\!\!/ H)\|_{L^{2r}_t L^1_{x'}}, \end{aligned}$$

where we used in the last inequality the finite band property, the dual of the sharp Bernstein inequality for tensors (4.37) for P_j and the bound $\|K\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon$ for K. We obtain

$$\begin{aligned} & \|P_j(FP_l(\nabla\!\!\!/ H)\|_{L^r_t L^2_{x'}} \qquad (D.66)\\ \lesssim\ & 2^{j-2l}\|F\|_{L^{2r}_t L^\infty_{x'}}\|\nabla\!\!\!/ P_l(\nabla\!\!\!/ H)\|_{L^{2r}_t L^2_{x'}} + 2^{j-2l}\|\nabla\!\!\!/ F\|_{L^{4r}_t L^2_{x'}}\|\nabla\!\!\!/ P_l(\nabla\!\!\!/ H)\|_{L^{4r}_t L^2_{x'}} \\ \lesssim\ & 2^{j-l}\varepsilon\|P_l(\nabla\!\!\!/ H)\|_{L^{4r}_t L^2_{x'}}, \end{aligned}$$

where we used in the last inequality the finite band property for P_l, and the assumptions on F. Finally, using (D.65) for $l \leq j$ and (D.66) for $l > j$, we obtain

$$\|P_j(FP_l(\not\nabla H)\|_{L^r_t L^2_{x'}} \lesssim 2^{-|j-l|}\varepsilon \|P_l(\not\nabla H)\|_{L^{4r}_t L^2_{x'}},$$

which together with (D.64) and the assumption on H implies

$$\|F\not\nabla H\|_{L^r_t B^0_{2,1}(P_{t,u})} \lesssim \varepsilon.$$

This concludes the proof of the lemma.

D.9 Proof of Lemma 8.18

Note that it suffices to prove for any $l \geq 0$ the estimate

$$\|[\partial_\omega, P_l]f\|_{L^\infty_{u'} L^2(\mathcal{H}_{u'})} \lesssim 2^{\frac{l}{2}}\varepsilon, \tag{D.67}$$

provided f satisfies the assumptions of Lemma 8.18. Let $W(\tau)$ solution of

$$(\partial_\tau - \not\Delta)W(\tau) = [\partial_\omega, \not\Delta]U(\tau)f,\ W(0) = 0. \tag{D.68}$$

Then, we have

$$[\partial_\omega, P_l]f = \int_0^\infty m_l(\tau)W(\tau)d\tau. \tag{D.69}$$

Assume that we have the following decomposition for W:

$$W = W_1 + W_2, \tag{D.70}$$

where W_1 and W_2 satisfy respectively for $0 \leq \tau \leq 1$

$$\|W_1(\tau)\|_{L^2(\mathcal{H}_u)} + \|\not\nabla W_1(\cdot)\|_{L^2_\tau L^2(\mathcal{H}_u)} \lesssim \varepsilon, \tag{D.71}$$

and

$$\|W_2(\tau)\|_{L^2(\mathcal{H}_u)} + \sqrt{\tau}\|\not\nabla W_2(\tau)\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{D.72}$$

Then, (D.70), (D.71), (D.72) together with (D.69) yields in view of the fact that $l \geq 0$ and m is supported in $(0, 1)$:

$$\begin{aligned}
\|[\partial_\omega, P_l]f\|_{L^\infty_u L^2(\mathcal{H}_u)} &\lesssim \sup_u \int_0^\infty m_l(\tau)\|W(\tau)\|_{L^2(\mathcal{H}_u)}d\tau \\
&\lesssim \sup_u \int_0^1 m_l(\tau)\|W_1(\tau)\|_{L^2(\mathcal{H}_u)}d\tau + \sup_u \int_0^1 m_l(\tau)\|W_2(\tau)\|_{L^2(\mathcal{H}_u)}d\tau \\
&\lesssim \varepsilon \int_0^1 m_l(\tau)d\tau \\
&\lesssim \varepsilon,
\end{aligned} \tag{D.73}$$

and

$$\begin{aligned}
&\|\nabla\!\!\!/[\partial_\omega, P_l]f\|_{L^\infty_u L^2(\mathcal{H}_u)} \qquad (D.74)\\
\lesssim\ & \sup_u \int_0^\infty m_l(\tau)\|\nabla\!\!\!/ W(\tau)\|_{L^2(\mathcal{H}_u)}d\tau\\
\lesssim\ & \sup_u \int_0^1 m_l(\tau)\|\nabla\!\!\!/ W_1(\tau)\|_{L^2(\mathcal{H}_u)}d\tau + \sup_u \int_0^1 m_l(\tau)\|\nabla\!\!\!/ W_2(\tau)\|_{L^2(\mathcal{H}_u)}d\tau\\
\lesssim\ & \varepsilon\left(\int_0^1 m_l(\tau)^2 d\tau\right)^{\frac{1}{2}} + \varepsilon\int_0^1 m_l(\tau)\tau^{-\frac{1}{2}}d\tau\\
\lesssim\ & 2^l\varepsilon.
\end{aligned}$$

(D.73) and (D.74) together with Lemma 8.9 yield

$$\|[\partial_\omega, P_l]f\|_{L^\infty_{u'}L^2(\mathcal{H}_{u'})} \lesssim \|[\partial_\omega, P_l]f\|^{\frac{1}{2}}_{L^\infty_u L^2(\mathcal{H}_u)}\|\nabla\!\!\!/[\partial_\omega, P_l]f\|^{\frac{1}{2}}_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{\frac{l}{2}}\varepsilon,$$

which is (D.67). Thus it remains to prove (D.70) (D.71) (D.72).

We first precise our choice for W_1 and W_2. Let h a scalar on function on Σ_t. Then, we have the following commutator formula

$$[\partial_\omega, \Delta\!\!\!/]h = -2\nabla\!\!\!/_{\partial_\omega N}\nabla_N h + 2\theta(\partial_\omega N, \nabla\!\!\!/ h) - \mathrm{tr}\theta\nabla\!\!\!/_{\partial_\omega N}h - \partial_\omega \mathrm{tr}\theta\nabla_N h. \qquad (D.75)$$

(D.75) is in the spirit of section 6.1. We refer to section 5.1.1 of [21] for a proof. We have

$$\nabla_{bN}U(\tau)f = U(\tau)\nabla_{bN}f + V(\tau), \qquad (D.76)$$

where $V(\tau)$ is the solution of

$$(\partial_\tau - \Delta\!\!\!/)V(\tau) = [\nabla_{bN}, \Delta\!\!\!/]U(\tau)f,\ V(0) = 0. \qquad (D.77)$$

In view of (D.75) and (D.76), we deduce

$$\begin{aligned}
[\partial_\omega, \Delta\!\!\!/]U(\tau)f \ =\ & -2\nabla\!\!\!/_{\partial_\omega N}(b^{-1}U(\tau)\nabla_{bN}f) - 2\nabla\!\!\!/_{\partial_\omega N}(b^{-1}V(\tau)) + 2\theta(\partial_\omega N, \nabla\!\!\!/ U(\tau)f)\\
& -\mathrm{tr}\theta\nabla\!\!\!/_{\partial_\omega N}U(\tau)f - b^{-1}\partial_\omega\mathrm{tr}\theta U(\tau)\nabla_{bN}f - b^{-1}\partial_\omega\mathrm{tr}\theta V(\tau). \qquad (D.78)
\end{aligned}$$

We choose W_1 and W_2 solution of the following equations

$$\begin{aligned}
(\partial_\tau - \Delta\!\!\!/)W_1(\tau) \ =\ & -2\mathrm{div}\!\!\!/(\partial_\omega N b^{-1}U(\tau)\nabla_{bN}f) - 2\nabla\!\!\!/_{\partial_\omega N}(b^{-1}V(\tau))\\
& +2\theta(\partial_\omega N, \nabla\!\!\!/ U(\tau)f) - \mathrm{tr}\theta\nabla\!\!\!/_{\partial_\omega N}U(\tau)f - b^{-1}\partial_\omega\mathrm{tr}\theta V(\tau),\\
W_1(0) \ =\ & 0, \qquad (D.79)
\end{aligned}$$

and

$$\begin{aligned}
(\partial_\tau - \Delta\!\!\!/)W_2(\tau) \ =\ & b^{-1}(2\mathrm{div}\!\!\!/(\partial_\omega N) - \partial_\omega\mathrm{tr}\theta)U(\tau)\nabla_{bN}f, \qquad (D.80)\\
W_2(0) \ =\ & 0.
\end{aligned}$$

In view of (D.68), (D.78), (D.79) and (D.80), we have (D.70). Thus, it remains to prove the estimate (D.71) for W_1 and the estimate (D.72) for W_2. We start with the estimate (D.71). The energy estimate (3.12) implies:

$$\|W_1(\tau)\|^2_{L^2(\mathcal{H}_u)} + \int_0^\tau \|\nabla\!\!\!/ W_1(\tau')\|^2_{L^2(\mathcal{H}_u)} d\tau' = \int_0^\tau \int_{\mathcal{H}_u} W_1(\tau')(\partial_\tau - \Delta\!\!\!/) W_1(\tau') d\mu_{t,u} dt d\tau'. \tag{D.81}$$

In view of (D.79), we obtain after integration by parts

$$\begin{aligned}
&\int_0^\tau \int_{\mathcal{H}_u} W_1(\tau')(\partial_\tau - \Delta\!\!\!/) W_1(\tau') f d\mu_{t,u} dt d\tau' \\
\lesssim\ & \|b^{-1}\partial_\omega N\|_{L^\infty} \int_0^\tau \|\nabla\!\!\!/ W_1(\tau')\|_{L^2(\mathcal{H}_u)} \|U(\tau')\nabla_{bN} f\|_{L^2(\mathcal{H}_u)} d\tau' \\
&+ \|b^{-1}\|_{L^\infty} (\|\partial_\omega N\|_{L^\infty} + \|\mathrm{div}\!\!\!/(\partial_\omega N)\|_{L^\infty_t L^2_{x'}} + \|\partial_\omega \mathrm{tr}\theta\|_{L^\infty_t L^2_{x'}}) \\
&\times \int_0^\tau \|\nabla\!\!\!/ W_1(\tau')\|_{L^2(\mathcal{H}_u)} \|\nabla\!\!\!/ V(\tau')\|_{L^2(\mathcal{H}_u)} d\tau' \\
&+ \|\partial_\omega N\|_{L^\infty} \|\theta\|_{L^\infty_t L^4_{x'}} \int_0^\tau \|\nabla\!\!\!/ W_1(\tau')\|_{L^2(\mathcal{H}_u)} \|\nabla\!\!\!/ U(\tau') f\|_{L^2(\mathcal{H}_u)} d\tau' \\
\lesssim\ & \int_0^\tau \|\nabla\!\!\!/ W_1(\tau')\|_{L^2(\mathcal{H}_u)} (\|U(\tau')\nabla_{bN} f\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/ V(\tau')\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/ U(\tau') f\|_{L^2(\mathcal{H}_u)}) d\tau',
\end{aligned}$$

where we used in the last inequality the fact that $\theta = \chi + \eta$ (see (4.61)), the estimates (2.70) (2.71) for χ, the estimate (2.68) for k, the estimate (2.69) for b, the estimates (2.76) (2.77) for $\partial_\omega N$, and the estimate (2.77) for $\partial_\omega \chi$. Together with (D.81), we deduce

$$\begin{aligned}
&\|W_1(\tau)\|^2_{L^2(\mathcal{H}_u)} + \int_0^\tau \|\nabla\!\!\!/ W_1(\tau')\|^2_{L^2(\mathcal{H}_u)} d\tau' \\
\lesssim\ & \int_0^\tau (\|U(\tau')\nabla_{bN} f\|^2_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/ U(\tau') f\|^2_{L^2(\mathcal{H}_u)}) d\tau'.
\end{aligned} \tag{D.82}$$

Next, we evaluate the right-hand side of (D.82). The heat flow estimate (3.8) yields

$$\int_0^\tau \|\nabla\!\!\!/ U(\tau') f\|^2_{L^2(\mathcal{H}_u)} d\tau' \lesssim \|f\|^2_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon^2, \tag{D.83}$$

where we used in the last inequality the assumptions on f. A heat flow estimate yields

$$\int_0^\tau \|U(\tau')\nabla_{bN} f\|^2_{L^2(\mathcal{H}_u)} d\tau' \lesssim \|\Lambda^{-1}(\nabla_{bN} f)\|^2_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon^2, \tag{D.84}$$

where we used in the last inequality the assumptions on f. Also, as a consequence of the estimate (E.17) which will be proved later, we have

$$\int_0^\tau \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(\mathcal{H}_u)} d\tau' \lesssim \mathcal{N}_1(f)^2 \lesssim \varepsilon^2, \tag{D.85}$$

where we used in the last inequality the assumptions on f. Finally, (D.82), (D.83), (D.84) and (D.85) imply the desired estimate (D.71).

It remains to prove the estimate (D.72). Using (D.80) together with Duhamel's formula, we have

$$W_2(\tau) = \int_0^\tau U(\tau-\sigma)\left[b^{-1}(2\underset{\backslash}{\mathrm{div}}(\partial_\omega N) - \partial_\omega \mathrm{tr}\theta)U(\sigma)\nabla_{bN} f\right] d\sigma. \tag{D.86}$$

Using the Gagliardo-Nirenberg inequality (3.3) and the heat flow estimate (3.10), we have for any scalar h and any $2 \leq p < +\infty$ and $0 \leq \tau \leq 1$

$$\|U(\tau)h\|_{L^p(P_{t,u})} \lesssim \frac{1}{\tau^{\frac{1}{2}-\frac{1}{p}}} \|h\|_{L^2(P_{t,u})}. \tag{D.87}$$

In view of the formula (D.86) of W_2, and using the dual of (D.87), we have:

$$\begin{aligned}
\|W_2(\tau)\|_{L^2(\mathcal{H}_u)} &\lesssim \int_0^\tau \|U(\tau-\sigma)\left[b^{-1}(2\underset{\backslash}{\mathrm{div}}(\partial_\omega N) - \partial_\omega \mathrm{tr}\theta)U(\sigma)\nabla_{bN} f\right]\|_{L^2(\mathcal{H}_u)} d\sigma \\
&\lesssim \int_0^\tau \frac{1}{(\tau-\sigma)^{\frac{1}{4}}} \|b^{-1}(2\underset{\backslash}{\mathrm{div}}(\partial_\omega N) - \partial_\omega \mathrm{tr}\theta)U(\sigma)\nabla_{bN} f\|_{L^2_t L^{\frac{4}{3}}_{x'}} d\sigma \\
&\lesssim \int_0^\tau \frac{1}{(\tau-\sigma)^{\frac{1}{4}}} \|b^{-1}(2\underset{\backslash}{\mathrm{div}}(\partial_\omega N) - \partial_\omega \mathrm{tr}\theta)\|_{L^\infty_t L^2_{x'}} \|U(\sigma)\nabla_{bN} f\|_{L^2_t L^4_{x'}} d\sigma \\
&\lesssim \varepsilon \int_0^\tau \frac{1}{(\tau-\sigma)^{\frac{1}{4}}} \frac{1}{\sigma^{\frac{1}{4}}} \left\| U\left(\frac{\sigma}{2}\right) \nabla_{bN} f \right\|_{L^2(\mathcal{H}_u)} d\sigma,
\end{aligned} \tag{D.88}$$

where we used in the last inequality the fact that

$$U(\sigma) = U\left(\frac{\sigma}{2}\right) U\left(\frac{\sigma}{2}\right),$$

(D.87) with $p = 4$, the fact that $\theta = \chi + \eta$ (see (4.61)), the estimates (2.70) (2.71) for χ, the estimate (2.68) for k, the estimate (2.69) for b, the estimates (2.76) (2.77) for $\partial_\omega N$, and the estimate (2.77) for $\partial_\omega \chi$. The heat flow estimate (3.29) and (D.88) yield

$$\begin{aligned}
\|W_2(\tau)\|_{L^2(\mathcal{H}_u)} &\lesssim \varepsilon \left(\int_0^\tau \frac{1}{(\tau-\sigma)^{\frac{1}{4}}} \frac{1}{\sigma^{\frac{3}{4}}} d\sigma \right) \|\Lambda^{-1} \nabla_{bN} f\|_{L^2(\mathcal{H}_u)} \\
&\lesssim \varepsilon,
\end{aligned} \tag{D.89}$$

where we used in the last inequality the assumptions for f. Next, we estimate $\underset{\backslash}{\nabla} W_2$. Using the fact that

$$U(\tau-\sigma) = U\left(\frac{\tau}{2} - \frac{\sigma}{2}\right) U\left(\frac{\tau}{2} - \frac{\sigma}{2}\right),$$

we obtain

$$\begin{aligned}
\|\underset{\backslash}{\nabla} W_2(\tau)\|_{L^2(\mathcal{H}_u)} &\lesssim \int_0^\tau \|\underset{\backslash}{\nabla} U(\tau-\sigma)\left[b^{-1}(2\underset{\backslash}{\mathrm{div}}(\partial_\omega N) - \partial_\omega \mathrm{tr}\theta)U(\sigma)\nabla_{bN} f\right]\|_{L^2(\mathcal{H}_u)} d\sigma \\
&\lesssim \int_0^\tau \frac{1}{\sqrt{\tau-\sigma}} \left\| U\left(\frac{\tau}{2} - \frac{\sigma}{2}\right) \left[b^{-1}(2\underset{\backslash}{\mathrm{div}}(\partial_\omega N) - \partial_\omega \mathrm{tr}\theta)U(\sigma)\nabla_{bN} f\right] \right\|_{L^2(\mathcal{H}_u)} d\sigma,
\end{aligned}$$

where we used in the last inequality the estimate (3.10) for the heat flow. Then, arguing as for the proof of (D.89), and noticing that we have:

$$\int_0^\tau \frac{1}{(\tau-\sigma)^{\frac{3}{4}}}\frac{1}{\sigma^{\frac{3}{4}}}d\sigma \lesssim \tau^{-\frac{1}{2}},$$

we obtain:

$$\|\nabla\!\!\!/ W_2(\tau)\|_{L^2(\mathcal{H}_u)} \lesssim \tau^{-\frac{1}{2}}\varepsilon. \tag{D.90}$$

Finally, (D.89) and (D.90) imply the desired estimate (D.72). This concludes the proof of Lemma 8.18.

D.10 Proof of Lemma 8.19

We start with the estimate for $\nabla\!\!\!/ b$. We have

$$\nabla\!\!\!/_{bN}\nabla\!\!\!/ b = \nabla\!\!\!/(\nabla_{bN}(b)) + [\nabla\!\!\!/_{bN}, \nabla\!\!\!/]b, \tag{D.91}$$

with

$$h_1 = \nabla_{bN}(b) \text{ and } H_2 = [\nabla\!\!\!/_{bN}, \nabla\!\!\!/]b.$$

In view of the commutator formula (2.50), we have

$$\begin{aligned}
\|h_1\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|H_2\|_{L^2_t L^{\frac{4}{3}}_{x'}} &\lesssim \|\mathbf{D}b\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|b(\chi+k)\nabla\!\!\!/ b\|_{L^2_t L^{\frac{4}{3}}_{x'}} \\
&\lesssim \|\mathbf{D}b\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|b\|_{L^\infty}(\|\chi\|_{L^\infty_t L^4_{x'}} + \|k\|_{L^\infty_t L^4_{x'}})\|\nabla\!\!\!/ b\|_{L^2(\mathcal{H}_u)} \\
&\lesssim \varepsilon,
\end{aligned} \tag{D.92}$$

where we used in the last inequality the estimate (2.69) for b, the estimate (2.68) for k and the estimates (2.70) (2.71) for χ.

Next, we consider the estimate for ζ. In view of the identity (2.26), we have

$$\nabla\!\!\!/_{bN}\zeta = \nabla\!\!\!/ h_3 + H_4, \tag{D.93}$$

with

$$h_3 = b^{-1}\nabla_{bN}(b) = b^{-1}h_1 \text{ and } H_4 = b^{-1}[\nabla\!\!\!/_{bN}, \nabla\!\!\!/]b + \nabla\!\!\!/_{bN}\epsilon = b^{-1}H_2 + \nabla\!\!\!/_{bN}\epsilon.$$

We have

$$\begin{aligned}
\|h_3\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|H_4\|_{L^2_t L^{\frac{4}{3}}_{x'}} &\lesssim \|b^{-1}h_1\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|b^{-1}H_2\|_{L^2_t L^{\frac{4}{3}}_{x'}} + \|\nabla\!\!\!/_{bN}\epsilon\|_{L^2_t L^{\frac{4}{3}}_{x'}} \\
&\lesssim \|b^{-1}\|_{L^\infty}(\|h_1\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|H_2\|_{L^2_t L^{\frac{4}{3}}_{x'}}) + \|b\|_{L^\infty}\|\nabla\!\!\!/_N\epsilon\|_{L^2(\mathcal{H}_u)} \\
&\lesssim \varepsilon,
\end{aligned} \tag{D.94}$$

where we used in the last inequality the estimate (2.68) for ϵ and the estimate (D.92) for h_1 and H_2. Finally, (D.91)-(D.94) yields the desired decompositions. This concludes the proof of Lemma 8.19.

D.11 Proof of Lemma 8.20

Recall the transport equation (6.39) for $\partial_\omega b$

$$L(\partial_\omega b) = -b\zeta_{\partial_\omega N} - \partial_\omega(b)\overline{\delta} - \overline{\epsilon}_{\partial_\omega N} b.$$

We differentiate with respect to $\underline{L}$. This yields

$$\begin{aligned} L(\underline{L}\partial_\omega b) + [\underline{L}, L]\partial_\omega b &= -b\nabla\!\!\!/_{\underline{L}}\zeta_{\partial_\omega N} - \underline{L}(b)\zeta_{\partial_\omega N} - b\zeta_{\nabla\!\!\!/_{\underline{L}}\partial_\omega N} - \underline{L}(\partial_\omega(b))\overline{\delta} - \partial_\omega(b)\underline{L}(\overline{\delta}) \\ &\quad -\nabla\!\!\!/_{\underline{L}}\overline{\epsilon}_{\partial_\omega N} b - \overline{\epsilon}_{\nabla\!\!\!/_{\underline{L}}\partial_\omega N} b - \overline{\epsilon}_{\partial_\omega N}\underline{L}(b). \end{aligned}$$

Together with the commutator formula (2.46), we obtain

$$L(\underline{L}\partial_\omega b) = -b\nabla\!\!\!/_{\underline{L}}\zeta_{\partial_\omega N} + f, \tag{D.95}$$

where the scalar f is given by

$$\begin{aligned} f &= -\underline{L}(b)\zeta_{\partial_\omega N} - b\zeta_{\nabla\!\!\!/_{\underline{L}}\partial_\omega N} - \partial_\omega(b)\underline{L}(\overline{\delta}) - \nabla\!\!\!/_{\underline{L}}\overline{\epsilon}_{\partial_\omega N} b - \overline{\epsilon}_{\nabla\!\!\!/_{\underline{L}}\partial_\omega N} b - \overline{\epsilon}_{\partial_\omega N}\underline{L}(b) \\ &\quad -(\delta + n^{-1}N(n))L(\partial_\omega b) - 2(\zeta - \underline{\zeta})\cdot\nabla\!\!\!/\partial_\omega b. \end{aligned}$$

f satisfies the following estimate

$$\begin{aligned} &\|f\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ \lesssim\ & \Big(\|\underline{L}(b)\|_{L^2_{x'}L^\infty_t} + \|\mathbf{D}\partial_\omega N\|_{L^2_{x'}L^\infty_t} + \|L(\partial_\omega b)\|_{L^2_{x'}L^\infty_t} + \|\nabla\!\!\!/\partial_\omega b\|_{L^2_{x'}L^\infty_t}\Big) \\ &\times\Big(1 + \|\zeta\|_{L^\infty_{x'}L^2_t} + \|\overline{\epsilon}\|_{L^\infty_{x'}L^2_t} + \|\delta\|_{L^\infty_{x'}L^2_t} + \|n^{-1}N(n)\|_{L^\infty} + \|\underline{\zeta}\|_{L^\infty_{x'}L^2_t}\Big) \\ &\times\Big(1 + \|\partial_\omega N\|_{L^\infty} + \|b\|_{L^\infty} + \|\partial_\omega(b)\|_{L^\infty}\Big) + \|\nabla\!\!\!/_{\underline{L}}\overline{\epsilon}\|_{L^\infty_u L^2(\mathcal{H}_u)}\|\partial_\omega N\|_{L^\infty}\|b\|_{L^\infty} \\ \lesssim\ & \varepsilon, \end{aligned} \tag{D.96}$$

where we used in the last inequality the estimates (2.67)-(2.69) for n, $\overline{\epsilon}$, δ, $\underline{\zeta}$ and b, the estimate (2.72) for ζ, the estimate (2.76) for $\partial_\omega N$, and the estimate (2.77) for $\partial_\omega N$ and $\partial_\omega b$.

In view of the identity (2.26), we have

$$\begin{aligned} b\nabla\!\!\!/_{\underline{L}}\zeta_{\partial_\omega N} &= b\nabla\!\!\!/_{\underline{L}}(b^{-1}\nabla\!\!\!/ b + \epsilon)_{\partial_\omega N} \\ &= (\nabla\!\!\!/_{\underline{L}}\nabla\!\!\!/ b)_{\partial_\omega N} - b^{-2}\underline{L}(b)\nabla\!\!\!/_{\partial_\omega N} b + \nabla\!\!\!/_{\underline{L}}\epsilon_{\partial_\omega N} \\ &= \mathrm{div}\!\!\!/(\underline{L}(b)\partial_\omega N) + f_1, \end{aligned} \tag{D.97}$$

where the scalar f_1 is given by

$$f_1 = ([\nabla\!\!\!/_{\underline{L}}, \nabla\!\!\!/]b)_{\partial_\omega N} - \underline{L}(b)\mathrm{div}\!\!\!/(\partial_\omega N) - b^{-2}\underline{L}(b)\nabla\!\!\!/_{\partial_\omega N} b + \nabla\!\!\!/_{\underline{L}}\epsilon_{\partial_\omega N}.$$

In view of the definition of f_1, we have

$$\begin{aligned} \|f_1\|_{L^2_t L^{\frac{4}{3}}_{x'}} &\lesssim \|[\nabla\!\!\!/_{\underline{L}}, \nabla\!\!\!/]b\|_{L^2_t L^{\frac{4}{3}}_{x'}}\|\partial_\omega N\|_{L^\infty} + \|\underline{L}(b)\|_{L^\infty_t L^4_{x'}}\|\mathrm{div}\!\!\!/(\partial_\omega N)\|_{L^\infty_t L^2_{x'}} \\ &\quad + \|b^{-2}\partial_\omega N\|_{L^\infty}\|\underline{L}(b)\|_{L^\infty_t L^2_{x'}}\|\nabla\!\!\!/ b\|_{L^\infty_t L^4_{x'}} + \|\nabla\!\!\!/_{\underline{L}}\epsilon\|_{L^\infty_u L^2(\mathcal{H}_u)}\|\partial_\omega N\|_{L^\infty} \\ &\lesssim \|[\nabla\!\!\!/_{\underline{L}}, \nabla\!\!\!/]b\|_{L^2_t L^{\frac{4}{3}}_{x'}} + \varepsilon, \end{aligned}$$

where we used in the last inequality the estimate (2.69) for b, the estimate (2.68) for ϵ and the estimates (2.76) (2.77) for $\partial_\omega N$. Together with the commutator formula (2.45), we deduce

$$\begin{aligned} \|f_1\|_{L_t^2 L_{x'}^{\frac{4}{3}}} &\lesssim \|(\underline{\chi}, \underline{\xi}, b^{-1}\nabla\!\!\!/ b)\|_{L_t^\infty L_{x'}^2} \|b\|_{L_t^\infty L_{x'}^4} + \varepsilon \\ &\lesssim \varepsilon, \end{aligned} \tag{D.98}$$

where we used in the last inequality the estimates (2.67)-(2.72) together with the identities (2.25) for $b, \underline{\chi}$ and $\underline{\xi}$.

In view of the transport equation (D.95) and (D.97), we have

$$nL(\underline{L}\partial_\omega b) = -\mathrm{div}\!\!\!/(n\underline{L}(b)\partial_\omega N) + f_2, \tag{D.99}$$

where f_2 is given by

$$f_2 = \underline{L}(b)\nabla\!\!\!/_{\partial_\omega N} n - nf_1 + nf.$$

In view of the definition of f_2, we have

$$\begin{aligned} \|f_2\|_{L_t^2 L_{x'}^{\frac{4}{3}}} &\lesssim \|\underline{L}(b)\nabla\!\!\!/_{\partial_\omega N} n\|_{L_t^2 L_{x'}^{\frac{4}{3}}} + \|nf_1\|_{L_t^2 L_{x'}^{\frac{4}{3}}} + \|nf\|_{L_t^2 L_{x'}^{\frac{4}{3}}} \\ &\lesssim \|\underline{L}(b)\|_{L_t^\infty L_{x'}^2} \|\nabla\!\!\!/ n\|_{L_t^2 L_{x'}^4} \|\partial_\omega N\|_{L^\infty} + \|n\|_{L^\infty}(\|f_1\|_{L_t^2 L_{x'}^{\frac{4}{3}}} + \|f\|_{L_t^2 L_{x'}^{\frac{4}{3}}}) \\ &\lesssim \varepsilon, \end{aligned} \tag{D.100}$$

where we used in the last inequality the estimate (2.69) for b, the estimate (2.67) for n, the estimate (2.76) for $\partial_\omega N$, the estimate (D.96) for f and the estimate (D.98) for f_1.

In view of the transport equation (D.99), we have

$$\underline{L}\partial_\omega b = b_0 - \int_0^t \mathrm{div}\!\!\!/(\underline{L}(b)\partial_\omega N) + \int_0^t f_2, \tag{D.101}$$

where b_0 is the solution to

$$nL(b_0) = 0, \quad b_0(0) = \underline{L}\partial_\omega b,$$

where $b_0(0)$ satisfies

$$\|P_j b_0(0)\|_{L^2(P_{0,u})} \lesssim 2^{\frac{j}{2}}\varepsilon$$

in view of (2.63). This implies for $j \geq 0$

$$\|P_j b_0\|_{L_t^\infty L_{x'}^2} \lesssim 2^{\frac{j}{2}}\varepsilon$$

in view of Lemma 5.11, and hence

$$\|\Lambda^{-1} b_0\|_{L_t^\infty L_{x'}^2} \lesssim \varepsilon. \tag{D.102}$$

(D.101) and (D.102) imply

$$\|\Lambda^{-1}(b\underline{L}\partial_\omega b)\|_{L_t^\infty L_{x'}^2} \lesssim \varepsilon + \left\|\Lambda^{-1}\left(b\int_0^t \mathrm{div}\!\!\!/(\underline{L}(b)\partial_\omega N)\right)\right\|_{L_t^\infty L_{x'}^2} + \left\|\Lambda^{-1}\left(b\int_0^t f_2\right)\right\|_{L_t^\infty L_{x'}^2}. \tag{D.103}$$

Using the estimate (3.27) for Λ^{-1} and the estimate (3.66) for transport equations, we have

$$\left\|\Lambda^{-1}\left(b\int_0^t f_2\right)\right\|_{L^\infty_t L^2_{x'}} \lesssim \left\|b\int_0^t f_2\right\|_{L^\infty_t L^{\frac{4}{3}}_{x'}} \lesssim \|b\|_{L^\infty}\|f_2\|_{L^1_t L^{\frac{4}{3}}_{x'}} \lesssim \varepsilon, \tag{D.104}$$

where we used (D.100) and the estimate (2.69) for b in the last inequality. Finally, we define

$$w = \int_0^t \not{\mathrm{div}}(\underline{L}(b)\partial_\omega N)$$

and the tensor W solution to the following transport equation

$$\not{\nabla}_{nL} W - n\chi \cdot W = \underline{L}(b)\partial_\omega N,\ W = 0 \text{ on } P_{0,u}.$$

Then, Lemma 5.2 implies

$$\begin{aligned} \|\not{\mathrm{div}}(W) - w\|_{L^\infty_t L^{\frac{4}{3}}_{x'}} &\lesssim \varepsilon\|\underline{L}(b)\partial_\omega N\|_{L^1_t L^4_{x'}} \\ &\lesssim \varepsilon\|\underline{L}b\|_{L^\infty_t L^4_{x'}}\|\partial_\omega N\|_{L^\infty} \\ &\lesssim \varepsilon, \end{aligned} \tag{D.105}$$

where we used in the last inequality the estimate (2.69) for b and the estimate (2.76) for $\partial_\omega N$. Also, in view of the transport equation satisfied by W, the estimate (3.66) for transport equations yields

$$\begin{aligned} \|W\|_{L^\infty_{x'} L^2_t} &\lesssim \|n\chi \cdot W\|_{L^2_{x'} L^1_t} + \|\underline{L}(b)\partial_\omega N\|_{L^2_{x'} L^1_t} \\ &\lesssim \|n\|_{L^\infty}\|\chi\|_{L^\infty_{x'} L^2_t}\|W\|_{L^2(\mathcal{H}_u)} + \|\underline{L}(b)\|_{L^\infty_t L^2_{x'}}\|\partial_\omega N\|_{L^\infty} \\ &\lesssim \varepsilon\|W\|_{L^\infty_t L^2_{x'}} + \varepsilon, \end{aligned}$$

where we used in the last inequality the estimate (2.67) for n, the estimates (2.70) (2.71) for χ, the estimate (2.69) for b and the estimate (2.76) for $\partial_\omega N$. We deduce

$$\|W\|_{L^\infty_t L^2_{x'}} \lesssim \varepsilon. \tag{D.106}$$

Using the estimates (3.25) and (3.27) for Λ^{-1}, we have

$$\begin{aligned} \|\Lambda^{-1}(bw)\|_{L^\infty_t L^2_{x'}} &\lesssim \|\Lambda^{-1}(b(w - \not{\mathrm{div}}(W)))\|_{L^\infty_t L^2_{x'}} + \|\Lambda^{-1}(b\not{\mathrm{div}}(W))\|_{L^\infty_t L^2_{x'}} \\ &\lesssim \|b(w - \not{\mathrm{div}}(W))\|_{L^\infty_t L^{\frac{4}{3}}_{x'}} + \|\Lambda^{-1}(\not{\nabla}(b) \cdot W)\|_{L^\infty_t L^2_{x'}} + \|\Lambda^{-1}\not{\mathrm{div}}(bW)\|_{L^\infty_t L^2_{x'}} \\ &\lesssim \|b\|_{L^\infty}\|w - \not{\mathrm{div}}(W)\|_{L^\infty_t L^{\frac{4}{3}}_{x'}} + \|\not{\nabla}(b) \cdot W\|_{L^\infty_t L^{\frac{4}{3}}_{x'}} + \|bW\|_{L^\infty_t L^2_{x'}} \\ &\lesssim \|b\|_{L^\infty}\|w - \not{\mathrm{div}}(W)\|_{L^\infty_t L^{\frac{4}{3}}_{x'}} + \|\not{\nabla}b\|_{L^\infty_t L^4_{x'}}\|W\|_{L^\infty_t L^2_{x'}} + \|b\|_{L^\infty}\|W\|_{L^\infty_t L^2_{x'}} \\ &\lesssim \varepsilon, \end{aligned}$$

where we used in the last inequality the estimate (2.69) for b and the estimates (D.105) and (D.106). In view of the definition of w, and together with (D.103) and (D.104), we finally obtain

$$\|\Lambda^{-1}(b\underline{L}\partial_\omega b)\|_{L^\infty_t L^2_{x'}} \lesssim \varepsilon. \tag{D.107}$$

On the other hand, we have

$$\begin{aligned} \|\Lambda^{-1}(bL\partial_\omega b)\|_{L_t^\infty L^2_{x'}} &\lesssim \|bL\partial_\omega b\|_{L_t^\infty L^2_{x'}} \\ &\lesssim \|b\|_{L^\infty}\|L\partial_\omega b\|_{L_t^\infty L^2_{x'}} \\ &\lesssim \varepsilon, \end{aligned} \tag{D.108}$$

where we used in the last inequality the estimate (2.69) for b and the estimate (6.41) for $\partial_\omega b$. Recall that

$$N = \frac{1}{2}(L - \underline{L}),$$

which together with (D.107) and (D.108) implies

$$\|\Lambda^{-1}(bN\partial_\omega b)\|_{L_t^\infty L^2_{x'}} \lesssim \varepsilon.$$

This concludes the proof of Lemma 8.20.

E Appendix to section 9

E.1 Proof of Proposition 9.1

Using the definition (3.15) of P_j, we have:

$$[\nabla\!\!\!/_{bN}, P_j]F = \int_0^\infty m_j(\tau)V(\tau)d\tau, \tag{E.1}$$

where $V(\tau)$ satisfies:

$$(\partial_\tau - \Delta\!\!\!/)V(\tau) = [\nabla\!\!\!/_{bN}, \Delta\!\!\!/]U(\tau)F,\ V(0) = 0. \tag{E.2}$$

(E.1) yields:

$$\|[\nabla\!\!\!/_{bN}, P_j]F\|_{L_u^{\frac{4}{3}}L^2(P_{t,u})} \lesssim \left\| \int_0^\infty m_j(\tau)\|V(\tau)\|_{L^2(P_{t,u})}d\tau \right\|_{L_u^{\frac{4}{3}}}. \tag{E.3}$$

In view of (9.1) and (E.3), we have to estimate $\|V(\tau)\|_{L^2(P_{t,u})}$. Let a, p real numbers satisfying:

$$0 < a < \frac{1}{4},\ 2 < p < +\infty,\ \text{such that } p < \min\left(\frac{8}{3}, \frac{4}{2-a}\right). \tag{E.4}$$

The energy estimate (3.30) implies:

$$\begin{aligned} &\|\Lambda^{-a}V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^2_{L^2(P_{t,u})}d\tau' \\ \lesssim\ & \int_0^\tau \int_{P_{t,u}} \Lambda^{-2a}V(\tau')[\nabla\!\!\!/_{bN}, \Delta\!\!\!/]U(\tau')Fd\mu_{t,u}d\tau'. \end{aligned} \tag{E.5}$$

We need to estimate the commutator term $[\nabla\!\!\!/_{bN}, \Delta\!\!\!/]U$. Using the definition of θ (4.61), we may rewrite the commutator formula (2.50) for any m-covariant tensor $\Pi_{\underline{A}}$ tangent to $P_{t,u}$ as:

$$\begin{aligned}\nabla\!\!\!/_B\nabla\!\!\!/_{bN}\Pi_{\underline{A}} - \nabla\!\!\!/_{bN}\nabla\!\!\!/_B\Pi_{\underline{A}} &= b\theta_{BC}\nabla\!\!\!/_C\Pi_{\underline{A}} \\ &+ b\sum_i(-\theta_{BC}b^{-1}\nabla\!\!\!/_C b + \theta_{BC}b^{-1}\nabla\!\!\!/_C b - k_{AB}k_{CN} + k_{BC}k_{AN} \\ &\quad -\frac{1}{2}\in_{A_iC}{}^*(\beta_B + \underline{\beta}_B))\Pi_{A_1..\check{C}..A_m}.\end{aligned} \tag{E.6}$$

Using twice the commutator formula (E.6), we have:

$$[\nabla\!\!\!/_{bN}, \Delta\!\!\!/]U = H\nabla\!\!\!/^2U + G\nabla\!\!\!/U + \nabla\!\!\!/(GU) \tag{E.7}$$

where the tensors H and G are given by $H = b\theta$ and $G = \theta\cdot\nabla\!\!\!/ b + k\cdot k + b^*(\beta + \underline{\beta})$. Using the curvature bound (2.59) for $\beta, \underline{\beta}$, the L^∞ bound (2.69) for b, the estimate (4.43) for k on Σ_t, and the bounds (C.89)-(C.91) for b and θ on Σ_t, we obtain the following bound for H and G:

$$\begin{aligned}\|\nabla\!\!\!/ H\|_{L^\infty_t L^2(\Sigma_t)} + \|G\|_{L^\infty_t L^2(\Sigma_t)} &\lesssim \|b\|_{L^\infty}\|\nabla\!\!\!/\theta\|_{L^\infty_t L^2(\Sigma_t)} + \|\theta\|_{L^\infty_t L^4(\Sigma_t)}\|\nabla\!\!\!/ b\|_{L^\infty_t L^4(\Sigma_t)} \\ &\quad + \|k\|^2_{L^\infty_t L^4(\Sigma_t)} + \|b\|_{L^\infty}(\|\beta\|_{L^\infty_t L^2(\Sigma_t)} + \|\underline{\beta}\|_{L^\infty_t L^2(\Sigma_t)}) \\ &\lesssim \varepsilon.\end{aligned} \tag{E.8}$$

Notice that the structure (E.7) (E.8) is completely analogous to (C.74) (C.75). Therefore, proceeding as in (C.76), we obtain:

$$\begin{aligned}&\int_0^\tau\int_{P_{t,u}} \Lambda^{-2a}V(\tau')[\nabla\!\!\!/_{nL}, \Delta\!\!\!/]U(\tau')F d\mu_{t,u}d\tau' \\ &\lesssim (\|\nabla\!\!\!/ H\|_{L^2(P_{t,u})} + \|G\|_{L^2(P_{t,u})})\int_0^\tau \|\nabla\!\!\!/ U(\tau')F\|_{L^p(P_{t,u})}\|\nabla\!\!\!/\Lambda^{-2a}V(\tau')\|_{L^2(P_{t,u})}d\tau'.\end{aligned} \tag{E.9}$$

The Gagliardo-Nirenberg inequality (3.3), the properties (3.23) and (3.22) of Λ, and the Bochner inequality (3.7) for tensors yield:

$$\begin{aligned}&\int_0^\tau \|\nabla\!\!\!/ U(\tau')\|_{L^p(P_{t,u})}\|\nabla\!\!\!/\Lambda^{-2a}V(\tau')\|_{L^2(P_{t,u})}d\tau' \\ &\lesssim \int_0^\tau \|\nabla\!\!\!/ U(\tau')\|^{\frac{2}{p}}_{L^2(P_{t,u})}\|\nabla\!\!\!/^2U(\tau')\|^{1-\frac{2}{p}}_{L^2(P_{t,u})}\|\Lambda^{-a}V(\tau')\|^a_{L^{p'}(P_{t,u})}\|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^{1-a}_{L^{p'}(P_{t,u})}d\tau' \\ &\lesssim \int_0^\tau \|\nabla\!\!\!/ U(\tau')\|^{\frac{2}{p}}_{L^2(P_{t,u})}\big(\|\Delta\!\!\!/ U(\tau')\|_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})}\|\nabla\!\!\!/ U(\tau')\|_{L^2(P_{t,u})} \\ &+\|K\|^2_{L^2(P_{t,u})}\|U(\tau')\|_{L^2(P_{t,u})}\big)^{1-\frac{2}{p}}\|\Lambda^{-a}V(\tau')\|^a_{L^{p'}(P_{t,u})}\|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^{1-a}_{L^{p'}(P_{t,u})}d\tau' \\ &\lesssim \left(\left(1 + \|K\|^{2(1-\frac{2}{p})}_{L^2(P_{t,u})}\right)\int_0^\tau \|\nabla\!\!\!/ U(\tau')\|^2_{L^2(P_{t,u})}d\tau' + \int_0^\tau \tau'\|\Delta\!\!\!/ U(\tau')\|^2_{L^2(P_{t,u})}d\tau'\right)^{\frac{1}{2}} \\ &\left(\frac{1}{2}\int_0^\tau \|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^2_{L^2(P_{t,u})}d\tau' + \int_0^\tau \tau'^{-\frac{2(p-2)}{ap}}\|\Lambda^{-a}V(\tau')\|^2_{L^2(P_{t,u})}d\tau'\right)^{\frac{1}{2}}\end{aligned} \tag{E.10}$$

which together with the estimates for the heat flow (3.8) and (3.10) implies:

$$\begin{aligned}
&\int_0^\tau \|\nabla\!\!\!/ U(\tau')\|_{L^p(P_{t,u})}\|\nabla\!\!\!/\Lambda^{-2a}V(\tau')\|_{L^2(P_{t,u})}d\tau' \\
\lesssim\ & \left(1+\|K\|^{2(1-\frac{2}{p})}_{L^2(P_{t,u})}\right)\|F\|_{L^2(P_{t,u})} \\
&\times\left(\int_0^\tau \|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^2_{L^2(P_{t,u})}d\tau' + \int_0^\tau \tau'^{-\frac{2(p-2)}{ap}}\|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^2_{L^2(P_{t,u})}d\tau'\right)^{\frac{1}{2}}.
\end{aligned} \tag{E.11}$$

Finally, the choice of p (A.17), (C.73), (C.76) and (C.78) implies:

$$\begin{aligned}
&\|\Lambda^{-a}V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^2_{L^2(P_{t,u})}d\tau' \\
&\lesssim (\|\nabla\!\!\!/ H\|_{L^2(P_{t,u})} + \|G\|_{L^2(P_{t,u})})\left(1+\|K\|^{2(1-\frac{2}{p})}_{L^2(P_{t,u})}\right)\|F\|_{L^\infty_u L^2(P_{t,u})}.
\end{aligned} \tag{E.12}$$

Using the interpolation inequality (3.22), we obtain:

$$\begin{aligned}
&\int_0^{+\infty}\|V(\tau)\|^{\frac{2}{a}}_{L^2(P_{t,u})}d\tau \lesssim \int_0^\tau \|\Lambda^{-a}V(\tau')\|^{\frac{2(1-a)}{a}}_{L^2(P_{t,u})}\|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^2_{L^2(P_{t,u})}d\tau' \\
&\lesssim (\|\nabla\!\!\!/ H\|_{L^2(P_{t,u})} + \|G\|_{L^2(P_{t,u})})\left(1+\|K\|^{2(1-\frac{2}{p})}_{L^2(P_{t,u})}\right)\|F\|_{L^\infty_u L^2(P_{t,u})}.
\end{aligned} \tag{E.13}$$

The estimate (E.8) for H and G and the choice (E.4) for p, yields:

$$\begin{aligned}
&\left\|\int_0^{+\infty} m_j(\tau)\|V(\tau)\|_{L^2(P_{t,u})}d\tau\right\|_{L^{\frac{4}{3}}_u} \lesssim 2^{ja}\left\|\left(\int_0^{+\infty}\|V(\tau)\|^{\frac{2}{a}}_{L^2(P_{t,u})}d\tau\right)^{\frac{a}{2}}\right\|_{L^{\frac{4}{3}}_u} \\
&\lesssim 2^{ja}(\|\nabla\!\!\!/ H\|_{L^\infty_t L^2(\Sigma_t)} + \|G\|_{L^\infty_t L^2(\Sigma_t)})\left(1+\|K\|^{2(1-\frac{2}{p})}_{L^\infty_t L^2(\Sigma_t)}\right)\|\nabla F\|_{L^\infty_u L^2(P_{t,u})} \\
&\lesssim 2^{ja}\|\nabla F\|_{L^\infty_u L^2(P_{t,u})}.
\end{aligned} \tag{E.14}$$

(E.14) and (E.3) yield

$$\|[\nabla\!\!\!/_{bN}, P_j]F\|_{L^{\frac{4}{3}}_u L^2(P_{t,u})} \lesssim 2^{ja}\|\nabla F\|_{L^infty_u L^2(P_{t,u})}.$$

Taking the supremum in t yields the desired estimate (9.1). This concludes the proof of the proposition.

E.2 Proof of Proposition 9.2

The proof of the estimate (9.3) being similar and slightly easier than the proof of (9.2), we focus on (9.2). In view of (E.1) (E.2), we have:

$$[bN, P_l]f = \int_0^\infty m_l(\tau)V(\tau)d\tau, \tag{E.15}$$

where $V(\tau)$ satisfies:

$$(\partial_\tau - \Delta\!\!\!/)V(\tau) = [bN, \Delta\!\!\!/]U(\tau)f,\ V(0) = 0. \tag{E.16}$$

Assume that V satisfies for all τ

$$\|V(\tau)\|^2_{L^2(\mathcal{H}_u)} + \int_0^\tau \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(\mathcal{H}_u)} d\tau' \lesssim \varepsilon^2 (\mathcal{N}_1(f))^2. \tag{E.17}$$

Then, in view of (E.15), we obtain

$$\begin{aligned}
& \|[bN, P_l]f\|_{L^2(\mathcal{H}_u)} + 2^{-l}\|\nabla\!\!\!/[bN, P_l]f\|_{L^2(\mathcal{H}_u)} \\
\lesssim\ & \int_0^\infty |m_l(\tau)| \|V(\tau)\|_{L^2(\mathcal{H}_u)} d\tau + 2^{-l}\int_0^\infty |m_l(\tau)| \|\nabla\!\!\!/ V(\tau)\|_{L^2(\mathcal{H}_u)} d\tau \\
\lesssim\ & \varepsilon \mathcal{N}_1(f) \left(\int_0^\infty |m_l(\tau)| d\tau + 2^{-l} \left(\int_0^\infty m_l^2(\tau) d\tau \right) \right) \\
\lesssim\ & \varepsilon \mathcal{N}_1(f),
\end{aligned}$$

which after taking the supremum in u yields (9.2). Thus, it remains to prove (E.17).

The energy estimate (3.12) implies after integration along null geodesics:

$$\|V(\tau)\|^2_{L^2(\mathcal{H}_u)} + \int_0^\tau \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(\mathcal{H}_u)} d\tau' \lesssim \int_0^\tau \int_{\mathcal{H}_u} V(\tau')[bN, \Delta\!\!\!/]U(\tau') f d\mu_{t,u} d\tau'. \tag{E.18}$$

We need to estimate the commutator term $[bN, \Delta\!\!\!/]U$. Using twice the commutator formula (2.50) together with the fact that $U(\tau)f$ is a scalar function, we have:

$$[bN, \Delta\!\!\!/]U = H\nabla\!\!\!/^2 U + G\nabla\!\!\!/ U \tag{E.19}$$

where the tensors H and G are given by $H = b(\chi + k)$ and $G = b\nabla\!\!\!/\chi + b\nabla\!\!\!/ k + (\chi + k)\nabla\!\!\!/ b + \chi(\epsilon + \underline{\xi}) + \underline{\chi}\zeta + b^*(\beta + \underline{\beta})$. Using the curvature estimate (2.59), and the estimates (2.67)-(2.72) for $k, b, \chi, \zeta, \underline{\xi}$ and $\underline{\chi}$, we obtain the following bound for H and G:

$$\begin{aligned}
\mathcal{N}_1(H) + \|G\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim\ & \|b\|_{L^\infty}(\mathcal{N}_1(\chi) + \mathcal{N}_1(k)) + \|(L, \nabla\!\!\!/)b\|_{L^2_t L^4_{x'}}(\|\chi\|_{L^\infty_t L^4_{x'}} + \|k\|_{L^\infty_t L^4_{x'}}) \\
& + \|\chi\|_{L^\infty_t L^4_{x'}}(\|\epsilon\|_{L^\infty_t L^4_{x'}} + \|\underline{\xi}\|_{L^\infty_t L^4_{x'}}) + \|\underline{\chi}\|_{L^\infty_t L^4_{x'}} \|\zeta\|_{L^\infty_t L^4_{x'}} \\
& + \|\beta\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\underline{\beta}\|_{L^\infty_u L^2(\mathcal{H}_u)} \\
\lesssim\ & \varepsilon.
\end{aligned} \tag{E.20}$$

Using (E.20), we obtain:

$$\begin{aligned}
&\int_0^\tau \int_{\mathcal{H}_u} V(\tau')[bN, \not{\Delta}]U(\tau') f d\mu_{t,u} d\tau' \\
\lesssim\ & \int_0^\tau \|H\|_{L_t^\infty L_{x'}^4} \|\not{\nabla}^2 U(\tau') f\|_{L^2(\mathcal{H}_u)} \|V(\tau')\|_{L_t^2 L_{x'}^4} d\tau' \\
&+ \int_0^1 \int_0^\tau \|G\|_{L^2(P_{t,u})} \|\not{\nabla} U(\tau') f\|_{L^4(P_{t,u})} \|V(\tau')\|_{L^4(P_{t,u})} d\tau' dt \\
\lesssim\ & \varepsilon \int_0^\tau \|\not{\nabla}^2 U(\tau') f\|_{L^2(\mathcal{H}_u)} \|\not{\nabla} V(\tau')\|_{L^2(\mathcal{H}_u)} d\tau' \\
&+ \|G\|_{L^2(\mathcal{H}_u)} \left\| \int_0^\tau \|\not{\nabla} U(\tau') f\|_{L^4(P_{t,u})}^2 d\tau' \right\|_{L_t^\infty}^{\frac{1}{2}} \left(\int_0^\tau \|V(\tau')\|_{L_t^2 L_{x'}^4}^2 d\tau' \right)^{\frac{1}{2}} \\
\lesssim\ & \varepsilon \int_0^\tau \|\not{\nabla}^2 U(\tau') f\|_{L^2(\mathcal{H}_u)} \|\not{\nabla} V(\tau')\|_{L^2(\mathcal{H}_u)} d\tau' \\
&+ \varepsilon \left\| \int_0^\tau \|\not{\nabla} U(\tau') f\|_{L^4(P_{t,u})}^2 d\tau' \right\|_{L_t^\infty}^{\frac{1}{2}} \left(\int_0^\tau \|\not{\nabla} V(\tau')\|_{L^2(\mathcal{H}_u)}^2 d\tau' \right)^{\frac{1}{2}}
\end{aligned}$$

where we have used in particular the following inequality

$$\begin{aligned}
&\int_0^1 \int_0^\tau \|G\|_{L^2(P_{t,u})} \|\not{\nabla} U(\tau') f\|_{L^4(P_{t,u})} \|V(\tau')\|_{L^4(P_{t,u})} d\tau' dt \\
=\ & \int_0^1 \|G\|_{L^2(P_{t,u})} \int_0^\tau \|\not{\nabla} U(\tau') f\|_{L^4(P_{t,u})} \|V(\tau')\|_{L^4(P_{t,u})} d\tau' dt \\
\lesssim\ & \|G\|_{L^2(\mathcal{H}_u)} \left(\int_0^1 \left(\int_0^\tau \|\not{\nabla} U(\tau') f\|_{L^4(P_{t,u})} \|V(\tau')\|_{L^4(P_{t,u})} d\tau' \right)^2 dt \right)^{\frac{1}{2}} \\
\lesssim\ & \|G\|_{L^2(\mathcal{H}_u)} \left(\int_0^1 \left(\int_0^\tau \|\not{\nabla} U(\tau') f\|_{L^4(P_{t,u})}^2 d\tau' \right) \left(\int_0^\tau \|V(\tau')\|_{L^4(P_{t,u})}^2 d\tau' \right) dt \right)^{\frac{1}{2}} \\
\lesssim\ & \|G\|_{L^2(\mathcal{H}_u)} \left\| \int_0^\tau \|\not{\nabla} U(\tau') f\|_{L^4(P_{t,u})}^2 d\tau' \right\|_{L_t^\infty}^{\frac{1}{2}} \left(\int_0^1 \int_0^\tau \|V(\tau')\|_{L^4(P_{t,u})}^2 d\tau' dt \right)^{\frac{1}{2}} \\
=\ & \|G\|_{L^2(\mathcal{H}_u)} \left\| \int_0^\tau \|\not{\nabla} U(\tau') f\|_{L^4(P_{t,u})}^2 d\tau' \right\|_{L_t^\infty}^{\frac{1}{2}} \left(\int_0^\tau \|V(\tau')\|_{L_t^2 L_{x'}^4}^2 d\tau' \right)^{\frac{1}{2}}.
\end{aligned}$$

Together with (E.18), we infer:

$$\begin{aligned}
&\|V(\tau)\|_{L^2(\mathcal{H}_u)}^2 + \int_0^\tau \|\not{\nabla} V(\tau')\|_{L^2(\mathcal{H}_u)}^2 d\tau' \qquad \text{(E.21)} \\
\lesssim\ & \varepsilon^2 \int_0^\tau \|\not{\nabla}^2 U(\tau') f\|_{L^2(\mathcal{H}_u)}^2 d\tau' + \varepsilon^2 \left\| \int_0^\tau \|\not{\nabla} U(\tau') f\|_{L^4(P_{t,u})}^2 d\tau' \right\|_{L_t^\infty} \\
\lesssim\ & \varepsilon^2 \int_0^\tau \|\not{\Delta} U(\tau') f\|_{L^2(\mathcal{H}_u)}^2 d\tau' + \varepsilon^2 \left\| \int_0^\tau \|\not{\nabla} U(\tau') f\|_{L^4(P_{t,u})}^2 d\tau' \right\|_{L_t^\infty},
\end{aligned}$$

where we used the Bochner inequality for scalars (4.34) in the last inequality. Now, the energy estimates (3.8) and (3.9) yield:

$$\int_0^\tau \|\not\Delta U(\tau')f\|^2_{L^2(\mathcal{H}_u)}d\tau' + \left\|\int_0^\tau \|\not\nabla U(\tau')f\|^2_{L^4(P_{t,u})}d\tau'\right\|_{L^\infty_t} \lesssim \|\not\nabla f\|^2_{L^2(\mathcal{H}_u)} + \|f\|^2_{L^\infty_t L^4_{x'}} \lesssim (\mathcal{N}_1(f))^2,$$

Together with (E.21), we obtain

$$\|V(\tau)\|^2_{L^2(\mathcal{H}_u)} + \int_0^\tau \|\not\nabla V(\tau')\|^2_{L^2(\mathcal{H}_u)}d\tau' \lesssim \varepsilon^2(\mathcal{N}_1(f))^2,$$

which is the desired estimate (E.17). This concludes the proof of the proposition.

E.3 Proof of Proposition 9.3

The estimate of the first term in the right-hand side of (9.4) being similar and slightly easier, we focus on the estimate of the second term involving $[bN, P_q]f$. In view of (E.15) and (E.16), we have:

$$\|[bN, P_q]f\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \left\|\int_0^\infty m_q(\tau)\|V(\tau)\|_{L^2(\mathcal{H}_u)}d\tau\right\|_{L^\infty_u}, \tag{E.22}$$

where $V(\tau)$ satisfies:

$$(\partial_\tau - \not\Delta)V(\tau) = [bN, \not\Delta]U(\tau)f,\ V(0) = 0. \tag{E.23}$$

In view of (E.22), we have to estimate $\|V(\tau)\|_{L^2(\mathcal{H}_u)}$. Let a, δ real numbers satisfying:

$$\frac{1}{2} < a < 1, \text{ and } 0 < \delta < a - \frac{1}{2}. \tag{E.24}$$

The energy estimate (3.30) implies:

$$\begin{aligned} &\|\Lambda^{-a}V(\tau)\|^2_{L^2(\mathcal{H}_u)} + \int_0^\tau \|\not\nabla\Lambda^{-a}V(\tau')\|^2_{L^2(\mathcal{H}_u)}d\tau' \\ \lesssim\ & \int_0^\tau\int_0^1\int_{P_{t,u}} \Lambda^{-2a}V(\tau')[bN, \not\Delta]U(\tau')f d\mu_{t,u}dtd\tau'. \end{aligned} \tag{E.25}$$

As in (E.19), we need to estimate the commutator term $[bN, \not\Delta]U$. Using twice the commutator formula (2.50) together with the fact that $U(\tau)f$ is a scalar function, we have:

$$[bN, \not\Delta]U = H\not\nabla^2 U + \not\nabla(H\not\nabla U) + G\not\nabla U \tag{E.26}$$

where the tensors H and G are given by $H = b(\chi + k)$ and $G = (\chi + k)\not\nabla b + \chi(\epsilon + \underline{\xi}) + \underline{\chi}\zeta + b^*(\beta + \underline{\beta})$. Using Lemma 5.9, Lemma 6.21, and the estimates (2.67)-(2.72) for $k, b, \chi, \zeta, \underline{\xi}$ and $\underline{\chi}$, we obtain the following bound for H and G:

$$\begin{aligned} &\sup_j \left(2^{\frac{j}{2}}\|P_jH\|_{L^\infty_t L^2_{x'}} + 2^{-\frac{j}{2}}\|P_jG\|_{L^\infty_t L^2_{x'}}\right) \\ \lesssim\ & \mathcal{N}_1(b(\chi + k)) + \|\not\nabla b\|_{L^2_t L^4_{x'}}(\|\chi\|_{L^\infty_t L^4_{x'}} + \|k\|_{L^\infty_t L^4_{x'}}) + \|\chi\|_{L^\infty_t L^4_{x'}}(\|\epsilon\|_{L^\infty_t L^4_{x'}} + \|\underline{\xi}\|_{L^\infty_t L^4_{x'}}) \\ &+ \|\underline{\chi}\|_{L^\infty_t L^4_{x'}}\|\zeta\|_{L^\infty_t L^4_{x'}} + \varepsilon \\ \lesssim\ & \varepsilon. \end{aligned} \tag{E.27}$$

Using the property of P_j, and in view of (E.26) and (E.27), we have:

$$\begin{aligned}
&\int_0^\tau \int_0^1 \int_{P_{t,u}} \Lambda^{-2a} V(\tau')[bN, \not\Delta] U(\tau') f d\mu_{t,u} dt d\tau' \\
= \;& \sum_j \int_0^\tau \int_0^1 \int_{P_{t,u}} (P_j(H) P_j(\not\nabla(\not\nabla U f \Lambda^{-2a} V(\tau'))) + P_j(H) P_j(\not\nabla U f \not\nabla \Lambda^{-2a} V(\tau'))) d\mu_{t,u} dt d\tau' \\
&+ \int_0^\tau \int_0^1 \int_{P_{t,u}} P_j(G) P_j(\not\nabla U f \Lambda^{-2a} V(\tau')) d\mu_{t,u} dt d\tau' \\
\lesssim \;& \sum_j 2^{-\frac{j}{2}} \varepsilon \int_0^\tau (\| P_j(\not\nabla(\not\nabla U f \Lambda^{-2a} V(\tau')))\|_{L^2(\mathcal{H}_u)} + \| P_j(\not\nabla U f \not\nabla \Lambda^{-2a} V(\tau'))\|_{L^2(\mathcal{H}_u)}) d\tau' \\
&+ \sum_j 2^{\frac{j}{2}} \varepsilon \int_0^\tau \| P_j(\not\nabla U f \Lambda^{-2a} V(\tau'))\|_{L^2(\mathcal{H}_u)} d\tau'. \qquad \text{(E.28)}
\end{aligned}$$

In order to estimate the right-hand side of (E.28), we derive three product estimates. Let h_1, h_2 two scalar functions. Let $\delta > 0$ a small constant to be chosen later on. Using the finite band property for P_j, the weak Bernstein inequality, the Gagliardo Nirenberg inequality (3.3), and the Bochner inequality for scalars (4.34), we obtain:

$$\begin{aligned}
&\| P_j \not\nabla((\not\nabla h_1) h_2)\|_{L^2(P_{t,u})} \qquad \text{(E.29)} \\
\lesssim \;& \| P_j \not\nabla((\not\nabla h_1) h_2)\|_{L^2(P_{t,u})}^{\frac{1}{2}+\delta} \| P_j \not\nabla((\not\nabla h_1) h_2)\|_{L^2(P_{t,u})}^{\frac{1}{2}-\delta} \\
\lesssim \;& (2^{j0_+} \|(\not\nabla^2 h_1) h_2)\|_{L^{2-}(P_{t,u})} + 2^{j0_+} \|(\not\nabla h_1)(\not\nabla h_2)\|_{L^{2-}(P_{t,u})})^{\frac{1}{2}+\delta} (2^j \|(\not\nabla h_1) h_2)\|_{L^2(P_{t,u})})^{\frac{1}{2}-\delta} \\
\lesssim \;& 2^{j(\frac{1}{2}-\frac{\delta}{4})} (\|\not\nabla^2 h_1\|_{L^2(P_{t,u})} \|\not\nabla h_2\|_{L^2(P_{t,u})})^{\frac{1}{2}+\delta} (\|\not\nabla h_1\|_{L^{2+}(P_{t,u})} \|\not\nabla h_2\|_{L^2(P_{t,u})})^{\frac{1}{2}-\delta} \\
\lesssim \;& 2^{j(\frac{1}{2}-\frac{\delta}{4})} \|\not\Delta h_1\|_{L^2(P_{t,u})}^{\frac{1}{2}+\delta} \|\not\nabla h_1\|_{L^2(P_{t,u})}^{\frac{1}{2}-\delta} \|\not\nabla h_2\|_{L^2(P_{t,u})}.
\end{aligned}$$

Also, the weak Bernstein inequality, the Gagliardo Nirenberg inequality (3.3), and the Bochner inequality for scalars (4.34) yields:

$$\begin{aligned}
\|P_j((\not\nabla h_1)(\not\nabla h_2))\|_{L^2(P_{t,u})} &\lesssim 2^{j(\frac{1}{2}-\delta)} \|(\not\nabla h_1)(\not\nabla h_2)\|_{L^{(\frac{4}{3})+}(P_{t,u})} \qquad \text{(E.30)} \\
&\lesssim 2^{j(\frac{1}{2}-\delta)} \|\not\nabla h_1\|_{L^{\frac{4}{1-2\delta}}(P_{t,u})} \|\not\nabla h_2\|_{L^2(P_{t,u})} \\
&\lesssim 2^{j(\frac{1}{2}-\delta)} \|\not\nabla^2 h_1\|_{L^2(P_{t,u})}^{\frac{1}{2}+\delta} \|\not\nabla h_1\|_{L^2(P_{t,u})}^{\frac{1}{2}-\delta} \|\not\nabla h_2\|_{L^2(P_{t,u})} \\
&\lesssim 2^{j(\frac{1}{2}-\delta)} \|\not\Delta h_1\|_{L^2(P_{t,u})}^{\frac{1}{2}+\delta} \|\not\nabla h_1\|_{L^2(P_{t,u})}^{\frac{1}{2}-\delta} \|\not\nabla h_2\|_{L^2(P_{t,u})}.
\end{aligned}$$

Finally, we have:

$$\|P_j((\not\nabla h_1) h_2)\|_{L^2(P_{t,u})} \lesssim \sum_{l,q} \|P_j(\not\nabla(P_l(h_1)) P_q(h_2))\|_{L^2(P_{t,u})}. \qquad \text{(E.31)}$$

If $j \geq \max(l, q)$, we obtain using the finite band property for P_j, P_l and P_q, the strong Bernstein inequality (4.32) for P_q, the Gagliardo Nirenberg inequality (3.3), and the

Bochner inequality for scalars (4.34):

$$\begin{aligned}
&\|P_j(\nabla\!\!\!/(P_l(h_1))P_q(h_2))\|_{L^2(P_{t,u})} \qquad \text{(E.32)}\\
\lesssim\ & 2^{-j}\|\nabla\!\!\!/^2(P_l(h_1))P_q(h_2)\|_{L^2(P_{t,u})} + 2^{-j}\|\nabla\!\!\!/(P_l(h_1))\nabla\!\!\!/(P_q(h_2))\|_{L^2(P_{t,u})}\\
\lesssim\ & 2^{-j}\|\nabla\!\!\!/^2(P_l(h_1))\|_{L^2(P_{t,u})}\|P_q(h_2))\|_{L^\infty(P_{t,u})} + 2^{-j}\|\nabla\!\!\!/(P_l(h_1))\|_{L^6(P_{t,u})}\|\nabla\!\!\!/(P_q(h_2))\|_{L^3(P_{t,u})}\\
\lesssim\ & (2^{-j+q+2l} + 2^{-j+\frac{4q}{3}+\frac{5l}{3}})\|P_l(h_1)\|_{L^2(P_{t,u})}\|P_q(h_2)\|_{L^2(P_{t,u})}\\
\lesssim\ & (2^{-j+l(\frac{1}{2}-\delta)} + 2^{-j+\frac{q}{3}+l(\frac{1}{6}-\delta)})\|\Delta\!\!\!/h_1\|^{\frac{1}{2}+\delta}_{L^2(P_{t,u})}\|\nabla\!\!\!/h_1\|^{\frac{1}{2}-\delta}_{L^2(P_{t,u})}\|\nabla\!\!\!/h_2\|_{L^2(P_{t,u})}\\
\lesssim\ & 2^{-j(\frac{1}{2}+\delta)}\|\Delta\!\!\!/h_1\|^{\frac{1}{2}+\delta}_{L^2(P_{t,u})}\|\nabla\!\!\!/h_1\|^{\frac{1}{2}-\delta}_{L^2(P_{t,u})}\|\nabla\!\!\!/h_2\|_{L^2(P_{t,u})}.
\end{aligned}$$

Next, if $l \geq \max(j,q)$, we obtain using the finite band property for P_j and P_l, the strong Bernstein inequality (4.32) for P_q, the Gagliardo Nirenberg inequality (3.3), and the Bochner inequality for scalars (4.34):

$$\begin{aligned}
\|P_j(\nabla\!\!\!/(P_l(h_1))P_q(h_2))\|_{L^2(P_{t,u})} &\lesssim \|\nabla\!\!\!/(P_l(h_1))P_q(h_2)\|_{L^2(P_{t,u})} \qquad \text{(E.33)}\\
&\lesssim \|\nabla\!\!\!/(P_l(h_1))\|_{L^2(P_{t,u})}\|P_q(h_2)\|_{L^\infty(P_{t,u})}\\
&\lesssim 2^{-l(\frac{1}{2}+\delta)}\|\Delta\!\!\!/h_1\|^{\frac{1}{2}+\delta}_{L^2(P_{t,u})}\|\nabla\!\!\!/h_1\|^{\frac{1}{2}-\delta}_{L^2(P_{t,u})}\|\nabla\!\!\!/h_2\|_{L^2(P_{t,u})}.
\end{aligned}$$

Finally, if $q \geq \max(j,l)$, we obtain using the finite band property for P_j, P_l and P_q, the weak Bernstein inequality for P_q, the Gagliardo Nirenberg inequality (3.3), and the Bochner inequality for scalars (4.34) :

$$\begin{aligned}
\|P_j(\nabla\!\!\!/(P_l(h_1))P_q(h_2))\|_{L^2(P_{t,u})} &\lesssim \|\nabla\!\!\!/(P_l(h_1))P_q(h_2)\|_{L^2(P_{t,u})}\\
&\lesssim \|\nabla\!\!\!/(P_l(h_1))\|_{L^6(P_{t,u})}\|P_q(h_2)\|_{L^3(P_{t,u})}\\
&\lesssim 2^{\frac{q}{3}+\frac{5l}{3}}\|P_l(h_1)\|_{L^2(P_{t,u})}\|P_q(h_2)\|_{L^2(P_{t,u})}\\
&\lesssim 2^{-\frac{2q}{3}+l(\frac{1}{6}-\delta)}\|\Delta\!\!\!/h_1\|^{\frac{1}{2}+\delta}_{L^2(P_{t,u})}\|\nabla\!\!\!/h_1\|^{\frac{1}{2}-\delta}_{L^2(P_{t,u})}\|\nabla\!\!\!/h_2\|_{L^2(P_{t,u})}\\
&\lesssim 2^{-q(\frac{1}{2}+\delta)}\|\Delta\!\!\!/h_1\|^{\frac{1}{2}+\delta}_{L^2(P_{t,u})}\|\nabla\!\!\!/h_1\|^{\frac{1}{2}-\delta}_{L^2(P_{t,u})}\|\nabla\!\!\!/h_2\|_{L^2(P_{t,u})},
\end{aligned}$$

which together with (E.31)-(E.33) yields:

$$\|P_j((\nabla\!\!\!/h_1)h_2)\|_{L^2(P_{t,u})} \lesssim 2^{-j(\frac{1}{2}+\delta)}\|\Delta\!\!\!/h_1\|^{\frac{1}{2}+\delta}_{L^2(P_{t,u})}\|\nabla\!\!\!/h_1\|^{\frac{1}{2}-\delta}_{L^2(P_{t,u})}\|\nabla\!\!\!/h_2\|_{L^2(P_{t,u})}. \qquad \text{(E.34)}$$

Now, we use (E.29) (E.30) and (E.34) with $h_1 = U(\tau')f$ and $h_2 = \Lambda^{-2a}V(\tau')$ to estimate respectively the first, second and third term in the right-hand side of (E.28). We obtain:

$$\begin{aligned}
&\int_0^\tau\int_0^1\int_{P_{t,u}} \Lambda^{-2a}V(\tau')[bN,\Delta\!\!\!/]U(\tau')f d\mu_{t,u}dtd\tau' \qquad \text{(E.35)}\\
\lesssim\ & \varepsilon\left(\sum_j 2^{-\frac{j\delta}{4}}\right)\int_0^\tau \|\Delta\!\!\!/U(\tau')f\|^{\frac{1}{2}+\delta}_{L^2(\mathcal{H}_u)}\|\nabla\!\!\!/U(\tau')f\|^{\frac{1}{2}-\delta}_{L^2(\mathcal{H}_u)}\|\nabla\!\!\!/\Lambda^{-2a}V(\tau')\|_{L^2(\mathcal{H}_u)}d\tau'\\
\lesssim\ & \varepsilon\int_0^\tau \|\Delta\!\!\!/U(\tau')f\|^{\frac{1}{2}+\delta}_{L^2(\mathcal{H}_u)}\|\nabla\!\!\!/U(\tau')f\|^{\frac{1}{2}-\delta}_{L^2(\mathcal{H}_u)}\|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^{1-a}_{L^2(\mathcal{H}_u)}\|\Lambda^{-a}V(\tau')\|^a_{L^2(\mathcal{H}_u)}d\tau',
\end{aligned}$$

where we used the interpolation estimate (3.22), (3.23), and the fact that $\delta > 0$ in the last inequality. Next, (E.25) and (E.35) yield:

$$\begin{aligned}
&\|\Lambda^{-a}V(\tau)\|^2_{L^2(\mathcal{H}_u)} + \int_0^\tau \|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^2_{L^2(\mathcal{H}_u)}d\tau' \\
\lesssim\ & \varepsilon\int_0^\tau \|\Delta\!\!\!/U(\tau')f\|^{\frac12+\delta}_{L^2(\mathcal{H}_u)}\|\nabla\!\!\!/U(\tau')f\|^{\frac12-\delta}_{L^2(\mathcal{H}_u)}\|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^{1-a}_{L^2(\mathcal{H}_u)}\|\Lambda^{-a}V(\tau')\|^{a}_{L^2(\mathcal{H}_u)}d\tau', \\
\lesssim\ & \varepsilon\int_0^\tau (\sqrt{\tau'}\|\Delta\!\!\!/U(\tau')f\|_{L^2(\mathcal{H}_u)})^{\frac12+\delta}\|\nabla\!\!\!/U(\tau')f\|^{\frac12-\delta}_{L^2(\mathcal{H}_u)} \\
&\times\|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^{1-a}_{L^2(\mathcal{H}_u)}(\tau'^{-\frac{\frac12+\delta}{2a}}\|\Lambda^{-a}V(\tau')\|_{L^2(\mathcal{H}_u)})^a d\tau', \\
\lesssim\ & \varepsilon\left(\int_0^\tau \tau'\|\Delta\!\!\!/U(\tau')f\|^2_{L^2(\mathcal{H}_u)}d\tau'\right)^{\frac14+\frac{\delta}{2}}\left(\int_0^\tau \|\nabla\!\!\!/U(\tau')f\|^2 d\tau'\right)^{\frac14-\frac{\delta}{2}}_{L^2(\mathcal{H}_u)} \\
&\times\left(\int_0^\tau \|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^2_{L^2(\mathcal{H}_u)}\right)^{\frac12-\frac{a}{2}}\left(\int_0^\tau \tau'^{-\frac{\frac12+\delta}{a}}\|\Lambda^{-a}V(\tau')\|^2_{L^2(\mathcal{H}_u)}d\tau'\right)^{\frac{a}{2}}.
\end{aligned}$$

Using Gronwall and the restriction (E.24), we infer for $0 \le \tau \le 1$

$$\begin{aligned}
&\|\Lambda^{-a}V(\tau)\|^2_{L^2(\mathcal{H}_u)} + \int_0^\tau \|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^2_{L^2(\mathcal{H}_u)}d\tau' \qquad \text{(E.36)}\\
\lesssim\ & \varepsilon\int_0^\tau (\tau')\|\Delta\!\!\!/U(\tau')\|^2_{L^2(\mathcal{H}_u)}d\tau' + \varepsilon\int_0^\tau \|\nabla\!\!\!/U(\tau')\|^2_{L^2(\mathcal{H}_u)}d\tau' \\
\lesssim\ & \varepsilon\|f\|^2_{L^2(\mathcal{H}_u)},
\end{aligned}$$

where we used the heat flow estimates (3.8) and (3.10) in the last inequality.

Using the interpolation inequality (3.22) and (E.36), we obtain:

$$\begin{aligned}
\int_0^1 \|V(\tau)\|^{\frac{2}{a}}_{L^2(P_{t,u})}d\tau &\lesssim \int_0^1 \|\Lambda^{-a}V(\tau')\|^{\frac{2(1-a)}{a}}_{L^2(P_{t,u})}\|\nabla\!\!\!/\Lambda^{-a}V(\tau')\|^2_{L^2(P_{t,u})}d\tau' \\
&\lesssim \varepsilon\|f\|^{\frac{2}{a}}_{L^\infty_u L^2(\mathcal{H}_u)}.
\end{aligned}$$

Together with (E.22), we obtain, in view of the fact that m is supported in $(0,1)$, for all $q \ge 0$:

$$\begin{aligned}
\|[bN, P_q]f\|_{L^\infty_u L^2(\mathcal{H}_u)} &\lesssim \left\|\int_0^\infty m_q(\tau)\|V(\tau)\|_{L^2(\mathcal{H}_u)}d\tau\right\|_{L^\infty_u} \qquad \text{(E.37)}\\
&\lesssim 2^{qa}\left\|\left(\int_0^1 \|V(\tau)\|^{\frac{2}{a}}_{L^2(\mathcal{H}_u)}d\tau\right)^{\frac{a}{2}}\right\|_{L^\infty_u} \\
&\lesssim 2^{qa}\|f\|_{L^\infty_u L^2(\mathcal{H}_u)}.
\end{aligned}$$

Since $a < 1$ in view of (E.24), (E.37) yields (9.4). This concludes the proof of the proposition.

E.4 Proof of Proposition 9.4

In view of the analog of (E.1) (E.2), we have:

$$[nL, P_j]\mathrm{tr}\chi = \int_0^\infty m_j(\tau)V(\tau)d\tau, \tag{E.38}$$

where $V(\tau)$ satisfies:

$$(\partial_\tau - \not{\Delta})V(\tau) = [nL, \not{\Delta}]U(\tau)\mathrm{tr}\chi,\ V(0) = 0. \tag{E.39}$$

Assume that $U(\tau)\mathrm{tr}\chi$ satisfies the following estimates

$$\|\chi\not{\nabla}^2 U(\tau)\mathrm{tr}\chi\|_{L^\infty_u L^1_t L^2_\tau L^2(P_{t,u})} \lesssim \varepsilon, \tag{E.40}$$

and

$$\|\not{\nabla}U(\tau)\mathrm{tr}\chi\|_{L^\infty_u L^2_t L^2_\tau L^\infty(P_{t,u})} \lesssim \varepsilon. \tag{E.41}$$

Then, in view of the commutator estimate (2.49), we have

$$\begin{aligned} & \|[nL, \not{\Delta}]U(\tau)\mathrm{tr}\chi\|_{L^\infty_u L^1_t L^2_\tau L^2(P_{t,u})} \qquad (E.42)\\ \lesssim\ & \|n\|_{L^\infty}\Big(\|\chi\not{\nabla}^2 U(\tau)\mathrm{tr}\chi\|_{L^\infty_u L^1_t L^2_\tau L^2(P_{t,u})} + (\|\chi\|_{L^\infty_t L^4_{x'}}\|\overline{\epsilon}\|_{L^\infty_t L^4_{x'}}\\ & +\|n^{-1}\not{\nabla}n\|_{L^\infty_t L^2_{x'}}\|\mathrm{tr}\chi\|_{L^\infty} + \|\not{\nabla}\mathrm{tr}\chi\|_{L^\infty_t L^2_{x'}})\|\not{\nabla}U(\tau)\mathrm{tr}\chi\|_{L^\infty_u L^1_t L^2_\tau L^\infty(P_{t,u})}\Big)\\ \lesssim\ & \varepsilon, \end{aligned}$$

where we used in the last inequality the estimate (2.67) for n, the estimates (2.70) (2.71) for χ, the estimates (2.67) (2.68) for $\overline{\epsilon}$, and the estimates (E.40) and (E.41). The energy estimate (3.11) implies

$$\|\not{\nabla}V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\not{\Delta}V(\tau')\|^2_{L^2(P_{t,u})}d\tau' \lesssim \int_0^\tau \|[nL, \not{\Delta}]U(\tau')\mathrm{tr}\chi\|^2_{L^2(P_{t,u})}d\tau'.$$

Taking the $L^\infty_u L^1_t$ norm, and using the estimate (E.42), we obtain

$$\|\not{\nabla}V\|_{L^\infty_u L^1_t L^\infty_\tau L^2(P_{t,u})} \lesssim \varepsilon,$$

which together with (E.38) yields the second part of the estimate (9.5)

$$\|\not{\nabla}[nL, P_j]\mathrm{tr}\chi\|_{L^1_t L^2_{x'}} \lesssim \varepsilon. \tag{E.43}$$

Also, the energy estimate (3.12) implies

$$\|V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\not{\nabla}V(\tau')\|^2_{L^2(P_{t,u})}d\tau' \lesssim \int_0^\tau \|V(\tau')\|_{L^2(P_{t,u})}\|[nL, \not{\Delta}]U(\tau')\mathrm{tr}\chi\|_{L^2(P_{t,u})}d\tau'. \tag{E.44}$$

Let

$$Y(\tau) = \int_0^\tau \|V(\tau')\|_{L^2(P_{t,u})}\|[nL, \not{\Delta}]U(\tau')\mathrm{tr}\chi\|_{L^2(P_{t,u})}d\tau'.$$

Then, (E.44) yields

$$Y'(\tau) \lesssim \sqrt{Y(\tau)} \|[nL, \not\Delta] U(\tau) \mathrm{tr}\chi\|_{L^2(P_{t,u})}.$$

Integrating in τ and using $Y(0) = 0$, we obtain

$$\|V(\tau)\|_{L^2(P_{t,u})} \lesssim \int_0^\tau \|[nL, \not\Delta] U(\tau') \mathrm{tr}\chi\|_{L^2(P_{t,u})} d\tau' \lesssim \sqrt{\tau} \|[nL, \not\Delta] U(\cdot) \mathrm{tr}\chi\|_{L^2_\tau L^2(P_{t,u})}.$$

Together with (E.38), this implies

$$\begin{aligned} \|[nL, P_j] \mathrm{tr}\chi\|_{L^2(P_{t,u})} &\lesssim \left(\int_0^\infty m_j(\tau) \sqrt{\tau} d\tau \right) \|[nL, \not\Delta] U(\tau) \mathrm{tr}\chi\|_{L^2_\tau L^2(P_{t,u})} \\ &\lesssim 2^{-j} \|[nL, \not\Delta] U(\cdot) \mathrm{tr}\chi\|_{L^2_\tau L^2(P_{t,u})}. \end{aligned}$$

Taking the $L^\infty_u L^1_t$ norm, and using the estimate (E.42), we obtain the first part of the estimate (9.5)

$$\|[nL, P_j] \mathrm{tr}\chi\|_{L^1_t L^2_{x'}} \lesssim 2^{-j} \varepsilon. \tag{E.45}$$

Finally, (E.43) and (E.45) yield the desired estimate (9.5). Thus, it remains to prove the estimates (E.40) and (E.41).

We start with the proof of (E.41). We have

$$\|\not\nabla U(\tau) \mathrm{tr}\chi\|_{L^\infty(P_{t,u})} \lesssim \sum_{j,l} \|P_j \not\nabla U(\tau) P_l \mathrm{tr}\chi\|_{L^\infty(P_{t,u})}. \tag{E.46}$$

We first consider the case $j < l$. Using the sharp Bernstein inequality for tensors (4.37) and the finite band property for P_j, we have

$$\begin{aligned} \|P_j \not\nabla U(\tau) P_l \mathrm{tr}\chi\|_{L^\infty(P_{t,u})} &\lesssim 2^j (1 + \|K\|_{L^2(P_{t,u})}) \|P_j \not\nabla U(\tau) P_l \mathrm{tr}\chi\|_{L^2(P_{t,u})} \\ &\lesssim 2^{2j} (1 + \|K\|_{L^2(P_{t,u})}) \|U(\tau) P_l \mathrm{tr}\chi\|_{L^2(P_{t,u})}. \end{aligned}$$

Taking the $L^2_t L^2_\tau$ norm, we obtain

$$\begin{aligned} \|P_j \not\nabla U(\cdot) P_l \mathrm{tr}\chi\|_{L^2_t L^2_\tau L^\infty(P_{t,u})} &\lesssim 2^{2j} (1 + \|K\|_{L^\infty_u L^2(\mathcal{H}_u)}) \|U(\cdot) P_l \mathrm{tr}\chi\|_{L^\infty_t L^2_\tau L^2(P_{t,u})} \\ &\lesssim 2^{2j} \|\Lambda^{-1} P_l \mathrm{tr}\chi\|_{L^\infty_t L^2_{x'}}, \end{aligned}$$

where we used in the last inequality the estimate (4.29) for K and a heat flow estimate for $U(\tau)\mathrm{tr}\chi$. Together with the finite band property for P_j and the assumption $j < l$, we obtain

$$\|P_j \not\nabla U(\cdot) P_l \mathrm{tr}\chi\|_{L^2_t L^2_\tau L^\infty(P_{t,u})} \lesssim 2^{-2|l-j|} (2^l \|P_l \mathrm{tr}\chi\|_{L^\infty_t L^2_{x'}}). \tag{E.47}$$

Next, we consider the case $l \geq j$. Using the sharp Bernstein inequality for tensors (4.37) and the finite band property for P_j, we have

$$\begin{aligned} \|P_j \not\nabla U(\tau) P_l \mathrm{tr}\chi\|_{L^\infty(P_{t,u})} &\lesssim 2^j (1 + \|K\|^{\frac{1}{2}}_{L^2(P_{t,u})}) \|P_j \not\nabla U(\tau) P_l \mathrm{tr}\chi\|_{L^2(P_{t,u})} \\ &\lesssim (1 + \|K\|^{\frac{1}{2}}_{L^2(P_{t,u})}) \|\not\nabla^2 U(\tau) P_l \mathrm{tr}\chi\|_{L^2(P_{t,u})} \\ &\lesssim (1 + \|K\|^{\frac{1}{2}}_{L^2(P_{t,u})}) \|\not\Delta U(\tau) P_l \mathrm{tr}\chi\|_{L^2(P_{t,u})}, \end{aligned} \tag{E.48}$$

where we used in the last inequality the Bochner inequality for scalars (4.34). Also, using the sharp Bernstein inequality for tensors (4.37) and the finite band property for P_j, we have

$$\begin{aligned} & \|P_j \nabla\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|_{L^\infty(P_{t,u})} \qquad \text{(E.49)} \\ \lesssim\ & 2^j (1+\|K\|^{\frac{1}{2}}_{L^2(P_{t,u})}) \|P_j \nabla\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|_{L^2(P_{t,u})} \\ \lesssim\ & 2^{-j} (1+\|K\|^{\frac{1}{2}}_{L^2(P_{t,u})}) \|P_j \Delta\!\!\!/ \nabla\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|_{L^2(P_{t,u})} \\ \lesssim\ & 2^{-j} (1+\|K\|^{\frac{1}{2}}_{L^2(P_{t,u})}) (\|\nabla\!\!\!/ \Delta\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|_{L^2(P_{t,u})} + \|P_j([\Delta\!\!\!/, \nabla\!\!\!/] U(\tau) P_l \mathrm{tr}\chi)\|_{L^2(P_{t,u})}). \end{aligned}$$

Using the commutator formula (B.87), the Bernstein inequality for P_j, the Gagliardo-Nirenberg inequality (3.3), and the Bochner inequality for scalars (4.34), we obtain

$$\begin{aligned} \|P_j(K \nabla\!\!\!/ U(\tau) P_l \mathrm{tr}\chi)\|_{L^2(P_{t,u})} \lesssim\ & 2^{\frac{j}{2}} \|K \nabla\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|_{L^{\frac{4}{3}}(P_{t,u})} \\ \lesssim\ & 2^{\frac{j}{2}} \|K\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|_{L^4(P_{t,u})} \\ \lesssim\ & 2^{\frac{j}{2}} \|K\|_{L^2(P_{t,u})} \|\nabla\!\!\!/^2 U(\tau) P_l \mathrm{tr}\chi\|^{\frac{1}{2}}_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|^{\frac{1}{2}}_{L^2(P_{t,u})} \\ \lesssim\ & 2^{\frac{j}{2}} \|K\|_{L^2(P_{t,u})} \|\Delta\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|^{\frac{1}{2}}_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|^{\frac{1}{2}}_{L^2(P_{t,u})}. \end{aligned}$$

Together with (E.49), this yields

$$\begin{aligned} \|P_j \nabla\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|_{L^\infty(P_{t,u})} \lesssim\ & 2^{-j} (1+\|K\|^{\frac{1}{2}}_{L^2(P_{t,u})}) \Big(\|\nabla\!\!\!/ \Delta\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|_{L^2(P_{t,u})} \\ & + 2^{\frac{j}{2}} \|K\|_{L^2(P_{t,u})} \|\Delta\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|^{\frac{1}{2}}_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|^{\frac{1}{2}}_{L^2(P_{t,u})} \Big). \end{aligned}$$

Interpolating with (E.48), we deduce

$$\begin{aligned} \|P_j \nabla\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|_{L^\infty(P_{t,u})} \lesssim\ & 2^{-\frac{j}{2}} (1+\|K\|^{\frac{1}{2}}_{L^2(P_{t,u})}) \|\Delta\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|^{\frac{1}{2}}_{L^2(P_{t,u})} \Big(\|\nabla\!\!\!/ \Delta\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|_{L^2(P_{t,u})} \\ & + 2^{\frac{j}{2}} \|K\|_{L^2(P_{t,u})} \|\Delta\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|^{\frac{1}{2}}_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|^{\frac{1}{2}}_{L^2(P_{t,u})} \Big)^{\frac{1}{2}}. \end{aligned}$$

Taking the $L^2_t L^2_\tau$ norm, we obtain

$$\begin{aligned} & \|P_j \nabla\!\!\!/ U(\cdot) P_l \mathrm{tr}\chi\|_{L^2_t L^2_\tau L^\infty(P_{t,u})} \\ \lesssim\ & 2^{-\frac{j}{2}} (1+\|K\|^{\frac{1}{2}}_{L^\infty_u L^2(\mathcal{H}_u)}) \|\Delta\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|^{\frac{1}{2}}_{L^\infty_t L^2_\tau L^2(P_{t,u})} \Big(\|\nabla\!\!\!/ \Delta\!\!\!/ U(\cdot) P_l \mathrm{tr}\chi\|_{L^\infty_t L^2_\tau L^2(P_{t,u})} \\ & + 2^{\frac{j}{2}} \|K\|_{L^\infty_u L^2(\mathcal{H}_u)} \|\Delta\!\!\!/ U(\cdot) P_l \mathrm{tr}\chi\|^{\frac{1}{2}}_{L^\infty_t L^2_\tau L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau) P_l \mathrm{tr}\chi\|^{\frac{1}{2}}_{L^\infty_t L^2_\tau L^2(P_{t,u})} \Big)^{\frac{1}{2}} \\ \lesssim\ & 2^{-\frac{j}{2}} \|\nabla\!\!\!/ P_l \mathrm{tr}\chi\|^{\frac{1}{2}}_{L^\infty_t L^2_{x'}} \Big(\|\Delta\!\!\!/ P_l \mathrm{tr}\chi\|_{L^\infty_u L^2(\mathcal{H}_u)} + 2^{\frac{j}{2}} \|\nabla\!\!\!/ P_l \mathrm{tr}\chi\|^{\frac{1}{2}}_{L^\infty_t L^2_{x'}} \|P_l \mathrm{tr}\chi\|^{\frac{1}{2}}_{L^\infty_t L^2_{x'}} \Big)^{\frac{1}{2}}, \end{aligned}$$

where we used in the last inequality the estimate (4.29) for K and a heat flow estimate for $U(\tau)\mathrm{tr}\chi$. Together with the finite band property for P_j and the assumption $l \leq j$, we obtain

$$\|P_j \nabla\!\!\!/ U(\cdot) P_l \mathrm{tr}\chi\|_{L^2_t L^2_\tau L^\infty(P_{t,u})} \lesssim 2^{-\frac{|l-j|}{4}} (2^l \|P_l \mathrm{tr}\chi\|_{L^\infty_t L^2_{x'}}). \qquad \text{(E.50)}$$

Finally, (E.46), (E.47) for $l > j$ and (E.50) for $l \leq j$ yield

$$\|\nabla\!\!\!/ U(\tau)\mathrm{tr}\chi\|_{L^2_t L^2_\tau L^\infty(P_{t,u})} \lesssim \sum_{j,l} 2^{-\frac{|l-j|}{4}} (2^l \|P_l \mathrm{tr}\chi\|_{L^\infty_t L^2_{x'}}) \lesssim \|\mathrm{tr}\chi\|_{\mathcal{B}^1}, \tag{E.51}$$

where the Besov space $\mathcal{B}^1$ has been defined in (5.5). Now, in view of the estimate (5.9), and the estimates (2.70) (5.7) for $\mathrm{tr}\chi$, we have

$$\|\mathrm{tr}\chi\|_{\mathcal{B}^1} \lesssim \|\mathrm{tr}\chi\|_{L^\infty_t L^2_{x'}} + \|\nabla\!\!\!/ \mathrm{tr}\chi\|_{\mathcal{B}^0} \lesssim \varepsilon.$$

Together with (E.51), this implies (E.41).

Next, we prove (E.40). Recall the Bochner identity for scalars on $P_{t,u}$ which is a 2-surface. For any scalar f on $P_{t,u}$, we have

$$\Delta\!\!\!/(|\nabla\!\!\!/ f|^2) = \nabla\!\!\!/(\Delta\!\!\!/ f) \cdot \nabla\!\!\!/ f + K|\nabla\!\!\!/ f|^2 + |\nabla\!\!\!/^2 f|^2.$$

Choosing $f = U(\tau)\mathrm{tr}\chi$, multiplying by $|\chi|^2$ and integrating over $P_{t,u}$ yields

$$\begin{aligned}
&\int_{P_{t,u}} |\chi|^2 \Delta\!\!\!/(|\nabla\!\!\!/ U(\tau)\mathrm{tr}\chi|^2) \\
=\ & \int_{P_{t,u}} |\chi|^2 \nabla\!\!\!/(\Delta\!\!\!/ U(\tau)\mathrm{tr}\chi) \cdot \nabla\!\!\!/ U(\tau)\mathrm{tr}\chi + \int_{P_{t,u}} K|\chi|^2 |\nabla\!\!\!/ U(\tau)\mathrm{tr}\chi|^2 + \int_{P_{t,u}} |\chi|^2 |\nabla\!\!\!/^2 U(\tau)\mathrm{tr}\chi|^2,
\end{aligned}$$

which implies after integration by parts

$$\begin{aligned}
&\|\chi\nabla\!\!\!/^2 U(\tau)\mathrm{tr}\chi\|^2_{L^2(P_{t,u})} \\
=\ & \|\chi\Delta\!\!\!/ U(\tau)\mathrm{tr}\chi\|^2_{L^2(P_{t,u})} - \int_{P_{t,u}} K|\chi|^2 |\nabla\!\!\!/ U(\tau)\mathrm{tr}\chi|^2 + 2\int_{P_{t,u}} \chi \cdot \nabla\!\!\!/\chi \Delta\!\!\!/ U(\tau)\mathrm{tr}\chi \cdot \nabla\!\!\!/ U(\tau)\mathrm{tr}\chi \\
& -2\int_{P_{t,u}} \chi \cdot \nabla\!\!\!/\chi \nabla\!\!\!/^2 U(\tau)\mathrm{tr}\chi \cdot \nabla\!\!\!/ U(\tau)\mathrm{tr}\chi.
\end{aligned}$$

We deduce

$$\begin{aligned}
\|\chi\nabla\!\!\!/^2 U(\tau)\mathrm{tr}\chi\|^2_{L^2(P_{t,u})} \lesssim\ & \|\chi\Delta\!\!\!/ U(\tau)\mathrm{tr}\chi\|^2_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})} \|\chi\|^2_{L^\infty_t L^4_{x'}} \|\nabla\!\!\!/ U(\tau)\mathrm{tr}\chi\|^2_{L^\infty(P_{t,u})} \\
& + \|\chi\Delta\!\!\!/ U(\tau)\mathrm{tr}\chi\|_{L^2(P_{t,u})} \|\nabla\!\!\!/\chi\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau)\mathrm{tr}\chi\|_{L^\infty(P_{t,u})} \\
& + \|\chi\nabla\!\!\!/^2 U(\tau)\mathrm{tr}\chi\|_{L^2(P_{t,u})} \|\nabla\!\!\!/\chi\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau)\mathrm{tr}\chi\|_{L^\infty(P_{t,u})},
\end{aligned}$$

which yields

$$\begin{aligned}
\|\chi\nabla\!\!\!/^2 U(\tau)\mathrm{tr}\chi\|_{L^2(P_{t,u})} \lesssim\ & \|\chi\Delta\!\!\!/ U(\tau)\mathrm{tr}\chi\|_{L^2(P_{t,u})} + \|K\|^{\frac{1}{2}}_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau)\mathrm{tr}\chi\|_{L^\infty(P_{t,u})} \\
& + \|\nabla\!\!\!/\chi\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau)\mathrm{tr}\chi\|_{L^\infty(P_{t,u})},
\end{aligned}$$

where we used in the last inequality the estimates (2.70) (2.71) for χ. Taking the $L^1_t L^2_\tau$ norm, we obtain

$$\begin{aligned}
& \|\chi\nabla\!\!\!/^2 U(\tau)\mathrm{tr}\chi\|_{L^1_t L^2_\tau L^2(P_{t,u})} \qquad\qquad (\text{E.52}) \\
\lesssim\ & \|\chi\Delta\!\!\!/ U(\tau)\mathrm{tr}\chi\|_{L^1_t L^2_\tau L^2(P_{t,u})} + \|K\|^{\frac{1}{2}}_{L^\infty_u L^2(\mathcal{H}_u)} \|\nabla\!\!\!/ U(\tau)\mathrm{tr}\chi\|_{L^2_t L^2_\tau L^\infty(P_{t,u})} \\
& + \|\nabla\!\!\!/\chi\|_{L^\infty_u L^2(\mathcal{H}_u)} \|\nabla\!\!\!/ U(\tau)\mathrm{tr}\chi\|_{L^2_t L^2_\tau L^\infty(P_{t,u})} \\
\lesssim\ & \|\chi\Delta\!\!\!/ U(\tau)\mathrm{tr}\chi\|_{L^1_t L^2_\tau L^2(P_{t,u})} + \varepsilon,
\end{aligned}$$

where we used in the last inequality the estimates (2.70) (2.71) for χ, the estimate (4.29) for K, and the estimate (E.41) for $\nabla\!\!\!/ U(\tau)\mathrm{tr}\chi$. Next, we estimate the right-hand side of (E.52). We multiply the heat equation satisfied by $U(\tau)\mathrm{tr}\chi$ by $|\chi|^2\Delta\!\!\!/ U(\tau)\mathrm{tr}\chi$ and we integrate over $P_{t,u}$. We obtain

$$\frac{1}{2}\frac{d}{d\tau}\|\chi\nabla\!\!\!/ U(\tau)\mathrm{tr}\chi\|^2_{L^2(P_{t,u})}+\|\chi\Delta\!\!\!/ U(\tau)\mathrm{tr}\chi\|^2_{L^2(P_{t,u})} = -2\int_{P_{t,u}}\chi\cdot\nabla\!\!\!/\chi\cdot\nabla\!\!\!/ U(\tau)\mathrm{tr}\chi\cdot\Delta\!\!\!/ U(\tau)\mathrm{tr}\chi d\mu_{t,u}.$$

This yields

$$\begin{aligned}&\|\chi\Delta\!\!\!/ U(\cdot)\mathrm{tr}\chi\|^2_{L^2_\tau L^2(P_{t,u})}\\ \lesssim\ &\|\chi\nabla\!\!\!/\mathrm{tr}\chi\|^2_{L^2(P_{t,u})}+\|\nabla\!\!\!/\chi\|_{L^2(P_{t,u})}\|\nabla\!\!\!/ U(\cdot)\mathrm{tr}\chi\|_{L^2_\tau L^\infty(P_{t,u})}\|\chi\Delta\!\!\!/ U(\cdot)\mathrm{tr}\chi\|_{L^2_\tau L^2(P_{t,u})}.\end{aligned}$$

We infer

$$\|\chi\Delta\!\!\!/ U(\cdot)\mathrm{tr}\chi\|_{L^2_\tau L^2(P_{t,u})}\ \lesssim\ \|\chi\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2(P_{t,u})}+\|\nabla\!\!\!/\chi\|_{L^2(P_{t,u})}\|\nabla\!\!\!/ U(\cdot)\mathrm{tr}\chi\|_{L^2_\tau L^\infty(P_{t,u})}$$

and hence, taking the L^1_t norm

$$\|\chi\Delta\!\!\!/ U(\cdot)\mathrm{tr}\chi\|_{L^1_t L^2_\tau L^2(P_{t,u})}\ \lesssim\ \varepsilon+\varepsilon\|\nabla\!\!\!/ U(\cdot)\mathrm{tr}\chi\|_{L^2_t L^2_\tau L^\infty(P_{t,u})} \tag{E.53}$$

where we used the estimates (2.70) (2.71) for χ which yield in particular

$$\begin{aligned}\|\chi\nabla\!\!\!/\mathrm{tr}\chi\|_{L^1_t L^2_{x'}}\ &\lesssim\ \|\chi\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2(\mathcal{H}_u)}\\ &\lesssim\ \|\chi\|_{L^\infty_{x'}L^2_t}\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2_{x'}L^\infty_t}\\ &\lesssim\ \varepsilon.\end{aligned}$$

Together with (E.41), we infer

$$\|\chi\Delta\!\!\!/ U(\cdot)\mathrm{tr}\chi\|_{L^1_t L^2_\tau L^2(P_{t,u})}\ \lesssim\ \varepsilon.$$

Together with (E.52), we finally obtain

$$\|\chi\nabla\!\!\!/^2 U(\tau)\mathrm{tr}\chi\|_{L^1_t L^2_\tau L^2(P_{t,u})}\lesssim\varepsilon.$$

Taking the supremum in u yields (E.40). This concludes the proof of the proposition.

E.5 Proof of Proposition 9.5

The proof of the estimate (9.7) being similar and slightly easier than the proof of (9.6), we focus on (9.6). In view of (E.1) (E.2), we have:

$$[bN,P_j]\mathrm{tr}\chi=\int_0^\infty m_j(\tau)V(\tau)d\tau, \tag{E.54}$$

where $V(\tau)$ satisfies:

$$(\partial_\tau-\Delta\!\!\!/)V(\tau)=[bN,\Delta\!\!\!/]U(\tau)\mathrm{tr}\chi,\ V(0)=0. \tag{E.55}$$

Assume that V satisfies for all $0 \leq \tau \leq 1$

$$\|V(\tau)\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon \tau^{\frac{1}{4}}, \tag{E.56}$$

and

$$\|\Lambda^{\frac{1}{2}} V(\tau)\|^2_{L^2(\mathcal{H}_u)} + \int_0^\tau \|\nabla\!\!\!/ \Lambda^{\frac{1}{2}} V(\tau')\|^2_{L^2(\mathcal{H}_u)} d\tau' \lesssim \varepsilon^2. \tag{E.57}$$

Then, first note in view of the interpolation inequality (3.22), that

$$\|\nabla\!\!\!/ V(\tau)\|_{L^2(P_{t,u})} \lesssim \|\Lambda^{\frac{1}{2}} V(\tau)\|^{\frac{1}{2}}_{L^2(P_{t,u})} \|\nabla\!\!\!/ \Lambda^{\frac{1}{2}} V(\tau)\|^{\frac{1}{2}}_{L^2(P_{t,u})}$$

which together with (E.57) implies

$$\|\nabla\!\!\!/ V(\cdot)\|_{L^4_\tau L^2(\mathcal{H}_u)} \lesssim \|\Lambda^{\frac{1}{2}} V(\cdot)\|^{\frac{1}{2}}_{L^\infty_\tau L^2(\mathcal{H}_u)} \|\nabla\!\!\!/ \Lambda^{\frac{1}{2}} V(\cdot)\|^{\frac{1}{2}}_{L^2_\tau L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{E.58}$$

Then, in view of (E.54), (E.56) and (E.58), we obtain for $j \geq 0$, using also that m is supported in $(0,1)$

$$\begin{aligned}
& 2^{\frac{j}{2}} \|[bN, P_j] \mathrm{tr}\chi\|_{L^2(\mathcal{H}_u)} + 2^{-\frac{j}{2}} \|\nabla\!\!\!/ [bN, P_j] \mathrm{tr}\chi\|_{L^2(\mathcal{H}_u)} \\
\lesssim\ & 2^{\frac{j}{2}} \int_0^\infty |m_j(\tau)| \|V(\tau)\|_{L^2(\mathcal{H}_u)} d\tau + 2^{-\frac{j}{2}} \int_0^\infty |m_j(\tau)| \|\nabla\!\!\!/ V(\tau)\|_{L^2(\mathcal{H}_u)} d\tau \\
=\ & 2^{\frac{j}{2}} \int_0^1 |m_j(\tau)| \|V(\tau)\|_{L^2(\mathcal{H}_u)} d\tau + 2^{-\frac{j}{2}} \int_0^1 |m_j(\tau)| \|\nabla\!\!\!/ V(\tau)\|_{L^2(\mathcal{H}_u)} d\tau \\
\lesssim\ & 2^{\frac{j}{2}} \varepsilon \left(\int_0^1 |m_j(\tau)| \tau^{\frac{1}{4}} d\tau \right) + 2^{-\frac{j}{2}} \varepsilon \left(\int_0^1 |m_j(\tau)|^{\frac{4}{3}} d\tau \right)^{\frac{3}{4}} \\
\lesssim\ & \varepsilon,
\end{aligned}$$

which after taking the supremum in u yields (9.6). Thus, it remains to prove (E.56) and (E.57).

We start with the proof of (E.56). The energy estimate (3.12) implies

$$\|V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(P_{t,u})} d\tau' \lesssim \int_0^\tau \int_{P_{t,u}} V(\tau') [bN, \Delta\!\!\!/] U(\tau') \mathrm{tr}\chi d\mu_{t,u}. \tag{E.59}$$

We need to estimate the commutator term $[bN, \Delta\!\!\!/] U$. Recall from (E.19) and (E.20) that we have

$$[bN, \Delta\!\!\!/] U = H \nabla\!\!\!/^2 U + G \nabla\!\!\!/ U \tag{E.60}$$

where the tensors H and G satisfy

$$\mathcal{N}_1(H) + \|G\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{E.61}$$

In view of (E.60), and integrating by parts the term $\nabla\!\!\!/^2 U$, we obtain:

$$\begin{aligned}
& \int_0^\tau \int_{P_{t,u}} V(\tau') [bN, \Delta\!\!\!/] U(\tau') \mathrm{tr}\chi d\mu_{t,u} d\tau' \\
\lesssim\ & \int_0^\tau \|H\|_{L^\infty_t L^4_{x'}} \|\nabla\!\!\!/ U(\tau') \mathrm{tr}\chi\|_{L^4(P_{t,u})} \|\nabla\!\!\!/ V(\tau')\|_{L^2(P_{t,u})} d\tau' \\
& + \int_0^\tau (\|\nabla\!\!\!/ H\|_{L^2(P_{t,u})} + \|G\|_{L^2(P_{t,u})}) \|\nabla\!\!\!/ U(\tau') \mathrm{tr}\chi\|_{L^4(P_{t,u})} \|V(\tau')\|_{L^4(P_{t,u})} d\tau'.
\end{aligned}$$

Together with (E.59) and the Gagliardo-Nirenberg inequality (3.3), as well as Gronwall's Lemma, this yields for $0 \leq \tau \leq 1$

$$\begin{aligned} & \|V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(P_{t,u})} d\tau' \qquad \text{(E.62)} \\ \lesssim\ & (\|H\|^2_{L^\infty_t L^4_{x'}} + \|\nabla\!\!\!/ H\|^2_{L^2(P_{t,u})} + \|G\|^2_{L^2(P_{t,u})}) \int_0^\tau \|\nabla\!\!\!/^2 U(\tau')\mathrm{tr}\chi\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau')\mathrm{tr}\chi\|_{L^2(P_{t,u})} d\tau' \\ \lesssim\ & (\varepsilon^2 + \|\nabla\!\!\!/ H\|^2_{L^2(P_{t,u})} + \|G\|^2_{L^2(P_{t,u})}) \int_0^\tau \|\Delta\!\!\!/ U(\tau')\mathrm{tr}\chi\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau')\mathrm{tr}\chi\|_{L^2(P_{t,u})} d\tau', \end{aligned}$$

where we used in the last inequality the estimate (E.61) and the Bochner inequality for scalars (4.34). Now, the heat flow estimate (3.9) yield:

$$\|\nabla\!\!\!/ U(\tau)\mathrm{tr}\chi\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\Delta\!\!\!/ U(\tau')\mathrm{tr}\chi\|^2_{L^2(P_{t,u})} d\tau' \lesssim \|\nabla\!\!\!/ \mathrm{tr}\chi\|^2_{L^\infty_t L^2_{x'}} \lesssim \varepsilon^2, \qquad \text{(E.63)}$$

where we used in the last inequality the estimate (2.70) for $\mathrm{tr}\chi$. Together with (E.62), we obtain

$$\|V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(P_{t,u})} d\tau' \lesssim \varepsilon^2 \tau^{\frac{1}{2}} (\varepsilon + \|\nabla\!\!\!/ H\|^2_{L^2(P_{t,u})} + \|G\|^2_{L^2(P_{t,u})}).$$

Integrating in time, and using the estimate (E.61) yields (E.56).

Next, we prove (E.57). The energy estimate (3.30) implies:

$$\begin{aligned} & \|\Lambda^{\frac{1}{2}} V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/ \Lambda^{\frac{1}{2}} V(\tau')\|^2_{L^2(P_{t,u})} d\tau' \\ =\ & \int_0^\tau \int_{P_{t,u}} \Lambda V(\tau')[bN, \Delta\!\!\!/] U(\tau')\mathrm{tr}\chi d\mu_{t,u} d\tau' \\ \lesssim\ & \int_0^\tau \|\Lambda^{\frac{3}{2}} V(\tau')\|_{L^2(P_{t,u})} \|\Lambda^{-\frac{1}{2}}([bN, \Delta\!\!\!/] U(\tau')\mathrm{tr}\chi)\|_{L^2(P_{t,u})} d\tau' \\ \lesssim\ & \int_0^\tau \|\nabla\!\!\!/ \Lambda^{\frac{1}{2}} V(\tau')\|_{L^2(P_{t,u})} \|[bN, \Delta\!\!\!/] U(\tau')\mathrm{tr}\chi\|_{L^{\frac{3}{2}}(P_{t,u})} d\tau', \end{aligned}$$

where we used (3.28) in the last inequality. This yields

$$\|\Lambda^{\frac{1}{2}} V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/ \Lambda^{\frac{1}{2}} V(\tau')\|^2_{L^2(P_{t,u})} d\tau' \lesssim \int_0^\tau \|[bN, \Delta\!\!\!/] U(\tau')\mathrm{tr}\chi\|^2_{L^{\frac{3}{2}}(P_{t,u})} d\tau'. \qquad \text{(E.64)}$$

In view of (E.60), we have

$$\begin{aligned} & \|[bN, \Delta\!\!\!/] U(\tau')\mathrm{tr}\chi\|_{L^{\frac{3}{2}}(P_{t,u})} \\ \lesssim\ & \|H\|_{L^6(P_{t,u})} \|\nabla\!\!\!/^2 U(\tau')\mathrm{tr}\chi\|_{L^2(P_{t,u})} + \|G\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau')\mathrm{tr}\chi\|_{L^6(P_{t,u})} \\ \lesssim\ & \|H\|_{L^6(P_{t,u})} \|\nabla\!\!\!/^2 U(\tau')\mathrm{tr}\chi\|_{L^2(P_{t,u})} + \|G\|_{L^2(P_{t,u})} \|\nabla\!\!\!/^2 U(\tau')\mathrm{tr}\chi\|^{\frac{2}{3}}_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau')\mathrm{tr}\chi\|^{\frac{1}{3}}_{L^2(P_{t,u})} \\ \lesssim\ & \|H\|_{L^6(P_{t,u})} \|\Delta\!\!\!/ U(\tau')\mathrm{tr}\chi\|_{L^2(P_{t,u})} + \|G\|_{L^2(P_{t,u})} \|\Delta\!\!\!/ U(\tau')\mathrm{tr}\chi\|^{\frac{2}{3}}_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau')\mathrm{tr}\chi\|^{\frac{1}{3}}_{L^2(P_{t,u})}, \end{aligned}$$

where we used the Gagliardo-Nirenberg inequality (3.3) and the Bochner inequality for scalars (4.34). Taking the L^2_τ norm and using (E.63) implies

$$\|[bN, \not\Delta]U(\tau')\mathrm{tr}\chi\|_{L^2_\tau L^{\frac{3}{2}}(P_{t,u})} \lesssim \varepsilon(\|H\|_{L^6(P_{t,u})} + \|G\|_{L^2(P_{t,u})}).$$

Now, taking the L^2_t norm and using (E.61) yields

$$\|[bN, \not\Delta]U(\tau')\mathrm{tr}\chi\|_{L^2_t L^2_\tau L^{\frac{3}{2}}(P_{t,u})} \lesssim \varepsilon. \tag{E.65}$$

Finally, integrating (E.64) in t, and injecting (E.65), we obtain (E.57). This concludes the proof of the proposition.

E.6 Proof of Proposition 9.6

We have:

$$[\not\nabla, P_j]\mathrm{tr}\chi = \int_0^\infty m_j(\tau)V(\tau)d\tau, \tag{E.66}$$

where $V(\tau)$ satisfies:

$$(\partial_\tau - \not\Delta)V(\tau) = [\not\nabla, \not\Delta]U(\tau)\mathrm{tr}\chi,\ V(0) = 0. \tag{E.67}$$

Assume that V satisfies for all τ

$$\|\Lambda^{\frac{3}{4}}V(\tau)\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{E.68}$$

Then, using the Bernstein inequality for P_j, the properties of Λ and the finite band property for P_j, we have

$$\begin{aligned}
\|V(\tau)\|_{L^2_t L^4_{x'}} &\lesssim \sum_j \|P_j V(\tau)\|_{L^2_t L^4_{x'}} \\
&\lesssim \sum_j 2^{\frac{j}{2}} \|P_j V(\tau)\|_{L^2(\mathcal{H}_u)} \\
&\lesssim \sum_j 2^{\frac{j}{2}} \|\Lambda^{-\frac{3}{4}} P_j\|_{\mathcal{L}(L^2(P_{t,u}))} \|\Lambda^{\frac{3}{4}} V(\tau)\|_{L^2(\mathcal{H}_u)} \\
&\lesssim \sum_j 2^{\frac{j}{2}} \|P_j\|^{\frac{1}{4}}_{\mathcal{L}(L^2(P_{t,u}))} \|\Lambda^{-1} P_j\|^{\frac{3}{4}}_{\mathcal{L}(L^2(P_{t,u}))} \|\Lambda^{\frac{3}{4}} V(\tau)\|_{L^2(\mathcal{H}_u)} \\
&\lesssim \left(\sum_j 2^{-\frac{j}{4}}\right) \|\Lambda^{\frac{3}{4}} V(\tau)\|_{L^2(\mathcal{H}_u)} \\
&\lesssim \varepsilon,
\end{aligned}$$

where we used in the last inequality (E.68). Together with (E.66), we obtain

$$\|[\not\nabla, P_j]\mathrm{tr}\chi\|_{L^2_t L^4_{x'}} \lesssim \int_0^\infty |m_j(\tau)| \|V(\tau)\|_{L^2_t L^4_{x'}} d\tau \lesssim \varepsilon,$$

which is the desired estimate (9.8). Thus, it remains to prove (E.68).

The energy estimate (3.30) implies:

$$\begin{aligned}
&\|\Lambda^{\frac{3}{4}}V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/\Lambda^{\frac{3}{4}}V(\tau')\|^2_{L^2(P_{t,u})}d\tau' \\
= &\int_0^\tau \int_{P_{t,u}} \Lambda^{\frac{3}{2}}V(\tau')[\nabla\!\!\!/, \Delta\!\!\!/]U(\tau')\mathrm{tr}\chi d\mu_{t,u}d\tau' \\
\lesssim &\int_0^\tau \|\Lambda^{1+\frac{3}{4}}V(\tau')\|_{L^2(P_{t,u})}\|\Lambda^{-\frac{1}{4}}([\nabla\!\!\!/, \Delta\!\!\!/]U(\tau')\mathrm{tr}\chi)\|_{L^2(P_{t,u})}d\tau' \\
\lesssim &\int_0^\tau \|\nabla\!\!\!/\Lambda^{\frac{3}{4}}V(\tau')\|_{L^2(P_{t,u})}\|[\nabla\!\!\!/, \Delta\!\!\!/]U(\tau')\mathrm{tr}\chi\|_{L^{\frac{5}{3}}(P_{t,u})}d\tau',
\end{aligned}$$

where we used (3.28) in the last inequality. This yields

$$\|\Lambda^{\frac{3}{4}}V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/\Lambda^{\frac{3}{4}}V(\tau')\|^2_{L^2(P_{t,u})}d\tau' \lesssim \int_0^\tau \|[\nabla\!\!\!/, \Delta\!\!\!/]U(\tau')\mathrm{tr}\chi\|^2_{L^{\frac{5}{3}}(P_{t,u})}d\tau'. \quad \text{(E.69)}$$

Now, in view of the commutator formula (B.87), we have

$$\begin{aligned}
\|[\nabla\!\!\!/, \Delta\!\!\!/]U(\tau')\mathrm{tr}\chi\|_{L^{\frac{5}{3}}(P_{t,u})} &\lesssim \|K\nabla\!\!\!/U(\tau')\mathrm{tr}\chi\|_{L^{\frac{5}{3}}(P_{t,u})} \\
&\lesssim \|K\|_{L^2(P_{t,u})}\|\nabla\!\!\!/U(\tau')\mathrm{tr}\chi\|_{L^{10}(P_{t,u})} \\
&\lesssim \|K\|_{L^2(P_{t,u})}\|\nabla\!\!\!/^2U(\tau')\mathrm{tr}\chi\|^{\frac{4}{5}}_{L^2(P_{t,u})}\|\nabla\!\!\!/U(\tau')\mathrm{tr}\chi\|^{\frac{1}{5}}_{L^2(P_{t,u})} \\
&\lesssim \|K\|_{L^2(P_{t,u})}\|\Delta\!\!\!/U(\tau')\mathrm{tr}\chi\|^{\frac{4}{5}}_{L^2(P_{t,u})}\|\nabla\!\!\!/U(\tau')\mathrm{tr}\chi\|^{\frac{1}{5}}_{L^2(P_{t,u})},
\end{aligned}$$

where we used the Gagliardo-Nirenberg inequality (3.3) and the Bochner inequality for scalars (4.34). Taking the L^2_τ norm and using (E.63) implies

$$\|[\nabla\!\!\!/, \Delta\!\!\!/]U(\tau')\mathrm{tr}\chi\|_{L^2_\tau L^{\frac{5}{3}}(P_{t,u})} \lesssim \varepsilon\|K\|_{L^2(P_{t,u})}.$$

Now, taking the L^2_t norm and using the estimate (4.29) for K yields

$$\|[\nabla\!\!\!/, \Delta\!\!\!/]U(\tau')\mathrm{tr}\chi\|_{L^2_tL^2_\tau L^{\frac{5}{3}}(P_{t,u})} \lesssim \varepsilon. \quad \text{(E.70)}$$

Finally, integrating (E.69) in t, and injecting (E.70), we obtain (E.68). This concludes the proof of the proposition.

E.7 Proof of Lemma 9.7

In this section, we will use the fact that the function m in Lemma 3.8 can be constructed such that

$$\int_0^\infty |m_{\geq j}(\tau)|d\tau + \int_0^\infty |m_{<j}(\tau)|d\tau < +\infty \quad \text{(E.71)}$$

for any j. Indeed, following the construction in Lemma 5.4 of [10], we may choose m such that

$$m(\tau) = \psi'(\tau), \quad \psi(\tau) = \phi(\tau) - \phi(4\tau), \quad \phi(\tau) = 1 \text{ for } \tau \leq 1, \quad \phi(\tau) = 0 \text{ for } \tau \geq 4,$$

where ϕ is a smooth function. Then, we have

$$m_j(\tau) = \frac{d}{d\tau}(\psi(2^{2j}\tau)),$$

and since we have for $0 < \tau < +\infty$

$$\sum_{l\geq j} \psi(2^{2l}\tau) = \phi(2^{2j}\tau), \quad \sum_{l<j} \psi(2^{2l}\tau) = -\phi(2^{2j}\tau),$$

we infer on $(0, +\infty)$

$$m_{\geq j}(\tau) = 2^{2j}\phi'(2^{2j}\tau), \quad m_{<j}(\tau) = -2^{2j}\phi'(2^{2j}\tau).$$

Hence, we have

$$\int_0^{+\infty} |m_{\geq j}(\tau)|d\tau = \int_0^{+\infty} |m_{<j}(\tau)|d\tau = \int_0^{+\infty} |\phi'(\tau)|d\tau < +\infty$$

since ϕ' is compactly supported in $(0, +\infty)$. This shows that this m satisfies indeed (E.71).

We are now ready to start the proof of Lemma 9.7. We have:

$$[P_{>j}, P_{\leq j}(h)]F = \int_0^{\infty} m_{>j}(\tau)V(\tau)d\tau, \tag{E.72}$$

where $V(\tau)$ satisfies:

$$(\partial_\tau - \not{\Delta})V(\tau) = \not{\Delta}P_{\leq j}(h)U(\tau)F + 2\not{\nabla}P_{\leq j}(h) \cdot \not{\nabla}U(\tau)F, \quad V(0) = 0. \tag{E.73}$$

Assume that V satisfies for all τ

$$\|V(\tau)\|_{L^2(P_{t,u})} \lesssim (1 + 2^j\sqrt{\tau} + 2^{\frac{3j}{2}}\tau^{\frac{3}{4}})\|\not{\nabla}h\|_{L^2(P_{t,u})}\|F\|_{L^2(P_{t,u})}. \tag{E.74}$$

Then, (E.72) and (E.74) imply in view of (E.71)

$$\begin{aligned}
\|[P_{>j}, P_{\leq j}(h)]F\|_{L^2(P_{t,u})} &\lesssim \int_0^{\infty} |m_{>j}(\tau)|\|V(\tau)\|_{L^2(P_{t,u})}d\tau \\
&\lesssim \left(\int_0^{\infty} |m_{>j}(\tau)|(1 + 2^j\sqrt{\tau} + 2^{\frac{3j}{2}}\tau^{\frac{3}{4}})d\tau\right)\|\not{\nabla}h\|_{L^2(P_{t,u})}\|F\|_{L^2(P_{t,u})} \\
&\lesssim \|\not{\nabla}h\|_{L^2(P_{t,u})}\|F\|_{L^2(P_{t,u})}
\end{aligned}$$

which is the desired estimate (9.11). Thus, it remains to prove (E.74).

The energy estimate (3.12) implies

$$\begin{aligned}
&\|V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^{\tau} \|\not{\nabla}V(\tau')\|^2_{L^2(P_{t,u})}d\tau' \\
\lesssim\ & \int_0^{\tau}\int_{P_{t,u}} V(\tau')\Big(\not{\Delta}P_{\leq j}(h)U(\tau)F + \not{\nabla}P_{\leq j}(h) \cdot \not{\nabla}U(\tau)F\Big)d\mu_{t,u} \\
\lesssim\ & \int_0^{\tau} \|\not{\Delta}P_{\leq j}(h)\|_{L^4(P_{t,u})}\|U(\tau)F\|_{L^4(P_{t,u})}\|V\|_{L^2(P_{t,u})} + \|\not{\nabla}P_{\leq j}(h)\|_{L^4(P_{t,u})}\|\not{\nabla}U(\tau)F\|_{L^2(P_{t,u})}\|V\|_{L^4(P_{t,u})} \\
\lesssim\ & \int_0^{\tau} \|\not{\nabla}\not{\Delta}P_{\leq j}(h)\|^{\frac{1}{2}}_{L^2(P_{t,u})}\|\not{\Delta}P_{\leq j}(h)\|^{\frac{1}{2}}_{L^2(P_{t,u})}\|\not{\nabla}U(\tau)F\|^{\frac{1}{2}}_{L^2(P_{t,u})}\|U(\tau)F\|^{\frac{1}{2}}_{L^2(P_{t,u})}\|V\|_{L^2(P_{t,u})} \\
&+ \int_0^{\tau} \|\not{\nabla}^2P_{\leq j}(h)\|^{\frac{1}{2}}_{L^2(P_{t,u})}\|\not{\nabla}P_{\leq j}(h)\|^{\frac{1}{2}}_{L^2(P_{t,u})}\|\not{\nabla}U(\tau)F\|_{L^2(P_{t,u})}\|\not{\nabla}V\|^{\frac{1}{2}}_{L^2(P_{t,u})}\|V\|^{\frac{1}{2}}_{L^2(P_{t,u})}
\end{aligned}$$

where we used in the last inequality the Gagliardo-Nirenberg inequality (3.3). Together with the Bochner inequality for scalars (4.34) and the finite band property for P_j, we obtain

$$\begin{aligned}
&\|V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(P_{t,u})} d\tau' \\
\lesssim\ & 2^{\frac{3j}{2}} \int_0^\tau \|\nabla\!\!\!/ h\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau)F\|^{\frac12}_{L^2(P_{t,u})} \|U(\tau)F\|^{\frac12}_{L^2(P_{t,u})} \|V\|_{L^2(P_{t,u})} \\
&+ \int_0^\tau \|\Delta\!\!\!/ P_{\le j}(h)\|^{\frac12}_{L^2(P_{t,u})} \|\nabla\!\!\!/ h\|^{\frac12}_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau)F\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ V\|^{\frac12}_{L^2(P_{t,u})} \|V\|^{\frac12}_{L^2(P_{t,u})} \\
\lesssim\ & 2^{\frac{3j}{2}} \int_0^\tau \|\nabla\!\!\!/ h\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau)F\|^{\frac12}_{L^2(P_{t,u})} \|U(\tau)F\|^{\frac12}_{L^2(P_{t,u})} \|V\|_{L^2(P_{t,u})} \\
&+ 2^{\frac{j}{2}} \int_0^\tau \|\nabla\!\!\!/ h\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau)F\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ V\|^{\frac12}_{L^2(P_{t,u})} \|V\|^{\frac12}_{L^2(P_{t,u})} \\
\lesssim\ & 2^{\frac{3j}{2}} \int_0^\tau \|\nabla\!\!\!/ h\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau)F\|^{\frac12}_{L^2(P_{t,u})} \|U(\tau)F\|^{\frac12}_{L^2(P_{t,u})} \|V\|_{L^2(P_{t,u})} \\
&+ 2^{j} \int_0^\tau \|\nabla\!\!\!/ h\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau)F\|_{L^2(P_{t,u})} \|V\|_{L^2(P_{t,u})} \\
&+ \int_0^\tau \|\nabla\!\!\!/ h\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau)F\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ V\|_{L^2(P_{t,u})}
\end{aligned}$$

This yields

$$\begin{aligned}
&\|V(\tau)\|^2_{L^2(P_{t,u})} \\
\lesssim\ & \|\nabla\!\!\!/ h\|^2_{L^2(P_{t,u})} \int_0^\tau \|\nabla\!\!\!/ U(\tau)F\|^2_{L^2(P_{t,u})} \\
&+ \int_0^\tau \|\nabla\!\!\!/ h\|_{L^2(P_{t,u})} \Big(2^{\frac{3j}{2}} \|\nabla\!\!\!/ U(\tau)F\|^{\frac12}_{L^2(P_{t,u})} \|U(\tau)F\|^{\frac12}_{L^2(P_{t,u})} + 2^j \|\nabla\!\!\!/ U(\tau)F\|_{L^2(P_{t,u})} \Big) \|V\|_{L^2(P_{t,u})}
\end{aligned}$$

which together with the heat flow estimate (3.8) and the fact that $U(0)F = F$ implies

$$\begin{aligned}
&\|V(\tau)\|^2_{L^2(P_{t,u})} \\
\lesssim\ & \|\nabla\!\!\!/ h\|^2_{L^2(P_{t,u})} \|F\|^2_{L^2(P_{t,u})} \\
&+ \int_0^\tau \|\nabla\!\!\!/ h\|_{L^2(P_{t,u})} \Big(2^{\frac{3j}{2}} \|\nabla\!\!\!/ U(\tau)F\|^{\frac12}_{L^2(P_{t,u})} \|U(\tau)F\|^{\frac12}_{L^2(P_{t,u})} + 2^j \|\nabla\!\!\!/ U(\tau)F\|_{L^2(P_{t,u})} \Big) \|V\|_{L^2(P_{t,u})}.
\end{aligned}$$

Integrating this differential inequality, we obtain

$$\begin{aligned}
\|V(\tau)\|^2_{L^2(P_{t,u})} \lesssim\ & \|\nabla\!\!\!/ h\|^2_{L^2(P_{t,u})} \|F\|^2_{L^2(P_{t,u})} \\
&+ 2^{2j} \|\nabla\!\!\!/ h\|^2_{L^2(P_{t,u})} \bigg(\int_0^\tau \Big(2^{\frac{j}{2}} \|\nabla\!\!\!/ U(\tau)F\|^{\frac12}_{L^2(P_{t,u})} \|U(\tau)F\|^{\frac12}_{L^2(P_{t,u})} + \|\nabla\!\!\!/ U(\tau)F\|_{L^2(P_{t,u})} \Big) \bigg)^2 \\
\lesssim\ & \|\nabla\!\!\!/ h\|^2_{L^2(P_{t,u})} \|F\|^2_{L^2(P_{t,u})} \\
&+ 2^{2j} \|\nabla\!\!\!/ h\|^2_{L^2(P_{t,u})} \tau \bigg(\int_0^\tau \Big(2^{j} \|\nabla\!\!\!/ U(\tau)F\|_{L^2(P_{t,u})} \|U(\tau)F\|_{L^2(P_{t,u})} + \|\nabla\!\!\!/ U(\tau)F\|^2_{L^2(P_{t,u})} \Big) \bigg).
\end{aligned} \tag{E.75}$$

Now, the heat flow estimate (3.8) and the fact that $U(0) = F$ implies

$$\begin{aligned}
&\int_0^\tau \Big(2^j \|\nabla\!\!\!/ U(\tau)F\|_{L^2(P_{t,u})} \|U(\tau)F\|_{L^2(P_{t,u})} + \|\nabla\!\!\!/ U(\tau)F\|^2_{L^2(P_{t,u})}\Big) \\
\lesssim\ & 2^j \sqrt{\tau} \sup_\tau \|U(\tau)\|_{L^2(P_{t,u})} \left(\int_0^\tau \|\nabla\!\!\!/ U(\tau)F\|^2_{L^2(P_{t,u})}\right) + \|F\|^2_{L^2(P_{t,u})} \\
\lesssim\ & (1 + 2^j\sqrt{\tau})\|F\|^2_{L^2(P_{t,u})}
\end{aligned}$$

which together with (E.75) yields the desired estimate (E.74). This concludes the proof of the lemma.

E.8 Proof of Lemma 9.8

Let $V(\tau)$ defined in (E.73). Assume that V satisfies for all τ

$$\|\nabla\!\!\!/ V(\tau)\|_{L^2(P_{t,u})} \lesssim 2^j((1 + 2^{\frac{j}{2}}\tau^{\frac{1}{4}})\|\nabla\!\!\!/ h\|_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})}\|h\|_{L^2(P_{t,u})})\|F\|_{L^2(P_{t,u})}. \quad \text{(E.76)}$$

Then, (E.72) and (E.76) imply

$$\begin{aligned}
&\|\nabla\!\!\!/ [P_{>j}, P_{\le j}(h)]F\|_{L^2(P_{t,u})} \\
\lesssim\ & \int_0^\infty m_{>j}(\tau)\|\nabla\!\!\!/ V(\tau)\|_{L^2(P_{t,u})} d\tau \\
\lesssim\ & 2^j \left(\int_0^\infty m_{>j}(\tau)((1 + 2^{\frac{j}{2}}\tau^{\frac{1}{4}})\|\nabla\!\!\!/ h\|_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})}\|h\|_{L^2(P_{t,u})})d\tau\right)\|F\|_{L^2(P_{t,u})} \\
\lesssim\ & 2^j(\|\nabla\!\!\!/ h\|_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})}\|h\|_{L^2(P_{t,u})})\|F\|_{L^2(P_{t,u})}
\end{aligned}$$

which is the desired estimate (9.12). Thus, it remains to prove (E.76).

The energy estimate (3.11) implies

$$\begin{aligned}
&\|\nabla\!\!\!/ V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\Delta\!\!\!/ V(\tau')\|^2_{L^2(P_{t,u})} d\tau' \\
\lesssim\ & \int_0^\tau \int_{P_{t,u}} \Delta\!\!\!/ V(\tau')\Big(\Delta\!\!\!/ P_{\le j}(h)U(\tau)F + \nabla\!\!\!/ P_{\le j}(h) \cdot \nabla\!\!\!/ U(\tau)F\Big) d\mu_{t,u} d\tau.
\end{aligned}$$

This yields

$$\begin{aligned}
&\|\nabla\!\!\!/ V(\tau)\|^2_{L^2(P_{t,u})} \\
\lesssim\ & \int_0^\tau \Big(\|\Delta\!\!\!/ P_{\le j}(h)\|^2_{L^4(P_{t,u})}\|U(\tau)F\|^2_{L^4(P_{t,u})} + \|\nabla\!\!\!/ P_{\le j}(h)\|^2_{L^\infty(P_{t,u})}\|\nabla\!\!\!/ U(\tau)F\|^2_{L^2(P_{t,u})}\Big) \\
\lesssim\ & \int_0^\tau \Big(\|\nabla\!\!\!/\Delta\!\!\!/ P_{\le j}(h)\|_{L^2(P_{t,u})}\|\Delta\!\!\!/ P_{\le j}(h)\|_{L^2(P_{t,u})}\|\nabla\!\!\!/ U(\tau)F\|_{L^2(P_{t,u})}\|U(\tau)F\|_{L^2(P_{t,u})} \\
&+\|\nabla\!\!\!/ P_{\le j}(h)\|^2_{L^\infty(P_{t,u})}\|\nabla\!\!\!/ U(\tau)F\|^2_{L^2(P_{t,u})}\Big)
\end{aligned}$$

where we used in the last inequality the Gagliardo-Nirenberg inequality (3.3). Together with the Bochner inequality for scalars (4.34) and the finite band property for P_j, we

obtain

$$\begin{aligned}\|\not\nabla V(\tau)\|^2_{L^2(P_{t,u})} &\lesssim \int_0^\tau \Big(2^{3j}\|\not\nabla h\|^2_{L^2(P_{t,u})}\|\not\nabla U(\tau)F\|_{L^2(P_{t,u})}\|U(\tau)F\|_{L^2(P_{t,u})} \\ &\quad +\|\not\nabla P_{\leq j}(h)\|^2_{L^\infty(P_{t,u})}\|\not\nabla U(\tau)F\|^2_{L^2(P_{t,u})}\Big).\end{aligned}$$

Together with the heat flow estimate (3.8) and the fact that $U(0)F = F$, this yields

$$\begin{aligned}\|\not\nabla V(\tau)\|^2_{L^2(P_{t,u})} &\lesssim 2^{3j}\|\not\nabla h\|^2_{L^2(P_{t,u})}\sqrt{\tau}\sup_\tau\|U(\tau)F\|_{L^2(P_{t,u})}\left(\int_0^\tau\|\not\nabla U(\tau)F\|^2_{L^2(P_{t,u})}\right)^{\frac{1}{2}} \\ &\quad +\|\not\nabla P_{\leq j}(h)\|^2_{L^\infty(P_{t,u})}\|F\|^2_{L^2(P_{t,u})} \\ &\lesssim \Big(2^{3j}\sqrt{\tau}\|\not\nabla h\|^2_{L^2(P_{t,u})} + \|\not\nabla P_{\leq j}(h)\|^2_{L^\infty(P_{t,u})}\Big)\|F\|^2_{L^2(P_{t,u})}. \qquad \text{(E.77)}\end{aligned}$$

Now, using (9.15) with the choice $f = P_{\leq j}(h)$ yields

$$\begin{aligned}\|\not\nabla P_{\leq j}(h)\|^2_{L^\infty(P_{t,u})} &\lesssim \|\not\Delta P_{\leq j}(h)\|_{L^2(P_{t,u})} + \|\not\nabla\not\Delta P_{\leq j}(h)\|^{\frac{1}{2}}_{L^2(P_{t,u})}\|\not\nabla P_{\leq j}(h)\|^{\frac{1}{2}}_{L^2(P_{t,u})} \\ &\quad +\|K\|_{L^2(P_{t,u})}\|\not\nabla P_{\leq j}(h)\|_{L^2(P_{t,u})} \\ &\lesssim 2^j\Big(\|\not\nabla h\|_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})}\|h\|_{L^2(P_{t,u})}\Big),\end{aligned}$$

where we used in the last inequality the finite band property for P_j. Together with (E.77), this yields the desired estimate (E.76). This concludes the proof of the lemma.

E.9 Proof of Lemma 9.9

We have:

$$[\not\nabla, P_{\leq j}]h = \int_0^\infty m_{\leq j}(\tau)V(\tau)d\tau, \qquad \text{(E.78)}$$

where $V(\tau)$ satisfies:

$$(\partial_\tau - \not\Delta)V(\tau) = [\not\nabla, \not\Delta]U(\tau)h,\ V(0) = 0. \qquad \text{(E.79)}$$

Assume that V satisfies for all τ and for all $a > 0$

$$\|V(\tau)\|_{L^2(P_{t,u})} \lesssim \|K\|_{L^2(P_{t,u})}(\|K\|_{L^2(P_{t,u})}\|h\|_{L^2(P_{t,u})} + \|\Lambda^a h\|_{L^2(P_{t,u})}). \qquad \text{(E.80)}$$

Then, in view of (E.78) and (E.71), we obtain for all $a > 0$

$$\begin{aligned}\|[P_{\leq j}, \not\nabla]h\|_{L^2(P_{t,u})} &\lesssim \int_0^{+\infty}|m_{\leq j}(\tau)|\|V(\tau)\|_{L^2(P_{t,u})}d\tau \\ &\lesssim \left(\int_0^{+\infty}|m_{\leq j}(\tau)|d\tau\right)\|K\|_{L^2(P_{t,u})}(\|K\|_{L^2(P_{t,u})}\|h\|_{L^2(P_{t,u})} + \|\Lambda^a h\|_{L^2(P_{t,u})}) \\ &\lesssim \|K\|_{L^2(P_{t,u})}(\|K\|_{L^2(P_{t,u})}\|h\|_{L^2(P_{t,u})} + \|\Lambda^a h\|_{L^2(P_{t,u})})\end{aligned}$$

which is the desired estimate (9.13). Thus, it remains to prove (E.80).

The energy estimate (3.12), together with the commutator formula (B.87), implies

$$\begin{aligned}
&\|V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(P_{t,u})} d\tau' \\
\lesssim\ & \int_0^\tau \|V(\tau')\|_{L^2(P_{t,u})} \|K\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau')h\|_{L^\infty(P_{t,u})} d\tau'.
\end{aligned}$$

Integrating this differential inequality, we obtain

$$\begin{aligned}
\|V(\tau)\|^2_{L^2(P_{t,u})} + \int_0^\tau \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(P_{t,u})} &\lesssim \|K\|^2_{L^2(P_{t,u})} \left(\int_0^\tau \|\nabla\!\!\!/ U(\tau')h\|_{L^\infty(P_{t,u})} d\tau' \right)^2 \qquad \text{(E.81)} \\
&\lesssim \|K\|^2_{L^2(P_{t,u})} \int_0^\tau \tau'^{1-\delta} \|\nabla\!\!\!/ U(\tau')h\|^2_{L^\infty(P_{t,u})} d\tau',
\end{aligned}$$

where $0 < \delta < 1$ will be chosen later. In view of the estimate (9.15), we have

$$\begin{aligned}
&\int_0^\tau \tau'^{1-\delta} \|\nabla\!\!\!/ U(\tau')h\|^2_{L^\infty(P_{t,u})} d\tau' \qquad \text{(E.82)} \\
\lesssim\ & \int_0^\tau \tau'^{1-\delta} (\|\Delta\!\!\!/ U(\tau')h\|^2_{L^2(P_{t,u})} + \|\nabla\!\!\!/ \Delta\!\!\!/ U(\tau')h\|_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau')h\|_{L^2(P_{t,u})} \\
&+ \|K\|^2_{L^2(P_{t,u})} \|\nabla\!\!\!/ U(\tau')h\|^2_{L^2(P_{t,u})}) d\tau' \\
\lesssim\ & \int_0^\tau \tau'^{1-\delta} \|\Delta\!\!\!/ U(\tau')h\|^2_{L^2(P_{t,u})} d\tau' + \int_0^\tau \tau'^{2-2\delta} \|\nabla\!\!\!/ \Delta\!\!\!/ U(\tau')h\|^2_{L^2(P_{t,u})} d\tau' \\
&+ (1 + \|K\|^2_{L^2(P_{t,u})}) \int_0^\tau \|\nabla\!\!\!/ U(\tau')h\|^2_{L^2(P_{t,u})} d\tau' \\
\lesssim\ & \int_0^\tau \tau'^{1-\delta} \|\Delta\!\!\!/ U(\tau')h\|^2_{L^2(P_{t,u})} d\tau' + \int_0^\tau \tau'^{2-2\delta} \|\Delta\!\!\!/^{\frac{3}{2}} U(\tau')h\|^2_{L^2(P_{t,u})} d\tau' \\
&+ (1 + \|K\|^2_{L^2(P_{t,u})}) \|h\|^2_{L^2(P_{t,u})},
\end{aligned}$$

where we used in the last inequality the heat flow estimate (3.8).

Next, we estimate the two first terms in the right-hand side of (E.82). We have

$$\begin{aligned}
&\left(\int_0^\tau \tau'^{1-\delta} \|\Delta\!\!\!/ U(\tau')h\|^2_{L^2(P_{t,u})} d\tau' \right)^{\frac{1}{2}} \\
\lesssim\ & \sum_{j \geq 0} \left(\int_0^\tau \tau'^{1-\delta} \|\Delta\!\!\!/ P_j U(\tau')h\|^2_{L^2(P_{t,u})} d\tau' \right)^{\frac{1}{2}} \\
\lesssim\ & \sum_{j \geq 0} \left(\int_0^\tau \tau' \|\Delta\!\!\!/ P_j U(\tau')h\|^2_{L^2(P_{t,u})} d\tau' \right)^{\frac{1-\delta}{2}} \left(\int_0^\tau \|\Delta\!\!\!/ P_j U(\tau')h\|^2_{L^2(P_{t,u})} d\tau' \right)^{\frac{\delta}{2}} \\
\lesssim\ & \sum_{j \geq 0} \|P_j h\|^{1-\delta}_{L^2(P_{t,u})} \|\nabla\!\!\!/ P_j h\|^\delta_{L^2(P_{t,u})},
\end{aligned}$$

where we used in the last inequality the heat flow estimates (3.9) and (3.10). Together

with the finite band property for P_j, we obtain

$$\begin{aligned}\left(\int_0^\tau \tau'^{1-\delta}\|\not{\Delta}U(\tau')h\|^2_{L^2(P_{t,u})}d\tau'\right)^{\frac{1}{2}} &\lesssim \sum_{j\geq 0}2^{\delta j}\|P_jh\|_{L^2(P_{t,u})} \\ &\lesssim \left(\sum_{j\geq 0}2^{-\delta j}\right)\|\Lambda^{2\delta}h\|_{L^2(P_{t,u})} \\ &\lesssim \|\Lambda^{2\delta}h\|_{L^2(P_{t,u})}.\end{aligned} \tag{E.83}$$

Also, we have

$$\begin{aligned}&\left(\int_0^\tau \tau'^{2-2\delta}\|\not{\Delta}^{\frac{3}{2}}U(\tau')h\|^2_{L^2(P_{t,u})}d\tau'\right)^{\frac{1}{2}} \\ \lesssim\ &\sum_{j\geq 0}\left(\int_0^\tau \tau'^{2-2\delta}\|\not{\Delta}^{\frac{3}{2}}P_jU(\tau')h\|^2_{L^2(P_{t,u})}d\tau'\right)^{\frac{1}{2}} \\ \lesssim\ &\sum_{j\geq 0}\left(\int_0^\tau (\tau')^2\|\not{\nabla}\not{\Delta}P_jU(\tau')h\|^2_{L^2(P_{t,u})}d\tau'\right)^{\frac{1-\delta}{2}}\left(\int_0^\tau \|\not{\nabla}\not{\Delta}P_jU(\tau')h\|^2_{L^2(P_{t,u})}d\tau'\right)^{\frac{\delta}{2}} \\ \lesssim\ &\sum_{j\geq 0}\|P_jh\|^{1-\delta}_{L^2(P_{t,u})}\|\not{\Delta}P_jh\|^{\delta}_{L^2(P_{t,u})},\end{aligned}$$

where we used in the last inequality heat flow estimates. Together with the finite band property for P_j, we obtain

$$\begin{aligned}\left(\int_0^\tau \tau'^{2-2\delta}\|\not{\Delta}^{\frac{3}{2}}U(\tau')h\|^2_{L^2(P_{t,u})}d\tau'\right)^{\frac{1}{2}} &\lesssim \sum_{j\geq 0}2^{2\delta j}\|P_jh\|_{L^2(P_{t,u})} \\ &\lesssim \left(\sum_{j\geq 0}2^{-\delta j}\right)\|\Lambda^{3\delta}h\|_{L^2(P_{t,u})} \\ &\lesssim \|\Lambda^{3\delta}h\|_{L^2(P_{t,u})}.\end{aligned} \tag{E.84}$$

Finally, (E.82), (E.83) and (E.84) imply for all $0<\delta<1$

$$\int_0^\tau \tau'^{1-\delta}\|\not{\nabla}U(\tau')h\|^2_{L^\infty(P_{t,u})}d\tau' \lesssim \|\Lambda^{3\delta}h\|^2_{L^2(P_{t,u})}+\|K\|^2_{L^2(P_{t,u})}\|h\|^2_{L^2(P_{t,u})}. \tag{E.85}$$

Injecting (E.85) in (E.81), we obtain

$$\|V(\tau)\|_{L^2(P_{t,u})} \lesssim \|K\|_{L^2(P_{t,u})}(\|K\|_{L^2(P_{t,u})}\|h\|_{L^2(P_{t,u})}+\|\Lambda^{3\delta}h\|_{L^2(P_{t,u})}). \tag{E.86}$$

Choosing $\delta=\frac{a}{3}$ in (E.86) yields the desired estimate (E.80). This concludes the proof of the lemma.

E.10 Proof of Lemma 9.10

We have:

$$[\nabla\!\!\!/, P_j]h = \int_0^\infty m_j(\tau)V(\tau)d\tau, \tag{E.87}$$

where $V(\tau)$ satisfies:

$$(\partial_\tau - \Delta\!\!\!/)V(\tau) = [\nabla\!\!\!/, \Delta\!\!\!/]U(\tau)h,\ V(0) = 0.$$

Assume that V satisfies for all $a > 0$

$$\left(\int_0^{+\infty} \|\nabla\!\!\!/ V(\tau)\|^2_{L^2(P_{t,u})}d\tau\right)^{\frac{1}{2}} \lesssim \|K\|_{L^2(P_{t,u})}(\|K\|_{L^2(P_{t,u})}\|h\|_{L^2(P_{t,u})} + \|\Lambda^a h\|_{L^2(P_{t,u})}). \tag{E.88}$$

Then, in view of (E.87), we obtain for all $a > 0$

$$\begin{aligned}
\|\nabla\!\!\!/[P_j, \nabla\!\!\!/]h\|_{L^2(P_{t,u})} &\lesssim \int_0^{+\infty} |m_j(\tau)|\|\nabla\!\!\!/ V(\tau)\|_{L^2(P_{t,u})}d\tau \\
&\lesssim \left(\int_0^{+\infty} m_j(\tau)^2 d\tau\right)^{\frac{1}{2}} \|K\|_{L^2(P_{t,u})}(\|K\|_{L^2(P_{t,u})}\|h\|_{L^2(P_{t,u})} + \|\Lambda^a h\|_{L^2(P_{t,u})}) \\
&\lesssim 2^j\|K\|_{L^2(P_{t,u})}(\|K\|_{L^2(P_{t,u})}\|h\|_{L^2(P_{t,u})} + \|\Lambda^a h\|_{L^2(P_{t,u})})
\end{aligned}$$

which is the desired estimate (9.14). Thus, it remains to prove (E.88).

Injecting (E.85) in (E.81), we obtain

$$\int_0^{+\infty} \|\nabla\!\!\!/ V(\tau)\|^2_{L^2(P_{t,u})}d\tau \lesssim \|K\|^2_{L^2(P_{t,u})}(\|K\|^2_{L^2(P_{t,u})}\|h\|^2_{L^2(P_{t,u})} + \|\Lambda^{3\delta} h\|^2_{L^2(P_{t,u})}). \tag{E.89}$$

Choosing $\delta = \frac{a}{3}$ in (E.89) yields the desired estimate (E.88). This concludes the proof of the lemma.

E.11 Proof of Lemma 9.11

We have in view of (3.38)

$$\|\nabla\!\!\!/ f\|_{L^\infty(P_{t,u})} \lesssim \|\nabla\!\!\!/^3 f\|^{\frac{1}{2}}_{L^2(P_{t,u})}\|\nabla\!\!\!/ f\|^{\frac{1}{2}}_{L^2(P_{t,u})} + \|\nabla\!\!\!/^2 f\|_{L^2(P_{t,u})}. \tag{E.90}$$

Now, using the Bochner inequality for tensors (3.7), we have

$$\begin{aligned}
\|\nabla\!\!\!/^3 f\|_{L^2(P_{t,u})} &\lesssim \|\Delta\!\!\!/\nabla\!\!\!/ f\|_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})}\|\nabla\!\!\!/^2 f\|_{L^2(P_{t,u})} + \|K\|^2_{L^2(P_{t,u})}\|\nabla\!\!\!/ f\|_{L^2(P_{t,u})} \\
&\lesssim \|\nabla\!\!\!/\Delta\!\!\!/ f\|_{L^2(P_{t,u})} + \|[\nabla\!\!\!/, \Delta\!\!\!/]f\|_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})}\|\nabla\!\!\!/^2 f\|_{L^2(P_{t,u})} \\
&\quad + \|K\|^2_{L^2(P_{t,u})}\|\nabla\!\!\!/ f\|_{L^2(P_{t,u})}.
\end{aligned}$$

In view of the commutator formula (B.87), we obtain

$$\begin{aligned}
\|\nabla\!\!\!/^3 f\|_{L^2(P_{t,u})} &\lesssim \|\nabla\!\!\!/\Delta\!\!\!/ f\|_{L^2(P_{t,u})} + \|K\nabla\!\!\!/ f\|_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})}\|\nabla\!\!\!/^2 f\|_{L^2(P_{t,u})} \\
&\quad + \|K\|^2_{L^2(P_{t,u})}\|\nabla\!\!\!/ f\|_{L^2(P_{t,u})} \\
&\lesssim \|\nabla\!\!\!/\Delta\!\!\!/ f\|_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})}\|\nabla\!\!\!/ f\|_{L^\infty(P_{t,u})} + \|K\|_{L^2(P_{t,u})}\|\nabla\!\!\!/^2 f\|_{L^2(P_{t,u})} \\
&\quad + \|K\|^2_{L^2(P_{t,u})}\|\nabla\!\!\!/ f\|_{L^2(P_{t,u})}
\end{aligned}$$

which together with (E.90) yields

$$\begin{aligned}\|\nabla\!\!\!/ f\|_{L^\infty(P_{t,u})} &\lesssim \|\nabla\!\!\!/ \Delta\!\!\!/ f\|^{\frac{1}{2}}_{L^2(P_{t,u})}\|\nabla\!\!\!/ f\|^{\frac{1}{2}}_{L^2(P_{t,u})} + \|K\|^{\frac{1}{2}}_{L^2(P_{t,u})}\|\nabla\!\!\!/ f\|^{\frac{1}{2}}_{L^\infty(P_{t,u})}\|\nabla\!\!\!/ f\|^{\frac{1}{2}}_{L^2(P_{t,u})}\\ &\quad +\|\nabla\!\!\!/^2 f\|_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})}\|\nabla\!\!\!/ f\|_{L^2(P_{t,u})}.\end{aligned}$$

We deduce

$$\|\nabla\!\!\!/ f\|_{L^\infty(P_{t,u})} \lesssim \|\nabla\!\!\!/ \Delta\!\!\!/ f\|^{\frac{1}{2}}_{L^2(P_{t,u})}\|\nabla\!\!\!/ f\|^{\frac{1}{2}}_{L^2(P_{t,u})} + \|\nabla\!\!\!/^2 f\|_{L^2(P_{t,u})} + \|K\|_{L^2(P_{t,u})}\|\nabla\!\!\!/ f\|_{L^2(P_{t,u})},$$

which together with the Bochner inequality for scalars (4.34) yields (9.15). This concludes the proof of the lemma.

Index

References

[1] Michael T. Anderson. Cheeger-Gromov theory and applications to general relativity. In *The Einstein equations and the large scale behavior of gravitational fields*, pages 347–377. Birkhäuser, Basel, 2004.

[2] Hajer Bahouri and Jean-Yves Chemin. Équations d'ondes quasilinéaires et effet dispersif. *Internat. Math. Res. Notices*, (21):1141–1178, 1999.

[3] Hajer Bahouri and Jean-Yves Chemin. Équations d'ondes quasilinéaires et estimations de Strichartz. *Amer. J. Math.*, 121(6):1337–1377, 1999.

[4] Demetrios Christodoulou and Sergiu Klainerman. *The global nonlinear stability of the Minkowski space*, volume 41 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993.

[5] Justin Corvino. Scalar curvature deformation and a gluing construction for the Einstein constraint equations. *Comm. Math. Phys.*, 214(1):137–189, 2000.

[6] Justin Corvino and Richard M. Schoen. On the asymptotics for the vacuum Einstein constraint equations. *J. Differential Geom.*, 73(2):185–217, 2006.

[7] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

[8] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time*. Cambridge University Press, London, 1973. Cambridge Monographs on Mathematical Physics, No. 1.

[9] S. Klainerman and I. Rodnianski. Improved local well-posedness for quasilinear wave equations in dimension three. *Duke Math. J.*, 117(1):1–124, 2003.

[10] S. Klainerman and I. Rodnianski. A geometric approach to the Littlewood-Paley theory. *Geom. Funct. Anal.*, 16(1):126–163, 2006.

[11] S. Klainerman and I. Rodnianski. Sharp trace theorems for null hypersurfaces on Einstein metrics with finite curvature flux. *Geom. Funct. Anal.*, 16(1):164–229, 2006.

[12] Sergiu Klainerman. PDE as a unified subject. *Geom. Funct. Anal.*, (Special Volume, Part I):279–315, 2000. GAFA 2000 (Tel Aviv, 1999).

[13] Sergiu Klainerman and Igor Rodnianski. Bilinear estimates on curved space-times. *J. Hyperbolic Differ. Equ.*, 2(2):279–291, 2005.

[14] Sergiu Klainerman and Igor Rodnianski. Causal geometry of Einstein-vacuum space-times with finite curvature flux. *Invent. Math.*, 159(3):437–529, 2005.

[15] Sergiu Klainerman and Igor Rodnianski. Rough solutions of the Einstein-vacuum equations. *Ann. of Math. (2)*, 161(3):1143–1193, 2005.

[16] Sergiu Klainerman and Igor Rodnianski. The causal structure of microlocalized rough Einstein metrics. *Ann. of Math. (2)*, 161(3):1195–1243, 2005.

[17] Sergiu Klainerman, Igor Rodnianski, and Jérémie Szeftel. The bounded L^2 curvature conjecture. Invent. Math. 202 (2015) 91-216.

[18] Peter Petersen. Convergence theorems in Riemannian geometry. In *Comparison geometry (Berkeley, CA, 1993–94)*, volume 30 of *Math. Sci. Res. Inst. Publ.*, pages 167–202. Cambridge Univ. Press, Cambridge, 1997.

[19] Hart F. Smith and Daniel Tataru. Sharp local well-posedness results for the nonlinear wave equation. *Ann. of Math. (2)*, 162(1):291–366, 2005.

[20] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[21] Jérémie Szeftel. Parametrix for wave equations on a rough background I: Regularity of the phase at initial time. *arXiv:1204.1768, 145 p*, 2012.

[22] Jérémie Szeftel. Parametrix for wave equations on a rough background II: Construction and control at initial time. *arXiv:1204.1769, 84 p*, 2012.

[23] Jérémie Szeftel. Parametrix for wave equations on a rough background III: Spacetime regularity of the phase. *arXiv:1204.1770, 276 p*, 2012.

[24] Jérémie Szeftel. Parametrix for wave equations on a rough background IV: Control of the error term. *arXiv:1204.1771, 284 p*, 2012.

[25] Jérémie Szeftel. Sharp Strichartz estimates for the wave equation on a rough background. Annales Scientifiques de l'Ecole Normale Suprieure 49 (6), 1279-1309, 2016.

[26] Daniel Tataru. Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation. *Amer. J. Math.*, 122(2):349–376, 2000.

[27] Daniel Tataru. Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients. III. *J. Amer. Math. Soc.*, 15(2):419–442 (electronic), 2002.

[28] Qian Wang. On the geometry of null cones in Einstein Vacuum Spacetimes, Ann.Inst. H. Poincaré Anal. Non Linéaire, 26 (2009), no. 1,285–328.

[29] Qian Wang. On Ricci coefficients of null hypersurfaces with time foliation in Einstein vacuum space-time: Part I , Calculus of Variations and Partial Differential Equations, March 2013, Volume 46, Issue 3-4, pp 461-503.