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Complex Modeling: Insights into our body through computer tomography – perspectives of a project day on inverse problems

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How does an autonomous vehicle recognize a traffic sign? How can seismic measurements be used to determine the origin of an earthquake? How do you get a three-dimensional tomographic reconstruction of a body part from a medical scanner? The methods used to answer all these questions come from the field of so-called inverse problems and are central to our society. They are omnipresent in technology, science and in everyday life and include mathematics without any people being aware. The process of mathematical modeling is used to solve inverse problems. Complex problems are simplified in such a way that mathematical models are developed and then accurate solutions can be calculated. The question now arises as to whether high school students are able to independently and accurately solve these kind of questions while only using their mathematical school knowledge and the computer. An approach to show how this could be done is given in this paper.

Keywords: High school mathematics, inverse problems, mathematical modeling, MATLAB, problem-solving.

Introduction

When taking a look at maths classes in school you often find students asking: “Why do I need to study this?” Schoolbooks are very method-oriented which has the effect that students can use certain algorithms and formulas but do not recognize the usefulness of mathematics for everyday life. This is why the development of interactive, problem-oriented material for high school students should be increased. Using this kind of material as an addition to the existing units students see the potential of mathematics to solve real life problems¹.

The project presented here will be part of the CAMMP (Computational and mathematical modeling program) project of the Karlsruhe Institute of Technology² and the RWTH in Aachen³. For several years, similar projects have been carried out in one-week and one-day formats with students of different ages in Aachen (One example can be found in Greefrath & Siller, 2018, p. 137-163). Until now over 400 students have participated in a modeling week and over 2900 in a modeling day. Since the beginning of 2018 the project is also being realized in Karlsruhe.

CAMMP wants to offer ways to innovative mathematics teaching in terms of content, didactics and methodology. Concepts for this were developed and tested. CAMMP is not only involved in

¹ We understand the word problems in the sence of Heinrich, Bruder and Bauer (2015).

² <http://www.scc.kit.edu/forschung/CAMMP>

³ <https://blog.rwth-aachen.de/cammp/>

mathematical modelling projects with pupils and teachers, but also in research and teaching in the field of mathematical modelling didactics. The goal is a didactics of problem-oriented mathematical research, in particular a creative handling of mathematics. Tasks and projects are always as problem-oriented as possible and actively dealt with by the students. Adapted to the given framework conditions (e.g. modeling week, -day, regular school lessons), innovative concepts are developed to balance the dimensions active-passive and method-oriented⁴ problem-oriented.

These projects are needed because a change in teaching mathematics must take place. Many graduated students do not know a lot about mathematical modeling even though it is one of the process-related competencies of the core curriculum. And according to the educational standards of the Conference of Education Ministers, all students should be able to model mathematically by the end of their school career. That means they should be able to go through the elementary steps of the mathematical modeling cycle independently (Kultusministerkonferenz der Länder in der Bundesrepublik, 2012). Real life is not about calculating as much as possible to exercise a certain algorithm (Maaß, 2015, p. 70). Mathematics should not be used for its own sake, but rather as a tool for solving problems (Consortium for Mathematics and Its Applications, 2016). It is clear that the necessary competence for the creation of models can only be built up if the pupils carry out their own modeling independently and regularly during their school years (Maaß, 2011, p. 8). However, modeling is new and unknown to many teachers and learners. Each step of the modeling cycle represents a cognitive hurdle for the students (Heiliö & Pohjolainen 2016). On the one hand they find it difficult to understand the real situation, on the other they expect teachers to evaluate the correctness of their solution, neglecting the validation step. If teachers face each of these challenges, studies have shown that the treatment of authentic modelling has a significant positive impact on their opinion of mathematics (Borromeo Ferri, Greefrath & Kaiser (Hrsg.), 2013, p. 30-34).

The project presented in this paper makes an important contribution to developing interactive, problem-oriented material by presenting a way to include computer tomography as an example of inverse problems into maths classes. It shows that the abilities of students should not be underestimated. The basics needed to understand inverse problems are already taught during school mathematics.

Inverse problems in school and everyday life

When having a look at cause-effect relationships one can either calculate the effect with given causes or turn the problem around and draw conclusions from given effects on the cause. If you have a look at image processing you can ask yourself what a clear image would look like if the camera were focused incorrectly. When turning the problem around you get the inverse problem, which means trying to deblur an image that was taken by an incorrectly focused camera. In both cases you have to solve a system of equations, but in mathematics there is a difference between the

⁴ By method-oriented we mean the application and practice of mathematical methods to solve purely mathematical problems without reference to application. In the problem-oriented treatment of tasks, certain mathematical methods are only applied when they are needed to solve a real extra-mathematical problem (Heinrich, Bruder & Bauer. 2015).

equations you use for the two problems. In real life you never have flawless data, which means that by using some kind of measuring tool you get errors in your data. If the error stays in the same magnitude while computing the solution, everything is perfectly fine. However, what happens if a tiny error in the given data leads to enormous errors in the solution?

These problems are so called ill-posed inverse problems. Jaques Hadamard described three conditions through which one can determine whether the problem at hand is well or ill-posed (Engl, Hanke & Neubauer, 2000, p. 31): There must be at least one solution (existence). There may be no more than one solution (uniqueness). The solution must depend continuously on the data (stability).

At the moment schools mainly deal with tasks in which direct problems occur. Data is presented and a function must be applied to it. This means that you start with the cause and calculate the consequence (Lasanen, 2014). This type of task only requires students to perform prescribed operations. Students are therefore often limited in their mathematical thinking (Gardiner, 2016). In the long term, however, the inverse problems are important, since through them students learn to work more freely, flexibly, playfully and application-oriented with the mathematics they have learned (Gardiner, 2016). From the teacher's point of view it is more challenging to discuss inverse problems in school because the approaches of students to inverse problems are far more open and are therefore not well predictable. These open tasks make it more difficult to have a joint discussion in class since there are many different calculations that lead to a correct solution. On top inverse problems are a lot more complex to grade in an exam. Eventually, due to the lack of discussion, confrontation and exercise, students achieve results a lot worse when having to solve inverse problems than when dealing with direct problems (Gardiner, 2016). Thus to prepare their students and give them the opportunity to think freely in terms of mathematics, teachers should include more inverse problems into their classes especially ones that are similar to the one presented in the following.

Perspectives of a project day about computer tomography

The aim of the project day for one is to show the working principle of computer tomography, but moreover it should also show the relevance of mathematics and simulation sciences in society in general. The students should realize that already with their knowledge they can understand and compute the basic methods behind complex processes. The material is formulated in a way that different approaches are possible and welcome. By using the software MATLAB the requirements regarding programming are set very low. The students are given a prepared code in the form of a gap text in which they only have to enter missing formulas or equations, which are then checked as they run the code. This means that students receive feedback on their considerations at all times. On top it is possible to include worksheets into MATLAB, so that the surface looks rather like a digital working sheet with code lines in between than a programming software. Through all of this, one hope of the developer is that students become more involved with mathematics.

Computer tomography

For computer tomography, the internal structure of an object is determined depending on the entry and exit intensity of the X-ray radiation and conclusively depicted (Grumme, Kluge, Lange, Meese & Ringel, 1988). The objects are irradiated with parallel beams at different fixed angle settings to

determine the absorption coefficients $f(x_1, x_2)$ of the different materials of the object (Mueller & Siltanen, 2012). Depending on the material the intensity of the incoming X-ray beam decreases differently. Thus, the out coming intensity depends on the absorption property $f(x_1, x_2)$ but also on the layer thickness z of the material through which the X-ray beam passes. If a single layer is penetrated by a beam with the intensity I_0 , the output intensity is calculated by the following equation:

$$I(z) = I_0 \cdot \exp(-f(x_1, x_2) \cdot z).$$

With the help of the a transformation, the parallel beams passing through the object are described mathematically as straight lines. By doing so, the function that was originally dependent on the Cartesian coordinates x_1 and x_2 is transformed so that the function is now described along a straight line with the angle θ and the distance to the origin s (see Figure 1). The points lying on the beams are described with $\vec{x} \cdot \vec{\theta} = s$, with $\vec{\theta} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$.

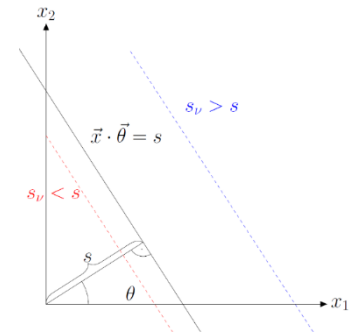


Figure 1: Illustration of the radon transformation

If you now look at different distances s_v from the line to the origin, the straight line moves to the right for $s_v > s$ (see Figure 1, blue line) and to the left for $s_v < s$ (see Figure 1, red line). This is of great importance since an object gets scanned by parallel beams. So the mathematical model also has to include several beams with a constant distance between one another. In addition, the X-ray tube is rotated around the object so that the beams are sent through the object at different angles. Due to the symmetry, it is sufficient to only look at the angles between 0° and 180° and therefore only to run a semicircle. Considering both factors one gets $N \cdot J$ beams and therefore $N \cdot J$ equations in total when having N parallel beams and J angle positions.

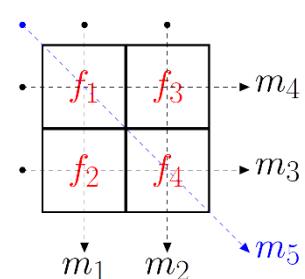


Figure 2: Five beams through four pixels

Assumptions and simplifications

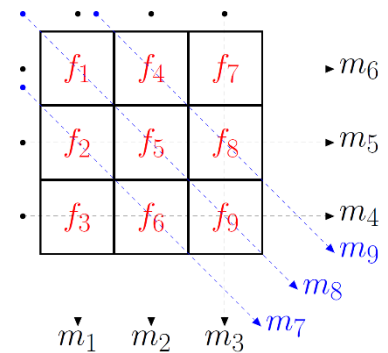
Before the students establish a system of equations and have a look at its solubility and the reconstructed inner structure of the scanned object, simplifications must be made. Strictly speaking, an object, like a human body, consists of an infinite number of different absorption coefficients. For a first model however, it firstly is sufficient to approximate the cross-section of an object as a square and secondly to allow only a small number of different absorption coefficients. The idea is to cut the square into smaller squares called pixels and to assume that each pixel only consists of one particular material and therefore only has one absorption coefficient f_i , see Figure 2. In addition, the absorption law is simplified. Instead of looking at the influences of the absorption coefficients, where the integral would have to be used, the object is described by gray values and these are weighted and added along one beam. Thus, the measured values, depicted as m_i with $i = 1, \dots, 5$ in Figure 2, do not represent intensities but simply weighted and added gray values f_i , $i = 1, \dots, 4$, which must be correctly distributed to the respective pixels for the reconstruction of the object.

Worksheet 1: First radiography of a simplified object

With the first worksheet the students get familiar with what actually happens during computer tomography. The students start into the task, where they try to reconstruct the gray values of the pixels for the first time. Because of the simplifications and assumptions that were made beforehand the students have a first impression of how the gray values of the pixels can be reconstructed, namely simply to solve a linear system of equations. So the students are given four values for m_i with $i = 1, \dots, 4$. By doing so a lot of questions arise: Can numerical values for the pixels be found so that the row and column sums are satisfied? If there are such entries, how are they determined and can they be determined in a unique way? If they do not exist, why not?

With the help of these questions, the students can independently work out the conditions of the concepts of existence and uniqueness and work out connections for solubility. They notice that with these values they do not have enough information to uniquely determine the gray values for the reconstruction of the object even though the students have four equations and four variables. The solution is not unique because the equations are not linearly independent. This first approximation to the mathematical description of the problem is done without formulas but instead using examples of possible gray values that the pixels could have with the given values m_i . Students should find a gray scale distribution and then either by comparing their own solution with other groups of students or by realizing mathematical considerations recognize that their distribution is not a unique solution. After this realization, the students are faced with the challenge of extending the problem in such a way that the gray values of the pixels can be explicitly solved. To do this, they must transmit a further beam through the object and determine the measured values of these new beams. For example it is possible to add a diagonal beam (see Figure 2), which leads to the system of equations (1) which has a unique solution.

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \sqrt{2} & 0 & 0 & \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \quad (1)$$



When you increase the number of pixels you keep having the same problem. Looking at the object subdivided into nine pixels (see Figure 3) it is again not enough to send nine beams through the object. The problem why this system of equations also is not uniquely solvable is that you only have eight linearly independent equations. So even more beams have to be used. Since the system of equations keeps growing it makes sense to write an algorithm that describes the beams mathematically and creates the matrix one needs to solve the system of equations. At this point of the project the algorithm can be developed, as the students have now completely penetrated the problem. In addition, the number of equations is still manageable, which also means that drawings of the object and the beams can be made without being confusing. This way errors and new solutions can be found more easily.

Figure 3: Nine beams through nine pixels

Worksheet 2: Mathematical description of the beams

The main goal of this worksheet is to find a mathematical model to describe the beams more generally for any number of parallel beams and angle settings, since so far the students only had the situation shown in Figure 3. The idea is to describe the beams as straight lines in \mathbb{R}^2 and then come up with an algorithm that calculates the length that a beam spends in each pixel. The last important step then is to not only get the individual lengths but to know where in the matrix the value has to be written.

For this task the students get the starting vectors $(X_{1,s} \ X_{2,s})^T$ of the points where the beams enter the object including the value of the associated angle θ for a given number of parallel beams and different angles. The standardized direction of the beam is described through the vector $\vec{r} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$. Therefore the beam itself can be described as $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_{1,s} \\ X_{2,s} \end{pmatrix} + s \cdot \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$. Students know this way of writing straight line equations from school. Nevertheless, help cards are also available to the students as support, since they may have difficulties, for example, with the use of the cosine and sine.

The points of interest are always those where either the x_1 or x_2 coordinates are equal to one of the pixel borders, namely $-1.5, -0.5, 0.5$ or 1.5 (see Figure 3). Thus to determine the interceptions the equation for the beam needs to be separated coordinate wise and solved for $s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$:

$$s_1 = \frac{X_1 - X_{1,s}}{\cos(\theta)}, \quad s_2 = \frac{X_2 - X_{2,s}}{\sin(\theta)}.$$

Again at this point the students have the possibility of using help cards, since they have to think of separating the coordinates and solving the equations. Some students have difficulties with deformation calculation steps. As feedback and an output of MATLAB the students get a depiction of their description of the beams through the object, so that they themselves can see whether the result seems to be consistent.

Once the students achieved this task they can easily calculate the length between two neighboring interceptions which is the length a beam spends in a pixel. The hard part is to figure out in which column and row of the matrix the length has to be written. One approach is to calculate the distance between the center of the pixels and the middle of the considered distance (see Figure 4). The center to which the distance is smallest is the pixel in which the beam spends the particular length. A second approach would be to have a look at the x_1 or x_2 coordinates of both interception points. After having numbered the pixels and defined the possible values for the x_1 and x_2 one can check to which both points belong and therefore determine the pixel number. While defining the pixels you have to include the border of the pixels each time.

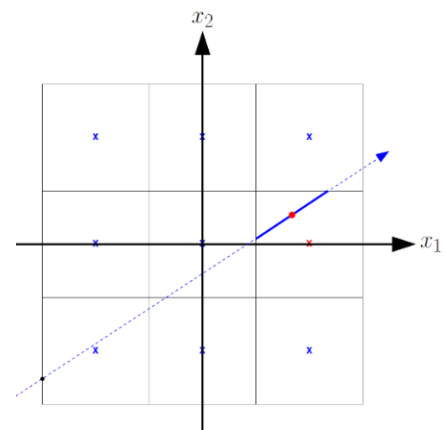


Figure 4: Illustration of the pixel number determination

Worksheet 3: Reconstruction of an object

After this mathematical model for the beams the students get the chance to reconstruct an image such as the first one in Figure 5. To get this first reconstruction the students have to find the gray values f_i so that the difference between $A \cdot \vec{f}$ and the measured values get minimized. To make it a little less complicated for the students they calculate with a small matrix A and small vectors \vec{f} and \vec{m} :

$$h(\vec{f}) = \|A \cdot \vec{f} - \vec{m}\|^2 = (A_{1,1} \cdot f_1 + A_{1,2} \cdot f_2 - m_1)^2 + (A_{2,1} \cdot f_1 + A_{2,2} \cdot f_2 - m_2)^2$$

From school they know that they have to differentiate $h(\vec{f})$ after f_1 and f_2 which leads the students to the gradient of $h(\vec{f})$. They then have to set the gradient to zero to find the \vec{f} that minimizes the equation:

$$\nabla_f h(\vec{f}) = 2 \cdot A^T \cdot A \cdot \vec{f} - 2A^T \cdot \vec{m} = 0.$$

The formula that results for the gray values that is, when multiplied with A , closest to the measured data, is the following:

$$\vec{f} = (A^T \cdot A)^{-1} \cdot A^T \cdot \vec{m}$$

If one uses data without errors the reconstruction is perfect. However, using flawless data does not depict reality. Every measuring device creates some kind of error so that you have to use data with a small error. Besides that in reality the scanned objects consist of more than nine different materials and therefore more than nine different absorption coefficients or in our case gray values. So on top of adding noise one has to increase the number of rows and columns in the object, which also leads to a higher number of parallel beams and more angles. If one adds 5 % noise one can already no longer recognize the original object (see Figure 5, picture 3). This is due to the fact that the problem is not stable since the matrices are singular and therefore badly conditioned. Thus the problem has to be slightly modified, so that one gets a more stable problem. This can be achieved by different mathematical methods which deliver results that come very close to the original picture (see Figure 5, picture 4 and 5).

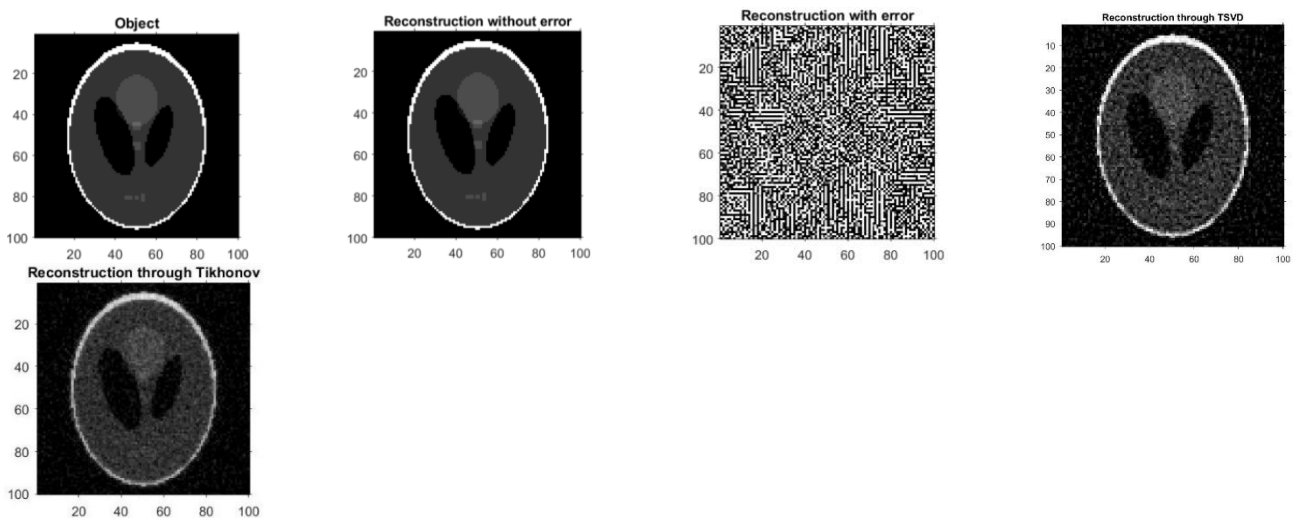


Figure 5: The object and different forms of reconstructions with 5 % noise

The aim is that the students should be guided as little as possible during the project. It would be optimal if they could independently develop a method to improve the reconstruction. At the moment this seems very hard to achieve but will be improved once the project has been realized with students.

Outlook

In the future, the existing material will be realized with high school students. Furthermore the workshop will be evaluated in order to assess what the students learned and how the material could be changed. The material will then be iteratively improved. Moreover material will be developed so that the implementation of the project can be adapted to school hours and thus be carried out in units. In addition, a draft of the described problem for an exam task will be developed.

In conclusion this project shows a possibility to treat authentic and relevant problems with the help of mathematical modeling in such a way that students are not only able to understand them but can also solve these. This should show that it is also possible to treat modelling competence in such open questions at school. The students' knowledge of mathematics is sufficient to understand the basics of these problems from their everyday life. This example should encourage teachers to address more complex problems in class in order to train and further develop their students' mathematical modeling skills. For example, parts of this workshop can be incorporated into the classroom, allowing students to directly see an authentic application of mathematics.

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