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RECOVERING THE NORMAL FORM OF AN ELASTICITY TENSOR

S. ABRAMIAN, B. DESMORAT, R. DESMORAT, B. KOLEV, AND M. OLIVE

ABSTRACT. We propose an effective geometrical approach to recover the normal form of a given Elasticity tensor, once we know its symmetry class. In other words, we produce a rotation which brings an Elasticity tensor onto its normal form, given its components in any orthonormal frame, and this for any tensor of any symmetry class. Our methodology relies on the use of specific covariants and on the geometric characterization of each symmetry class using these covariants.

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1. INTRODUCTION

The linear elastic properties of a given material are encoded into an Elasticity tensor \mathbf{E} , a fourth-order tensor which relates linearly the stress tensor to the strain tensor. As it was clearly emphasized by Boehler and coworkers [7], any rotated Elasticity tensor encodes the same material properties (in a different orientation). One shall say that the rotated tensor and initial one are in the same orbit.

The elastic materials are classified by their eight symmetry classes [15] (isotropic, transversely-isotropic, cubic, trigonal, tetragonal, orthotropic, monoclinic, triclinic). Any non triclinic Elasticity tensor has a *normal form*. An orthonormal frame in which the matrix representation of this tensor belongs to such a normal form is called a *proper or natural basis* for \mathbf{E} [14]. For instance, consider a cubic Elasticity tensor which is given in an arbitrary frame by its Voigt

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representation as

$$(1.1) \quad [\mathbf{E}] = \begin{pmatrix} E_{1111} & E_{1122} & E_{1133} & E_{1123} & E_{1113} & E_{1112} \\ E_{2211} & E_{2222} & E_{2233} & E_{2223} & E_{2213} & E_{2212} \\ E_{3311} & E_{3322} & E_{3333} & E_{3323} & E_{3313} & E_{3312} \\ E_{2311} & E_{2322} & E_{2333} & E_{2323} & E_{2313} & E_{2312} \\ E_{1311} & E_{1322} & E_{1333} & E_{1323} & E_{1313} & E_{1312} \\ E_{1211} & E_{1222} & E_{1233} & E_{1223} & E_{1213} & E_{1212} \end{pmatrix}$$

Then, there exists a rotation g such that the rotated Elasticity tensor, denoted by $g \star \mathbf{E}$ and where

$$(g \star \mathbf{E})_{ijkl} = g_{ip}g_{jq}g_{kr}g_{ls}E_{pqrs},$$

has the following Voigt representation

$$(1.2) \quad [g \star \mathbf{E}] = \begin{pmatrix} (g \star \mathbf{E})_{1111} & (g \star \mathbf{E})_{1112} & (g \star \mathbf{E})_{1112} & 0 & 0 & 0 \\ (g \star \mathbf{E})_{1112} & (g \star \mathbf{E})_{1111} & (g \star \mathbf{E})_{1112} & 0 & 0 & 0 \\ (g \star \mathbf{E})_{1112} & (g \star \mathbf{E})_{1112} & (g \star \mathbf{E})_{1111} & 0 & 0 & 0 \\ 0 & 0 & 0 & (g \star \mathbf{E})_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & (g \star \mathbf{E})_{1212} & 0 \\ 0 & 0 & 0 & 0 & 0 & (g \star \mathbf{E})_{1212} \end{pmatrix}$$

The problem is that it is not always easy to compute explicitly such a rotation. For instance, given a cubic Elasticity tensor in its normal form (1.2) and applying a rotation of angle $\frac{\pi}{6}$ around axis $\langle 111 \rangle$, it is not an easy matter, if not aware of this transformation, to find a way back. Moreover, measured tensors are in practice triclinic, due to numerical errors and experimental discrepancy [1, 16, 13, 18]. Hence, the problem may also be numerically difficult.

Partial answers concerning the explicit determination of a proper basis have already been investigated in [11, 10, 19, 5, 9] for the monoclinic and the orthotropic symmetry classes. To do so, the authors construct a basis of eigenvectors for the second-order symmetric tensors that inherit (part) of the symmetry of \mathbf{E} ; the dilatation and Voigt's tensors [11, 10], defined as

$$(1.3) \quad \begin{aligned} \mathbf{d} &:= \text{tr}_{12} \mathbf{E} & (d_{ij} &= E_{kkij}), \\ \mathbf{v} &:= \text{tr}_{13} \mathbf{E} & (v_{ij} &= E_{kikj}). \end{aligned}$$

The cornerstone of this approach is that $\mathbf{d} = \mathbf{d}(\mathbf{E})$ and $\mathbf{v} = \mathbf{v}(\mathbf{E})$ are covariants of \mathbf{E} , meaning that one has the *covariance property*

$$\mathbf{d}(g \star \mathbf{E}) = g \star \mathbf{d}(\mathbf{E}), \quad \mathbf{v}(g \star \mathbf{E}) = g \star \mathbf{v}(\mathbf{E}),$$

where $(g \star \mathbf{a})_{ij} = g_{ik}g_{jl}a_{kl}$, for a second-order tensor \mathbf{a} . In some non-degenerate cases, this leads to the answer. The weakness of this approach is that \mathbf{d} and \mathbf{v} have at least the symmetry of \mathbf{E} but they may have more symmetry. For instance, in the cubic case, the pair (\mathbf{d}, \mathbf{v}) is isotropic. Such loss of information has to be handled, as they can be experimentally encountered, for example from the ultrasonic measurements made on a Ni base single crystal superalloy, close to be cubic [17] (studied in section 5.1).

A natural idea is to extend the idea of using covariants of \mathbf{E} , which naturally inherit the symmetry of \mathbf{E} , but different from \mathbf{d} and \mathbf{v} . Note, however, that second-order covariants cannot always encode all the geometric information carried by a fourth-order tensor [6, 22] (for example when \mathbf{E} is cubic). Taking account this observation, it has been tried by some authors to use the *harmonic factorization*, according to Sylvester's theorem [26] and Maxwell's multipoles [27]. However, this involves roots' computations of polynomials of degree 4 and 8 [4, 5, 8], in order to build a set of 8 unit vectors (Maxwell's multipoles), without any clue of how to organize such data. Besides, Maxwell's multipoles are not, strictly speaking, first-order covariant of \mathbf{E} and are very sensitive to conditioning.

The purpose of the present work is to obtain an explicit normal form of an Elasticity tensor \mathbf{E} once we know its symmetry class. Note, by the way, that the problem of determining the symmetry class of a given elasticity tensor \mathbf{E} , using polynomial covariant equations, has already been solved explicitly in [23]. Of course, our goal can be achieved numerically, as we can compute

$\bar{\mathbf{E}} = g \star \mathbf{E}$ for all $g \in \text{SO}(3)$ and try to find rotation g such that Voigt's representation $[\bar{\mathbf{E}}]$ has the good shape [16, 17]. A more geometrical approach, initiated in [11, 10, 19, 5], relying on *covariants*, is possible and will be described in this work. We shall formulate new effective and fast algorithms to calculate a natural basis for a given Elasticity tensor, once we know its symmetry class.

An important tool, introduced in [12, 23] and which will be used many times in this paper, is the *generalized cross product* between two *totally symmetric tensors* of any order $\mathbf{A} = \mathbf{A}^s$ and $\mathbf{B} = \mathbf{B}^s$. It is defined as follows

$$(1.4) \quad \mathbf{A} \times \mathbf{B} = (\mathbf{B} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{A})^s = -\mathbf{B} \times \mathbf{A},$$

where $(\cdot)^s$ means the total symmetrization (over all subscripts) and where $\boldsymbol{\varepsilon}$ is Levi-Civita third order tensor ($\varepsilon_{ijk} = \det(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)$ in any orthonormal frame (\mathbf{e}_i)).

The outline of the paper is as follows. We first recall some mathematical materials on the normal form of an Elasticity tensor in section 2 and the harmonic decomposition and covariants in section 3. Then, in section 4, we formulate and prove theorems that are the cornerstones to build our algorithms. In section 5, we provide and analyze experimental data, issued from the literature and which will be used to illustrate our methodology. Finally, in section 6, we explicit our algorithms, which for any given elasticity tensor \mathbf{E} produce a natural basis for it (and thus a rotation which brings it back to its normal form).

2. NORMAL FORM OF AN ELASTICITY TENSOR

An Elasticity tensor \mathbf{E} represents a material in a specific orientation, but the *same material* is represented in another orientation by a rotated tensor $g \star \mathbf{E}$. In mathematical terms, this means that the rotation group $\text{SO}(3)$ acts linearly on the space $\mathbb{E}la$ of Elasticity tensors, which we write as

$$\mathbf{E} \mapsto \bar{\mathbf{E}} = g \star \mathbf{E},$$

where

$$\bar{E}_{ijkl} = g_{ip}g_{jq}g_{kr}g_{ls}E_{pqrs},$$

in any orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. The subset

$$\{g \star \mathbf{E}; g \in \text{SO}(3)\}$$

is called the *orbit* of \mathbf{E} . A linear elastic material is thus represented by an orbit of an Elasticity tensor and not by an individual Elasticity tensor.

The *symmetry group* of a tensor $\mathbf{E} \in \mathbb{E}la$ is the subgroup of $\text{SO}(3)$ defined as

$$G_{\mathbf{E}} := \{g \in \text{SO}(3); g \star \mathbf{E} = \mathbf{E}\}.$$

Note that the symmetry group of $g \star \mathbf{E}$ is

$$(2.1) \quad G_{\bar{\mathbf{E}}} = gG_{\mathbf{E}}g^{-1}.$$

Therefore, the classification of *symmetries of materials* relies on the *conjugacy classes*

$$[G_{\mathbf{E}}] := \{gG_{\mathbf{E}}g^{-1}, g \in \text{SO}(3)\}$$

rather than on the symmetry groups of their respective tensors in a specific orientation. These are known as *symmetry classes*.

It was shown in [15] that there are exactly eight Elasticity symmetry classes: triclinic [$\mathbb{1}$], monoclinic [\mathbb{Z}_2], orthotropic [\mathbb{D}_2], tetragonal [\mathbb{D}_4], trigonal [\mathbb{D}_3], transversely-isotropic [$\mathbb{O}(2)$], cubic [\mathbb{O}] and isotropic [$\text{SO}(3)$]. Here, the subgroup $\mathbb{1}$ contains only the identity element of $\text{SO}(3)$, while all the others (except $\text{SO}(3)$) are represented in the canonical basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of \mathbb{R}^3 as follows.

- \mathbb{Z}_2 is generated by the second-order rotation $\mathbf{r}(\mathbf{e}_3, \pi)$. It has order 2;
- \mathbb{D}_2 is generated by the second-order rotations $\mathbf{r}(\mathbf{e}_3, \pi)$ and $\mathbf{r}(\mathbf{e}_1, \pi)$. It has order 4;
- \mathbb{D}_3 is generated by the third order rotation $\mathbf{r}(\mathbf{e}_3, \frac{2\pi}{3})$ and the second-order rotation $\mathbf{r}(\mathbf{e}_1, \pi)$. It has order 6;

- \mathbb{D}_4 is generated by the fourth-order rotation $\mathbf{r}(\mathbf{e}_3, \frac{\pi}{4})$ and the second-order rotation $\mathbf{r}(\mathbf{e}_1, \pi)$. It has order 8;
- \mathbb{O} is the *octahedral* group, the orientation-preserving symmetry group of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, which has order 24;
- $O(2)$ is the group generated by all rotations $\mathbf{r}(\mathbf{e}_3, \theta)$ ($\theta \in [0; 2\pi[$) and the second-order rotation $\mathbf{r}(\mathbf{e}_1, \pi)$. It is of infinite order.

There exists a *partial order* on symmetry classes, induced by inclusion between subgroups, defined as follows:

$$(2.2) \quad [G_1] \preceq [G_2] \iff \exists g \in \text{SO}(3), \quad G_1 \subset g G_2 g^{-1}.$$

We can thus say that a tensor has “at least” or “at most” such or such symmetry. For example, a tensor \mathbf{E} is said to be at least orthotropic if it is either orthotropic, tetragonal, transversely-isotropic, cubic or isotropic. A tensor \mathbf{E} is said to be at least trigonal if it is either trigonal, transversely-isotropic, cubic or isotropic. This order is however *partial*, which means that two classes cannot necessarily be compared (for example the trigonal and the tetragonal classes). The symmetry classes and their relations are summarized in Figure 1, where an arrow $[G_1] \rightarrow [G_2]$ means that $[G_1] \preceq [G_2]$.

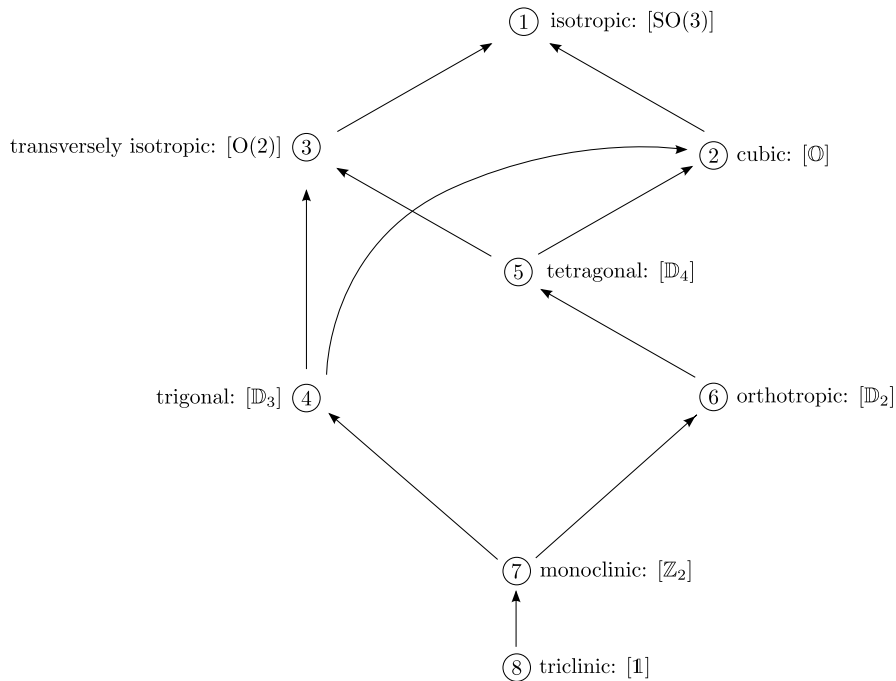


FIGURE 1. The eight symmetry classes of the Elasticity tensor [15, 3].

For any subgroup G of $\text{SO}(3)$ in the list defined above and defining a symmetry class $[G]$, consider the *fixed point set*

$$\mathbb{E}la^G := \{\mathbf{E} \in \mathbb{E}la; g \star \mathbf{E} = \mathbf{E}, \forall g \in G\}.$$

This linear subspace of $\mathbb{E}la$ is called a *linear slice*. It meets all the orbits of tensors which have at least the symmetry class $[G]$. In other words, given an Elasticity tensor \mathbf{E} in the symmetry class $[G]$, there exists a rotation $g \in \text{SO}(3)$ such that the symmetry group of $g \star \mathbf{E}$ is exactly the subgroup G , which means that $g \star \mathbf{E} \in \mathbb{E}la^G$. We say then that the Elasticity tensor $g \star \mathbf{E}$ is a *normal form* of \mathbf{E} .

Remark 2.1. When G is a finite group, the linear slice $\mathbb{E}la^G$ is the subspace of solutions of the linear system $g_k \star \mathbf{E} = \mathbf{E}$ ($k = 1, \dots, r$), where the g_k generate G .

We recall now, for each (non trivial) symmetry class $[G]$ of $\mathbb{E}la$, a normal form for each class in the Voigt representation. An orthonormal basis in which the Voigt representation of an elasticity tensor \mathbf{E} is a normal form is called a *proper basis* or a *natural basis* for \mathbf{E} .

- The cubic normal form has 3 independent parameters and writes

$$(2.3) \quad [\mathbf{E}_O] = \begin{pmatrix} E_{1111} & E_{1112} & E_{1112} & 0 & 0 & 0 \\ E_{1112} & E_{1111} & E_{1112} & 0 & 0 & 0 \\ E_{1112} & E_{1112} & E_{1111} & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{1212} & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{1212} \end{pmatrix}$$

- The transversely-isotropic normal form has 5 independent parameters and writes

$$(2.4) \quad [\mathbf{E}_{O(2)}] = \begin{pmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 \\ E_{1122} & E_{1111} & E_{1133} & 0 & 0 & 0 \\ E_{1133} & E_{1133} & E_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{1313} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(E_{1111} - E_{1122}) \end{pmatrix}$$

- The trigonal normal form has 6 independent parameters and writes

$$(2.5) \quad [\mathbf{E}_{\mathbb{D}_3}] = \begin{pmatrix} E_{1111} & E_{1122} & E_{1133} & E_{1123} & 0 & 0 \\ E_{1122} & E_{2222} & E_{1133} & -E_{1123} & 0 & 0 \\ E_{1133} & E_{1133} & E_{3333} & 0 & 0 & 0 \\ E_{1123} & -E_{1123} & 0 & E_{1313} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{1313} & E_{1123} \\ 0 & 0 & 0 & 0 & E_{1123} & \frac{1}{2}(E_{1111} - E_{1122}) \end{pmatrix}$$

- The tetragonal normal form has 6 independent parameters and writes

$$(2.6) \quad [\mathbf{E}_{\mathbb{D}_4}] = \begin{pmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 \\ E_{1122} & E_{1111} & E_{1133} & 0 & 0 & 0 \\ E_{1133} & E_{1133} & E_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{1313} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{1212} \end{pmatrix}$$

- The orthotropic normal form has 9 independent parameters and writes

$$(2.7) \quad [\mathbf{E}_{\mathbb{D}_2}] = \begin{pmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 \\ E_{1122} & E_{2222} & E_{2233} & 0 & 0 & 0 \\ E_{1133} & E_{2233} & E_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{1212} \end{pmatrix}$$

- The monoclinic normal form has 13 independent parameters and writes

$$(2.8) \quad [\mathbf{E}_{\mathbb{Z}_2}] = \begin{pmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & E_{1112} \\ E_{1122} & E_{2222} & E_{2233} & 0 & 0 & E_{2212} \\ E_{1133} & E_{2233} & E_{3333} & 0 & 0 & E_{3312} \\ 0 & 0 & 0 & E_{2323} & E_{1323} & 0 \\ 0 & 0 & 0 & E_{1323} & E_{1313} & 0 \\ E_{1112} & E_{2212} & E_{3312} & 0 & 0 & E_{1212} \end{pmatrix}$$

3. COVARIANTS OF THE ELASTICITY TENSOR

A *polynomial covariant* $\mathbf{C}(\mathbf{E})$ of the Elasticity tensor \mathbf{E} , is a tensor \mathbf{C} which is a polynomial function of \mathbf{E} and such that

$$g \star \mathbf{C}(\mathbf{E}) = \mathbf{C}(g \star \mathbf{E}), \quad \forall g \in \text{SO}(3).$$

Examples of covariants are the dilatation and Voigt's second-order covariants $\mathbf{d}(\mathbf{E})$, $\mathbf{v}(\mathbf{E})$, defined by (1.3). Fourth-order covariants appear in the *harmonic decomposition*

$$\mathbf{E} = (\text{tr } \mathbf{d}, \text{tr } \mathbf{v}, \mathbf{d}', \mathbf{v}', \mathbf{H})$$

of an elasticity tensor (see [4, 24, 2]), where \mathbf{d}' and \mathbf{v}' are the *deviatoric parts* of \mathbf{d} and \mathbf{v} , defined as

$$(\cdot)' := (\cdot) - \frac{1}{3} \text{tr}(\cdot) \mathbf{1}.$$

More precisely, we can write

$$(3.1) \quad \mathbf{E} = \mathbf{E}_{iso} + \mathbf{E}_{dv} + \mathbf{H}$$

where the isotropic part of \mathbf{E} is defined as

$$\mathbf{E}_{iso} := \frac{1}{9} (\text{tr } \mathbf{d}) \mathbf{1} \otimes \mathbf{1} + \frac{1}{15} (3 \text{tr } \mathbf{v} - \text{tr } \mathbf{d}) \mathbf{J}, \quad \mathbf{J} := \mathbf{1} \overline{\otimes} \mathbf{1} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1},$$

and its dilatation-Voigt part as

$$\mathbf{E}_{dv} := \frac{1}{7} (\mathbf{1} \otimes (5\mathbf{d}' - 4\mathbf{v}') + (5\mathbf{d}' - 4\mathbf{v}') \otimes \mathbf{1}) + \frac{2}{7} (\mathbf{1} \overline{\otimes} (3\mathbf{v}' - 2\mathbf{d}') + (3\mathbf{v}' - 2\mathbf{d}') \overline{\otimes} \mathbf{1}).$$

The remaining part

$$\mathbf{H} := \mathbf{E} - \mathbf{E}_{dv} - \mathbf{E}_{iso}$$

is a fourth-order harmonic tensor (*i.e.* totally symmetric and traceless).

Remark 3.1. In these formulas, we have used the tensor products \otimes and $\overline{\otimes}$ of two symmetric second-order tensors \mathbf{a} and \mathbf{b}

$$(\mathbf{a} \otimes \mathbf{b})_{ijkl} = a_{ij} b_{kl}, \quad (\mathbf{a} \overline{\otimes} \mathbf{b})_{ijkl} = \frac{1}{2} (a_{ik} b_{jk} + a_{il} b_{jk}).$$

The covariants $\mathbf{d}(\mathbf{E})$, $\mathbf{v}(\mathbf{E})$ and $\mathbf{H}(\mathbf{E})$ depend linearly on \mathbf{E} but there are other non linear covariants which are extremely useful to study the geometry of \mathbf{E} and they have been extensively used in [23] to formulate simple characterizations of the Elasticity symmetry classes. One of them is the following second-order, quadratic covariant, first introduced by Boehler and coworkers [7]:

$$(3.2) \quad \mathbf{d}_2 := \text{tr}_{13}(\mathbf{H} : \mathbf{H}) = \mathbf{H} : \mathbf{H}, \quad (\mathbf{d}_2)_{ij} = H_{ipqr} H_{pqrj}.$$

It depends on \mathbf{E} through \mathbf{H} . A full set of 70 polynomial covariants of \mathbf{H} which generates the polynomial covariant algebra of \mathbf{H} has been produced in [23].

4. RECOVERING NORMAL FORMS USING COVARIANTS

Covariants are useful to characterize the symmetry class of a tensor [23]. For instance, we have introduced in (1.4), the generalized cross product which writes

$$(\mathbf{a} \times \mathbf{b})_{ijk} = \frac{1}{6} (b_{ip} \varepsilon_{pjqa} a_{qk} + b_{ip} \varepsilon_{pkqa} a_{qj} + b_{jp} \varepsilon_{piqa} a_{qk} + b_{jp} \varepsilon_{pkqa} a_{qi} + b_{kp} \varepsilon_{piqa} a_{qj} + b_{kp} \varepsilon_{pjqa} a_{qi}),$$

for two second-order symmetric tensors $\mathbf{A} = \mathbf{a}$ and $\mathbf{B} = \mathbf{b}$, and we have the following result [23].

Lemma 4.1. *A second-order symmetric tensor \mathbf{a} is orthotropic if and only if the third order covariant $\mathbf{a}^2 \times \mathbf{a}$ is non-vanishing.*

Remark 4.2. $\mathbf{a} \times \mathbf{1} = 0$, $\mathbf{a}' \times \mathbf{b} = \mathbf{a} \times \mathbf{b}' = \mathbf{a}' \times \mathbf{b}'$.

Consider now a family of second-order symmetric tensors $\mathcal{F} = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$, with $n \geq 2$. Recall that the symmetry class $[G_{\mathcal{F}}]$ of \mathcal{F} is the conjugacy class of the subgroup

$$G_{\mathcal{F}} = \bigcap_i G_{\mathbf{a}_i} = \{g \in \text{SO}(3); g \star \mathbf{a}_i = \mathbf{a}_i, \forall i\},$$

and that such a family is either isotropic, transversely-isotropic, orthotropic or monoclinic. We have moreover the following result [23].

Theorem 4.3. *Let $(\mathbf{a}_1, \dots, \mathbf{a}_n)$ be an n -tuple of second-order symmetric tensors. Then:*

- (1) $(\mathbf{a}_1, \dots, \mathbf{a}_n)$ is isotropic if and only if

$$\mathbf{a}_k' = 0, \quad 1 \leq k \leq n,$$

where \mathbf{a}_k' is the deviatoric part of \mathbf{a}_k .

- (2) $(\mathbf{a}_1, \dots, \mathbf{a}_n)$ is transversely-isotropic if and only if there exists \mathbf{a}_j such that

$$\mathbf{a}_j' \neq 0, \quad \mathbf{a}_j \times \mathbf{a}_j^2 = 0,$$

and

$$\mathbf{a}_j \times \mathbf{a}_k = 0, \quad 1 \leq k \leq n.$$

- (3) $(\mathbf{a}_1, \dots, \mathbf{a}_n)$ is orthotropic if and only if

$$\text{tr}(\mathbf{a}_k \times \mathbf{a}_l) = 0, \quad 1 \leq k, l \leq n,$$

and

- either there exists \mathbf{a}_j such that

$$\mathbf{a}_j \times \mathbf{a}_j^2 \neq 0;$$

- or there exists a pair $(\mathbf{a}_i, \mathbf{a}_j)$ such that

$$\mathbf{a}_i \times \mathbf{a}_j \neq 0.$$

- (4) $(\mathbf{a}_1, \dots, \mathbf{a}_n)$ is monoclinic if and only if there exists a pair $(\mathbf{a}_i, \mathbf{a}_j)$ such that

$$\boldsymbol{\omega} := \text{tr}(\mathbf{a}_i \times \mathbf{a}_j) \neq 0, \quad \text{and} \quad (\mathbf{a}_k \boldsymbol{\omega}) \times \boldsymbol{\omega} = 0, \quad 1 \leq k \leq n.$$

Remark 4.4. If we define the commutator of \mathbf{a}_i and \mathbf{a}_j by $[\mathbf{a}_i, \mathbf{a}_j] = \mathbf{a}_i \mathbf{a}_j - \mathbf{a}_j \mathbf{a}_i$, then we have

$$\text{tr}(\mathbf{a}_i \times \mathbf{a}_j) = -\frac{1}{6} \boldsymbol{\varepsilon} : [\mathbf{a}_i, \mathbf{a}_j].$$

Theorem 4.3 is the key point to recover the natural basis of a family $\mathcal{F} = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ of second-order symmetric tensors as follows. A natural basis for the family \mathcal{F} is one in which all the members of the family have the same matrix-shape with a maximum of zero (see Figure 2).

$$\begin{array}{ccc} \begin{pmatrix} \circ & 0 & 0 \\ 0 & \circ & 0 \\ 0 & 0 & * \end{pmatrix} & \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} & \begin{pmatrix} * & \circ & 0 \\ \circ & * & 0 \\ 0 & 0 & * \end{pmatrix} \\ \text{(transversely-isotropic)} & \text{(orthotropic)} & \text{(monoclinic)} \end{array}$$

FIGURE 2. Shapes of normal forms for families of second-order symmetric tensors.

The problem of finding a normal form for \mathcal{F} is meaningful only when it is transversely-isotropic, orthotropic or monoclinic. As an illustration of our purpose, we shall now detail how to find a rotation which brings \mathcal{F} into a normal form in each of these three cases.

- **\mathcal{F} is transversely-isotropic:** Find a member \mathbf{a}_j as in point (2) of theorem 4.3. Any basis in which \mathbf{a}_j is diagonal, the last vector corresponding to its simple eigenvalue, will achieve the task.

- **\mathcal{F} is orthotropic:** If there exists \mathbf{a}_j orthotropic in \mathcal{F} (i.e. $\mathbf{a}_j^2 \times \mathbf{a}_j \neq 0$), then just diagonalize \mathbf{a}_j and this will answer the question. Otherwise, by point (3) of theorem 4.3, we can find an orthotropic pair $(\mathbf{a}_i, \mathbf{a}_j)$ in \mathcal{F} where both \mathbf{a}_i and \mathbf{a}_j are transversely-isotropic. In that case, the eigenspaces of \mathbf{a}_i and \mathbf{a}_j corresponding to single eigenvalues are one-dimensional and mutually orthogonal. A natural basis for \mathcal{F} is obtained by choosing a unit vector \mathbf{u} in the first space, \mathbf{v} in the second space and completing to a (direct) basis by adding $\mathbf{u} \times \mathbf{v}$.
- **\mathcal{F} is monoclinic:** In that case, construct $\boldsymbol{\omega}$ as in point (4) of theorem 4.3. Normalize it to a unit vector and complete it into a basis by adding an orthonormal basis of the plane $\boldsymbol{\omega}^\perp$. Permute, if necessary, the vectors to obtain a direct basis and we are done.

Remark 4.5. If \mathbf{t} is a non-vanishing transversely-isotropic deviator, we do not need to solve a polynomial equation to compute its unique simple eigenvalue. It is given by

$$\lambda = 2 \frac{\text{tr}(\mathbf{t}^3)}{\text{tr}(\mathbf{t}^2)},$$

and the main axis of \mathbf{t} (eigenspace of the simple eigenvalue) corresponds to the one-dimensional subspace

$$(4.1) \quad \ker \left(\mathbf{t} - 2 \frac{\text{tr}(\mathbf{t}^3)}{\text{tr}(\mathbf{t}^2)} \mathbf{1} \right).$$

Remark 4.6. If $(\mathbf{a}_1, \mathbf{a}_2)$ is an orthotropic couple where both \mathbf{a}_1 and \mathbf{a}_2 are transversely-isotropic, then, their respective main axes are orthogonal and correspond respectively to

$$\ker \left(\mathbf{t} - 2 \frac{\text{tr}(\mathbf{a}_1^3)}{\text{tr}(\mathbf{a}_1^2)} \mathbf{1} \right) \quad \text{and} \quad \ker \left(\mathbf{t} - 2 \frac{\text{tr}(\mathbf{a}_2^3)}{\text{tr}(\mathbf{a}_2^2)} \mathbf{1} \right).$$

The methodology developed above for a family \mathcal{F} of second-order symmetric tensors will allow us to find a natural basis of all elasticity tensors \mathbf{E} , provided they are either transversely-isotropic, tetragonal, trigonal, orthotropic or monoclinic. The isotropic case is trivial and the triclinic case will not be considered in this paper (even if it also possible to define some kind of normal form for a triclinic tensor). The cubic case will be treated at the end of this section. To start with, we recall the following result which was obtained in [23]. It allows us to solve the problem when \mathbf{E} is either transversely-isotropic, tetragonal or trigonal (details will be provided in section 6).

Theorem 4.7. *Let \mathbf{E} be a transversely-isotropic, tetragonal or trigonal Elasticity tensor. Then, the triplet $(\mathbf{d}', \mathbf{v}', \mathbf{d}_2')$ is transversely-isotropic.*

To be able to reduce the case of an elasticity tensor \mathbf{E} to a family of second-order symmetric tensors, when \mathbf{E} is either orthotropic or monoclinic, we need more second-order symmetric covariants which we shall introduce now. First, let us recall that the 2-contraction between \mathbf{H} and a second-order tensor \mathbf{a} is defined as

$$(\mathbf{H} : \mathbf{a})_{ij} := H_{ijpq} a_{pq}.$$

Using this operation, we produce first the following two covariants

$$\mathbf{c}_3 := \mathbf{H} : \mathbf{d}_2, \quad \text{and} \quad \mathbf{c}_4 := \mathbf{H} : \mathbf{c}_3,$$

and introduce two families of symmetric second-order covariants of \mathbf{E} , which will allow us to solve the problem when \mathbf{E} is either orthotropic or monoclinic. The first family

$$(4.2) \quad \mathcal{F}_o := \{ \mathbf{d}', \mathbf{v}', \mathbf{d}_2', \mathbf{c}_3, \mathbf{c}_4, \mathbf{H} : \mathbf{d}, \mathbf{H} : \mathbf{v}, \mathbf{H} : \mathbf{d}^2, \mathbf{H} : \mathbf{v}^2 \}$$

will be used in the orthotropic case and the second family

$$(4.3) \quad \mathcal{F}_m := \{ \mathbf{d}', \mathbf{v}', \mathbf{d}_2', \mathbf{c}_3, \mathbf{c}_4, \mathbf{H} : \mathbf{d}, \mathbf{H} : \mathbf{v}, \mathbf{H} : \mathbf{d}^2, \mathbf{H} : \mathbf{v}^2, \mathbf{H} : (\mathbf{d}\mathbf{v})^s, \mathbf{H} : (\mathbf{d}\mathbf{d}_2)^s, \mathbf{H} : (\mathbf{v}\mathbf{d}_2)^s \}$$

will be used in the monoclinic case. Here, $(\cdot)^s$ stands for the symmetric part of a second-order tensor. The key-point to conclude is the following result [23, Theorem 10.2].

Theorem 4.8. *For any Elasticity tensor \mathbf{E} :*

- (1) *If \mathbf{E} is orthotropic then the family \mathcal{F}_o of second-order tensors is orthotropic.*
- (2) *If \mathbf{E} is monoclinic then the family \mathcal{F}_m of second-order tensors is monoclinic.*

It remains to solve the problem when \mathbf{E} is cubic. In that case, each second-order covariant of \mathbf{E} is isotropic [23]. Therefore its fourth-order covariant \mathbf{H} is necessarily cubic (and thus non-vanishing). A natural basis for \mathbf{H} is therefore also one for \mathbf{E} . The key-point to calculate such a natural basis is then provided by the following theorem.

Theorem 4.9. *Let \mathbf{H} be a fourth-order cubic harmonic tensor. Then, the solutions of the linear equation*

$$(4.4) \quad \text{tr}(\mathbf{H} \times \mathbf{a}) = 0,$$

where \mathbf{a} is a second-order symmetric tensor, is a three-dimensional vector space. Moreover, orthotropic tensors \mathbf{a} which are solution of (4.4) form a dense open set and the natural basis of any such orthotropic tensor is a natural basis for \mathbf{H} .

Remark 4.10. This means that solving the linear system (4.4) and picking-up randomly a solution among them provides us with an orthotropic second-order symmetric tensor \mathbf{a} which eigenvectors define a proper basis for \mathbf{H} .

Remark 4.11. In an orthonormal basis, the 10 components of the totally symmetric three-order tensor $\text{tr}(\mathbf{H} \times \mathbf{a})$ write

$$(\text{tr}(\mathbf{H} \times \mathbf{a}))_{ijk} = \frac{1}{10} \sum_{p,q,r} (\varepsilon_{ipq} H_{jkpr} + \varepsilon_{j pq} H_{ikpr} + \varepsilon_{kpq} H_{ijpr}) a_{qr}.$$

Proof. The binary operation $\text{tr}(\mathbf{H} \times \mathbf{a})$ being covariant, solutions \mathbf{a} of $\text{tr}(\mathbf{H} \times \mathbf{a}) = 0$ write as $g \star \mathbf{a}_0$, where \mathbf{a}_0 are the solutions of $\text{tr}(\mathbf{H}_0 \times \mathbf{a}_0) = 0$, and where \mathbf{H}_0 is the normal form of \mathbf{H} . This normal form \mathbf{H}_0 (see for instance [3]) writes, in Voigt's representation (1.1), as

$$[\mathbf{H}_0] = \delta \begin{pmatrix} 8 & -4 & -4 & 0 & 0 & 0 \\ -4 & 8 & -4 & 0 & 0 & 0 \\ -4 & -4 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{pmatrix}$$

where $\delta \neq 0$. It can be checked that the space of solutions of the equation $\text{tr}(\mathbf{H}_0 \times \mathbf{a}_0) = 0$ corresponds exactly to the three-dimensional vector space of diagonal tensors, in which orthotropic tensors are a dense open set. Hence, any natural basis for \mathbf{H}_0 (there are 24 such ones) is a natural basis for any solution \mathbf{a}_0 of $\text{tr}(\mathbf{H}_0 \times \mathbf{a}_0) = 0$. Conversely, any natural basis of an *orthotropic* solution \mathbf{a}_0 corresponds to a natural basis of \mathbf{H}_0 , since there are only 24 such bases. Therefore, any natural basis for \mathbf{H} corresponds to a natural basis of \mathbf{a} , an orthotropic solution of $\text{tr}(\mathbf{H} \times \mathbf{a}) = 0$, and *vice-versa*, which ends the proof. \square

5. EXPERIMENTAL DATA

As pointed out in the introduction, natural bases have been obtained in the literature for some non-degenerate situations, using the dilatation or Voigt's tensors [11, 10, 19, 5, 9]. We present here some data which will be used in section 6 to illustrate our methodology. They consist in a *Ni base single crystal superalloy* (with its cubic, tetragonal, orthotropic and monoclinic approximations), an *α -quartz* [25, 27] (with its trigonal approximations), and a transversely-isotropic approximation issued from [20]. Note that the *Ni base single crystal superalloy* has a microstructure close to be cubic [17]. Thus, we consider relevant approximations of its associated Elasticity tensor to be cubic (and thus $\mathbf{d}' = \mathbf{v}' = 0$). These approximations have exactly the corresponding symmetry. Finally, in this section, all the experimental tensors, and their linear covariants, are expressed in GPa and their fourth-order harmonic components are given in Appendix A.

5.1. Ni base single crystal superalloy. The Voigt representation of the measured Elasticity tensor \mathbf{E}^γ of a *Ni base single crystal superalloy*, obtained by ultrasonic measurements in [17], writes as

$$[\mathbf{E}^\gamma] = \begin{pmatrix} 243 & 136 & 135 & 22 & 52 & -17 \\ 136 & 239 & 137 & -28 & 11 & 16 \\ 135 & 137 & 233 & 29 & -49 & 3 \\ 22 & -28 & 29 & 133 & -10 & -4 \\ 52 & 11 & -49 & -10 & 119 & -2 \\ -17 & 16 & 3 & -4 & -2 & 130 \end{pmatrix}$$

We get

$$\text{tr } \mathbf{d} = 1531, \quad \text{tr } \mathbf{v} = 1479,$$

and

$$\mathbf{d}' = \begin{pmatrix} \frac{11}{3} & 2 & 14 \\ 2 & \frac{5}{3} & 23 \\ 14 & 23 & -\frac{16}{3} \end{pmatrix}, \quad \mathbf{v}' = \begin{pmatrix} -1 & -11 & -1 \\ -11 & 9 & -1 \\ -1 & -1 & -8 \end{pmatrix}$$

The harmonic fourth-order part of \mathbf{E}^γ has Voigt's representation

$$[\mathbf{H}] = \begin{pmatrix} -\frac{1986}{35} & \frac{1093}{35} & \frac{893}{35} & 5 & \frac{352}{7} & -\frac{99}{7} \\ \frac{1093}{35} & -\frac{2306}{35} & \frac{1213}{35} & -31 & \frac{3}{7} & \frac{132}{7} \\ \frac{893}{35} & \frac{1213}{35} & -\frac{2106}{35} & 26 & -\frac{355}{7} & -\frac{33}{7} \\ 5 & -31 & 26 & \frac{1213}{35} & -\frac{33}{7} & \frac{3}{7} \\ \frac{352}{7} & \frac{3}{7} & -\frac{355}{7} & -\frac{33}{7} & \frac{893}{35} & 5 \\ -\frac{99}{7} & \frac{132}{7} & -\frac{33}{7} & \frac{3}{7} & 5 & \frac{1093}{35} \end{pmatrix}$$

We get thus

$$\|\mathbf{E}^\gamma\|^2 = \|\mathbf{E}_{iso}\|^2 + \|\mathbf{E}_{dv}\|^2 + \|\mathbf{H}\|^2$$

where $\|\mathbf{E}^\gamma\|^2 := E_{ijkl}^\gamma E_{ijkl}^\gamma$. Moreover:

- the square of the norm of the isotropic part \mathbf{E}_{iso} of \mathbf{E}^γ writes

$$\|\mathbf{E}_{iso}\|^2 = \frac{1}{15}(2(\text{tr } \mathbf{d})^2 - 2 \text{tr } \mathbf{d} \text{tr } \mathbf{v} + 3(\text{tr } \mathbf{v})^2).$$

It corresponds to the contribution of the isotropic parts of the dilatation and the Voigt tensors.

- the square of the norm of the dilatation-Voigt parts \mathbf{E}_{dv} of \mathbf{E}^γ writes

$$\|\mathbf{E}_{dv}\|^2 = \frac{2}{21}\|\mathbf{d}' + 2\mathbf{v}'\|^2 + \frac{4}{3}\|\mathbf{d}' - \mathbf{v}'\|^2.$$

It corresponds to the contribution of the deviatoric parts of the dilatation and the Voigt tensors.

- the square of the norm of the harmonic part \mathbf{H} of \mathbf{E}^γ writes

$$(5.1) \quad \|\mathbf{H}\|^2 = \text{tr } \mathbf{d}_2.$$

For this single crystal superalloy the isotropic contribution is the largest,

$$\frac{\|\mathbf{E}_{iso}\|^2}{\|\mathbf{E}^\gamma\|^2} = 0.880438$$

while the anisotropic dilatational-Voigt contribution is negligible, as

$$\frac{\|\mathbf{E}_{dv}\|^2}{\|\mathbf{E}^\gamma\|^2} = 0.005826$$

and the fourth-order harmonic contribution is second in magnitude,

$$\frac{\|\mathbf{H}\|^2}{\|\mathbf{E}^\gamma\|^2} = 0.113736$$

All the following approximations have identical isotropic parts.

Cubic approximation: $\|\mathbf{E}^\gamma - \mathbf{E}_{cubic}^\gamma\|/\|\mathbf{E}^\gamma\| = 0.105$ and we have

$$(5.2) \quad [\mathbf{E}_{cubic}^\gamma] = \begin{pmatrix} 240.131 & 144.442 & 125.760 & 6.39666 & 41.9737 & -21.1614 \\ 144.442 & 223.957 & 141.935 & -27.7808 & 2.27754 & 16.6041 \\ 125.760 & 141.935 & 242.638 & 21.3841 & -44.2512 & 4.55736 \\ 6.39666 & -27.7808 & 21.3841 & 133.268 & 4.55736 & 2.27754 \\ 41.9737 & 2.27754 & -44.2512 & 4.55736 & 117.094 & 6.39666 \\ -21.1614 & 16.6041 & 4.55736 & 2.27754 & 6.39666 & 135.776 \end{pmatrix}.$$

Tetragonal approximation: $\|\mathbf{E}^\gamma - \mathbf{E}_{tetra}^\gamma\|/\|\mathbf{E}^\gamma\| = 0.0996$ and we have

$$(5.3) \quad [\mathbf{E}_{tetra}^\gamma] = \begin{pmatrix} 239.333 & 141.383 & 129.618 & 5.8239 & 46.7414 & -21.0897 \\ 141.383 & 229.664 & 139.287 & -25.7103 & 1.8896 & 15.3674 \\ 129.618 & 139.287 & 241.429 & 19.8864 & -48.631 & 5.7223 \\ 5.8239 & -25.7103 & 19.8864 & 130.62 & 5.7223 & 1.8896 \\ 46.7414 & 1.8896 & -48.631 & 5.7223 & 120.951 & 5.8239 \\ -21.0897 & 15.3674 & 5.7223 & 1.8896 & 5.8239 & 132.716 \end{pmatrix}.$$

First orthotropic approximation: $\|\mathbf{E}^\gamma - \mathbf{E}_{orth}^\gamma\|/\|\mathbf{E}^\gamma\| = 0.0988$ and we have

$$(5.4) \quad [\mathbf{E}_{orth}^\gamma] = \begin{pmatrix} 241.3079 & 139.4923 & 129.533 & 5.3342 & 46.3021 & -20.3543 \\ 139.4923 & 229.6671 & 141.1738 & -26.2801 & 2.8831 & 14.4327 \\ 129.533 & 141.1738 & 239.6264 & 20.9459 & -49.1853 & 5.9215 \\ 5.3342 & -26.2801 & 20.9459 & 132.5072 & 5.9215 & 2.8831 \\ 46.3021 & 2.8831 & -49.1853 & 5.9215 & 120.8663 & 5.3342 \\ -20.3543 & 14.4327 & 5.9215 & 2.8831 & 5.3342 & 130.8256 \end{pmatrix}.$$

Second orthotropic approximation: $\|\mathbf{E}^\gamma - \mathbf{E}_{orth}^{\gamma'}\|/\|\mathbf{E}^\gamma\| = 0.1029$ and we have

$$(5.5) \quad [\mathbf{E}_{orth}^{\gamma'}] = \begin{pmatrix} 237.183 & 141.391 & 128.075 & 5.73042 & 42.1382 & -21.4889 \\ 141.391 & 228.24 & 142.671 & -27.3681 & 3.32484 & 16.2765 \\ 128.075 & 142.671 & 241.302 & 21.7967 & -44.0867 & 4.55057 \\ 5.73042 & -27.3681 & 21.7967 & 133.637 & 4.397 & 1.83613 \\ 42.1382 & 3.32484 & -44.0867 & 4.397 & 114.865 & 6.93608 \\ -21.4889 & 16.2765 & 4.55057 & 1.83613 & 6.93608 & 137.635 \end{pmatrix}.$$

Monoclinic approximation: $\|\mathbf{E}^\gamma - \mathbf{E}_{mono}^\gamma\|/\|\mathbf{E}^\gamma\| = 0.0883$ and we have

$$(5.6) \quad [\mathbf{E}_{mono}^\gamma] = \begin{pmatrix} 240.532 & 140.501 & 129.3 & 0.6715 & 47.7714 & -18.1515 \\ 140.501 & 231.37 & 138.463 & -26.3969 & 4.4758 & 18.2628 \\ 129.3 & 138.463 & 242.57 & 25.7254 & -52.2472 & -0.1113 \\ 0.6715 & -26.3969 & 25.7254 & 129.796 & -0.1113 & 4.4758 \\ 47.7714 & 4.4758 & -52.2472 & -0.1113 & 120.634 & 0.6715 \\ -18.1515 & 18.2628 & -0.1113 & 4.4758 & 0.6715 & 131.834 \end{pmatrix}.$$

5.2. Trigonal α -quartz. We consider now the trigonal Elasticity tensor of the α -quartz. It was given in [27] and determined from experimental values issued from [25]. In Voigt's notation, it writes

$$(5.7) \quad [\mathbf{E}_{trig}^\alpha] = \begin{pmatrix} 7.9122 & 0.7161 & 2.1801 & -0.0778 & -0.054 & -0.6541 \\ 0.7161 & 10.808 & -0.0235 & 0.4322 & -0.0725 & 2.1674 \\ 2.1801 & -0.0235 & 10.2544 & 0.8267 & 0.5779 & -1.2035 \\ -0.0778 & 0.4322 & 0.8267 & 4.3259 & -1.0971 & 0.0825 \\ -0.054 & -0.0725 & 0.5779 & -1.0971 & 6.2917 & 0.3278 \\ -0.6541 & 2.1674 & -1.2035 & 0.0825 & 0.3278 & 4.5151 \end{pmatrix}.$$

5.3. Transversely isotropic Elasticity tensor. Finally, we will consider the transversely-isotropic Elasticity tensor \mathbf{E}_{TI}^{KS} , obtained in [20]. In Voigt's representation, it writes

$$(5.8) \quad [\mathbf{E}_{TI}^{KS}] = \begin{pmatrix} 1.4373 & 0.5382 & 0.2699 & -0.0879 & -0.181 & -0.054 \\ 0.5382 & 1.4577 & 0.2634 & -0.1688 & -0.1009 & -0.0553 \\ 0.2699 & 0.2634 & 1.0327 & 0.0295 & 0.0324 & 0.0348 \\ -0.0879 & -0.1688 & 0.0295 & 0.4046 & 0.0588 & -0.0209 \\ -0.181 & -0.1009 & 0.0324 & 0.0588 & 0.4156 & -0.015 \\ -0.054 & -0.0553 & 0.0348 & -0.0209 & -0.015 & 0.4615 \end{pmatrix}.$$

It was obtained as the closest transversely-isotropic tensor with relative error

$$\|\mathbf{E}^{KS} - \mathbf{E}_{TI}^{KS}\| / \|\mathbf{E}^{KS}\| = 0.1278$$

to the following raw Elasticity tensor

$$[\mathbf{E}^{KS}] = \begin{pmatrix} 1.3045 & 0.6327 & 0.2592 & -0.1039 & -0.2385 & -0.1215 \\ 0.6327 & 1.4131 & 0.2648 & -0.1261 & -0.0705 & -0.0301 \\ 0.2592 & 0.2648 & 1.0389 & 0.0395 & 0.045 & 0.0317 \\ -0.1039 & -0.1261 & 0.0395 & 0.4794 & 0.019 & -0.0514 \\ -0.2385 & -0.0705 & 0.045 & 0.019 & 0.3747 & -0.016 \\ -0.1215 & -0.0301 & 0.0317 & -0.0514 & -0.016 & 0.5128 \end{pmatrix}.$$

6. EFFECTIVE COMPUTATIONS

For each symmetry class, we shall explain how to find a rotation which brings an Elasticity tensor \mathbf{E} whose components are given in an arbitrary orthonormal direct basis $\mathcal{B}_0 = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, into its normal form. More precisely, we will compute an orthonormal basis $\mathcal{B} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, and hence a rotation

$$(6.1) \quad g = \begin{pmatrix} \mathbf{u}_1 \cdot \mathbf{e}_1 & \mathbf{u}_1 \cdot \mathbf{e}_2 & \mathbf{u}_1 \cdot \mathbf{e}_3 \\ \mathbf{u}_2 \cdot \mathbf{e}_1 & \mathbf{u}_2 \cdot \mathbf{e}_2 & \mathbf{u}_2 \cdot \mathbf{e}_3 \\ \mathbf{u}_3 \cdot \mathbf{e}_1 & \mathbf{u}_3 \cdot \mathbf{e}_2 & \mathbf{u}_3 \cdot \mathbf{e}_3 \end{pmatrix}$$

such that the Voigt's representation (1.1) of $g \star \mathbf{E}$ is a normal form of the symmetry class of \mathbf{E} .

6.1. Cubic class. The proposed methodology for a cubic Elasticity tensor \mathbf{E}_{cubic} is the following.

- (1) Calculate the fourth-order harmonic tensor \mathbf{H} of \mathbf{E}_{cubic} from (3.1).
- (2) Solve the linear system

$$\text{tr}(\mathbf{H} \times \mathbf{a}) = 0,$$

where \mathbf{a} is a second-order symmetric tensor.

- (3) Pick-up randomly a solution \mathbf{a} among them. According to theorem 4.9, it will be orthotropic. This can be checked by verifying that $\mathbf{a}^2 \times \mathbf{a} \neq 0$.
- (4) Diagonalize \mathbf{a} and compute a direct orthonormal basis $\mathcal{B} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ of eigenvectors for \mathbf{a} .
- (5) The normal form (2.3) is given by $\mathbf{E}_{\mathbb{O}} = g \star \mathbf{E}_{cubic}$ with g defined by (6.1).

Remark 6.1. Since $\text{tr}(\mathbf{H} \times \mathbf{1}) = 0$ for every tensor \mathbf{H} , it is enough to solve the equation

$$\text{tr}(\mathbf{H} \times \mathbf{a}') = 0,$$

for deviatoric tensors \mathbf{a}' , which leads to solve a linear system in a five-dimensional space.

Example 6.2. Consider the cubic Elasticity tensor (5.2) for Ni base single crystal superalloy. It is such that $\text{tr} \mathbf{d} = 1531$, $\text{tr} \mathbf{v} = 1479$, $\mathbf{d}' = 0$, $\mathbf{v}' = 0$. Its fourth-order harmonic part \mathbf{H} is given by (A.1) and we get $\mathbf{d}'_2 = (\text{tr}_{13} \mathbf{H}^2)' = 0$. Setting arbitrarily $a'_{13} = 1$ and $a'_{12} = 1$, the solution of $\text{tr}(\mathbf{H} \times \mathbf{a}') = 0$ leads to

$$\mathbf{a}' = \begin{pmatrix} 3.485 & 1 & 1 \\ 1 & -9.93526 & -2.23089 \\ 1 & -2.23089 & 6.45025 \end{pmatrix}$$

which is orthotropic since $(\mathbf{a}')^2 \times \mathbf{a}' \neq 0$. Computing a direct orthonormal basis of eigenvectors for \mathbf{a}' , we get

$$g = \begin{pmatrix} 0.0813519 & -0.987342 & -0.136151 \\ 0.24438 & -0.112674 & 0.963111 \\ -0.966261 & -0.111623 & 0.232121 \end{pmatrix}$$

and we can check that $(\mathbf{E}_{cubic}^\gamma)_\circ = g \star \mathbf{E}_{cubic}^\gamma$ writes

$$[(\mathbf{E}_{cubic}^\gamma)_\circ] = \begin{pmatrix} 213.355 & 148.489 & 148.489 & 0 & 0 & 0 \\ 148.489 & 213.355 & 148.489 & 0 & 0 & 0 \\ 148.489 & 148.489 & 213.355 & 0 & 0 & 0 \\ 0 & 0 & 0 & 139.823 & 0 & 0 \\ 0 & 0 & 0 & 0 & 139.823 & 0 \\ 0 & 0 & 0 & 0 & 0 & 139.823 \end{pmatrix} \quad (\text{GPa}).$$

The normal form given in [17] is retrieved.

6.2. Transversely isotropic class. The proposed methodology for a transversely-isotropic Elasticity tensor \mathbf{E}_{TI} is the following.

- (1) Compute the triplet of covariant deviators $(\mathbf{d}', \mathbf{v}', \mathbf{d}_2')$ of \mathbf{E}_{TI} (see section 3). By theorem 4.7, the triplet $(\mathbf{d}', \mathbf{v}', \mathbf{d}_2')$ is transversely-isotropic. Thus, one of them, let us call it \mathbf{t} , is transversely-isotropic.
- (2) Let \mathbf{u}_3 be the unit eigenvector corresponding to the single eigenvalue of \mathbf{t} . By remark 4.5, \mathbf{u}_3 can be obtained by solving the linear system,

$$\left(\mathbf{t} - 2 \frac{\text{tr}(\mathbf{t}^3)}{\text{tr}(\mathbf{t}^2)} \mathbf{1} \right) \mathbf{u} = 0.$$

- (3) Complete \mathbf{u}_3 into a direct orthonormal basis $\mathcal{B} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ of \mathbb{R}^3 by choosing an orthonormal pair $(\mathbf{u}_1, \mathbf{u}_1)$ orthogonal to \mathbf{u}_3 . For instance, if $\mathbf{u}_3 \neq \pm \mathbf{e}_3$, one can choose

$$(6.2) \quad \mathbf{u}_1 = \frac{\mathbf{e}_3 \times \mathbf{u}_3}{\|\mathbf{e}_3 \times \mathbf{u}_3\|} \quad \text{and} \quad \mathbf{u}_2 = \mathbf{u}_3 \times \mathbf{u}_1.$$

- (4) The normal form (2.4) is given by $\mathbf{E}_{O(2)} = g \star \mathbf{E}_{TI}$ with g defined by (6.1).

Example 6.3. Consider the transversely-isotropic Elasticity tensor (5.8). We find

$$\mathbf{d}' = \begin{pmatrix} 0.221833 & -0.0745 & -0.2495 \\ -0.0745 & 0.235733 & -0.2272 \\ -0.2495 & -0.2272 & -0.457567 \end{pmatrix}$$

$$\mathbf{v}' = 0.679222 \mathbf{d}', \quad \mathbf{d}_2' = -0.0977232 \mathbf{d}'$$

and we can check that the triplet $(\mathbf{d}', \mathbf{v}', \mathbf{d}_2')$ is transversely-isotropic. We choose $\mathbf{t} = \mathbf{d}'$. Its simple eigenvalue is given by

$$\lambda = 2 \frac{\text{tr}(\mathbf{d}'^3)}{\text{tr}(\mathbf{d}'^2)} = -0.607173.$$

Solving the linear system (4.1) returns:

$$\mathbf{u}_3 = \begin{pmatrix} 0.29966 \\ 0.272898 \\ 0.914183 \end{pmatrix}.$$

Build a direct orthonormal basis $\mathcal{B} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ from (6.2) and compute g using (6.1):

$$g = \begin{pmatrix} -0.673321 & 0.73935 & 0 \\ -0.675902 & -0.615539 & 0.405301 \\ 0.29966 & 0.272898 & 0.914183 \end{pmatrix}.$$

Then, we can check that $(\mathbf{E}_{TI}^{KS})_{O(2)} = g \star \mathbf{E}_{TI}^{KS}$ writes

$$(6.3) \quad [(\mathbf{E}_{TI}^{KS})_{O(2)}] = \begin{pmatrix} 1.5641 & 0.6046 & 0.1583 & 0 & 0 & 0 \\ 0.6046 & 1.5642 & 0.1582 & 0 & 0 & 0 \\ 0.1583 & 0.1582 & 1.0997 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3258 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3257 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4799 \end{pmatrix} \quad (\text{GPa}).$$

which is the normal form of the optimal transversely-isotropic tensor obtained in [20].

6.3. Trigonal class. The proposed methodology for a trigonal Elasticity tensor \mathbf{E}_{trig} is the following. The first three steps are the same as the ones for the transversely-isotropic case.

- (1) Compute the transversely-isotropic triplet $(\mathbf{d}', \mathbf{v}', \mathbf{d}_2')$ from \mathbf{E}_{trig} (see proposition 4.7 and section 3). Extract from this triplet a transversely-isotropic deviator \mathbf{t} .
- (2) Let \mathbf{u}_3 be a unit vector, solution of the linear system

$$\left(\mathbf{t} - 2 \frac{\text{tr}(\mathbf{t}^3)}{\text{tr}(\mathbf{t}^2)} \mathbf{1} \right) \mathbf{u} = 0.$$

- (3) Complete \mathbf{u}_3 into a direct orthonormal basis $\mathcal{B}_1 = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{u}_3)$ of \mathbb{R}^3 , using (6.2), for instance, and define g_1 as the rotation given by (6.1).
- (4) Compute $\bar{\mathbf{E}} := g_1 \star \mathbf{E}_{trig}$ and define θ_0 to be one solution of the equation

$$(6.4) \quad \bar{E}_{1123} \sin 3\theta = \bar{E}_{1113} \cos 3\theta.$$

- (5) The normal form (2.5) of \mathbf{E}_{trig} is given by $\mathbf{E}_{\mathbb{D}_3} = \mathbf{r}(\mathbf{e}_3, \theta_0) \star \bar{\mathbf{E}}$, where $\mathbf{r}(\mathbf{e}_3, \theta_0)$ is the rotation of angle θ_0 around axis \mathbf{e}_3 .

Remark 6.4. Equation (6.4) derives from the observation that the matrix form of a trigonal Elasticity tensor with correct third axis \mathbf{u}_3 writes

$$[\bar{\mathbf{E}}_{trig}] = \begin{pmatrix} \bar{E}_{1111} & \bar{E}_{1122} & \bar{E}_{1133} & \bar{E}_{1123} & \bar{E}_{1113} & 0 \\ \bar{E}_{1122} & \bar{E}_{1111} & \bar{E}_{1133} & -\bar{E}_{1123} & -\bar{E}_{1113} & 0 \\ \bar{E}_{1133} & \bar{E}_{1133} & \bar{E}_{3333} & 0 & 0 & 0 \\ \bar{E}_{1123} & -\bar{E}_{1123} & 0 & \bar{E}_{1313} & 0 & -\bar{E}_{1113} \\ \bar{E}_{1113} & -\bar{E}_{1113} & 0 & 0 & \bar{E}_{1313} & \bar{E}_{1123} \\ 0 & 0 & 0 & -\bar{E}_{1113} & \bar{E}_{1123} & \frac{1}{2}(\bar{E}_{1111} - \bar{E}_{1122}) \end{pmatrix}$$

Thus, a rotation of $\bar{\mathbf{E}}_{trig}$ around \mathbf{u}_3 and of angle θ_0 , solution of (6.4), leads to the normal form (2.5).

Example 6.5. Consider the trigonal Elasticity tensor (5.7) for α -quartz. We compute

$$\text{tr } \mathbf{d} = 34.709, \quad \text{tr } \mathbf{v} = 59.249$$

and

$$\mathbf{v}' = \begin{pmatrix} -1.02767 & 0.4162 & 0.6064 \\ 0.4162 & -0.0976667 & 1.5867 \\ 0.6064 & 1.5867 & 1.12533 \end{pmatrix}.$$

We check that \mathbf{v}' is transversely-isotropic ($\mathbf{v}'^2 \times \mathbf{v}' = 0$) and observe that

$$\mathbf{d}' = 0.74434 \mathbf{v}', \quad \text{and} \quad \mathbf{d}_2' = -0.828279 \mathbf{v}'.$$

The simple eigenvalue of \mathbf{v}' is given by

$$2 \frac{\text{tr}(\mathbf{v}'^3)}{\text{tr}(\mathbf{v}'^2)} = 2.37334.$$

Solving the linear system (4.1) with $\mathbf{t} = \mathbf{v}'$ gives

$$\mathbf{u}_3 = \begin{pmatrix} 0.21137 \\ 0.553074 \\ 0.805873 \end{pmatrix}, \quad \|\mathbf{u}_3\| = 1.$$

We build then a direct orthonormal basis $\mathcal{B}_1 = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{u}_3)$ using (6.2) and define g_1 using (6.1),

$$g_1 = \begin{pmatrix} -0.934108 & 0.356991 & 0 \\ -0.287689 & -0.752773 & 0.592088 \\ 0.21137 & 0.553074 & 0.805873 \end{pmatrix}.$$

The Elasticity tensor $\bar{\mathbf{E}} = g_1 \star \mathbf{E}_{trig}$ writes

$$[\bar{\mathbf{E}}] = \begin{pmatrix} 8.76 & 0.6 & 1.33 & 1.706 & 0.289 & 0 \\ 0.6 & 8.76 & 1.33 & -1.706 & -0.288 & 0 \\ 1.33 & 1.33 & 10.68 & 0 & 0 & 0 \\ 1.706 & -1.706 & 0 & 5.72 & 0 & -0.288 \\ 0.289 & -0.288 & 0 & 0 & 5.72 & 1.706 \\ 0 & 0 & 0 & -0.288 & 1.706 & 4.08 \end{pmatrix}$$

We solve (6.4) and choose the solution

$$\theta_0 = \frac{1}{3} \arctan \left(\frac{\bar{E}_{1113}}{\bar{E}_{1123}} \right) = 0.0558614,$$

to define

$$\mathbf{r}(\mathbf{e}_3, \theta_0) = \begin{pmatrix} 0.99844 & -0.0558324 & 0 \\ 0.0558324 & 0.99844 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally, one can check that $(\mathbf{E}_{trig}^\alpha)_{\mathbb{D}_3} = \mathbf{r}(\mathbf{e}_3, \theta_0) \star \bar{\mathbf{E}}$ writes

$$[(\mathbf{E}_{trig}^\alpha)_{\mathbb{D}_3}] = \begin{pmatrix} 8.76 & 0.6 & 1.33 & 1.73 & 0 & 0 \\ 0.6 & 8.76 & 1.33 & -1.73 & 0 & 0 \\ 1.33 & 1.33 & 10.68 & 0 & 0 & 0 \\ 1.73 & -1.73 & 0 & 5.72 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.72 & 1.73 \\ 0 & 0 & 0 & 0 & 1.73 & 4.08 \end{pmatrix} \quad (\text{GPa}).$$

6.4. Tetragonal class. The methodology for a tetragonal Elasticity tensor \mathbf{E}_{tetra} is similar to the one used for the trigonal case.

- (1) Compute the transversely-isotropic triplet $(\mathbf{d}', \mathbf{v}', \mathbf{d}_2')$ from \mathbf{E}_{tetra} (see proposition 4.7 and section 3). Extract from this triplet a transversely-isotropic deviator \mathbf{t} .
- (2) Let \mathbf{u}_3 with $\|\mathbf{u}_3\| = 1$ be a solution of the linear system

$$\left(\mathbf{t} - 2 \frac{\text{tr}(\mathbf{t}^3)}{\text{tr}(\mathbf{t}^2)} \mathbf{1} \right) \mathbf{u} = 0$$

- (3) Complete \mathbf{u}_3 into a direct orthonormal basis $\mathcal{B}_1 = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{u}_3)$ of \mathbb{R}^3 , using (6.2), for instance and define g_1 as the rotation given by (6.1).
- (4) Compute $\bar{\mathbf{E}} := g_1 \star \mathbf{E}_{tetra}$ and define θ_0 to be one solution of

$$(6.5) \quad 4\bar{E}_{1112} \cos 4\theta = (2\bar{E}_{1212} + \bar{E}_{1122} - \bar{E}_{1111}) \sin 4\theta$$

which always exists if $\bar{\mathbf{E}}$ is tetragonal.

- (5) The normal form (2.6) is given by $\mathbf{E}_{\mathbb{D}_4} = \mathbf{r}(\mathbf{e}_3, \theta_0) \star \bar{\mathbf{E}}$, where $\mathbf{r}(\mathbf{e}_3, \theta_0)$ is the rotation of angle θ_0 around \mathbf{e}_3 .

Remark 6.6. Equation (6.5) derives from the observation that the matrix-form of a tetragonal Elasticity tensor with correct third axis \mathbf{u}_3 writes

$$[\bar{\mathbf{E}}_{tetra}] = \begin{pmatrix} \bar{E}_{1111} & \bar{E}_{1122} & \bar{E}_{1133} & 0 & 0 & \bar{E}_{1112} \\ \bar{E}_{1122} & \bar{E}_{1111} & \bar{E}_{1133} & 0 & 0 & -\bar{E}_{1112} \\ \bar{E}_{1133} & \bar{E}_{1133} & \bar{E}_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{E}_{1313} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{E}_{1313} & 0 \\ \bar{E}_{1112} & -\bar{E}_{1112} & 0 & 0 & 0 & \bar{E}_{1212} \end{pmatrix}.$$

Thus, a rotation of $\bar{\mathbf{E}}_{tetra}$ around \mathbf{u}_3 and of angle θ_0 , solution of (6.5), leads to the normal form (2.6).

Example 6.7. Consider the tetragonal Elasticity tensor (5.3) for Ni base single crystal superalloy. We get $\mathbf{d}' = 0$, $\mathbf{v}' = 0$ and

$$\mathbf{d}_2' = \begin{pmatrix} 1389.87 & 341.696 & 47.1186 \\ 341.696 & -2729.03 & -571.863 \\ 47.1186 & -571.863 & 1339.17 \end{pmatrix}$$

which is transversely-isotropic ($(\mathbf{d}_2')^2 \times \mathbf{d}_2' = 0$). Its simple eigenvalue is given by

$$2 \frac{\text{tr}(\mathbf{d}_2'^3)}{\text{tr}(\mathbf{d}_2'^2)} = -2836.05.$$

Solving the linear system (4.1) with $\mathbf{t} = \mathbf{d}_2'$ gives

$$\mathbf{u}_3 = \begin{pmatrix} 0.0813519 \\ -0.987342 \\ -0.136151 \end{pmatrix}, \quad \|\mathbf{u}_3\| = 1.$$

We build then a direct orthonormal basis $\mathcal{B}_1 = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{u}_3)$ using (6.2) and define g_1 using (6.1),

$$g_1 = \begin{pmatrix} 0.996623 & 0.0821166 & 0 \\ 0.0111802 & -0.135691 & 0.990688 \\ 0.0813519 & -0.987342 & -0.136151 \end{pmatrix}$$

We solve (6.5) and choose the solution

$$\theta_0 = \frac{1}{4} \arctan \left(\frac{4\bar{E}_{1112}}{2\bar{E}_{1212} + \bar{E}_{1122} - \bar{E}_{1111}} \right) = 0.236501.$$

Finally, we can check that $(\mathbf{E}_{tetra}^\gamma)_{\mathbb{D}_4} = \mathbf{r}(\mathbf{e}_3, \theta_0) \star \bar{\mathbf{E}}$ writes

$$[(\mathbf{E}_{tetra}^\gamma)_{\mathbb{D}_4}] = \begin{pmatrix} 210.103 & 154.993 & 145.237 & 0 & 0 & 0 \\ 154.993 & 210.103 & 145.237 & 0 & 0 & 0 \\ 145.237 & 145.237 & 219.858 & 0 & 0 & 0 \\ 0 & 0 & 0 & 136.571 & 0 & 0 \\ 0 & 0 & 0 & 0 & 136.571 & 0 \\ 0 & 0 & 0 & 0 & 0 & 146.326 \end{pmatrix}.$$

6.5. Orthotropic class. The methodology for an orthotropic Elasticity tensor \mathbf{E}_{ortho} is based on the deep investigation of the family \mathcal{F}_o of second-order symmetric covariants given by (4.2). This family is orthotropic by theorem 4.8 and we have to distinguish between two cases.

- (1) If there exists an orthotropic tensor \mathbf{a} in the family \mathcal{F}_o , then, a direct orthonormal basis of eigenvectors for \mathbf{a} is also a natural basis for \mathcal{F}_o .
- (2) Otherwise, we can find an orthotropic couple $(\mathbf{a}_1, \mathbf{a}_2)$ in \mathcal{F}_o . In that case, both \mathbf{a}_1 and \mathbf{a}_2 are transversely-isotropic and their respective main axis are orthogonal. Let \mathbf{u}_1 and \mathbf{u}_2 be unit vectors spanning these axes (they can be obtained using remark 4.6). Then, a natural basis for \mathbf{E}_{ortho} is $\mathcal{B} := (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_1 \times \mathbf{u}_2)$.

In both cases, the orthotropic normal form (2.7) is recovered by $(\mathbf{E}_{ortho})_{\mathbb{D}_2} = g \star \mathbf{E}_{ortho}$, where g is defined by (6.1).

In [23, Theorem 10.2], it was shown that if \mathbf{E}_{ortho} is orthotropic, then the triplet $(\mathbf{d}', \mathbf{v}', \mathbf{d}_2')$ is either orthotropic or transversely-isotropic. This observation leads to a possible optimization of the methodology proposed above.

- If this triplet is orthotropic, our methodology can be optimized, by looking for an orthotropic tensor or an orthotropic couple of transversely-isotropic tensors in this triplet rather than in the whole family \mathcal{F}_o .
- If the triplet is transversely-isotropic, an alternative methodology similar to the one used for a trigonal or a tetragonal tensor is still possible and is detailed below.
 - (1) Extract a transversely-isotropic deviator \mathbf{t} from the triplet $(\mathbf{d}', \mathbf{v}', \mathbf{d}_2')$.
 - (2) Compute \mathbf{u}_3 with $\|\mathbf{u}_3\| = 1$ as a solution of the linear system

$$\left(\mathbf{t} - 2 \frac{\text{tr}(\mathbf{t}^3)}{\text{tr}(\mathbf{t}^2)} \mathbf{1} \right) \mathbf{u} = 0,$$

as explained in remark 4.5.

- (3) Complete \mathbf{u}_3 into a direct orthonormal basis $\mathcal{B}_1 = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{u}_3)$ of \mathbb{R}^3 , using (6.2) for instance, and define g_1 as the rotation given by (6.1).
- (4) Compute $\bar{\mathbf{E}} := g_1 \star \mathbf{E}_{ortho}$ and let θ_0 be a solution of

$$(6.6) \quad 2\bar{E}_{3312} \cos 2\theta = (\bar{E}_{1133} - \bar{E}_{2233}) \sin 2\theta$$

which always exists as $\bar{\mathbf{E}}$ is an orthotropic tensor.

- (5) The normal form (2.7) is given by $(\mathbf{E}_{ortho})_{\mathbb{D}_2} = \mathbf{r}(\mathbf{e}_3, \theta_0) \star \bar{\mathbf{E}}$, where $\mathbf{r}(\mathbf{e}_3, \theta_0)$ is the rotation of angle θ_0 around \mathbf{e}_3 .

Remark 6.8. As in the trigonal and the tetragonal cases, equation (6.6) is derived from the observation that an orthotropic Elasticity tensor with one correct axis, say \mathbf{u}_3 , writes

$$[\bar{\mathbf{E}}_{ortho}] = \begin{pmatrix} \bar{E}_{1111} & \bar{E}_{1122} & \bar{E}_{1133} & 0 & 0 & \bar{E}_{1112} \\ \bar{E}_{1122} & \bar{E}_{2222} & \bar{E}_{2233} & 0 & 0 & \bar{E}_{2212} \\ \bar{E}_{1133} & \bar{E}_{2233} & \bar{E}_{3333} & 0 & 0 & \bar{E}_{3312} \\ 0 & 0 & 0 & \bar{E}_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{E}_{1313} & 0 \\ \bar{E}_{1112} & \bar{E}_{2212} & \bar{E}_{3312} & 0 & 0 & \bar{E}_{1212} \end{pmatrix}$$

Thus, a rotation of $\bar{\mathbf{E}}_{ortho}$ around \mathbf{u}_3 and of angle θ_0 , solution of (6.6), leads to the normal form (2.7).

Example 6.9. Consider the orthotropic Elasticity tensor (5.4) for Ni base single crystal superalloy. This example is interesting because both the dilatation and the Voigt second-order covariants of this Elasticity tensor are isotropic, $\mathbf{d}' = \mathbf{v}' = 0$. Hence simple methods to recover its normal form fail. However, one can check that its deviatoric second-order covariant

$$\mathbf{d}_2' = \begin{pmatrix} 523.33 & 207.103 & 500.816 \\ 207.103 & -2721.59 & -651.919 \\ 500.816 & -651.919 & 2198.26 \end{pmatrix}$$

is orthotropic. Its diagonalization defines using (6.1) the rotation

$$g = \begin{pmatrix} 0.0813478 & -0.987343 & -0.136151 \\ 0.244376 & -0.112676 & 0.963112 \\ -0.966262 & -0.111619 & 0.232117 \end{pmatrix}$$

and one can check that the Elasticity tensor $(\mathbf{E}_{ortho}^\gamma)_{\mathbb{D}_2} = g \star \mathbf{E}_{orth}^\gamma$ writes

$$[(\mathbf{E}_{ortho}^\gamma)_{\mathbb{D}_2}] = \begin{pmatrix} 219.858 & 147.607 & 142.867 & 0 & 0 & 0 \\ 147.607 & 207.732 & 154.992 & 0 & 0 & 0 \\ 142.867 & 154.992 & 212.473 & 0 & 0 & 0 \\ 0 & 0 & 0 & 146.326 & 0 & 0 \\ 0 & 0 & 0 & 0 & 134.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 138.94 \end{pmatrix} \quad (\text{GPa})$$

and is its orthotropic normal form (2.7).

Example 6.10. Consider now the second orthotropic Elasticity tensor approximation (5.5) for Ni base single crystal superalloy. This time $\mathbf{d}' = 0$ (since its fourth-order harmonic part is cubic) and both \mathbf{d}' and \mathbf{v}' are transversely-isotropic but not of the same axis. The pair $(\mathbf{d}', \mathbf{v}')$ is orthotropic. The unit eigenvectors \mathbf{u}_1 and \mathbf{u}_2 corresponding respectively to the simple eigenvalue of \mathbf{d}' and \mathbf{v}' are

$$\mathbf{u}_1 = \begin{pmatrix} -0.966261 \\ -0.111623 \\ 0.232121 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 0.0813519 \\ -0.987342 \\ -0.136151 \end{pmatrix}.$$

The rotation g build from (6.1), with $\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2$, is such that the Elasticity tensor $(\mathbf{E}_{ortho}^{\gamma'})_{\mathbb{D}_2} = g \star \mathbf{E}_{orth}^\gamma$ has the orthotropic normal form

$$[(\mathbf{E}_{ortho}^{\gamma'})_{\mathbb{D}_2}] = \begin{pmatrix} 217.806 & 149.478 & 145.095 & 0. & 0. & 0. \\ 149.478 & 212.006 & 150.896 & 0. & 0. & 0. \\ 145.095 & 150.896 & 210.252 & 0. & 0. & 0. \\ 0. & 0. & 0. & 137.507 & 0. & 0. \\ 0. & 0. & 0. & 0. & 141.857 & 0. \\ 0. & 0. & 0. & 0. & 0. & 140.104 \end{pmatrix} \quad (\text{GPa}).$$

Remark 6.11. Compared to previous works [11, 10, 19, 5, 9], our procedure, relying on theorem 4.8, is exhaustive and allows to handle all degenerate cases. It is based on the list \mathcal{F}_o of second-order covariants which carries all the information required to recover the normal form of an orthotropic Elasticity tensor.

6.6. Monoclinic class. The methodology for a monoclinic Elasticity tensor \mathbf{E}_{mono} is based on the investigation of the family \mathcal{F}_m of second-order symmetric covariants given by (4.3). This family is monoclinic by theorem 4.8. The algorithm is the following.

- (1) Find a common eigenvector $\boldsymbol{\omega}$ for all second-order covariants in the family \mathcal{F}_m , by computing the commutators ($\boldsymbol{\omega} = \boldsymbol{\varepsilon} : [\mathbf{a}_i, \mathbf{a}_j]$), as in theorem 4.3.
- (2) Set $\mathbf{u}_3 = \boldsymbol{\omega}/\|\boldsymbol{\omega}\|$ and complete it into a direct orthonormal basis $\mathcal{B} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, using (6.2) for instance.
- (3) The monoclinic normal form (2.8) is given by $g \star \mathbf{E}_{mono}$ where g is defined by (6.1).

Remark 6.12. In most (non degenerate) cases, the commutator $\boldsymbol{\varepsilon} : [\mathbf{d}, \mathbf{v}] = 2\boldsymbol{\varepsilon} : (\mathbf{d}\mathbf{v})$ of the dilatation and the Voigt tensors will allow to initiate the first step of the algorithm (as in [11, 10, 19]). But $\boldsymbol{\varepsilon} : [\mathbf{d}, \mathbf{v}]$ may vanish, as in the next example. In that case, another candidate is required (for instance $\boldsymbol{\omega} = \boldsymbol{\varepsilon} : (\mathbf{d}_2\mathbf{c}_3)$ in the next example). Our methodology relies on Theorem 4.8 and is exhaustive.

Example 6.13. Consider the degenerate monoclinic Elasticity tensor (5.6), where $\mathbf{d}' = \mathbf{v}' = 0$. A non-vanishing first-order covariant is $\boldsymbol{\omega} = \boldsymbol{\varepsilon} : (\mathbf{d}_2\mathbf{c}_3)$ which writes

$$\boldsymbol{\omega} = 10^7 \begin{pmatrix} 16.727 \\ -7.71214 \\ 65.9218 \end{pmatrix}.$$

Set

$$\mathbf{n} = \frac{1}{\|\boldsymbol{\omega}\|} \boldsymbol{\omega}, \quad \mathbf{u} = \frac{1}{\sqrt{n_1^2 + n_2^2}} \begin{pmatrix} -n_2 \\ n_1 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \mathbf{n} \times \mathbf{u}.$$

We get then

$$\mathbf{n} = \begin{pmatrix} 0.24438 \\ -0.112674 \\ 0.963111 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0.418699 \\ 0.908125 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -0.874625 \\ 0.403254 \\ 0.269104 \end{pmatrix}.$$

and

$$g = \begin{pmatrix} 0.4187 & 0.90812 & 0 \\ -0.87463 & 0.40325 & 0.2691 \\ 0.24438 & -0.11267 & 0.96311 \end{pmatrix}.$$

One can check that $(\mathbf{E}_{mono}^\gamma)_{\mathbb{Z}_2} = g \star \mathbf{E}_{mono}$ writes

$$[(\mathbf{E}_{mono}^\gamma)_{\mathbb{Z}_2}] = \begin{pmatrix} 299.7 & 68.6 & 142.1 & 0 & 0 & -42. \\ 68.6 & 281.3 & 160.5 & 0 & 0 & 41.1 \\ 142.1 & 160.5 & 207.7 & 0 & 0 & 0.9 \\ 0 & 0 & 0 & 151.8 & 0.9 & 0 \\ 0 & 0 & 0 & 0.9 & 133.4 & 0 \\ -42. & 41.1 & 0.9 & 0 & 0 & 59.9 \end{pmatrix} \quad (\text{GPa}),$$

which is the normal form (2.8).

Remark 6.14. Recall that, following [21, 14], an additional zero can be placed in the normal form $\widehat{\mathbf{E}} = \mathbf{E}_{\mathbb{Z}_2}$ of a monoclinic tensor. This is due to the fact that any rotation around the third axis $\mathbf{n} = \mathbf{e}_3$ of the normal form (2.8) does not change the shape of this normal form. If either

$$\widehat{E}_{2233} - \widehat{E}_{1133} \quad \text{or} \quad \widehat{E}_{2323} - \widehat{E}_{1313}$$

does not vanish, we can look for a rotation $\mathbf{r}(\mathbf{e}_3, \theta_{36}) \star \widehat{\mathbf{E}}$ where

$$\tan(2\theta_{36}) = 2\widehat{E}_{3312} / (\widehat{E}_{2233} - \widehat{E}_{1133})$$

so that the component $(\mathbf{r}(\mathbf{e}_3, \theta_{36}) \star \widehat{\mathbf{E}})_{3312}$, in row 3, column 6 and row 6, column 3 of the new normal form (2.8) vanishes. Or, we can look for a rotation $\mathbf{r}(\mathbf{e}_3, \theta_{45}) \star \widehat{\mathbf{E}}$ where

$$\tan(2\theta_{45}) = 2\widehat{E}_{1323} / (\widehat{E}_{2323} - \widehat{E}_{1313})$$

so that the component $(\mathbf{r}(\mathbf{e}_3, \theta_{45}) \star \widehat{\mathbf{E}})_{1323}$ in row 4, column 5 and row 5, column 4 of the new normal form (2.8) vanishes.

CONCLUSION

We have formulated effective algorithms to recover the normal form of an Elasticity tensor, measured in any basis, provided that we know to which symmetry class it belongs to (this other problem having been solved, by the way, in a previous work [23]). Thanks to the definition of the generalized cross product (1.4) between totally symmetric tensors, a quite simple method has been proposed for Elasticity tensors with cubic symmetry, which required only to solve a linear system in five variables and diagonalize a three-dimensional symmetric matrix.

Moreover, a simple algorithm has been provided for each symmetry class of the Elasticity tensor. These procedures are moreover *exhaustive*. In particular, all the degenerate cases (when second-order covariants, such as \mathbf{d}' and \mathbf{v}' , or first-order covariants such as $\boldsymbol{\varepsilon} : [\mathbf{d}, \mathbf{v}]$ vanish) are handled. To formulate and prove these results, we have used the families of covariants derived in [23], which were crucial to establish necessary and sufficient conditions for an Elasticity tensor to belong to a given symmetry class.

Besides, we have illustrated our methods, for each symmetry class, by applying them on experimental Elasticity tensors found in the literature. More generally, applying the above

procedure to a given Elasticity tensor \mathbf{E} allows to recognize a normal form for it and is a way to determine its symmetry class.

APPENDIX A. HARMONIC COMPONENTS OF CONSIDERED ELASTICITY TENSORS

In this section, all the linear covariants \mathbf{d} , \mathbf{v} , \mathbf{H} are given in GPa and the fourth-order harmonic part \mathbf{H} is expressed in Voigt's representation.

Cubic approximation (5.2): $\mathbf{d}' = \mathbf{v}' = 0$, $\mathbf{d}_2' = 0$, $\text{tr } \mathbf{d} = 1531$, $\text{tr } \mathbf{v} = 1479$ and

$$(A.1) \quad [\mathbf{H}_{cubic}^\gamma] = \begin{pmatrix} -59.1358 & 38.9089 & 20.2269 & 6.39666 & 41.9737 & -21.1614 \\ 38.9089 & -75.3102 & 36.4013 & -27.7808 & 2.27754 & 16.6041 \\ 20.2269 & 36.4013 & -56.6282 & 21.3841 & -44.2512 & 4.55736 \\ 6.39666 & -27.7808 & 21.3841 & 36.4013 & 4.55736 & 2.27754 \\ 41.9737 & 2.27754 & -44.2512 & 4.55736 & 20.2269 & 6.39666 \\ -21.1614 & 16.6041 & 4.55736 & 2.27754 & 6.39666 & 38.9089 \end{pmatrix}$$

Tetragonal approximation (5.3): $\mathbf{d}' = \mathbf{v}' = 0$, $\text{tr } \mathbf{d} = 1531$, $\text{tr } \mathbf{v} = 1479$ and

$$(A.2) \quad [\mathbf{H}_{tetra}^\gamma] = \begin{pmatrix} -59.9342 & 35.8495 & 24.0847 & 5.8239 & 46.7414 & -21.0897 \\ 35.8495 & -69.6028 & 33.7533 & -25.7103 & 1.8896 & 15.3674 \\ 24.0847 & 33.7533 & -57.8381 & 19.8864 & -48.631 & 5.7223 \\ 5.8239 & -25.7103 & 19.8864 & 33.7533 & 5.7223 & 1.8896 \\ 46.7414 & 1.8896 & -48.631 & 5.7223 & 24.0847 & 5.8239 \\ -21.0897 & 15.3674 & 5.7223 & 1.8896 & 5.8239 & 35.8495 \end{pmatrix}$$

First orthotropic approximation (5.4): $\mathbf{d}' = \mathbf{v}' = 0$, $\text{tr } \mathbf{d} = 1531$, $\text{tr } \mathbf{v} = 1479$ and

$$(A.3) \quad [\mathbf{H}_{orth}^\gamma] = \begin{pmatrix} -57.9586 & 33.959 & 23.9997 & 5.3342 & 46.3021 & -20.3543 \\ 33.959 & -69.5995 & 35.6405 & -26.2801 & 2.88311 & 14.4327 \\ 23.9997 & 35.6405 & -59.6402 & 20.9459 & -49.1853 & 5.92151 \\ 5.3342 & -26.2801 & 20.9459 & 35.6405 & 5.92151 & 2.88311 \\ 46.3021 & 2.88311 & -49.1853 & 5.92151 & 23.9997 & 5.3342 \\ -20.3543 & 14.4327 & 5.92151 & 2.88311 & 5.3342 & 33.959 \end{pmatrix}$$

Second orthotropic approximation (5.5): $\text{tr } \mathbf{d} = 1531$, $\text{tr } \mathbf{v} = 1479$,

$$\mathbf{d}' = \begin{pmatrix} -3.6837 & -0.661831 & 1.37627 \\ -0.661831 & 1.96893 & 0.158989 \\ 1.37627 & 0.158989 & 1.71477 \end{pmatrix}, \quad \mathbf{v}' = \begin{pmatrix} -3.31669 & -0.8154 & -0.112441 \\ -0.8154 & 6.51238 & 1.36466 \\ -0.112441 & 1.36466 & -3.19569 \end{pmatrix}$$

and $\mathbf{H} = \mathbf{H}_{cubic}^\gamma$ is given by (A.1) (in particular $\mathbf{d}_2' = 0$).

Monoclinic approximation (5.6): $\mathbf{d}' = \mathbf{v}' = 0$, $\text{tr } \mathbf{d} = 1531$, $\text{tr } \mathbf{v} = 1479$ and

$$(A.4) \quad [\mathbf{H}_{mono}^\gamma] = \begin{pmatrix} -58.7344 & 34.9674 & 23.767 & 0.6715 & 47.7714 & -18.1515 \\ 34.9674 & -67.8968 & 32.9294 & -26.3969 & 4.4758 & 18.2628 \\ 23.767 & 32.9294 & -56.6964 & 25.7254 & -52.2472 & -0.1113 \\ 0.6715 & -26.3969 & 25.7254 & 32.9294 & -0.1113 & 4.4758 \\ 47.7714 & 4.4758 & -52.2472 & -0.1113 & 23.767 & 0.6715 \\ -18.1515 & 18.2628 & -0.1113 & 4.4758 & 0.6715 & 34.9674 \end{pmatrix}$$

Trigonal approximation of α -quartz Elasticity tensor (5.7): $\text{tr } \mathbf{d} = 34.72$, $\text{tr } \mathbf{v} = 59.24$,

$$\mathbf{d}' = \begin{pmatrix} -0.764933 & 0.3098 & 0.4514 \\ 0.3098 & -0.0727333 & 1.1811 \\ 0.4514 & 1.1811 & 0.837667 \end{pmatrix}, \quad \mathbf{v}' = \begin{pmatrix} -1.02767 & 0.4162 & 0.6064 \\ 0.4162 & -0.0976667 & 1.5867 \\ 0.6064 & 1.5867 & 1.12533 \end{pmatrix}$$

and

$$(A.5) \quad [\mathbf{H}_{trig}^\alpha] = \begin{pmatrix} -1.4953 & -0.0086 & 1.504 & -0.0148 & -0.2917 & -0.8173 \\ -0.0086 & 0.6713 & -0.6626 & -0.1899 & -0.0484 & 2.0042 \\ 1.504 & -0.6626 & -0.8413 & 0.2046 & 0.3402 & -1.187 \\ -0.0148 & -0.1899 & 0.2046 & -0.6626 & -1.187 & -0.0484 \\ -0.2917 & -0.0484 & 0.3402 & -1.187 & 1.504 & -0.0148 \\ -0.8173 & 2.0042 & -1.187 & -0.0484 & -0.0148 & -0.0086 \end{pmatrix}$$

Transversely approximation of Elasticity tensor (5.8): $\text{tr } \mathbf{d} = 6.0707$, $\text{tr } \mathbf{v} = 6.4911$,

$$\mathbf{d}' = \begin{pmatrix} 0.221833 & -0.0745 & -0.2495 \\ -0.0745 & 0.235733 & -0.2272 \\ -0.2495 & -0.2272 & -0.457567 \end{pmatrix}, \quad \mathbf{v}' = \begin{pmatrix} 0.1507 & -0.0505 & -0.1695 \\ -0.0505 & 0.1601 & -0.1543 \\ -0.1695 & -0.1543 & -0.3108 \end{pmatrix}$$

and

$$[\mathbf{H}_{TI}^{KS}] = \begin{pmatrix} 0.0176 & 0.0123 & -0.0299 & -0.0138 & -0.0969 & -0.0289 \\ 0.0123 & 0.0287 & -0.0409 & -0.0923 & -0.0195 & -0.0302 \\ -0.0299 & -0.0409 & 0.0708 & 0.106 & 0.1165 & 0.0592 \\ -0.0138 & -0.0923 & 0.106 & -0.0409 & 0.0592 & -0.0195 \\ -0.0969 & -0.0195 & 0.1165 & 0.0592 & -0.0299 & -0.0138 \\ -0.0289 & -0.0302 & 0.0592 & -0.0195 & -0.0138 & 0.0123 \end{pmatrix}$$

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