



HAL
open science

Mathematical modeling of musical fountains and light organs – where is the M in interdisciplinary STEM projects?

Wolfgang Bock, Martin Bracke, Patrick Capraro

► To cite this version:

Wolfgang Bock, Martin Bracke, Patrick Capraro. Mathematical modeling of musical fountains and light organs – where is the M in interdisciplinary STEM projects?. Eleventh Congress of the European Society for Research in Mathematics Education, Utrecht University, Feb 2019, Utrecht, Netherlands. hal-02410321

HAL Id: hal-02410321

<https://hal.science/hal-02410321>

Submitted on 13 Dec 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Mathematical modeling of musical fountains and light organs – where is the M in interdisciplinary STEM projects?

Wolfgang Bock, Martin Bracke and Patrick Capraro

Technische Universität Kaiserslautern, Department of Mathematics, Germany;

bock@mathematik.uni-kl.de, bracke@mathematik.uni-kl.de, capraro@mathematik.uni-kl.de

We discuss a modeling project, in which students have to build a device that acts simultaneously to music – either a musical fountain or a light organ. We show how the incorporated university level mathematics can be reduced to the level of secondary education. Being product oriented, the modeling activity has a strong interdisciplinary character. We focus on two implementations of the project with high school students and discuss their results. Here we draw the attention to the role of mathematics in a STEM project, where the presence of programming and engineering seems more obvious than the mathematical content.

Keywords: Interdisciplinarity, modeling, product orientation.

Introduction

Some of the most important real-life applications of mathematical modeling we find in engineering and related activities, where products are developed and improved. At the university of Kaiserslautern industrial mathematics plays an important role in research and teaching. It also had an influence in mathematical modeling activities with high school students (see Bock & Bracke, 2015). In these cases, the process of mathematical modeling is closely connected to the product that should be developed.

In Kotler et.al. (2006), p. 230, the term *product* is defined from an economical point of view as “anything that can be offered to a market that might satisfy a want or need”. Hence a product can take various forms, it can be a device that is actually being assembled, it can be a computer software or just the answer to a question. Here the production process closely affects the modeling activity. Even a sophisticated mathematical model has its flaws, if it can’t be applied in the desired situation or if it doesn’t solve the given problem adequately. Furthermore, the more production steps are needed to advance from the mathematical model to the product, the more impact the production process has on the mathematical model (compare to Bock, Bracke & Capraro, 2017).

Product oriented mathematical modeling immediately calls for high interdisciplinarity and involves all aspects of STEM, especially if the product is not only a theoretical concept, but the implementation of the solution actually takes place (Bock, Bracke, Capraro & Lantau, 2017).

In our technology driven world, complex computations can easily be done by computers and other electronic devices. Often there is no need for the programmer (or engineer) to know all the details of the involved mathematical tools. Here the question arises, if product oriented mathematical modeling and engineering in general can be done with a minimum of mathematical activity (see also Tosmour-Bayazit & Ubuz, 2013).

Here we present a modeling project, in which students have to build a musical fountain or a light organ. Both projects contain similar mathematics at a university level (Fourier analysis), which can be hidden in numeric tools and used by students with a basic mathematical knowledge at a secondary school level. The project has a strong interdisciplinary character, since programming is essential for data processing, and building a working model of the product or at least creating a simulation is requested.

We have a look at two implementations of the project. We find that even if the computer is used in a large extent, there still are enough mathematical challenges for students.

The modeling task and mathematical background

The students are given the task of modeling the regulation of a musical fountain or a light organ (both lead to similar mathematical problems). The students get a short introduction in a programming language to be able to read audio files and get access to the raw data. Furthermore we teach basic ideas of Fourier analysis to enable the students to compute frequency spectra and analyze them.

```
from scipy.io import wavfile
from scipy.fftpack import fft
import matplotlib.pyplot as plt
import math
import numpy as np
[Fr,g]=wavfile.read('test.wav')           #Fr=frame rate, g=sample data
if(len(g.shape)==2):                       #if stereo, combine the channels
    channel1=np.array(g[:,0])
    channel2=np.array(g[:,1])
    g=channel1+channel2
G=fft(g)                                   #Fourier transform
p=plt.plot(abs(G))                         #plot sqrt(A_n^2+B_n^2)
plt.show()
```

Listing 1: A code example in python. The program reads a wav-file and plots the frequency spectrum.

Fundamental is the Fourier series, that states that a T -periodic function g (with certain conditions to integrability or Lipschitz-continuity) can be decomposed to a series of trigonometric functions

$$g(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(2\pi nft) + B_n \sin(2\pi nft), \quad f = \frac{1}{T}, \quad (1)$$

where the coefficients A_n , B_n give the amplitude of the corresponding frequency nf . By the orthogonality of trigonometric functions, we can derive formulae for the coefficients

$$A_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g(t) \cos(n2\pi ft) dt, \quad B_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g(t) \sin(n2\pi ft) dt. \quad (2)$$

From this the Fourier transform (3.1) arises, which gives the frequency spectrum G of the function g . If we process the data g on a computer, it isn't given as a function but rather as a vector with finitely many entries. In this case we have the discrete Fourier transform (3.2), where the integral becomes a sum

$$G(f) = \int_{-\infty}^{\infty} g(t) \exp(2\pi itf) dt, \quad (3.1) \quad G_m = \sum_{t=1}^n g_t \exp\left(\frac{2\pi i}{n}(t-1)(m-1)\right). \quad (3.2)$$

Two more results about discrete Fourier transform are substantial when the spectra are going to be interpreted. For a data vector g with n real valued entries, we have $G_m = G_{n+1-m}^*$, i.e. the second half

of the spectrum is just a (complex conjugate) copy of the first. Furthermore, we have the sampling theorem, which states that for a data sample g with frame rate Fr , the spectrum can be computed for frequencies up to $Fr/2$. For a data set of n samples, the value $G_{n/2}$ corresponds to the frequency $Fr/2$.

Breaking down Fourier analysis to high school level

Fourier analysis involves mathematical concepts that are located at the final years of high school education or even at university level, like integrals, complex numbers and convergence of functions. Nevertheless, if we see Fourier analysis as a numerical tool, the barriers to implement it in school are quite low. Fourier transform can be introduced as a black box. Certainly, for all well known programming languages there already exists an implementation of the code and that's all we need to know. On the other hand, if there is enough time to do so, the discrete Fourier transform (equation 3.2) can easily be understood and applied as soon as the students know the concept of trigonometric functions.

Understanding overtones and the Fourier series

The Fourier series is strongly connected to instruments and the concept of overtones: If we play a tone on an instrument, we do not only produce the fundamental frequency f of the tone itself, but also overtones with frequencies $2f$, $3f$, $4f$, etc. From a mathematical point of view this is obvious, since acoustic waves are represented by periodic functions. The period of the acoustic wave is the same as the period of the fundamental frequency. Hence the overtones do not disturb the pitch of the observed tone.

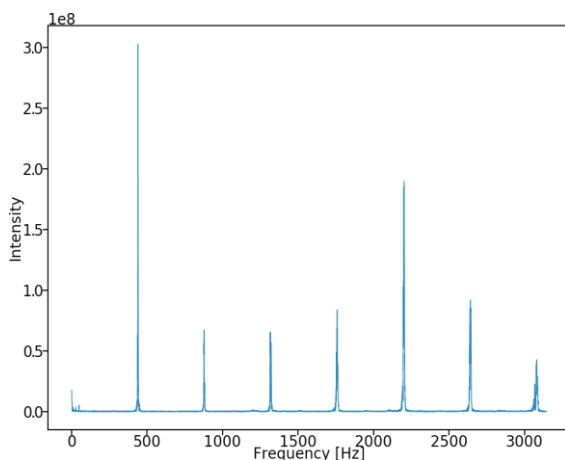


Figure 1: The frequency spectrum of a violin. The fundamental frequency is 440 Hz. The peaks around 0 Hz are possibly due to edge effects or noise.

The characteristic sound of an instrument is strongly determined by the relative intensities of these overtones. This real-life phenomenon can be used as a motivation to understand and use the ideas of Fourier series.

If the students do not know the concept of overtones, we could create and analyze frequency spectra (e.g. by playing tones on instruments, recording them and using computer software to obtain the spectra, compare to Fig. 1). The pattern in the graphs that are created could motivate a further investigation of the mathematical concepts behind. This could also be the starting point for an interdisciplinary project between mathematics, music and physics classes.

How to deal with complex numbers?

Fourier transform involves complex numbers and the complex exponential function. As we know, the exponential function with imaginary exponent decomposes in real and imaginary parts with trigonometric functions and real valued arguments. Hence, we have the interpretation, that the vector entries in G take 2-dimensional real values

$$G_m = \left(\sum_{t=1}^n g_t \cos\left(\frac{2\pi}{n}(t-1)(m-1)\right), \sum_{t=1}^n g_t \sin\left(\frac{2\pi}{n}(t-1)(m-1)\right) \right). \quad (4)$$

The intensity of the frequency represented by G_m is then given by the length of the vector.

Deriving formulae for the Fourier coefficients

Equation (2) is a consequence of the orthogonality relations of trigonometric functions

$$\int_{-\pi}^{\pi} \cos(nt) \cos(kt) dt = \int_{-\pi}^{\pi} \sin(nt) \sin(kt) dt = \begin{cases} \pi, & n = k, \\ 0, & n \neq k, \end{cases} \quad \int_{-\pi}^{\pi} \cos(nt) \sin(kt) dt = 0.$$

Older students who know the concept of integrals can prove these identities. The basic idea is to use integration by parts two times, until right hand side and left-hand side show the same integral (with a prefactor and an additive constant).

With equation (2) the definition of the Fourier transform can easily be understood. Hence in this case there is no need to see the Fourier transform as a black box tool or a formula given by the teacher without any theoretical background.

Sampling theorem

If the students are using a discrete Fourier transform to obtain frequency spectra, they surely will encounter two questions: Why is there an axial symmetry in the spectrum? And how can we read the frequencies on our axes? To answer these questions, we can give the students the above mentioned results about the Fourier transformed data vector G . A quantitative discussion why these results are true is omitted due to the complexity of the involved mathematics.

Two examples for interdisciplinary modeling projects

In Kaiserslautern, mathematical modeling activities involving acoustic phenomena have been implemented in various situations with high school students and university students. Here we present two cases.

Modeling of a musical fountain with a time frame of 15 hours

This implementation was done in August 2018 with 6 high school students who participated at a *Fraunhofer math talent school for girls*. The participating students (upper secondary level) came from different schools in Germany and worked on 4 different projects at the *Fraunhofer institute for industrial mathematics* in Kaiserslautern. The students had the opportunity to vote for their preferred projects. In the musical fountain project 5 of 6 students told us that they had a strong connection to music (all 5 played an instrument) and therefore chose this project. Also, the sixth student voted for the project as her first choice. Some of the students had programming experience, some of them didn't. The group was supervised by two university members.

There were 4 working phases (3 hours each) at the first two days. At the third day, there were 3 hours for preparation and execution of the presentations. Due to the restricted time frame, the students were given selected mathematical input at the beginning of the course.

First working phase: Introduction to Fourier analysis (frontal instruction); reading audio files and plotting spectra with python (frontal instruction and exercise); after a strongly teacher driven

opening, the students had the chance to create audio files and analyze them with python or other computer software, like audacity. They did this in groups of 2 students.

Second working phase: The students were asked to develop mathematical criteria that could be used for the regulation of the fountain, but had no further instructions. They chose to stay in the previously formed groups and analyze different aspects of the graphs they could create (group work). After first ideas were established, an additional programming course in python was given (frontal instruction and exercise).

Third working phase: The ideas of the second working phase were implemented in python code. Some ideas were dropped, others could be improved. One group found mathematical techniques to smooth the highly irregular amplitude-time-function, another one worked on an algorithm that could find an underlying rhythm in a song.

Fourth working phase: It was discussed, if the students wished to build a real-life model of a small fountain with 4 solenoid valves. The students preferred to do a computer visualization of their mathematical results instead. The students used the remaining time to further develop their python code and to decide how their result should be used to regulate the fountain. They also started to work on the visualization.

Presentation day: The students had 2 hours to prepare their presentations and finish the work on the visualization of the fountain regulation. Each of the 4 project groups had 15 min. for their presentation.

In the simulation, the analyzed song and the visualization of the processed data where replayed simultaneously. It became apparent that the data and the perceived sound matched adequately.

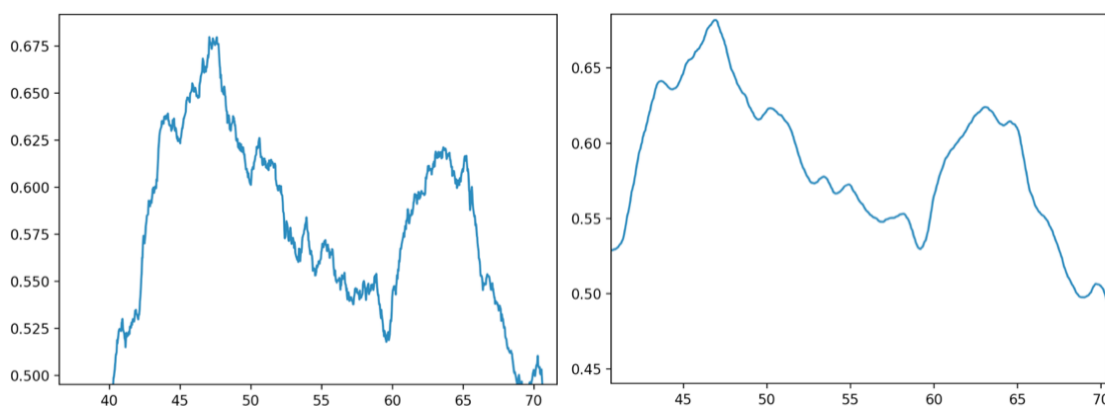


Figure 2: Loudness of a song vs. time. The students wanted to use *increasing and decreasing loudness* to control the fountain. Since the irregularity of the sample data caused problems, they found methods to smooth the data (left: first iteration of the smoothing process, right: second iteration).

Modeling of a light organ with a time frame of 5 days

The light organ was one of 8 projects at a mathematical modeling week in February 2018. At a modeling week the students (upper secondary level) spend 5 days at a youth hostel and work on their own responsibility for 4 days (from 9 am to 18:30 pm with several breaks for lunch and coffee as well as an excursion on one afternoon). On the 5th day the presentations are given.

The students can choose their preferred project. There were 4 students working on the light organ. They chose the project with different motivations, concentrating on different aspects of STEM. They were supervised by one university member and by 2 teachers. For the teachers, the participation was part of a teacher training.

With a rather generous time frame, the projects at the modeling week are open to a great extent. The students were asked to build a light organ that would work without a microphone. Instead they should get the data directly from the electronic device, where the song is played, or use digital information. They were provided with electronic components such as a Raspberry Pi, Arduino microcontrollers, LEDs, cables, resistors, etc.

One of the students immediately took the initiative and started to work on his idea of building electric circuits that would work as frequency filters. He started with some first experiments involving the Arduino microcontroller and the question how to record the audio signal. The other students helped him but didn't participate in the creative process.

During the last hours of the first day the group experienced a crisis: 2 of the students were unsatisfied with the progress that was mainly based on ideas from physics and electrical engineering. Both wanted to focus on mathematical and computational aspects of the project. They discussed their ideas with the supervisor. Together a solution was worked out that would be based on the audio information that is stored in music files. From this point on the students worked on two different solutions.

At the second day, the students with the mathematical and computational focus asked for some mathematical background about frequencies. They were given the same information as in the musical fountain project. Only this time not as frontal instructions, but rather in a *piece by piece* manner. The students were given some mathematical input and had the chance to do some experiments or online research. When they asked deeper questions, they received further details.

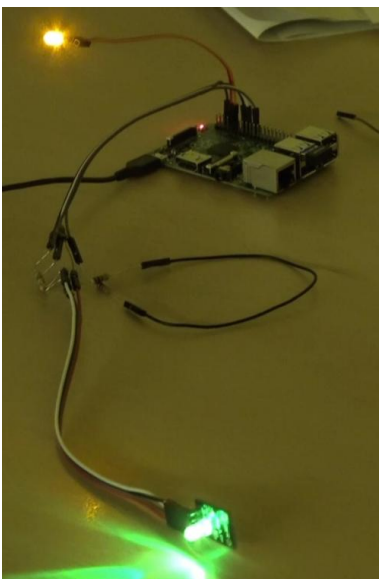


Figure 3: The light organ. The yellow LED is controlled by the loudness, the RGB LED by the frequency-spectrum.

Later they were able to implement a Fourier transform in python and run the program on a Raspberry Pi. At this point the students worked on 2 tasks simultaneously, one improving the python program, the other controlling an RGB LED on the Raspberry Pi. Questions to the supervisor were mainly about the involved programming languages and the operating system of the Raspberry Pi. At the end, they could split a piece of music in time intervals, compare the intensity of different frequency intervals and assign color and brightness of the LED to the frequency values.

From a mathematical point of view, the students had to deal with problems related to esthetical questions: How should the frequencies be assigned to different colors, such that the light effects won't be too monotonous? If the brightness of the lights is going to be related to the loudness of the music, which level of loudness should be defined as 100%? Are these thoughts independent of the music

genre? Is it sufficient to analyze the song locally (i.e. each time frame individually) or do we need global data?

To questions like these there is no right answer, but rather a wide range of possible approaches. Here mathematical experience helps us to give a structure to our problems and quantize the vague questions asked above. Mathematical skills help us to evaluate the information that is hidden in the given data

The other group with the focus on electrical engineering worked almost without supervision. Their main problem was the lack of capacitors with a suitable capacity. This could be solved when one of the supervising teachers asked at a nearby school if she could borrow some material.

In this group mathematical activities took place on a rather basic level, e.g. when the students had to choose the threshold of their frequency filters and compute the resistance and capacity of the electric components that were needed.

Misconceptions of the students about algorithms and computers

When confronted with a subject that most people know very well from their everyday life (like music in our case), the students come up rather quickly with interesting ideas. Some of them are based on the misconception that computers can do complicated tasks on their own, while the programmer doesn't need to fully understand the situation.

In the musical fountain project, there were ideas about recognizing different instruments in the music or finding certain patterns in the rhythm automatically. Those tasks are difficult to implement in an algorithm. The students were surprised how complicated their mathematical model had to be to find a quite simple rhythm in their song (and they didn't even elaborate further steps to make the algorithm work for arbitrary music). In the light organ project, similar situations took place.

The project clearly helped the students to get a better idea of what algorithms are capable of and where the limitations could be. One of the main consequences in the project was that if we want an automated solution to a problem we have to provide a detailed mathematical description.

Conclusion

Using acoustic phenomena in terms of mathematical modeling is a rich field for interdisciplinary activities. It naturally includes physics and music. The link between both topics we find in Fourier analysis, where concepts of music theory, like overtones, become obvious in the mathematical description of waves.

Computer science (or rather programming) is not only helpful, but essential to manage huge amounts of data and extensive computations. Technical aspects can be included by posing a product oriented problem. Hence all aspects of STEM can be addressed.

If all calculations are done by computers, one might fear that the mathematical aspects of such a project will only appear in theory lessons, but won't be applied by the students. In our examples, we have seen that there still is a strong potential to include mathematical aspects that go beyond the computational part. The raw data could be in a bad state for data processing (compare to Figure 2), where elaborate mathematical ideas are needed to overcome the problems. On the other hand, like

in the case of the light organ, assumptions have to be made and decisions have to be taken, where an intuitive evaluation of the given data is necessary.

Of course, with a modeling task that is openly posed, there always is the possibility that students find another way of solving the problem. One part of the light organ group used a completely technology based approach, where no mathematics beyond basic arithmetic was needed. On the other hand, from our experience it is essential to let the students choose their approach with the least amount of regulation. Otherwise, if the teacher tries to force certain solutions and mathematical tools, this could have a devastating impact on the students' motivation.

This can lead to serious difficulties for the teacher, since he or she must be able to adapt the ideas of the students quite fast and try to figure out possible outcomes or obstacles for an approach that wasn't considered beforehand. If this new approach requires expert knowledge from another discipline, either the teacher has to have that knowledge (or acquire it fast enough). If this isn't the case, the teacher must be confident that the students can work on their solution relying on their own expertise (why would they have chosen this approach if they don't have any knowledge about the topic?). Additionally, not knowing all the details of a solution for a given problem is an unfamiliar situation for most teachers. It requires a certain amount of self-confidence to let the students proceed anyway.

Acknowledgements

The close collaboration with the Fraunhofer Institute for Industrial Mathematics as well as the financial support of the European Social Fund for the modeling weeks are thankfully acknowledged.

References

Armstrong, G., Kotler, P., Harker, M. and Brennan, R. (2009). *Marketing: An introduction*. Harlow, England: Pearson Education.

Bock, W. and Bracke, M. (2015). *Applied school mathematics – made in Kaiserslautern*. In: Neunzert, H., Prätzel-Wolters D. (Eds.), *Currents in Industrial Mathematics: From Concepts to Research to Education*. Berlin, Heidelberg: Springer.

Bock, W., Bracke, M. and Capraro, P. (2017). Product orientation in modeling tasks. In Dooley, T. and Gueudet, G. (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education* (pp. 1028–1031). Dublin: ERME.

Bock, W., Bracke, M., Capraro, P. and Lantau, J.-M. (2017). How does a product-based and action-orientated modelling project promote interdisciplinary teaching and learning? To appear in: *Proceedings ICTMA-18*.

Tosmour-Bayazit, N. and Ubuz, B. (2013). Practicing engineer's perspective on mathematics and mathematics education in college. *Journal of STEM Education*, 14(3), 34–40.