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# Situation Specific-Skills Working Backward Reasoning: The Student's Perceptions and Affect Dimension

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*This study focuses on the perceptions of mathematical students on the use of backward reasoning. Since an integrative model of interactions between emotions and thinking is showed new understanding of cognitive phenomena of backward reasoning. Based on a questionnaire to assess the mathematical cognitive processes, difficulties of backward reasoning and emotions among mathematical undergraduate students, the rule structure obtained from the use of the CRT (Classification and Regression Tree) methodology analysis is reported.*

*Keywords: Backward reasoning, Heuristics, Cognition and affect, Mathematics, Perception, Regression trees*

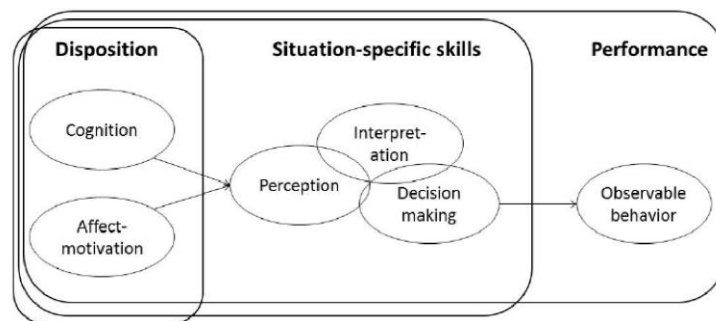
## Introduction

The purpose of this study was to determine the perceptions of mathematical students on the use of backward reasoning and how this perception may mediate between the disposition and performance of mathematical and cognitive processes involved in this type of reasoning. Backward reasoning has great potential in the study of mathematics - it can be used to improve student achievement and to help develop mathematical argumentation and proof processes. Among the challenges in university teaching we focus on mathematical thinking, where learning the method of analysis is a critical issue (Antonini, 2011, Peckhaus, 2002, Wickelgren, 1974, Xu, Xing & Van Der Schaar, 2016). Wickelgren (1974) analyses and systematizes the methods of problem solving, introducing contradiction and working backwards as two strategies that require operations to be performed on the expected result, as well as on the information given. Addressing mathematics and engineering students, he examines how the explicit teaching of theoretical and practical analyses of problems can overcome the difficulties experienced by the students. This author indicates it is more difficult to work backwards than to work forwards, so it is necessary to offer students a large class of problems to which the method of working backwards is appropriate, such as games of the Nim family of games presented here. In our previous research papers at university level (Gómez-Chacón, 2017; Barbero & Gómez-Chacón, 2018), it was identified that explicit teaching of theoretical aspects is not enough, that there might be factors in the cognitive and affect interplay, which would inevitably cause difficulties for students to construct and work backwards. These studies showed emotions continually exert numerous so-called operator effects, both linear and nonlinear, on attentional activity and on the ability to perceive relative to goal-path obstacles and overcoming them. Understanding is linked with the appraisal of their ability to influence (control dimension), with their ability to predict, and with mental flexibility.

Hence, a conscious integration of backward reasoning when learning mathematics at university, raises the need for an articulation between dispositional and cognitive aspects. In this study we try to contribute to a more comprehensive understanding of the origins of the difficulties and identify conceptual and methodological challenges involved in assessing competencies about backward reasoning in higher education, using strategy games as specific tasks.

When integrating a cognitive and a situated perspective, Blömeke et al. (2015) suggested considering competence as a continuum (cf. Fig. 1), they stated that “processes such as the perception and interpretation of a specific situation, together with decision-making may mediate between disposition and performance” (p. 7). This framework considers competence as a multidimensional construct, and resolves the dichotomy of “disposition versus performance” as

follows: “[...] our notion of competence includes ‘criterion behaviour’ as well as the knowledge, cognitive skills and affective-motivational dispositions that underlie that behaviour” (Blömeke et al. 2015, p. 3). Following this understanding, a key role is assigned to situation-specific skills, as in-between processes that explain how dispositions are translated into classroom performance. The aim of this article is to focus on perception, as the way in which something is regarded, understood, or interpreted, as a situation-specific skill that is linked to students knowledge about backward reasoning and its practice. A set of variables measuring difficulties and appraisal cognition helps us to identify these relationships.



**Fig. 1 Competence modelled as a continuum (Blömeke et al. 2015a, p. 5)**

A specific methodological challenge in the context of the affect-cognition interplay is to explain the gradualness of the processing of affective mechanisms. This paper explores the use of decision trees to analyse data from questionnaires, which focus on the use of backward reasoning. Thus, here we will report our results in the form of a tree structure, providing rules to assess the state on the use of backward reasoning (perception, difficulties and mathematical and cognitive processes) in mathematical undergraduate students.

### **Characteristics of backward reasoning**

The typical mathematical thinking process that is used in the discovery phases, is the backward reasoning. Pappus was the mathematician who has contributed substantially to the clarification and exemplification of the method. In the seventh book of his *Collectio* (~340 AD) he deals with the topic of Heuristics (methods to solve the problems). He exemplifies the method of analysis and the method of synthesis, therefore making the development of this reasoning clearer. Pappus defines the method of analysis as follows: “In analysis, we start from what is required, we take it for granted; and we draw correspondences (ακολουθον) from it, and correspondences from the correspondences, until we reach a point that we can use as a starting point in synthesis. That is to say, in analysis we assume what is sought as already found (what we have to prove as true).” (elaboration by Polya, 1965 by Hintikka and Remes, 1974).

The concept of Backward Reasoning involves some characteristics that allow us to identify its development throughout the resolution of a task. Different philosophers and mathematicians from the ancient Greeks, through the authors of the 17th and 18th centuries to the year 2000 have studied its characteristics. The main features are the following:

- *Direction vs cause-effect.* In Pappus’ definition, the backward direction of reasoning is highlighted. This entails going from the end of the problem to its beginning. By applying the method, the premises of a certain idea are sought. In the 17th and 18th centuries, authors such as Arnauld and Nicole interpreted the method as a search for cause-effect relationships between ideas. By these, the connection between the notions in background and the problem are identified. The knowledge of the development of the resolution of the task and the effects and causes of each notion involved in the process arise (Beaney, 2018; Peckhaus, 2002).

- *Decomposition*. According to Plato and Pappus, this kind of reasoning allows for the reduction of the problem to its simplest components. The properties that define the relationships between the most complex and the simplest objects involved in it are identified by extracting and investigating the principles that are at the base of the task. Aristotle, for example, underlines the fact that "sometimes, to solve a geometrical problem, you can only analyse a figure", breaking it down into its basic components and understanding the different parts of it (Beaney, 2018).

- *Auxiliary elements*. Kant, Polya and Hintikka, focus their attention on a fundamental part of the process: the introduction of new elements. In the progressive and deductive processes all the bases are given and from these, consequences are elaborated. Unlike the backward reasoning, new notions appear and develop throughout the resolution at specific moments, according to the needs of the solver (Beaney, 2018; Hintikka & Remes, 1974).

- *Strategies*. Backward reasoning involves different problem-solving techniques: backtracking heuristic, method of Diaeresis, Reduction ad Absurdum (Beaney, 2018).

We will focus on the *backtracking heuristic*. This is the strategy of working backwards, it consists rather of doing some steps backwards in the process. These steps can be done starting from, the end of the problem, or during the process of resolution in combination with progressive steps.

## **Research questions and methodology**

### **Research Questions**

We particularly pursued the following research questions: RQ 1: What are students' perception of the use of the *backtracking heuristic*? RQ 2: What difficulties are more prevalent in the development of the *backtracking heuristic*? RQ 3: What are the affective, mathematical and cognitive processes that have impact on the development of the *backtracking heuristic*?

### **Participants and instrument**

Data was collected from 32 (19 women and 13 men, aged between 21 and 23) Caucasian undergraduates working toward a BSc. in mathematics. All of the participants were in their last year of academic study. They were following advanced courses in several areas of geometry, algebra, probability and analysis. With regard to solving problems, the students had been introduced to the problem solving heuristics and they received training as students in one subject related to advanced professional knowledge, practice and relationship skills relevant to teaching. They had not received any special training about backtracking heuristics.

The work dynamic started with individuals being given paper and pencil, with which they needed to resolve two games, each lasting one and a half hours. Fig. 2 shows the problem which we will analyse in the results section. Strategy games allow for the natural development of regressive reasoning. These games are disconnected from the mathematical content which forces the student to use their mathematical knowledge acquired in their university degree.

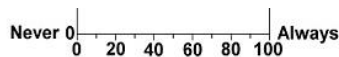
*The Triangular Solitaire* is a game for a single person that requires a board with 15 boxes, as the figure 2 shows.

	<p><i>These are the rules:</i></p> <ol style="list-style-type: none"> <li>1. Place the pawns in all boxes, except in the one marked in black.</li> <li>2. The player can move as many pawns as are the chances of jumping a pawn, adjacent to an empty box (along the line); at the same time, he "eats" and retreats from the table the pawn that was jumped. All pawns will move in this way. Pawns can move around the table.</li> </ol> <p><i>Target: The player wins when there is only one pawn on the table.</i></p>
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**Fig. 2. The Triangular Solitaire**

Students were given the game and asked to describe their approaches to solving the problem on protocols including: thought processes in the resolution, explanations of the difficulties they might face, and strategies they would use in order to solve with paper and pencil. Afterwards, each game was followed by a questionnaire focused on heuristics related to backward thinking and the difficulties that are generated during the process of solving problems, emotions and cognitive processes (Gómez-Chacón, 2017). The cognitive dimension refers to the characterization of the personal meanings of the subjects on the cognitive dimension of backtracking heuristic, or backward reasoning, and the cognitive appraisal processes of the interaction feeling puzzled.

1. Indicates how often you use the working backwards strategy when you are solving the problem



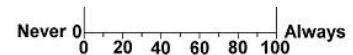
Now focus on the solving process of *Triangular Solitaire*

2- Have you used backward reasoning or backward strategy to solve the problem? Yes/No. Why?

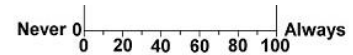
3- If you did not use the strategy, what type of processes did you consider to have failed to take account of in the *backtracking heuristic* in your solutions. You have to choose between the following items (You can justify your statements with "extracts of the problem".):

1. Actions involved basically in the determining of the mathematical model
2. Attainment of sufficient conditions
3. Actions of discovery
4. Recognition and explanation of the meaning of representational equivalence
5. Creation of the object solution
6. Formulation of axioms
7. Characterisation and establishment of relations
8. Attainment of justifications of adequate conditions in propositional equivalences.

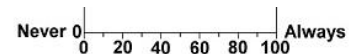
7. When you tried backward strategy, how well could you predict what was going to happen in this situation? (Ability to predict).



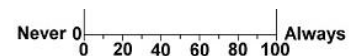
8. When you wanted to use backward strategy, how much did you feel there were problems that had to be solved before you could get what you wanted? (Goal-path obstacle).



9. When you were feeling puzzled, how much of an ability did you feel you had to switch between modes of thought and to simultaneously think about multiple concepts? (Mental flexibility).



10. Indicate the degree to which the following emotions were felt whilst solving the problem: confidence, pleasure, confusion, surprise.



**Fig. 3 Examples of some items from the questionnaire**

In Fig. 3 shows examples of some items from the questionnaire. Items 1, 2, 3 are focused on the perception of difficulties, (the difficulty in the creation of the solution, the actions of discovery, the characterisation and establishing of relationships, and the basic actions involved in determining the mathematical model can generate) while items 7, 8, 9 are referred to the processes of evaluation (cognitive appraisal) and item 10 about emotions. The cognitive dimensions of assessment were as follows: pleasantness, attention activity, control, certainty, goal-path obstacle, anticipated effort, and mental flexibility. Furthermore, emotions such as confidence, pleasure, confusion, surprise were assessed (Gómez-Chacón, 2017).

## Data analysis

Both qualitative and quantitative methods were used to address the subject of this study. This paper presents the quantitative analysis performed on the undergraduate students' written responses to the questionnaire. The CRT (*Classification and Regression Tree*) methodology is a data mining approach widely employed to develop 'IF-THEN' rule models in order to explain the behaviour of a variable of interest (the dependent variable) in terms of logical conditions over a set of explanatory or independent variables (see Breiman et al., 1984). Particularly, the CRT methodology allows for the determination of a subset of the available independent variables as well as a set of conditions over these variable's values that separate the data into groups as homogeneous as possible in terms of the values of the response or dependent variable. Regarding this aim, the CRT method performs successive dichotomous splits of the data by identifying both the independent variable and its cut-point that provide the greatest variability (i.e. variance) reduction at the split data groups verifying either condition (i.e. being greater or lower than such cut-point).

In this work we apply this regression tree methodology to develop a rule model capturing the relationships between a numerical dependent variable, measuring either the intensity of perception of using backward reasoning experienced by students while solving a mathematical problem, or a set of independent variables measuring the difficulties and mathematical and appraisal cognition. Different regression trees analysis were performed with SPSS to uncover the prescriptive nature of the variables. Two regression trees, together with their associated rule models, are reported next.

## Results

### Students' perception of the use of backtracking heuristics

In the comparison between the analysis of the protocols and the questionnaires, a discrepancy emerged between the real use of the backward strategy and the perception of its use by the students. 14 students (43.75%) were aware of using the strategy, 9 students (28.13%) were aware of not using the strategy, 4 students (12.5%) planned to use it, but it does not appear in the protocols and 5 students (15.63%) were not aware of having used it. This led us to track the students' perceptions in depth.

### Difficulties of mathematical cognition in backtracking heuristics

Table 1 shows the percentages of students (in the questionnaire) who, according to the cognitive aspects of reasoning have failed to consider the *backtracking heuristic* use in their solutions.

To analyse the relationships between the difficulties expressed by the students and their perception of the use of backtracking heuristics (BH), we will use the classification tree of Figure 4 (Classification and Regression Tree (CRT) analysis). For the interpretation of the classification tree, we should go looking at the nodes and branching them until the final leaves. First, we look at the root node 0 that describes the dependent variable: Perception of the use BH to solve the problem (Q1). It indicates that 56, 2% has this perception. Then, note that the data is split into nodes 1 and 2 depending on the variable "*I consider that my process lacked the characterisation and*

*establishment of relations*” (Q3\_7), indicating that this is the main predictor variable. The node 1 indicates that 75% of the students do not consider that the process *lacked the characterisation and establishment of relations*”. The 70,8% of these students have perception of the use of backtracking heuristics (BH)

Actions involved basically in the determining of the mathematical model	31%
Formulation of axioms	25%
Actions of discovery	22%
Characterisation and establishment of relations	22%
Attainment of sufficient conditions	19%
Creation of the object solution	16%
Attainment of justifications of adequate conditions in propositional equivalences.	16%
Recognition and explanation of the meaning of representational	13%

**Table 1. Mathematical cognition processes in backtracking heuristics**

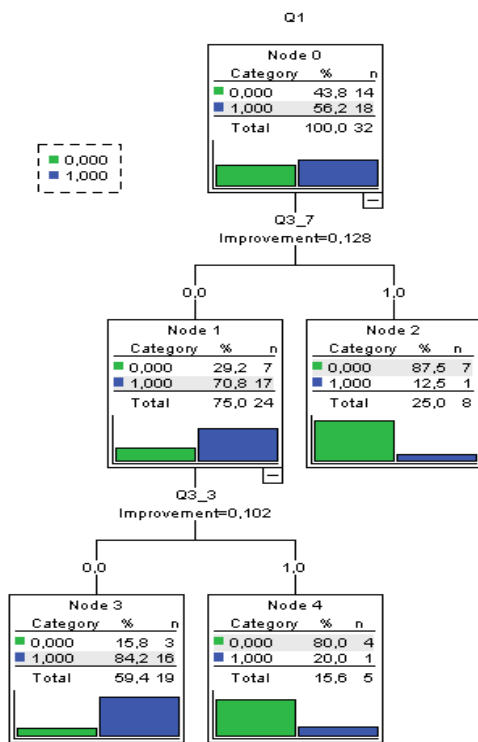
This node 1 is again split up into nodes 3 and 4 depending on variable Q3\_3 (*I consider that in my process there lacked the actions of discovery*”). We note in node 4 formed by the students that have difficulties in *the actions of discovery* (15,6% of the group) only have the perception of the use BH to solve the problem the 20%, , while students at node 3 that do not have this difficulty (59,4% of the group) have the perception of the use BH to solve the problem a 84,2%. These two nodes 3 and 4 are leaves that allow us to infer rules 2 and 3 below. Particularly, each path from the root of a decision tree to one of its leaves can be transformed into a rule simply by conjoining the conditions along the path to form the antecedent part, and taking the leaf’s mean as the rule prediction or consequent. Similarly, in order to define the rest of the rules, node 2 and the following ones are studied. The inferred rules are the following:

Rule 1 (node 2): IF *there are difficulties in the characterisation and establishment of relations* (Q3\_7) THEN the prediction of the use of BH is supported by 12,5% of students.

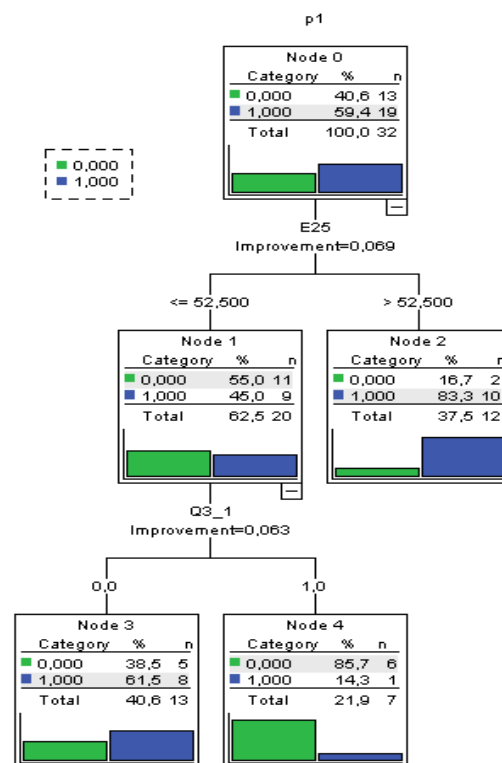
Rule 2 (node 3): IF *there are not difficulties in the characterisation and establishment of relations* (Q3\_7) AND *there are not difficulties in the actions of discovery* (Q3\_3) THEN the prediction of the use of BH is the 84,2% of students.

Rule 3 (node 4): IF *there are not difficulties in the characterisation and establishment of relations* (Q3\_7) AND *there are difficulties in the actions of discovery* (Q3\_3) THEN the prediction of the use of BH is the 20% of students.

As we described in section 2, *characterizations of backward reasoning*, there are some fundamental elements that characterise the development of this type of reasoning on which relationships are identified with the affective and cognitive aspects. These processes necessarily involve different cognitive processes and specific skills in the subject such as, mental flexibility, control or ability to identify the problems that need to be solved beforehand as well as feelings of confidence and pleasure.



**Fig. 4: Difficulties on mathematical cognition in backtracking heuristics according to the student's perception**



**Fig. 5: Cognitive appraisal dimensions affecting the real use of backtracking heuristics**

### Affective and cognitive appraisal dimensions influencing the real use of backtracking heuristics

In the process of solving the problem, results indicate that only a 47 % of students show confidence working backtracking heuristics, and a degree of pleasure of mean: 36.86, indicating that *actions involved basically in the determining of the mathematical model* is the main predictor variable. The pleasure is greater (mean: 47.81) for those who use models.

From the classification tree ((CRT) analysis), the exploration of the real use of backtracking heuristics and some of the cognitive appraisal dimensions (Fig.5), we can infer the following rules:

Rule 1 (node 2): IF (*Mental flexibility (E25)*(  $>52.50$ ) the prediction of the use of BH is the 83,3%.

Rule 2 (node 3): IF ((*Mental flexibility (E25)* $\leq 52.50$ )) AND (“*There are not difficulties in the actions involved basically in the determining of the mathematical model*” (*Q3\_1*)) THEN the prediction of the use of BH is the 61.5% of students.

Rule 2 (node 4): IF ((*Mental flexibility (E25)* $\leq 52.50$ )) AND (“*There are difficulties in the actions involved basically in the determining of the mathematical model*” (*Q3\_1*)) THEN the prediction of the use of BH is the 14,3% of students.

### Conclusions and discussion

The data shows that knowledge and use of backward reasoning are linked to situation-specific skills, namely, perception and planning of action through cognitive process. From the regression



trees analysis of the data, the following hierarchy of themes emerged as components of students' perceptions of the use of BH: associations between difficulties in the characterisation and establishment of mathematical relations and difficulties in the actions of discovery; and in the real use of BH: the person's ability of mental flexibility and the actions involved basically in the determining of the mathematical model. The perception of the use of BH is related to the difficulties while the real use of it is related to personal cognitive skills and specific concrete actions that entail the BH. The emotions, such as confidence and pleasure, arose out of information-oriented appraisals and the use of models or *actions involved basically in the determining of the mathematical model*.

These variables can significantly predict students' performance in backward reasoning in terms of: (a) identifying that backtracking heuristics are important in a problem solving situation; (b) using some characteristics of backward reasoning (mathematical processes, skills and the person's ability); and (c) making connections between broader features of backward reasoning to overcome specific difficulties.

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