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REDUCIBILITY OF ULTRA-DIFFERENTIABLE QUASIPERIODIC COCYCLES UNDER AN ADAPTED ARITHMETIC CONDITION

ABED BOUNEMOURA, CLAIRE CHAUDAUDRET, AND SHUQING LIANG

ABSTRACT. We prove a reducibility result for $sl(2, \mathbb{R})$ quasi-periodic cocycles close to a constant elliptic matrix in ultra-differentiable classes, under an adapted arithmetic condition which extends the Brjuno-Rüssmann condition in the analytic case. The proof is based on an elementary property of the fibered rotation number and deals with ultra-differentiable functions with a weighted Fourier norm. We also show that a weaker arithmetic condition is necessary for reducibility, and that it can be compared to a sufficient arithmetic condition.

1. INTRODUCTION

We will study the following time-quasiperiodic linear system, or quasiperiodic cocycle

$$\begin{cases} x'(t) = (A + F(\theta(t)))x(t), \\ \theta'(t) = \omega, \end{cases}$$

where $x \in \mathbb{R}^2$, $\theta \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ with an integer $d \geq 1$, $\omega \in \mathbb{R}^d$ is a non-resonant frequency vector, A is elliptic (meaning it is conjugated to some non-zero element in $so(2, \mathbb{R})$) and $F : \mathbb{T}^d \rightarrow sl(2, \mathbb{R})$ belongs to some ultra-differentiable class. Let us recall that $sl(2, \mathbb{R})$ is the Lie algebra of traceless matrices and $so(2, \mathbb{R})$ is the Lie sub-algebra of skew-symmetric matrices. Such a quasi-periodic cocycle is said to be reducible if the time-quasiperiodic linear system can be conjugated, by a time quasi-periodic transformation, to a constant (time-independent) linear system.

One of the main motivation for studying reducibility of quasi-periodic cocycles came from the Schrödinger equation

$$-y''(t) + q(\theta + \omega t)y(t) = Ey(t)$$

and the question of existence of so-called Floquet solutions (which always exist when $d = 1$); this is readily seen to be equivalent to the reducibility of a family of quasi-periodic cocycle depending on the "energy" parameter E . In a pioneering work, Dinaburg and Sinai [5] proved that for a small analytic potential q (or large energy E), for a set of positive

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(and asymptotically full) measure of energy E in the spectrum, the associated cocycle is reducible provided the frequency $\omega \in \mathbb{R}^d$ is Diophantine:

$$|k \cdot \omega| \geq \frac{\gamma}{|k|^\tau}, \quad k \in \mathbb{Z}^d \setminus \{0\},$$

for some constant $\gamma > 0$ and $\tau \geq d - 1$, where $|k|$ is the sum of the absolute values of the components and $k \cdot \omega$ the Euclidean inner product. For a fixed cocycle as we considered above, their result amounts to a reducibility result under a Diophantine condition (with respect to ω) on the so-called fibered rotation number $\rho = \rho(A + F)$, for a small analytic F , the smallness assumption depending on ρ . Rüssmann [12] later extended this result, under a more general arithmetic condition on ω and ρ (this condition, weaker than the Diophantine condition, is slightly stronger than the so-called the Brjuno-Rüssmann condition). Moser and Pöschel [9] further extended the result to include some rational fibered rotation numbers, using a technique of resonance-cancellation, but the breakthrough came from Eliasson [6]: by sharpening this resonance-cancellation technique he obtained the reducibility for a set of full measure of fibered rotation number.

Since then, many works have been devoted to the reducibility of quasi-periodic cocycles, in different regularity classes and for cocycles taking values in different Lie algebras. In particular, many non-perturbative results have been obtained but they are restricted to two-dimensional frequencies $\omega \in \mathbb{R}^2$; we should not try to review to state of the art as we will be interested only in perturbative results, but valid in any dimension $\omega \in \mathbb{R}^d$. More precisely, we will be interested in the interaction between the regularity of the cocycle and the arithmetic properties of the frequency vector ω .

For smooth cocycles, the Diophantine condition on ω is known to be sufficient, and it is not hard to see it is also necessary. The analytic case is more subtle. Chavaudret and Marmi [4] extended the result of Rüssmann to obtain reducibility under the Brjuno-Rüssmann condition; this sufficient arithmetic condition is not known to be optimal, but can be compared to a natural necessary condition (that we call the Rüssmann condition). The proof in [4] uses ideas of Rüssmann [13] and Pöschel [11], which deal with the corresponding results for respectively Hamiltonian systems and vector fields on the torus. Those results were later extended by Bounemoura and Féjoz [1], [2] for a more general class of systems with ultra-differentiable regularity. A particular case is the α -Gevrey regularity, for a real parameter $\alpha \geq 1$, for which the analytic case is recovered by setting $\alpha = 1$; an α -Brjuno-Rüssmann condition is introduced in [1] (an equivalent condition was independently obtained in [10] for vector fields on the torus) and exactly as for $\alpha = 1$, this sufficient condition is shown to be comparable to the natural necessary condition. Unfortunately the results in [2] do not allow such a comparison in general as the sufficient arithmetic condition is affected by a technical assumption, and so the results are not as accurate as those obtained in the Gevrey case in [1].

The purpose of this article is to improve the results of [2] within the context of quasi-periodic cocycles: we will obtain a result of reducibility valid for a larger class of ultra-differentiable systems, with a better sufficient arithmetic condition in the sense that it can be compared to the natural necessary condition.

2. STATEMENT OF THE MAIN RESULTS

Let us recall the setting. We have $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ for an integer $d \geq 1$ and we consider the cocycle

$$(2.1) \quad \begin{cases} x'(t) = (A + F(\theta(t)))x(t), \\ \theta'(t) = \omega, \end{cases}$$

where $x \in \mathbb{R}^2$, $\theta \in \mathbb{T}^d$ and $A \in sl(2, \mathbb{R})$ is an elliptic matrix, or equivalently, fixing $\theta(0) = 0 \in \mathbb{T}^d$, we consider

$$x'(t) = (A + F(t\omega))x(t).$$

Such a cocycle will be simply denoted by $(\omega, A+F)$. It is said to be reducible if there exists a (fibered) conjugacy between $(A+F)$ and a constant cocycle: there exist $Y : \mathbb{T}^d \rightarrow GL(2, \mathbb{R})$ and an elliptic matrix $B \in sl(2, \mathbb{R})$ such that

$$\partial_\omega Y = (A + F)Y - YB,$$

where $\partial_\omega Y(\theta) = \partial_\theta Y(\theta) \cdot \omega$. Since the matrices actually take values in $sl(2, \mathbb{R})$, a perhaps more natural definition would be to require that the conjugacy Y takes value in the corresponding Lie group $SL(2, \mathbb{R})$: it follows from the work in [3] that these two definitions are the same. In order to get such a reducibility, one need to impose regularity assumptions on F and an arithmetic condition on ω (we will also impose a similar arithmetic condition on the fibered rotation number).

To quantify the regularity of $F : \mathbb{T}^d \rightarrow sl(2, \mathbb{R})$, we introduce a weight function

$$\Lambda : [1, +\infty) \rightarrow [1, +\infty)$$

which we assume is increasing and differentiable. Expanding a smooth function $f \in C^\infty(\mathbb{T}^d, \mathbb{R})$ in Fourier series

$$f(\theta) = \sum_k \hat{f}(k) e^{2\pi i k \cdot \theta}$$

we will say it is Λ -ultra-differentiable if there exists $r > 0$ such that

$$(2.2) \quad |f|_r = |f|_{\Lambda, r} := \sum_{k \in \mathbb{Z}^d} |\hat{f}(k)| e^{2\pi \Lambda(|k|)r} < \infty.$$

The ultra-differentiable weighted norm of f is then defined by (2.2), and we say f belongs to the ultra-differentiable function class $U_{\Lambda, r}(\mathbb{T}^d, \mathbb{R})$. This defines a Banach space. We will require the function Λ to be subadditive, namely

$$(S) \quad \Lambda(x + y) \leq \Lambda(x) + \Lambda(y), \quad x, y \geq 1.$$

This assumption turns $U_{\Lambda, r}(\mathbb{T}^d, \mathbb{R})$ into a Banach algebra (for a proof of this elementary fact, see Appendix 8.1). Now for a matrix-valued function $M : \mathbb{T}^d \rightarrow M_2(\mathbb{R})$, one extends the definition of (2.2) in such a way that $U_{\Lambda, r}(\mathbb{T}^d, M_2(\mathbb{R}))$ becomes a Banach algebra for the product of matrices. In the sequel, we will use the notation $U_\Lambda = \bigcup_{r>0} U_{\Lambda, r}$ when convenient. Main examples of ultra-differentiable classes are the α -Gevrey class associated to $\Lambda_\alpha(v) = v^{\frac{1}{\alpha}}$ for $\alpha \geq 1$ and the real analytic class for $\Lambda_1(v) = v$, but many more examples are readily available. In particular, the quasi-analytic class, i.e the class of functions which

are uniquely determined by the sequence of their derivatives at a point, corresponds to a function Λ satisfying

$$(2.3) \quad \int_1^\infty \frac{\Lambda(v)}{v^2} dv = +\infty$$

The parameter r can be called "ultra-differentiable parameter" if (2.2) holds, and it is essentially the radius of convergence for real-analytic functions.

Next we need to quantify the non-resonance condition on $\omega \in \mathbb{R}^d$. To do so, we introduce an approximating function

$$\Psi : [1, +\infty) \rightarrow [1, +\infty)$$

which we assume, without loss of generality, to be increasing and differentiable, and for which

$$(2.4) \quad \Psi(K) = \max\{ |2\pi k \cdot \omega|^{-1} \mid 0 < |k| \leq K \}, \quad K \in \mathbb{N}.$$

We also need to quantify the non-resonance condition on the fibered rotation number $\rho = \rho(A + F)$, a definition of which is recalled in Appendix 8.2. Without loss of generality, we use the same approximating function and requires that

$$(2.5) \quad |2\rho \pm 2\pi k \cdot \omega| \geq \frac{1}{\Psi(K)}, \quad 0 < |k| \leq K.$$

The approximating function Ψ will be assumed to satisfy the following arithmetic condition adapted to the weight Λ , that we call the Λ -Brjuno-Rüssmann condition

$$(\Lambda\text{-BR}) \quad \int_1^{+\infty} \frac{\Lambda'(v) \ln \Psi(v) dv}{\Lambda^2(v)} < \infty.$$

One easily check that the last condition is equivalent to

$$(2.6) \quad \int_1^{+\infty} \frac{\Psi'(v) dv}{\Psi(v)\Lambda(v)} < \infty.$$

In the Gevrey case $\Lambda_\alpha(v) = v^{\frac{1}{\alpha}}$ (and thus in the analytic case when $\alpha = 1$), the Λ_α -Brjuno-Rüssmann condition is

$$(2.7) \quad \int_1^{+\infty} \frac{\ln \Psi(v) dv}{v^{1+\frac{1}{\alpha}}}$$

and one recovers the α -Brjuno-Rüssmann condition introduced in [1] (for $\alpha = 1$, this is the Brjuno-Rüssmann condition as in [4]).

Now we can state the main theorem of this paper.

Theorem 1. *Assume that Λ satisfy (S) and ω and ρ verify (2.4) and (2.5) with Ψ satisfying the Λ -Brjuno-Rüssmann condition (Λ -BR). Given any $r > 0$ and any quasiperiodic cocycle $(\omega, A + F)$ as in (2.1), with a non-zero elliptic matrix $A \in sl(2, \mathbb{R})$ and $F \in U_{\Lambda, r}(\mathbb{T}^d, sl(2, \mathbb{R}))$, there exists $\bar{\varepsilon}$ depending only on r, Λ, A, ω such that if $|F|_r \leq \bar{\varepsilon}$, the cocycle $(\omega, A + F)$ is reducible with a conjugacy $Y \in U_{\Lambda, r/2}(\mathbb{T}^d, GL(2, \mathbb{R}))$ which satisfies $|Y|_{\frac{r}{2}} \leq 2$ and $|Y^{-1}|_{\frac{r}{2}} \leq 2$.*

The Λ -Brjuno-Rüssmann condition (Λ -BR) is thus sufficient for the reducibility within the class U_Λ ; we do not know if the condition is necessary yet it implies the following Λ -Rüssmann condition

$$(\Lambda\text{-R}) \quad \lim_{v \rightarrow \infty} \frac{\log \Psi(v)}{\Lambda(v)} = 0$$

which is necessary as the next statement shows.

Theorem 2. *Assume that Λ satisfy (S), ω verify (2.4) with Ψ not satisfying the Λ -Rüssmann condition (Λ -R) and ρ is arbitrary. Then there exists $r > 0$ such that for all $\varepsilon > 0$, there exist a quasiperiodic cocycle $(\omega, A + F)$ as in (2.1), with a non-zero elliptic matrix $A \in sl(2, \mathbb{R})$ and $F \in U_{\Lambda, r}(\mathbb{T}^d, sl(2, \mathbb{R}))$ satisfying $|F|_r \leq \varepsilon$ which is not reducible by any continuous conjugacy $Y : \mathbb{T}^d \rightarrow GL(2, \mathbb{R})$.*

3. COROLLARIES AND APPLICATIONS

3.1. A special case: quasi-analytic functions. In the α -Gevrey case when $\Lambda_\alpha(v) = v^{\frac{1}{\alpha}}$, in view of (2.7) the condition (Λ -BR) holds true for the approximating function $\Psi(v) = e^{v^\beta}$ for any $\beta < 1/\alpha$; in particular in the analytic case when $\alpha = 1$ it holds true for any $\beta < 1$.

As a matter of fact, the latter also holds true for quasi-analytic functions, that is if Λ satisfies (2.3), then the condition (Λ -BR) holds true for $\Psi(v) = e^{v^\beta}$ for any $\beta < 1$. Indeed, there exists

$$\frac{1 + \beta}{2} < \gamma < 1, \quad 0 < \delta < \frac{1 - \beta}{2}$$

and an unbounded sequence (v_n) such that $1 \leq v_{n+1} - v_n \leq n^\delta$ and $\Lambda(v_n) > v_n^\gamma$. In particular $v_n \geq n$ for all n . Thus for all $n \geq 1$, recalling that (Λ -BR) is equivalent to (2.6), one has

$$\int_1^{v_{n+1}} \frac{\Psi'(v)dv}{\Psi(v)\Lambda(v)} \leq \beta \sum_{k \leq n} \frac{v_k^{\beta-1}}{\Lambda(v_k)} (v_{k+1} - v_k) \leq \beta \sum_{k \leq n} \frac{1}{k^{1+\gamma-\beta-\delta}}$$

and this last sum converges as $n \rightarrow +\infty$.

3.2. Regularity of the Lyapunov exponent. Let $\omega \in \mathbb{R}^d$ be fixed. For a matrix-valued function G , denote by $L(G)$ the maximal Lyapunov exponent of the cocycle (ω, G) .

Corollary 1. *Let $(\omega, A + F)$ satisfy the assumptions of Theorem 1. Then for all $A' \in U_\Lambda(\mathbb{T}^d, sl(2, \mathbb{R}))$,*

$$(3.1) \quad |L(A') - L(A + F)| \leq 4|A' - (A + F)|_0$$

Proof. Let Y be as in Theorem 1 and A_∞ the elliptic matrix such that

$$\partial_\omega Y = (A + F)Y - YA_\infty.$$

Let $A' \in U_{\Lambda, r}(\mathbb{T}^d, M_2(\mathbb{R}))$, then

$$\partial_\omega Y = A'Y - Y(A_\infty + Y^{-1}(A' - (A + F))Y).$$

Notice that the Lyapunov exponent of $(\omega, A + F)$ is the same as for (ω, A_∞) , thus it is zero. Denoting by L the Lyapunov exponent, one has

$$\begin{aligned} |L(A') - L(A + F)| &= L(A') = L(A_\infty + Y^{-1}(A' - (A + F))Y) \\ &\leq \ln |A_\infty + Y^{-1}(A' - (A + F))Y|_0 \\ &\leq \ln(1 + |Y^{-1}(A' - (A + F))Y|_0) \\ &\leq |Y^{-1}(A' - (A + F))Y|_0 \end{aligned}$$

The bound on Y gives the estimate (3.1). \square

Let us remark that in order to describe the regularity without any condition on the fibered rotation number, a statement about almost reducibility would be needed.

4. PRELIMINARY REDUCTIONS AND CHOICE OF THE SEQUENCES OF PARAMETERS

Let us start with some preliminary lemmas. Recall that the spectrum of a non-zero elliptic matrix $A \in sl(2, \mathbb{R})$ is of the form $\text{Spec}(A) = \{\pm i\alpha\}$ for some real number $\alpha > 0$ (such a number is well-defined up to a sign); this real number will be called the rotation number of A and denoted by $\rho(A)$. The following lemma gives us a real normal form for such an elliptic matrix and an estimate on the size of the transformation to normal form.

Lemma 4.1. *Given an elliptic matrix $A \in sl(2, \mathbb{R})$ with $\rho(A) = \alpha > 0$, there exist a matrix $P \in SL(2, \mathbb{R})$ such that*

$$PAP^{-1} = \alpha J, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$|P| \leq 2(|A|/\alpha)^{1/2}, \quad |P^{-1}| \leq 1.$$

For a proof, we refer to [8]. It will sometimes be useful to use a complex normal form in which the matrix is diagonal. To do this, we consider the matrix

$$(4.1) \quad M = \frac{1}{1=i} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \in U(2)$$

and we define the complex invertible matrix $Q = MP$ where P is the matrix given by Lemma 4.1, so that

$$QAQ^{-1} = i\alpha R, \quad R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$|Q| \leq 2(|A|/\alpha)^{1/2}, \quad |Q^{-1}| \leq 1.$$

This lemma allows us to reduce the proof of Theorem 1 to the case where $A = \alpha J$; indeed, it suffices to replace the smallness assumption on $|F|_r$ by a smallness assumption on $|PFP^{-1}|_r$ which is bounded by $2(|A|/\alpha)^{1/2}|F_r|$. In such a normal form, we have the following elementary lemma.

Lemma 4.2. *Assume that $A = \alpha J$. Then*

$$|\rho(A) - \rho(A + F)| \leq 4|F|_0.$$

For a proof, we refer to Appendix 8.2. Finally, a matrix $A = \alpha J$ will remain elliptic under a small perturbation by a matrix in $sl(2, \mathbb{R})$ but no longer in normal form. Yet Lemma 4.1 immediately implies the following.

Lemma 4.3. *Assume that $A = \alpha J$ with $\alpha > 0$ and $B \in sl(2, \mathbb{R})$ such that $|B| \leq \varepsilon$. If $\varepsilon \leq \alpha/4$, then the matrix $A + B$ is elliptic with $\rho(A + B) = \beta$ satisfying*

$$\alpha/2 \leq \beta \leq \alpha + \alpha/2$$

and thus there exists $P \in SL(2, \mathbb{R})$ such that $P(A + B)P^{-1} = \beta J$ with

$$|P| \leq 4 \quad |P^{-1}| \leq 1.$$

Upon these preliminary reductions, we can define a sequence of parameters for the iterations. Let $\rho(A) = \alpha$, $\rho(A + F) = \rho$, $\varepsilon = |F|_r$ and define for $\nu \in \mathbb{N}$

$$\varepsilon_\nu = 4^{-\nu} \varepsilon.$$

We now choose $N_0 \in \mathbb{N}$, depending on r and the approximating function Ψ , sufficiently large so that

$$(4.2) \quad \int_{N_0}^{+\infty} \frac{\Lambda'(v) \ln \Psi(v) dv}{\Lambda^2(v)} \leq \frac{\pi}{6} r.$$

Then we set

$$N_\nu = \Psi^{-1}(2^\nu \Psi(N_0))$$

and observe that for all $\nu \in \mathbb{N}$, we have

$$(4.3) \quad \Psi(N_\nu) \varepsilon_\nu = 2^{-\nu} \Psi(N_0) \varepsilon_0 = 2^{-\nu} \Psi(N_0) \varepsilon.$$

and thus the sequence $\Psi(N_\nu) \varepsilon_\nu$ is summable. We can define our threshold $\varepsilon \leq \bar{\varepsilon}$ by the requirements that

$$\varepsilon \leq \alpha/4, \quad 2^8 \Psi(N_0) \varepsilon \leq 1.$$

One then easily checks that

$$(4.4) \quad 2^8 \Psi(N_\nu) \varepsilon_\nu \leq 1$$

holds true for all $\nu \in \mathbb{N}$. Next we define another sequence $\sigma_\nu > 0$ for $\nu \in \mathbb{N}$ by

$$\sigma_\nu = \frac{3 \ln 2}{\pi \Lambda(N_\nu)}$$

so that for all $\nu \in \mathbb{N}$, we have the equality

$$(4.5) \quad 2^6 e^{-2\pi \Lambda(N_\nu) \sigma_\nu} = 1.$$

Finally we define recursively the sequence r_ν by setting $r_0 = r$ and $r_{\nu+1} = r_\nu - \sigma_\nu$; we will see later, as a consequence of (4.2) and of our choice of σ_ν , that this sequence is well-defined (in the sense that $\sigma_\nu < r_\nu$), $r_\mu \geq r/2$ and thus r_ν converges to its infimum $r^* \geq r/2$.

5. THE ITERATION STEP

Given $F : \mathbb{T}^d \rightarrow gl(2, \mathbb{R})$, we define $\text{tr}\langle F \rangle = \text{tr}(\hat{F}(0))$ where $\hat{F}(0)$ is the average of F with respect to Lebesgue measure. We also define truncation operators T_N and \dot{T}_N on $U_\Lambda(\mathbb{T}^d, gl(2, \mathbb{R}))$ as

$$(T_N F)(\theta) := \sum_{|k| \leq N} \hat{F}(k) e^{i2\pi k \cdot \theta}, \quad (\dot{T}_N F)(\theta) := \sum_{0 < |k| \leq N} \hat{F}(k) e^{i2\pi k \cdot \theta}.$$

In this section, for a fixed $\nu \in \mathbb{N}$, we consider a cocycle $(\omega, A_\nu + F_\nu)$ which satisfy

$$(H_\nu) \quad \begin{cases} A_\nu = \alpha_\nu J, \\ |F_\nu|_{r_\nu} \leq \varepsilon_\nu, \quad \varepsilon_\nu \leq \alpha_\nu/4, \quad \text{tr}\langle F_\nu \rangle = 0, \\ \rho(A_\nu + F_\nu) = \rho. \end{cases}$$

We will conjugate this cocycle, by a transformation which is homotopic to the identity, to a cocycle $(\omega, A_{\nu+1} + F_{\nu+1})$ which satisfy $H_{\nu+1}$ together with estimates on such a transformation. First we have the following obvious lemma.

Lemma 5.1. *For any $k \in \mathbb{Z}^d$ such that $0 < |k| \leq N_\nu$, we have*

$$|2\alpha_\nu \pm 2\pi k \cdot \omega| > \frac{1}{2\Psi(N_\nu)}.$$

Proof. Indeed we have $\rho(A_\nu) = \alpha_\nu$ and $\rho(A_\nu + F_\nu) = \rho$ hence

$$\begin{aligned} |2\alpha_\nu \pm 2\pi k \cdot \omega| &\geq |2\rho \mp 2\pi k \cdot \omega| - |2\rho(A_\nu + F_\nu) - 2\rho(A_\nu)| \\ &\geq \frac{1}{\Psi(N_\nu)} - 8\varepsilon_\nu \geq \frac{1}{2\Psi(N_\nu)} \end{aligned}$$

where we used Lemma 4.2 and the fact that $16\Psi(N_\nu)\varepsilon_\nu \leq 1$. \square

Next we solve an approximate cohomological equation.

Lemma 5.2. *If $G_\nu = \dot{T}_{N_\nu} F_\nu$, there is a unique X_ν such that $X_\nu = \dot{T}_{N_\nu} X_\nu$ satisfying the equation*

$$(5.1) \quad \partial_\omega X_\nu = [A_\nu, X_\nu] + G_\nu$$

with the estimates

$$|X_\nu|_{r_\nu} \leq \Psi(N_\nu)\varepsilon_\nu, \quad |(I + X_\nu)^{-1}|_{r_\nu} \leq 2.$$

Moreover $\text{tr}\langle X_\nu \rangle = 0$.

Proof. Observe that conjugating the cocycle by the complex matrix M defined in (4.1), it is sufficient to prove the statement for A_ν in complex normal form $i\alpha R$. Expanding F_ν and X_ν in Fourier series, the equation (5.1) yields

$$\sum_{0 < |k| \leq N_\nu} \partial_\omega \hat{X}_\nu(k) e^{2\pi i k \cdot \theta} = \sum_{0 < |k| \leq N_\nu} [A_\nu, \hat{X}_\nu(k)] e^{2\pi i k \cdot \theta} + \sum_{0 < |k| \leq N_\nu} \hat{F}_\nu(k) e^{2\pi i k \cdot \theta},$$

which is equivalent to

$$\partial_\omega \hat{X}_\nu(k) = [A_\nu, \hat{X}_\nu(k)] + \hat{F}_\nu(k), \quad 0 < |k| \leq N_\nu.$$

Since A_ν is diagonal, the solution of the above equation is

$$\hat{X}_\nu(k) = L_k^{-1} \hat{F}_\nu(k),$$

where L_k is the operator defined by

$$L_k : sl(2, \mathbb{R}) \rightarrow sl(2, \mathbb{R}), \quad \tilde{X} \mapsto 2\pi i k \cdot \omega \tilde{X} - [\tilde{A}_\nu, \tilde{X}].$$

The spectrum of L_k is $\{2\pi i k \cdot \omega \pm 2\alpha_\nu, 2\pi i k \cdot \omega\}$. By Lemma 5.1 and (2.4), the operator L_k for $0 < |k| \leq N_\nu$ is invertible with norm bounded by $2\Psi(N_\nu)$ so

$$|\hat{X}_\nu(k)| = |L_k^{-1} \hat{F}_\nu(k)| \leq 2\Psi(N_\nu) |\hat{F}_\nu(k)|$$

and thus

$$|X_\nu|_{r_\nu} = \sum_{0 < |k| \leq N_\nu} |\hat{X}_\nu(k)| e^{2\pi\Lambda(k)r} \leq 2\Psi(N_\nu) \sum_{0 < |k| \leq N_\nu} |\hat{F}_\nu(k)| e^{2\pi\Lambda(k)r} \leq 2\Psi(N_\nu) \varepsilon_\nu.$$

Since $4\Psi(N_\nu) \varepsilon_\nu \leq 1$ the estimate

$$|(I + X_\nu)^{-1}|_{r_\nu} \leq \frac{1}{1 - |X_\nu|_{r_\nu}} \leq 2$$

is obvious and so is $\text{tr}\langle X_\nu \rangle = 0$ because $\hat{X}_\nu(0) = 0$. This completes the proof. \square

We can finally state our main iterative proposition.

Proposition 5.3. *Let $(\omega, A_\nu + F_\nu)$ be as in (H_ν) . Then there exists a transformation Y_ν homotopic to the identity such that*

$$\partial_\omega Y_\nu = (A_\nu + F_\nu) Y_\nu - Y_\nu (A_{\nu+1} + F_{\nu+1}),$$

with $(\omega, A_{\nu+1} + F_{\nu+1})$ satisfying $(H_{\nu+1})$ and such that

$$|Y_\nu - I|_{r_\nu} \leq 8\Psi(N_\nu) \varepsilon_\nu$$

Proof. The transformation Y_ν will be the composition of a quasi-periodic linear transformation given by Lemma 5.2 and a constant transformation given by Lemma 4.3 to put back the constant elliptic part into normal form.

Let X_ν be given by Lemma 5.2, and let $Z_\nu = I + X_\nu$. Since X_ν solves

$$\partial_\omega X_\nu = [A_\nu, X_\nu] + G_\nu,$$

a computation leads to

$$\partial_\omega Z_\nu = (A_\nu + F_\nu) Z_\nu - Z_\nu (B_\nu + R_\nu),$$

with

$$B_\nu = A_\nu + \hat{F}_\nu(0), \quad R_\nu = (I + X_\nu)^{-1} [(F_\nu - \hat{F}_\nu(0) - G_\nu) + F_\nu X_\nu - X_\nu \hat{F}_\nu(0)].$$

We can estimate

$$\begin{aligned}
|R_\nu|_{r_\nu-\sigma_\nu} &= |(I + X_\nu)^{-1}[(F_\nu - \hat{F}_\nu(0) - G_\nu) + F_\nu X_\nu - X_\nu \hat{F}_\nu(0)]|_{r_\nu-\sigma_\nu} \\
&\leq 2|F_\nu - \hat{F}_\nu(0) - G_\nu|_{r_\nu-\sigma_\nu} + 2|F_\nu X_\nu|_{r_\nu-\sigma_\nu} + 2|X_\nu \hat{F}_\nu(0)|_{r_\nu-\sigma_\nu} \\
&\leq |F_\nu|_{r_\nu} 2e^{-2\pi\Lambda(N_\nu)\sigma_\nu} + 4|X_\nu|_r |F_\nu|_r \\
&\leq 2e^{-2\pi\Lambda(N_\nu)\sigma_\nu} \varepsilon_\nu + 8\Psi(N_\nu)\varepsilon_\nu \varepsilon_\nu \\
&\leq 2^{-5}\varepsilon_\nu + 2^{-5}\varepsilon_\nu
\end{aligned}$$

where we used (4.4) and (4.5) in the last inequality, and therefore

$$(5.2) \quad |R_\nu|_{r_\nu-\sigma_\nu} \leq 2^{-4}\varepsilon_\nu.$$

Let us now check that $\text{tr}\langle R_\nu \rangle = 0$. By the equality

$$\partial_\omega(I + X_\nu) = (A_\nu + F_\nu)(I + X_\nu) - (I + X_\nu)(B_\nu + R_\nu),$$

we know that

$$R_\nu = -(I + X_\nu)^{-1}\partial_\omega(I + X_\nu) + (I + X_\nu)^{-1}(A_\nu + F_\nu)(I + X_\nu) - B_\nu.$$

It follows from the assumptions $\text{tr}A_\nu = \text{tr}\hat{F}_\nu(0) = \text{tr}B_\nu = 0$ that

$$\text{tr}\langle (I + X_\nu)^{-1}(A_\nu + F_\nu)(I + X_\nu) - B_\nu \rangle = 0.$$

On the other hand, using $\text{tr}(AB) = \text{tr}(BA)$ we have

$$\begin{aligned}
&\text{tr}\langle (I + X_\nu)^{-1}\partial_\omega(I + X_\nu) \rangle \\
&= \text{tr}\langle \partial_\omega(I + X_\nu)(I + X_\nu)^{-1} \rangle \\
&= \text{tr}\langle (\partial_\omega X_\nu)(I + X_\nu)^{-1} \rangle \\
&= \text{tr}\langle \partial_\omega X_\nu - \frac{1}{2}((\partial_\omega X_\nu)X_\nu + X_\nu\partial_\omega X_\nu) \rangle + \frac{1}{3}((\partial_\omega X_\nu)X_\nu^2 + X_\nu(\partial_\omega X_\nu)X_\nu + (\partial_\omega X_\nu)X_\nu^2) + \dots \\
&= \text{tr}\langle \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \langle \partial_\omega(X_\nu^{k+1}) \rangle \rangle \\
&= 0.
\end{aligned}$$

Thus $\text{tr}\langle R_\nu \rangle = 0$. Now we want to apply Lemma 4.3. Observe that $\varepsilon_\nu \leq \alpha_\nu/4$, therefore Lemma 4.3 gives that $\rho(B_\nu) = \rho(A_\nu + \hat{F}_\nu(0)) = \alpha_{\nu+1}$ satisfy

$$\alpha_\nu/2 \leq \alpha_{\nu+1} \leq \alpha_\nu + \alpha_\nu/2$$

hence there exists $P_\nu \in SL(2, \mathbb{R})$ such that $P_\nu(B_\nu)P_\nu^{-1} = \alpha_{\nu+1}J$ with

$$(5.3) \quad |P_\nu| \leq 4 \quad |P_\nu^{-1}| \leq 1.$$

We can finally define

$$Y_\nu = P_\nu Z_\nu P_\nu^{-1}, \quad A_{\nu+1} = \alpha_{\nu+1}J, \quad F_{\nu+1} = P_\nu R_\nu P_\nu^{-1}$$

so that

$$\partial_\omega Y_\nu = (A_\nu + F_\nu)Y_\nu - Y_\nu(A_{\nu+1} + F_{\nu+1}).$$

Let us check that $(H)_{\nu+1}$ is satisfied. By definition $A_{\nu+1} = \alpha_{\nu+1}J$ and we know that

$$\varepsilon_{\nu+1} = \varepsilon_{\nu}/4 < \alpha_{\nu}/8 \leq \alpha_{\nu+1}/4.$$

Again by definition, $F_{\nu+1} = P_{\nu}R_{\nu}P_{\nu}^{-1}$ and from the estimates (5.2) and (5.3) we have

$$|F_{\nu+1}|_{r_{\nu}-\sigma_{\nu}} \leq 2^{-4}4\varepsilon_{\nu} = \varepsilon_{\nu+1}$$

whereas $\text{tr}\langle F_{\nu+1} \rangle = \text{tr}\langle R_{\nu} \rangle = 0$. Then, $Z_{\nu} = I + X_{\nu}$ is obviously homotopic to the identity and so is $P_{\nu} \in SL(2, \mathbb{R})$ hence

$$\rho(A_{\nu+1} + F_{\nu+1}) = \rho(A_{\nu} + F_{\nu}) = \rho.$$

To conclude, from (5.3) and the estimates on X_{ν} given by Lemma 5.2 we get

$$|Y_{\nu} - I|_{r_{\nu}} \leq 4|Z_{\nu} - I|_{r_{\nu}} \leq 4|X_{\nu}|_{r_{\nu}} \leq 8\Psi(N_{\nu})\varepsilon_{\nu}.$$

This concludes the proof. \square

6. PROOF OF THEOREM 1

In this section we finally prove Theorem 1. Letting $A_0 = A$ and $F_0 = F$, we can apply inductively Proposition 5.3 and for any $\nu \in \mathbb{N}$, if we define

$$Y^{\nu} = \prod_{\mu \leq \nu} Y_{\mu} \in U_{\Lambda, r_{\nu}}(\mathbb{T}^d, GL(2, \mathbb{R}))$$

we have

$$\partial_{\omega} Y^{\nu} = (A + F)Y^{\nu} - Y^{\nu}(A_{\nu+1} + F_{\nu+1}),$$

with

$$\rho(A_{\nu+1} + F_{\nu+1}) = \rho, \quad |F|_{r_{\nu+1}} \leq \varepsilon_{\nu+1}.$$

As the sequence ε_{ν} converges to zero, the only thing that remains to be proved is that r_{ν} converges to a non-zero limit and that Y^{ν} converges. To do so, let us first observe that

$$\sum_{\nu \geq 1} \frac{1}{\Lambda(N_{\nu})} = \sum_{\nu \geq 1} \frac{1}{\Lambda(\Psi^{-1}(2^{\nu}\Psi(N_0)))} \leq \int_0^{+\infty} \frac{dx}{\Lambda(\Psi^{-1}(2^x\Psi(N_0)))}$$

and changing variables $v = \Psi^{-1}(2^x\Psi(N_0))$, this gives

$$\sum_{\nu \geq 1} \frac{1}{\Lambda(N_{\nu})} \leq \frac{1}{\ln 2} \int_{N_0}^{\infty} \frac{\Psi'(v)dv}{\Lambda(v)\Psi(v)}$$

and finally, by an integration by parts this yields

$$\sum_{\nu \geq 0} \frac{1}{\Lambda(N_{\nu})} \leq \frac{1}{\ln 2} \int_{N_0}^{\infty} \frac{\Lambda'(v) \ln \Psi(v) dv}{\Lambda^2(v)} \leq \frac{\pi}{6 \ln 2} r$$

where the last inequality follows from (4.2). By definition of σ_{μ} this gives

$$\sum_{\nu \geq 0} \sigma_{\nu} = \frac{3 \ln 2}{\pi} \sum_{\nu \geq 0} \frac{1}{\Lambda(N_{\nu})} \leq r/2$$

and thus $r_\nu \geq r/2$ converges to some $r^* \geq r/2$. To conclude, $Y^\nu - I$ is easily seen to form a Cauchy sequence on the space $U_{\Lambda, r/2}(\mathbb{T}^d, GL(2, \mathbb{R}))$ and thus Y^ν converges to a limit Y which satisfies the bound

$$|Y - I|_{r/2} \leq 2 \sum_{\nu \geq 0} |Y_\nu - I| \leq 16 \sum_{\nu \geq 0} \Psi(N_\nu) \varepsilon_\nu \leq 32 \Psi(N_0) \varepsilon.$$

7. PROOF OF THEOREM 2

Let us start with the following lemma, which says that if ω does not satisfy the Λ -Rüssmann condition (Λ -R), then one cannot solve the cohomological equation in general.

Lemma 7.1. *Assume ω does not satisfy (Λ -R), that is*

$$\limsup_{v \rightarrow \infty} \frac{\log \Psi(v)}{\Lambda(v)} > 0.$$

Then there exist $r > 0$ such that for any $\varepsilon \geq 0$ and any $\rho \in \mathbb{R}$, there exists a function $u : \mathbb{T}^d \rightarrow \mathbb{R}$ for which

$$\int_{\mathbb{T}^d} u(\theta) d\theta = \rho, \quad |u - \rho|_r \leq \varepsilon$$

but such that the equation

$$(E) \quad \omega \cdot \partial_\theta v(\theta) = u(\theta) - \rho$$

has no continuous solution $v : \mathbb{T}^d \rightarrow \mathbb{R}$.

Proof. Let $r > 0$ be such that

$$\limsup_{v \rightarrow \infty} \frac{\log \Psi(v)}{\Lambda(v)} \geq 3\pi r$$

so that there exists a positive sequence $v_j \rightarrow +\infty$ for which

$$\Psi(v_j)^{-1} \leq e^{-3\pi r \Lambda(v_j)}$$

By definition of Ψ , there exists infinitely many $k_j \in \mathbb{Z}^d \setminus \{0\}$ (for which $|k_j| = v_j$) and

$$|2\pi k_j \cdot \omega| \leq e^{-3\pi r \Lambda(|k_j|)}.$$

Let us define a constant

$$C = \sum_{j \in \mathbb{N}} e^{-\pi r \Lambda(|k_j|)} < +\infty$$

and a function

$$u(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{u}(k) e^{2\pi i k \cdot \theta},$$

by setting

$$\hat{u}(0) = \rho, \quad \hat{u}(k_j) = \varepsilon C^{-1} 2\pi k_j \cdot \omega$$

and all other Fourier coefficients equal to zero. By construction we do have

$$\int_{\mathbb{T}^d} u(\theta) d\theta = \rho, \quad |u - \rho|_r \leq \varepsilon.$$

Now a function $v : \mathbb{T}^d \rightarrow \mathbb{R}$

$$v(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{v}(k) e^{2\pi i k \cdot \theta}$$

solves (E) if and only if

$$\hat{v}(k_j) = \frac{\hat{u}(k_j)}{2\pi i k_j \cdot \omega} = \varepsilon C^{-1}.$$

Clearly such Fourier coefficients do not define a function which is integrable, and therefore v cannot be continuous. \square

To conclude the proof of Theorem 2, let $u : \mathbb{T}^d \rightarrow \mathbb{R}$ be the function given by Lemma 7.1, and consider the $sl(2, \mathbb{R})$ cocycle $(\omega, A + F)$ defined by

$$A = \rho J, \quad F(\theta) = u(\theta)J - \rho J.$$

Its fibered rotation number is equal to ρ (see Appendix 8.2) and $|F| \leq \varepsilon$. Argue by contradiction that $(\omega, A + F)$ is reducible. Since it takes values in $so(2, \mathbb{R})$, it follows from [3] that it is reducible by a transformation that takes values in $SO(2, \mathbb{R})$, therefore there exists $v : \mathbb{T}^d \rightarrow \mathbb{R}$ and

$$Y(\theta) = \begin{pmatrix} \cos v(\theta) & \sin v(\theta) \\ -\sin v(\theta) & \cos v(\theta) \end{pmatrix} \in SO(2, \mathbb{R})$$

such that

$$\partial_\omega Y = MY - YB$$

for some constant matrix $B = \beta J$. But then necessarily $\beta = \rho$ (that is $B = A$) and the above equation is equivalent to (E) which has no continuous solution, which is a contradiction.

8. APPENDIX

8.1. Product estimates.

Lemma 8.1. *Suppose Λ satisfy the subadditivity condition (S). For any $f, g \in U_{\Lambda, r}(\mathbb{T}^d, \mathbb{R})$, we have $fg \in U_{\Lambda, r}(\mathbb{T}^d, \mathbb{R})$ and*

$$|fg|_r \leq |f|_r |g|_r.$$

Proof. Expanding in Fourier series we have

$$f(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{f}(k) e^{2\pi i k \cdot \theta}, \quad g(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{g}(k) e^{2\pi i k \cdot \theta}$$

and

$$\begin{aligned} f(\theta)g(\theta) &= \left(\sum_{m \in \mathbb{Z}^d} \hat{f}(m) e^{2\pi i m \cdot \theta} \right) \left(\sum_{n \in \mathbb{Z}^d} \hat{g}(n) e^{2\pi i n \cdot \theta} \right) \\ &= \sum_{k \in \mathbb{Z}^d} \left(\sum_{m+n=k} \hat{f}(m) \hat{g}(n) \right) e^{2\pi i k \cdot \theta}. \end{aligned}$$

On the one hand

$$\begin{aligned}
|fg|_r &= \left| \sum_{k \in \mathbb{Z}^d} \left(\sum_{m+n=k} \hat{f}(m) \hat{g}(n) \right) e^{2\pi i k \cdot \theta} \right|_r \\
&= \sum_{k \in \mathbb{Z}^d} \left| \sum_{m+n=k} \hat{f}(m) \hat{g}(n) \right| e^{2\pi \Lambda(|k|)r} \\
&\leq \sum_{k \in \mathbb{Z}^d} \left(\sum_{m+n=k} |\hat{f}(m) \hat{g}(n)| e^{2\pi \Lambda(|k|)r} \right)
\end{aligned}$$

and on the other hand

$$\begin{aligned}
|f|_r |g|_r &= \left| \sum_{m \in \mathbb{Z}^d} \hat{f}(m) e^{2\pi i m \cdot \theta} \right|_r \left| \sum_{n \in \mathbb{Z}^d} \hat{g}(n) e^{2\pi i n \cdot \theta} \right|_r \\
&= \left(\sum_{m \in \mathbb{Z}^d} |\hat{f}(m)| e^{2\pi \Lambda(|m|)r} \right) \left(\sum_{n \in \mathbb{Z}^d} |\hat{g}(n)| e^{2\pi \Lambda(|n|)r} \right) \\
&= \sum_{k \in \mathbb{Z}^d} \sum_{m+n=k} |\hat{f}(m) \hat{g}(n)| e^{2\pi r(\Lambda(|m|) + \Lambda(|n|))}.
\end{aligned}$$

Since $|k| = |m+n| \leq |m| + |n|$ we have $\Lambda(|k|) \leq \Lambda(|m|) + \Lambda(|n|)$ by subadditivity and thus $|f \cdot g|_r \leq |f|_r |g|_r$. \square

8.2. Fibered rotation number. Let us consider a quasi-periodic cocycle

$$x'(t) = M(t\omega)x(t),$$

where

$$M : \mathbb{T}^d \rightarrow sl(2, \mathbb{R}), \quad M(\theta) = \begin{pmatrix} a(\theta) & b(\theta) \\ c(\theta) & -a(\theta) \end{pmatrix}$$

with $\omega \in \mathbb{R}^d$ non-resonant. Following Eliasson [6], we define the fibered rotation number $\rho(M) \in \mathbb{R}$ as

$$\rho(M) = \lim_{t \rightarrow +\infty} \frac{\phi_M(t)}{t}$$

where $\phi_M(t) = \phi(t)$ is any solution of the equation

$$\phi'(t) = 2a(t\omega) \cos \phi(t) \sin \phi(t) - (b(t\omega) + c(t\omega)) \cos^2 \phi(t) + b(t\omega).$$

A first special case is

$$M(\theta) = u(\theta)J, \quad u : \mathbb{T}^d \rightarrow \mathbb{R}$$

we then have

$$\rho(M) = \lim_{t \rightarrow +\infty} \frac{\phi_M(t)}{t} = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \phi'_M(s) ds = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t u(s\omega) ds = \int_{\mathbb{T}^d} u(\theta) d\theta$$

where the last equality follows from the unique ergodicity of the translation flow by ω . Another special case is

$$M(\theta) = \alpha J + F(\theta), \quad \sup_{\theta \in \mathbb{T}^d} |F(\theta)| = \sup_{\theta \in \mathbb{T}^d} \left| \begin{pmatrix} f_1(\theta) & f_2(\theta) \\ f_3(\theta) & -f_1(\theta) \end{pmatrix} \right| \leq \varepsilon$$

from which one immediately obtains the estimate

$$|\rho(\alpha J) - \rho(M)| = |\alpha - \rho(M)| \leq \left| \frac{1}{t} \int_0^t \phi'_F(s) ds \right| \leq 4\varepsilon$$

which is nothing but Lemma 4.2.

REFERENCES

- [1] Abed Bounemoura and Jacques Féjoz. KAM, α -Gevrey regularity and the α -Brjuno-Rüssmann condition. *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze*, XIX, 1-55, 2019.
- [2] Abed Bounemoura and Jacques Féjoz. Hamiltonian perturbation theory for ultra-differentiable functions. *Memoirs of the American Mathematical Society*, to appear.
- [3] Claire Chavaudret. Reducibility of quasiperiodic cocycles in Linear Lie groups. *Ergodic Theory and Dynamical Systems*, 31(3):741–769, 2011.
- [4] Claire Chavaudret and Stefano Marmi. Reducibility of cocycles under a Brjuno-Rüssmann arithmetical condition. *Journal of Modern Dynamics*, 6(1):59–78, 2012.
- [5] E.I. Dinaburg and Ya. G. Sinai. The one-dimensional Schrödinger equation with a quasiperiodic potential. *Functional Analysis and Its Applications*, 9(4):279–289, 1975.
- [6] L Håkan Eliasson. Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation. *Communications in mathematical physics*, 146(3):447–482, 1992.
- [7] Hai-Long Her and Jiangong You. Full measure reducibility for generic one-parameter family of quasi-periodic linear systems. *Journal of Dynamics and Differential Equations*, 20(4):831, 2008.
- [8] Xuanji Hou and Jiangong You. Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems. *Inventiones mathematicae*, 190(1):209–260, 2012.
- [9] Jürgen Moser and Jürgen Pöschel. An extension of a result by Dinaburg and Sinai on quasi-periodic potentials. *Commentarii Mathematici Helvetici*, 59(1):39–85, 1984.
- [10] João Lopes Dias and José Pedro Gaivão. Linearization of Gevrey flows on \mathbb{T}^d with a Brjuno type arithmetical condition *Journal of Differential Equations*, 267:7167–7212, 2019.
- [11] Jürgen Pöschel. KAM à la R. *Regular and Chaotic Dynamics*, 16(1-2):17–23, 2011.
- [12] Helmut Rüssmann. On the one-dimensional Schrödinger equation with a quasi-periodic potential. *Annals of the New York Academy of Sciences*, 357(1):90–107, 1980.
- [13] Helmut Rüssmann. KAM iteration with nearly infinitely small steps in dynamical systems of polynomial character. *Discrete and Continuous Dynamical Systems-S*, 3(4):683–718, 2010.

CNRS - PSL RESEARCH UNIVERSITY, (UNIVERSITÉ PARIS-DAUPHINE AND OBSERVATOIRE DE PARIS)
Email address: abedbou@gmail.com

INSTITUT DE MATHÉMATIQUES DE JUSSIEU, UNIVERSITÉ PARIS DIDEROT
Email address: chavaudr@math.univ-paris-diderot.fr

SCHOOL OF MATHEMATICS, JILIN UNIVERSITY, 130012 CHANGCHUN, P. R. CHINA;

CNRS - PSL RESEARCH UNIVERSITY, (CEREMADE, UNIVERSITÉ PARIS-DAUPHINE)
Email address: liangshuqing@jlu.edu.cn; liang@ceremade.dauphine.fr