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REDUCIBILITY OF ULTRA-DIFFERENTIABLE QUASIPERIODIC COCYCLES UNDER AN ADAPTED ARITHMETIC CONDITION

ABED BOUNEMOURA, CLAIRE CHAVAUDRET, AND SHUQING LIANG

Abstract. We prove a reducibility result for \( sl(2, \mathbb{R}) \) quasi-periodic cocycles close to a constant elliptic matrix in ultra-differentiable classes, under an adapted arithmetic condition which extends the Brjuno-Rüssmann condition in the analytic case. The proof is based on an elementary property of the fibered rotation number and deals with ultra-differentiable functions with a weighted Fourier norm. We also show that a weaker arithmetic condition is necessary for reducibility, and that it can be compared to a sufficient arithmetic condition.

1. Introduction

We will study the following time-quasiperiodic linear system, or quasiperiodic cocycle

\[
\begin{align*}
  x'(t) &= (A + F(\theta(t)))x(t), \\
  \theta'(t) &= \omega,
\end{align*}
\]

where \( x \in \mathbb{R}^2, \theta \in \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d \) with an integer \( d \geq 1 \), \( \omega \in \mathbb{R}^d \) is a non-resonant frequency vector, \( A \) is elliptic (meaning it is conjugated to some non-zero element in \( so(2, \mathbb{R}) \)) and \( F : \mathbb{T}^d \to sl(2, \mathbb{R}) \) belongs to some ultra-differentiable class. Let us recall that \( sl(2, \mathbb{R}) \) is the Lie algebra of traceless matrices and \( so(2, \mathbb{R}) \) is the Lie sub-algebra of skew-symmetric matrices. Such a quasi-periodic cocycle is said to be reducible if the time-quasiperiodic linear system can be conjugated, by a time quasi-periodic transformation, to a constant (time-independent) linear system.

One of the main motivation for studying reducibility of quasi-periodic cocycles came from the Schrödinger equation

\[-y''(t) + q(\theta + \omega t)y(t) = Ey(t),\]

and the question of existence of so-called Floquet solutions (which always exist when \( d = 1 \)); this is readily seen to be equivalent to the reducibility of a family of quasi-periodic cocycle depending on the "energy" parameter \( E \). In a pioneering work, Dinaburg and Sinai [5] proved that for a small analytic potential \( q \) (or large energy \( E \)), for a set of positive
(and asymptotically full) measure of energy $E$ in the spectrum, the associated cocycle is reducible provided the frequency $\omega \in \mathbb{R}^d$ is Diophantine:

$$|k \cdot \omega| \geq \frac{\gamma}{|k|^\tau}, \quad k \in \mathbb{Z}^d \setminus \{0\},$$

for some constant $\gamma > 0$ and $\tau \geq d - 1$, where $|k|$ is the sum of the absolute values of the components and $k \cdot \omega$ the Euclidean inner product. For a fixed cocycle as we considered above, their result amounts to a reducibility result under a Diophantine condition (with respect to $\omega$) on the so-called fibered rotation number $\rho = \rho(A + F)$, for a small analytic $F$, the smallness assumption depending on $\rho$. Rüssmann [12] later extended this result, under a more general arithmetic condition on $\omega$ and $\rho$ (this condition, weaker than the Diophantine condition, is slightly stronger than the so-called the Brjuno-Rüssmann condition). Moser and Pöschel [9] further extended the result to include some rational fibered rotation numbers, using a technique of resonance-cancellation, but the breakthrough came from Eliasson [6]: by sharpening this resonance-cancellation technique he obtained the reducibility for a set of full measure of fibered rotation number.

Since then, many works have been devoted to the reducibility of quasi-periodic cocycles, in different regularity classes and for cocycles taking values in different Lie algebras. In particular, many non-perturbative results have been obtained but they are restricted to two-dimensional frequencies $\omega \in \mathbb{R}^2$; we should not try to review to state of the art as we will be interested only in perturbative results, but valid in any dimension $\omega \in \mathbb{R}^d$. More precisely, we will be interested in the interaction between the regularity of the cocycle and the arithmetic properties of the frequency vector $\omega$.

For smooth cocycles, the Diophantine condition on $\omega$ is known to be sufficient, and it is not hard to see it is also necessary. The analytic case is more subtle. Chavaudret and Marmi [4] extended the result of Rüssmann to obtain reducibility under the Brjuno-Rüssmann condition; this sufficient arithmetic condition is not know to be optimal, but can be compared to a natural necessary condition (that we call the Rüssmann condition). The proof in [4] uses ideas of Rüssmann [13] and Pöschel [11], which deal with the corresponding results for respectively Hamiltonian systems and vector fields on the torus. Those results were later extended by Bounemoura and Féjoz [1], [2] for a more general class of systems with ultra-differentiable regularity. A particular case is the $\alpha$-Gevrey regularity, for a real parameter $\alpha \geq 1$, for which the analytic case is recovered by setting $\alpha = 1$; an $\alpha$-Brjuno-Rüssmann condition is introduced in [1] (an equivalent condition was independently obtained in [10] for vector fields on the torus) and exactly as for $\alpha = 1$, this sufficient condition is shown to be comparable to the natural necessary condition. Unfortunately the results in [2] do not allow such a comparison in general as the sufficient arithmetic condition is affected by a technical assumption, and so the results are not as accurate as those obtained in the Gevrey case in [1].

The purpose of this article is to improve the results of [2] within the context of quasi-periodic cocycles: we will obtain a result of reducibility valid for a larger class of ultra-differentiable systems, with a better sufficient arithmetic condition in the sense that it can be compared to the natural necessary condition.
2. Statement of the main results

Let us recall the setting. We have $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ for an integer $d \geq 1$ and we consider the cocycle

\[
\begin{align*}
\begin{cases} 
  x'(t) = (A + F(\theta(t)))x(t), \\
  \theta'(t) = \omega,
\end{cases}
\end{align*}
\]

where $x \in \mathbb{R}^2$, $\theta \in \mathbb{T}^d$ and $A \in sl(2, \mathbb{R})$ is an elliptic matrix, or equivalently, fixing $\theta(0) = 0 \in \mathbb{T}^d$, we consider

\[
x'(t) = (A + F(t\omega))x(t).
\]

Such a cocycle will be simply denoted by $(\omega, A + F)$. It is said to be reducible if there exists a (fibered) conjugacy between $(A + F)$ and a constant cocycle: there exist $Y: \mathbb{T}^d \to GL(2, \mathbb{R})$ and an elliptic matrix $B \in sl(2, \mathbb{R})$ such that

\[
\partial_\omega Y = (A + F)Y - YB,
\]

where $\partial_\omega Y(\theta) = \partial_\theta Y(\theta) \cdot \omega$. Since the matrices actually take values in $sl(2, \mathbb{R})$, a perhaps more natural definition would be to require that the conjugacy $Y$ takes value in the corresponding Lie group $SL(2, \mathbb{R})$: it follows from the work in [3] that these two definitions are the same. In order to get such a reducibility, one need to impose regularity assumptions on $F$ and an arithmetic condition on $\omega$ (we will also impose a similar arithmetic condition on the fibered rotation number).

To quantify the regularity of $F: \mathbb{T}^d \to sl(2, \mathbb{R})$, we introduce a weight function

\[
\Lambda: [1, +\infty) \to [1, +\infty)
\]

which we assume is increasing and differentiable. Expanding a smooth function $f \in C^\infty(\mathbb{T}^d, \mathbb{R})$ in Fourier series

\[
f(\theta) = \sum_k \hat{f}(k)e^{2\pi ik \cdot \theta}
\]

we will say it is $\Lambda$-ultra-differentiable if there exists $r > 0$ such that

\[
|f|_r = |f|_{\Lambda, r} := \sum_{k \in \mathbb{Z}^d} |\hat{f}(k)|e^{2\pi \Lambda(|k|)}r < \infty.
\]

The ultra-differentiable weighted norm of $f$ is then defined by (2.2), and we say $f$ belongs to the ultra-differentiable function class $U_{\Lambda, r}(\mathbb{T}^d, \mathbb{R})$. This defines a Banach space. We will require the function $\Lambda$ to be subadditive, namely

\[
\Lambda(x + y) \leq \Lambda(x) + \Lambda(y), \quad x, y \geq 1.
\]

This assumption turns $U_{\Lambda, r}(\mathbb{T}^d, \mathbb{R})$ into a Banach algebra (for a proof of this elementary fact, see Appendix 8.1). Now for a matrix-valued function $M: \mathbb{T}^d \to M_2(\mathbb{R})$, one extends the definition of (2.2) in such a way that $U_{\Lambda, r}(\mathbb{T}^d, M_2(\mathbb{R}))$ becomes a Banach algebra for the product of matrices. In the sequel, we will use the notation $U_\Lambda = \bigcup_{r>0} U_{\Lambda, r}$ when convenient. Main examples of ultra-differentiable classes are the $\alpha$-Gevrey class associated to $\Lambda_\alpha(v) = v^\alpha$ for $\alpha \geq 1$ and the real analytic class for $\Lambda_1(v) = v$, but many more examples are readily available. In particular, the quasi-analytic class, i.e the class of functions which
are uniquely determined by the sequence of their derivatives at a point, corresponds to a function \( \Lambda \) satisfying
\[
\int_1^\infty \frac{\Lambda(v)}{v^2} dv = +\infty
\]
The parameter \( r \) can be called "ultra-differentiable parameter" if (2.2) holds, and it is essentially the radius of convergence for real-analytic functions.

Next we need to quantify the non-resonance condition on \( \omega \in \mathbb{R}^d \). To do so, we introduce an approximating function
\[
\Psi : [1, +\infty) \to [1, +\infty)
\]
which we assume, without loss of generality, to be increasing and differentiable, and for which
\[
\Psi(K) = \max\{|2\pi k \cdot \omega|^{-1} | 0 < |k| \leq K\}, \quad K \in \mathbb{N}.
\]
We also need to quantify the non-resonance condition on the fibered rotation number \( \rho = \rho(A + F) \), a definition of which is recalled in Appendix 8.2. Without loss of generality, we use the same approximating function and requires that
\[
|2\rho \pm 2\pi k \cdot \omega| \geq \frac{1}{\Psi(K)}, \quad 0 < |k| \leq K.
\]
The approximating function \( \Psi \) will be assumed to satisfy the following arithmetic condition adapted to the weight \( \Lambda \), that we call the \( \Lambda \)-Brjuno-Rüssmann condition
\[
(\Lambda\text{-BR}) \quad \int_1^{+\infty} \frac{\Lambda'(v) \ln \Psi(v) dv}{\Lambda^2(v)} < \infty.
\]
One easily check that the last condition is equivalent to
\[
\int_1^{+\infty} \frac{\Psi'(v) dv}{\Psi(v) \Lambda(v)} < \infty.
\]
In the Gevrey case \( \Lambda_\alpha(v) = v^\frac{1}{\alpha} \) (and thus in the analytic case when \( \alpha = 1 \)), the \( \Lambda_\alpha \)-Brjuno-Rüssmann condition is
\[
(\Lambda\text{-BR}_\alpha) \quad \int_1^{+\infty} \frac{\ln \Psi(v) dv}{v^{1+\frac{1}{\alpha}}} < \infty.
\]
and one recovers the \( \alpha \)-Brjuno-Rüssmann condition introduced in [1] (for \( \alpha = 1 \), this is the Brjuno-Rüssmann condition as in [4]).

Now we can state the main theorem of this paper.

**Theorem 1.** Assume that \( \Lambda \) satisfy (S) and \( \omega \) and \( \rho \) verify (2.4) and (2.5) with \( \Psi \) satisfying the \( \Lambda \)-Brjuno-Rüssmann condition (\( \Lambda\text{-BR} \)). Given any \( r > 0 \) and any quasiperiodic cocycle \( (\omega, A + F) \) as in (2.1), with a non-zero elliptic matrix \( A \in \text{sl}(2, \mathbb{R}) \) and \( F \in U_{\Lambda,r}(\mathbb{T}^d, \text{sl}(2, \mathbb{R})) \), there exists \( \varepsilon \) depending only on \( r, \Lambda, A, \omega \) such that if \( |F|_r \leq \varepsilon \), the cocycle \( (\omega, A + F) \) is reducible with a conjugacy \( Y \in U_{\Lambda,r/2}(\mathbb{T}^d, \text{GL}(2, \mathbb{R})) \) which satisfies \( |Y|_2 \leq 2 \) and \( |Y^{-1}|_2 \leq 2 \).
The Λ-Brjuno-Rüssmann condition (Λ-BR) is thus sufficient for the reducibility within the class $U_\Lambda$; we do not know if the condition is necessary yet it implies the following Λ-Rüssmann condition

$$(\Lambda-R) \quad \lim_{v \to \infty} \frac{\log \Psi(v)}{\Lambda(v)} = 0$$

which is necessary as the next statement shows.

**Theorem 2.** Assume that Λ satisfy (S), ω verify (2.4) with Ψ not satisfying the Λ-Rüssmann condition (Λ-R) and ρ is arbitrary. Then there exists $r > 0$ such that for all $\varepsilon > 0$, there exist a quasiperiodic cocycle $(\omega, A + F)$ as in (2.1), with a non-zero elliptic matrix $A \in sl(2, \mathbb{R})$ and $F \in U_{\Lambda,r}(\mathbb{T}^d, sl(2, \mathbb{R}))$ satisfying $|F|_r \leq \varepsilon$ which is not reducible by any continuous conjugacy $Y : \mathbb{T}^d \to GL(2, \mathbb{R})$.

### 3. Corollaries and applications

#### 3.1. A special case: quasi-analytic functions.

In the α-Gevrey case when $\Lambda_\alpha(v) = v^{1/\alpha}$, in view of (2.7) the condition (Λ-BR) holds true for the approximating function $\Psi(v) = e^{v^{1/\alpha}}$ for any $\beta < 1/\alpha$; in particular in the analytic case when $\alpha = 1$ it holds true for any $\beta < 1$.

As a matter of fact, the latter also holds true for quasi-analytic functions, that is if Λ satisfies (2.3), then the condition (Λ-BR) holds true for $\Psi(v) = e^{v^{1/\alpha}}$ for any $\beta < 1$. Indeed, there exists

$$\frac{1+\beta}{2} < \gamma < 1, \quad 0 < \delta < \frac{1-\beta}{2}$$

and an unbounded sequence $(v_n)$ such that $1 \leq v_{n+1} - v_n \leq n^\delta$ and $\Lambda(v_n) > v_n^\gamma$. In particular $v_n \geq n$ for all $n$. Thus for all $n \geq 1$, recalling that (Λ-BR) is equivalent to (2.6), one has

$$\int_1^{v_{n+1}} \frac{\Psi'(v)dv}{\Psi(v)\Lambda(v)} \leq \beta \sum_{k \leq n} \frac{v_k^{\beta-1}}{\Lambda(v_k)}(v_{k+1} - v_k) \leq \beta \sum_{k \leq n} \frac{1}{k^{1+\gamma-\beta-\delta}}$$

and this last sum converges as $n \to +\infty$.

#### 3.2. Regularity of the Lyapunov exponent.

Let $\omega \in \mathbb{R}^d$ be fixed. For a matrix-valued function $G$, denote by $L(G)$ the maximal Lyapunov exponent of the cocycle $(\omega, G)$.

**Corollary 1.** Let $(\omega, A + F)$ satisfy the assumptions of Theorem 1. Then for all $A' \in U_\Lambda(\mathbb{T}^d, sl(2, \mathbb{R}))$,

$$|L(A') - L(A + F)| \leq 4|A' - (A + F)|_0$$

**Proof.** Let $Y$ be as in Theorem 1 and $A_\infty$ the elliptic matrix such that

$$\partial_\omega Y = (A + F)Y - YA_\infty.$$

Let $A' \in U_{\Lambda,r}(\mathbb{T}^d, M_2(\mathbb{R}))$, then

$$\partial_\omega Y = A'Y - Y(A_\infty + Y^{-1}(A' - (A + F))Y).$$
Notice that the Lyapunov exponent of $(\omega, A + F)$ is the same as for $(\omega, A_\infty)$, thus it is zero. Denoting by $L$ the Lyapunov exponent, one has

$$|L(A') - L(A + F)| = L(A') = L(A_\infty + Y^{-1}(A' - (A + F))Y)$$

$$\leq \ln|A_\infty + Y^{-1}(A' - (A + F))Y|_0$$

$$\leq \ln(1 + |Y^{-1}(A' - (A + F))Y|_0)$$

$$\leq |Y^{-1}(A' - (A + F))Y|_0$$

The bound on $Y$ gives the estimate (3.1). □

Let us remark that in order to describe the regularity without any condition on the fibered rotation number, a statement about almost reducibility would be needed.

4. PRELIMINARY REDUCTIONS AND CHOICE OF THE SEQUENCES OF PARAMETERS

Let us start with some preliminary lemmas. Recall that the spectrum of a non-zero elliptic matrix $A \in sl(2, \mathbb{R})$ is of the form Spec($A$) = $\{\pm i\alpha\}$ for some real number $\alpha > 0$ (such a number is well-defined up to a sign); this real number will be called the rotation number of $A$ and denoted by $\rho(A)$. The following lemma gives us a real normal form for such an elliptic matrix and an estimate on the size of the transformation to normal form.

**Lemma 4.1.** Given an elliptic matrix $A \in sl(2, \mathbb{R})$ with $\rho(A) = \alpha > 0$, there exist a matrix $P \in SL(2, \mathbb{R})$ such that

$$PAP^{-1} = \alpha J, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$|P| \leq 2(|A|/\alpha)^{1/2}, \quad |P^{-1}| \leq 1.$$

For a proof, we refer to [8]. It will sometimes be useful to use a complex normal form in which the matrix is diagonal. To do this, we consider the matrix

$$M = \frac{1}{1-i} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \in U(2)$$

and we define the complex invertible matrix $Q = MP$ where $P$ is the matrix given by Lemma 4.1, so that

$$QAQ^{-1} = i\alpha R, \quad R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$|Q| \leq 2(|A|/\alpha)^{1/2}, \quad |P^{-1}| \leq 1.$$

This lemma allows us to reduce the proof of Theorem 1 to the case where $A = \alpha J$; indeed, it suffices to replace the smallness assumption on $|F|_r$ by a smallness assumption on $|PPP^{-1}|_r$ which is bounded by $2(|A|/\alpha)^{1/2}|F_r|$. In such a normal form, we have the following elementary lemma.

**Lemma 4.2.** Assume that $A = \alpha J$. Then

$$|\rho(A) - \rho(A + F)| \leq 4|F|_0.$$
For a proof, we refer to Appendix 8.2. Finally, a matrix $A = \alpha J$ will remain elliptic under a small perturbation by a matrix in $sl(2, \mathbb{R})$ but no longer in normal form. Yet Lemma 4.1 immediately implies the following.

**Lemma 4.3.** Assume that $A = \alpha J$ with $\alpha > 0$ and $B \in sl(2, \mathbb{R})$ such that $|B| \leq \varepsilon$. If $\varepsilon \leq \alpha/4$, then the matrix $A + B$ is elliptic with $\rho(A + B) = \beta$ satisfying

$$\frac{\alpha}{2} \leq \beta \leq \alpha + \alpha/2$$

and thus there exists $P \in SL(2, \mathbb{R}))$ such that $P(A + B)P^{-1} = \beta J$ with

$$|P| \leq 4 \quad |P^{-1}| \leq 1.$$

Upon these preliminary reductions, we can define a sequence of parameters for the iterations. Let $\rho(A) = \alpha$, $\rho(A + F) = \rho$, $\varepsilon = |F|_r$ and define for $\nu \in \mathbb{N}$

$$\varepsilon_\nu = 4^{-\nu}\varepsilon.$$

We now choose $N_0 \in \mathbb{N}$, depending on $r$ and the approximating function $\Psi$, sufficiently large so that

$$\int_{N_0}^{+\infty} \frac{\Gamma'(v) \ln \Psi(v) dv}{\Gamma^2(v)} \leq \frac{\pi}{6} r. \tag{4.2}$$

Then we set

$$N_\nu = \Psi^{-1}(2^\nu \Psi(N_0))$$

and observe that for all $\nu \in \mathbb{N}$, we have

$$\Psi(N_\nu)\varepsilon_\nu = 2^{-\nu}\Psi(N_0)\varepsilon_0 = 2^{-\nu}\Psi(N_0)\varepsilon. \tag{4.3}$$

and thus the sequence $\Psi(N_\nu)\varepsilon_\nu$ is summable. We can define our threshold $\varepsilon \leq \bar{\varepsilon}$ by the requirements that

$$\varepsilon \leq \alpha/4, \quad 2^8\Psi(N_0)\varepsilon \leq 1.$$

One then easily check that

$$2^8\Psi(N_\nu)\varepsilon_\nu \leq 1 \tag{4.4}$$

holds true for all $\nu \in \mathbb{N}$. Next we define another sequence $\sigma_\nu > 0$ for $\nu \in \mathbb{N}$ by

$$\sigma_\nu = \frac{3 \ln 2}{\pi \Lambda(N_\nu)}$$

so that for all $\nu \in \mathbb{N}$, we have the equality

$$2^6 e^{-2\pi \Lambda(N_\nu)\sigma_\nu} = 1. \tag{4.5}$$

Finally we define recursively the sequence $r_\nu$ by setting $r_0 = r$ and $r_{\nu+1} = r_\nu - \sigma_\nu$; we will see later, as a consequence of (4.2) and of our choice of $\sigma_\nu$, that this sequence is well-defined (in the sense that $\sigma_\nu < r_\nu$), $r_\mu \geq r/2$ and thus $r_\nu$ converges to its infimum $r^* \geq r/2$. 
5. The iteration step

Given $F : \mathbb{T}^d \to gl(2, \mathbb{R})$, we define $\text{tr}(F) = \text{tr}(\hat{F}(0))$ where $\hat{F}(0)$ is the average of $F$ with respect to Lebesgue measure. We also define truncation operators $T_N$ and $\dot{T}_N$ on $U_\Lambda(\mathbb{T}^d, gl(2, \mathbb{R}))$ as
\[
(T_N F)(\theta) := \sum_{|k| \leq N} \hat{F}(k)e^{i2\pi k \cdot \theta}, \quad (\dot{T}_N F)(\theta) := \sum_{0 < |k| \leq N} \hat{F}(k)e^{i2\pi k \cdot \theta}.
\]

In this section, for a fixed $\nu \in \mathbb{N}$, we consider a cocycle $(\omega, A_\nu + F_\nu)$ which satisfy $(H_\nu)$
\[
\begin{cases}
A_\nu = \alpha_\nu J, \\
|F_\nu|_{r_\nu} \leq \varepsilon_\nu, \quad \varepsilon_\mu \leq \alpha_\nu/4, \quad \text{tr}(F_\nu) = 0, \\
\rho(A_\nu + F_\nu) = \rho.
\end{cases}
\]

We will conjugate this cocycle, by a transformation which is homotopic to the identity, to a cocycle $(\omega, A_{\nu+1} + F_{\nu+1})$ which satisfy $H_{\nu+1}$ together with estimates on such a transformation. First we have the following obvious lemma.

**Lemma 5.1.** For any $k \in \mathbb{Z}^d$ such that $0 < |k| \leq N_\nu$, we have
\[
|2\alpha_\nu \pm 2\pi k \cdot \omega| > \frac{1}{2\Psi(N_\nu)}.
\]

**Proof.** Indeed we have $\rho(A_\nu) = \alpha_\nu$ and $\rho(A_\nu + F_\nu) = \rho$ hence
\[
|2\alpha_\nu \pm 2\pi k \cdot \omega| \geq |2\rho \mp 2\pi k \cdot \omega| - |2\rho(A_\nu + F_\nu) - 2\rho(A_\nu)|
\]
\[
\geq \frac{1}{\Psi(N_\nu)} - 8\varepsilon_\nu \geq \frac{1}{2\Psi(N_\nu)}
\]
where we used Lemma 4.2 and the fact that $16\Psi(N_\nu)\varepsilon_\nu \leq 1$. \(\square\)

Next we solve an approximate cohomological equation.

**Lemma 5.2.** If $G_\nu = \dot{T}_N F_\nu$, there is a unique $X_\nu$ such that $X_\nu = \dot{T}_N X_\nu$ satisfying the equation
\[
\partial_\omega X_\nu = [A_\nu, X_\nu] + G_\nu
\]
with the estimates
\[
|X_\nu|_{r_\nu} \leq \Psi(N_\nu)\varepsilon_\nu, \quad |(I + X_\nu)^{-1}|_{r_\nu} \leq 2.
\]
Moreover $\text{tr}(X_\nu) = 0$.

**Proof.** Observe that conjugating the cocycle by the complex matrix $M$ defined in (4.1), it is sufficient to prove the statement for $A_\nu$ in complex normal form $i\alpha R$. Expanding $F_\nu$ and $X_\nu$ in Fourier series, the equation (5.1) yields
\[
\sum_{0 < |k| \leq N_\nu} \partial_\omega \dot{X}_\nu(k)e^{2\pi i k \cdot \theta} = \sum_{0 < |k| \leq N_\nu} [A_\nu, \dot{X}_\nu(k)]e^{2\pi i k \cdot \theta} + \sum_{0 < |k| \leq N_\nu} \dot{F}_\nu(k)e^{2\pi i k \cdot \theta},
\]
which is equivalent to
\[ \partial_\omega \hat{X}_\nu(k) = [A_\nu, \hat{X}_\nu(k)] + \hat{F}_\nu(k), \quad 0 < |k| \leq N_\nu. \]

Since \( A_\nu \) is diagonal, the solution of the above equation is
\[ \hat{X}_\nu(k) = L_k^{-1}\hat{F}_\nu(k), \]
where \( L_k \) is the operator defined by
\[ L_k : sl(2, \mathbb{R}) \rightarrow sl(2, \mathbb{R}), \quad \tilde{X} \mapsto 2\pi ik \cdot \omega \tilde{X} - [\tilde{A}_\nu, \tilde{X}]. \]

The spectrum of \( L_k \) is \( \{ 2\pi ik \cdot \omega \pm 2\alpha_\nu, \ 2\pi ik \cdot \omega \} \). By Lemma 5.1 and (2.4), the operator \( L_k \) for \( 0 < |k| \leq N_\nu \) is invertible with norm bounded by \( 2\Psi(N_\nu) \) so
\[ |\hat{X}_\nu(k)| = |L_k^{-1}\hat{F}_\nu(k)| \leq 2\Psi(N_\nu)|\hat{F}_\nu(k)| \]
and thus
\[ |X_\nu|_{r_\nu} = \sum_{0 < |k| \leq N_\nu} |\hat{X}_\nu(k)|e^{2\pi \Lambda(k)r} \leq 2\Psi(N_\nu) \sum_{0 < |k| \leq N_\nu} |\hat{F}_\nu(k)|e^{2\pi \Lambda(k)r} \leq 2\Psi(N_\nu)e_\nu. \]

Since \( 4\Psi(N_\nu)e_\nu \leq 1 \) the estimate
\[ |(I + X_\nu)^{-1}|_{r_\nu} \leq \frac{1}{1 - |X_\nu|_{r_\nu}} \leq 2 \]
is obvious and so is \( \text{tr}(X_\nu) = 0 \) because \( \hat{X}_\nu(0) = 0 \). This completes the proof. \( \square \)

We can finally state our main iterative proposition.

**Proposition 5.3.** Let \( (\omega, A_\nu + F_\nu) \) be as in \( (H_\nu) \). Then there exists a transformation \( Y_\nu \) homotopic to the identity such that
\[ \partial_\omega Y_\nu = (A_\nu + F_\nu)Y_\nu - Y_\nu(A_{\nu+1} + F_{\nu+1}), \]
with \( (\omega, A_{\nu+1} + F_{\nu+1}) \) satisfying \( (H_{\nu+1}) \) and such that
\[ |Y_\nu - I|_{r_\nu} \leq 8\Psi(N_\nu)e_\nu. \]

**Proof.** The transformation \( Y_\nu \) will be the composition of a quasi-periodic linear transformation given by Lemma 5.2 and a constant transformation given by Lemma 4.3 to put back the constant elliptic part into normal form.

Let \( X_\nu \) be given by Lemma 5.2, and let \( Z_\nu = I + X_\nu \). Since \( X_\nu \) solves
\[ \partial_\omega X_\nu = [A_\nu, X_\nu] + G_\nu, \]
a computation leads to
\[ \partial_\omega Z_\nu = (A_\nu + F_\nu)Z_\nu - Z_\nu(B_\nu + R_\nu), \]
with
\[ B_\nu = A_\nu + \hat{F}_\nu(0), \quad R_\nu = (I + X_\nu)^{-1}[(F_\nu - \hat{F}_\nu(0) - G_\nu) + F_\nu X_\nu - X_\nu \hat{F}_\nu(0)]. \]
We can estimate
\[
|R_{\nu}|_{r_{\nu}-\sigma_{\nu}} = |(I + X_{\nu})^{-1}[(F_{\nu} - \hat{F}_{\nu}(0) - G_{\nu}) + F_{\nu}X_{\nu} - X_{\nu}\hat{F}_{\nu}(0)]|_{r_{\nu}-\sigma_{\nu}} \\
\leq 2|F_{\nu} - \hat{F}_{\nu}(0) - G_{\nu}|_{r_{\nu}-\sigma_{\nu}} + 2|F_{\nu}X_{\nu}|_{r_{\nu}-\sigma_{\nu}} + 2|X_{\nu}\hat{F}_{\nu}(0)|_{r_{\nu}-\sigma_{\nu}} \\
\leq |F_{\nu} - \hat{F}_{\nu}(0)|_{r_{\nu}} 2e^{-2\pi \Lambda(N_{\nu})\sigma_{\nu}} + 4|X_{\nu}|_{r_{\nu}} |F_{\nu}|_{r_{\nu}} \\
\leq 2e^{-2\pi \Lambda(N_{\nu})\sigma_{\nu}} \varepsilon_{\nu} + 8\Psi(N_{\nu}) \varepsilon_{\nu} \varepsilon_{\nu} \\
\leq 2^{-5} \varepsilon_{\nu} + 2^{-5} \varepsilon_{\nu}
\]
where we used (4.4) and (4.5) in the last inequality, and therefore
\[(5.2) \quad |R_{\nu}|_{r_{\nu}-\sigma_{\nu}} \leq 2^{-4} \varepsilon_{\nu}.
\]
Let us now check that \(\text{tr} R_{\nu} = 0\). By the equality
\[
\partial_\omega (I + X_{\nu}) = (A_{\nu} + F_{\nu})(I + X_{\nu}) - (I + X_{\nu})(B_{\nu} + R_{\nu}),
\]
we know that
\[
R_{\nu} = -(I + X_{\nu})^{-1} \partial_\omega (I + X_{\nu}) + (I + X_{\nu})^{-1} (A_{\nu} + F_{\nu})(I + X_{\nu}) - B_{\nu}.
\]
It follows from the assumptions \(\text{tr} A_{\nu} = \text{tr} \hat{F}_{\nu}(0) = \text{tr} B_{\nu} = 0\) that
\[
\text{tr}((I + X_{\nu})^{-1} (A_{\nu} + F_{\nu})(I + X_{\nu}) - B_{\nu}) = 0.
\]
On the other hand, using \(\text{tr}(AB) = \text{tr}(BA)\) we have
\[
\text{tr}((I + X_{\nu})^{-1} \partial_\omega (I + X_{\nu})) \\
= \text{tr}(\partial_\omega (I + X_{\nu})(I + X_{\nu})^{-1}) \\
= \text{tr}(\partial_\omega X_{\nu}^{-1}) \\
= \text{tr}(\partial_\omega X_{\nu} - \frac{1}{2}((\partial_\omega X_{\nu})X_{\nu} + X_{\nu}\partial_\omega X_{\nu})) + \frac{1}{3}((\partial_\omega X_{\nu})X_{\nu}^2 + X_{\nu}(\partial_\omega X_{\nu})X_{\nu} + (\partial_\omega X_{\nu})X_{\nu}^2) + \cdots \\
= \text{tr}(\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \partial_\omega (X_{\nu}^{k+1})) \\
= 0.
\]
Thus \(\text{tr} R_{\nu} = 0\). Now we want to apply Lemma 4.3. Observe that \(\varepsilon_{\nu} \leq \alpha_{\nu}/4\), therefore Lemma 4.3 gives that \(\rho(B_{\nu}) = \rho(A_{\nu} + \hat{F}_{\nu}(0)) = \alpha_{\nu+1}\) satisfy
\[
\alpha_{\nu}/2 \leq \alpha_{\nu+1} \leq \alpha_{\nu} + \alpha_{\nu}/2
\]
hence there exists \(P_{\nu} \in SL(2, \mathbb{R})\) such that \(P_{\nu}(B_{\nu}) P_{\nu}^{-1} = \alpha_{\nu+1} J\) with
\[(5.3) \quad |P_{\nu}| \leq 4 \quad |P_{\nu}^{-1}| \leq 1.
\]
We can finally define
\[
Y_{\nu} = P_{\nu} Z_{\nu} P_{\nu}^{-1}, \quad A_{\nu+1} = \alpha_{\nu+1} J, \quad F_{\nu+1} = P_{\nu} R_{\nu} P_{\nu}^{-1}
\]
so that
\[
\partial_\omega Y_{\nu} = (A_{\nu} + F_{\nu}) Y_{\nu} - Y_{\nu} (A_{\nu+1} + F_{\nu+1}).
\]
Let us check that \((H)_{\nu+1}\) is satisfied. By definition \(A_{\nu+1} = \alpha_{\nu+1}J\) and we know that
\[
\varepsilon_{\nu+1} = \varepsilon_{\nu}/4 < \alpha_{\nu}/8 \leq \alpha_{\nu+1}/4.
\]
Again by definition, \(F_{\nu+1} = P_{\nu}R_{\nu}P_{\nu}^{-1}\) and from the estimates (5.2) and (5.3) we have
\[
|F_{\nu+1}|_{r_{\nu}-\sigma_{\nu}} \leq 2^{-4}\varepsilon_{\nu} = \varepsilon_{\nu+1}
\]
whereas \(\text{tr}(F_{\nu+1}) = \text{tr}(R_{\nu}) = 0\). Then, \(Z_{\nu} = I + X_{\nu}\) is obviously homotopic to the identity and so is \(P_{\nu} \in SL(2, \mathbb{R})\) hence
\[
\rho(A_{\nu+1} + F_{\nu+1}) = \rho(A_{\nu} + F_{\nu}) = \rho.
\]
To conclude, from (5.3) and the estimates on \(X_{\nu}\) given by Lemma 5.2 we get
\[
|Y_{\nu} - I|_{r_{\nu}} \leq 4|Z_{\nu} - I|_{r_{\nu}} \leq 4|X_{\nu}|_{r_{\nu}} \leq 8\Psi(N_{\nu})\varepsilon_{\nu}.
\]
This concludes the proof. \(\square\)

6. PROOF OF THEOREM 1

In this section we finally prove Theorem 1. Letting \(A_0 = A\) and \(F_0 = F\), we can apply inductively Proposition 5.3 and for any \(\nu \in \mathbb{N}\), if we define
\[
Y^\nu = \prod_{\mu \leq \nu} Y_{\mu} \in U_{\Lambda,N_{\nu}}(\mathbb{T}^d, GL(2, \mathbb{R}))
\]
we have
\[
\partial_\nu Y^\nu = (A + F)Y^\nu - Y^\nu(A_{\nu+1} + F_{\nu+1}),
\]
with
\[
\rho(A_{\nu+1} + F_{\nu+1}) = \rho, \quad |F|_{r_{\nu+1}} \leq \varepsilon_{\nu+1}.
\]
As the sequence \(\varepsilon_{\nu}\) converges to zero, the only thing that remains to be proved is that \(r_{\nu}\) converges to a non-zero limit and that \(Y^\nu\) converges. To do so, let us first observe that
\[
\sum_{\nu \geq 1} \frac{1}{\Lambda(N_{\nu})} = \sum_{\nu \geq 1} \frac{1}{\Lambda(\Psi^{-1}(2^\nu \Psi(N_0)))} \leq \int_0^{+\infty} \frac{dx}{\Lambda(\Psi^{-1}(2^x \Psi(N_0)))}
\]
and changing variables \(v = \Psi^{-1}(2^x \Psi(N_0))\), this gives
\[
\sum_{\nu \geq 1} \frac{1}{\Lambda(N_{\nu})} \leq \frac{1}{\ln 2} \int_{N_0}^{\infty} \frac{\Psi'(v)dv}{\Lambda(v)\Psi(v)}
\]
and finally, by an integration by parts this yields
\[
\sum_{\nu \geq 0} \frac{1}{\Lambda(N_{\nu})} \leq \frac{1}{\ln 2} \int_{N_0}^{\infty} \frac{\Lambda'(v)\ln \Psi(v)dv}{\Lambda^2(v)} \leq \frac{\pi}{6\ln 2} \frac{r}{r^2}
\]
where the last inequality follows from (4.2). By definition of \(\sigma_{\nu}\), this gives
\[
\sum_{\nu \geq 0} \sigma_{\nu} = \frac{3\ln 2}{\pi} \sum_{\nu \geq 0} \frac{1}{\Lambda(N_{\nu})} \leq r/2
\]
and thus \( r_\nu \geq r/2 \) converges to some \( r^* \geq r/2 \). To conclude, \( Y^\nu - I \) is easily seen to form a Cauchy sequence on the space \( U_{\Lambda,r/2}(\mathbb{T}^d, GL(2, \mathbb{R})) \) and thus \( Y^\nu \) converges to a limit \( Y \) which satisfies the bound

\[
|Y - I|_{r/2} \leq 2 \sum_{\nu \geq 0} |Y_\nu - I| \leq 16 \sum_{\nu \geq 0} \Psi(N_\nu) \varepsilon_\nu \leq 32 \Psi(N_0) \varepsilon.
\]

7. Proof of Theorem 2

Let us start with the following lemma, which says that if \( \omega \) does not satisfy the \( \Lambda \)-Rüssmann condition (\( \Lambda \)-R), then one cannot solve the cohomological equation in general.

**Lemma 7.1.** Assume \( \omega \) does not satisfy \( \Lambda \)-R, that is

\[
\limsup_{v \to +\infty} \frac{\log \Psi(v)}{\Lambda(v)} > 0.
\]

Then there exist \( r > 0 \) such that for any \( \varepsilon \geq 0 \) and any \( \rho \in \mathbb{R} \), there exists a function \( u : \mathbb{T}^d \to \mathbb{R} \) for which

\[
\int_{\mathbb{T}^d} u(\theta) \, d\theta = \rho, \quad |u - \rho|_r \leq \varepsilon
\]

but such that the equation

\[
(\mathbb{E}) \quad \omega \cdot \partial_\theta v(\theta) = u(\theta) - \rho
\]

has no continuous solution \( v : \mathbb{T}^d \to \mathbb{R} \).

**Proof.** Let \( r > 0 \) be such that

\[
\limsup_{v \to +\infty} \frac{\log \Psi(v)}{\Lambda(v)} \geq 3\pi r
\]

so that there exists a positive sequence \( v_j \to +\infty \) for which

\[
\Psi(v_j)^{-1} \leq e^{-3\pi r \Lambda(v_j)}
\]

By definition of \( \Psi \), there exists infinitely many \( k_j \in \mathbb{Z}^d \setminus \{0\} \) (for which \( |k_j| = v_j \)) and

\[
|2\pi k_j \cdot \omega| \leq e^{-3\pi r \Lambda(|k_j|)}.
\]

Let us define a constant

\[
C = \sum_{j \in \mathbb{N}} e^{-\pi r \Lambda(|k_j|)} < +\infty
\]

and a function

\[
u(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{u}(k) e^{2\pi i k \cdot \theta},
\]

by setting

\[
\hat{u}(0) = \rho, \quad \hat{u}(k_j) = \varepsilon C^{-1} 2\pi k_j \cdot \omega
\]

and all other Fourier coefficients equal to zero. By construction we do have

\[
\int_{\mathbb{T}^d} u(\theta) \, d\theta = \rho, \quad |u - \rho|_r \leq \varepsilon.
\]
Now a function $v : \mathbb{T}^d \to \mathbb{R}$
\[ v(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{v}(k)e^{2\pi ik \cdot \theta} \]
solves (E) if and only if
\[ \hat{v}(k_j) = \frac{\hat{u}(k_j)}{2\pi ik_j \cdot \omega} = \varepsilon C^{-1}. \]
Clearly such Fourier coefficients do not define a function which is integrable, and therefore $v$ cannot be continuous. \hfill \Box

To conclude the proof of Theorem 2, let $u : \mathbb{T}^d \to \mathbb{R}$ be the function given by Lemma 7.1, and consider the $sl(2, \mathbb{R})$ cocycle $(\omega, A + F)$ defined by
\[ A = \rho J, \quad F(\theta) = u(\theta)J - \rho J. \]
Its fibered rotation number is equal to $\rho$ (see Appendix 8.2) and $|F| \leq \varepsilon$. Argue by contradiction that $(\omega, A + F)$ is reducible. Since it takes values in $so(2, \mathbb{R})$, it follows from [3] that it is reducible by a transformation that takes values in $SO(2, \mathbb{R})$, therefore there exists $v : \mathbb{T}^d \to \mathbb{R}$ and
\[ Y(\theta) = \begin{pmatrix} \cos v(\theta) & \sin v(\theta) \\ -\sin v(\theta) & \cos v(\theta) \end{pmatrix} \in SO(2, \mathbb{R}) \]
such that
\[ \partial_\omega Y = MY - YB \]
for some constant matrix $B = \beta J$. But then necessarily $\beta = \rho$ (that is $B = A$) and the above equation is equivalent to (E) which has no continuous solution, which is a contradiction.

8. Appendix

8.1. Product estimates.

**Lemma 8.1.** Suppose $\Lambda$ satisfy the subadditivity condition (S). For any $f, g \in U_{\Lambda,r}(\mathbb{T}^d, \mathbb{R})$, we have $fg \in U_{\Lambda,r}(\mathbb{T}^d, \mathbb{R})$ and
\[ |fg|_r \leq |f|_r |g|_r. \]

**Proof.** Expanding in Fourier series we have
\[ f(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{f}(k)e^{2\pi ik \cdot \theta}, \quad g(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{g}(k)e^{2\pi ik \cdot \theta} \]
and
\[ f(\theta)g(\theta) = \left( \sum_{m \in \mathbb{Z}^d} \hat{f}(m)e^{2\pi im \cdot \theta} \right) \left( \sum_{n \in \mathbb{Z}^d} \hat{g}(n)e^{2\pi in \cdot \theta} \right) \]
\[ = \sum_{k \in \mathbb{Z}^d} \left( \sum_{m+n=k} \hat{f}(m)\hat{g}(n) \right)e^{2\pi ik \cdot \theta}. \]
On the one hand
\[ |fg|_r = | \sum_{k \in \mathbb{Z}^d} \left( \sum_{m+n=k} \hat{f}(m) \hat{g}(n) \right) e^{2\pi ik \cdot \theta} |_r, \]
\[ = \sum_{k \in \mathbb{Z}^d} \left| \sum_{m+n=k} \hat{f}(m) \hat{g}(n) \right| e^{2\pi \Lambda(|k|)r} \]
\[ \leq \sum_{k \in \mathbb{Z}^d} \left( \sum_{m+n=k} |\hat{f}(m)\hat{g}(n)| e^{2\pi \Lambda(|k|)r} \right) \]
and on the other hand
\[ |f|_r |g|_r = | \sum_{m \in \mathbb{Z}^d} \hat{f}(m) e^{2\pi im \cdot \theta} | \sum_{n \in \mathbb{Z}^d} \hat{g}(n) e^{2\pi in \cdot \theta} |_r, \]
\[ = \left( \sum_{m \in \mathbb{Z}^d} |\hat{f}(m)| e^{2\pi \Lambda(|m|)r} \right) \left( \sum_{m \in \mathbb{Z}^d} |\hat{g}(n)| e^{2\pi \Lambda(|n|)r} \right) \]
\[ = \sum_{k \in \mathbb{Z}^d} \sum_{m+n=k} |\hat{f}(m)\hat{g}(n)| e^{2\pi \Lambda(|m|+|n|)}. \]

Since \(|k| = |m+n| \leq |m| + |n|\) we have \(\Lambda(|k|) \leq \Lambda(|m|) + \Lambda(|n|)\) by subadditivity and thus \(|f \cdot g|_r \leq |f|_r |g|_r|_r.

8.2. Fibered rotation number. Let us consider a quasi-periodic cocycle
\[ x'(t) = M(t\omega)x(t), \]
where
\[ M : \mathbb{T}^d \rightarrow sl(2, \mathbb{R}), \quad M(\theta) = \left( \begin{array}{cc} a(\theta) & b(\theta) \\ c(\theta) & -a(\theta) \end{array} \right), \]
with \(\omega \in \mathbb{R}^d\) non-resonant. Following Eliasson [6], we define the fibered rotation number \(\rho(M) \in \mathbb{R}\) as
\[ \rho(M) = \lim_{t \to +\infty} \frac{\phi_M(t)}{t} \]
where \(\phi_M(t) = \phi(t)\) is any solution of the equation
\[ \phi'(t) = 2a(t\omega) \cos \phi(t) \sin \phi(t) - (b(t\omega) + c(t\omega)) \cos^2 \phi(t) + b(t\omega). \]
A first special case is
\[ M(\theta) = u(\theta)J, \quad u : \mathbb{T}^d \rightarrow \mathbb{R} \]
we then have
\[ \rho(M) = \lim_{t \to +\infty} \frac{\phi_M(t)}{t} = \lim_{t \to +\infty} \frac{1}{t} \int_0^t \phi_M'(s)ds = \lim_{t \to +\infty} \frac{1}{t} \int_0^t u(s\omega)ds = \int_{\mathbb{T}^d} u(\theta)d\theta \]
where the last equality follows from the unique ergodicity of the translation flow by \(\omega\).
Another special case is
\[ M(\theta) = \alpha J + F(\theta), \quad \sup_{\theta \in \mathbb{T}^d} |F(\theta)| = \sup_{\theta \in \mathbb{T}^d} \left| \begin{pmatrix} f_1(\theta) & f_2(\theta) \\ f_3(\theta) & -f_1(\theta) \end{pmatrix} \right| \leq \varepsilon \]
from which one immediately obtains the estimate
\[ |\rho(\alpha J) - \rho(M)| = |\alpha - \rho(M)| \leq \frac{1}{t} \int_0^t \phi_F'(s) ds \leq 4\varepsilon \]
which is nothing but Lemma 4.2.

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