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# REDUCIBILITY OF ULTRA-DIFFERENTIABLE QUASIPERIODIC COCYCLES UNDER AN ADAPTED ARITHMETIC CONDITION

ABED BOUNEMOURA, CLAIRE CHAUDAUDRET, AND SHUQING LIANG

**ABSTRACT.** We prove a reducibility result for  $sl(2, \mathbb{R})$  quasi-periodic cocycles close to a constant elliptic matrix in ultra-differentiable classes, under an adapted arithmetic condition which extends the Brjuno-Rüssmann condition in the analytic case. The proof is based on an elementary property of the fibered rotation number and deals with ultra-differentiable functions with a weighted Fourier norm. We also show that a weaker arithmetic condition is necessary for reducibility, and that it can be compared to a sufficient arithmetic condition.

## 1. INTRODUCTION

We will study the following time-quasiperiodic linear system, or quasiperiodic cocycle

$$\begin{cases} x'(t) = (A + F(\theta(t)))x(t), \\ \theta'(t) = \omega, \end{cases}$$

where  $x \in \mathbb{R}^2$ ,  $\theta \in \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$  with an integer  $d \geq 1$ ,  $\omega \in \mathbb{R}^d$  is a non-resonant frequency vector,  $A$  is elliptic (meaning it is conjugated to some non-zero element in  $so(2, \mathbb{R})$ ) and  $F : \mathbb{T}^d \rightarrow sl(2, \mathbb{R})$  belongs to some ultra-differentiable class. Let us recall that  $sl(2, \mathbb{R})$  is the Lie algebra of traceless matrices and  $so(2, \mathbb{R})$  is the Lie sub-algebra of skew-symmetric matrices. Such a quasi-periodic cocycle is said to be reducible if the time-quasiperiodic linear system can be conjugated, by a time quasi-periodic transformation, to a constant (time-independent) linear system.

One of the main motivation for studying reducibility of quasi-periodic cocycles came from the Schrödinger equation

$$-y''(t) + q(\theta + \omega t)y(t) = Ey(t)$$

and the question of existence of so-called Floquet solutions (which always exist when  $d = 1$ ); this is readily seen to be equivalent to the reducibility of a family of quasi-periodic cocycle depending on the "energy" parameter  $E$ . In a pioneering work, Dinaburg and Sinai [5] proved that for a small analytic potential  $q$  (or large energy  $E$ ), for a set of positive

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(and asymptotically full) measure of energy  $E$  in the spectrum, the associated cocycle is reducible provided the frequency  $\omega \in \mathbb{R}^d$  is Diophantine:

$$|k \cdot \omega| \geq \frac{\gamma}{|k|^\tau}, \quad k \in \mathbb{Z}^d \setminus \{0\},$$

for some constant  $\gamma > 0$  and  $\tau \geq d - 1$ , where  $|k|$  is the sum of the absolute values of the components and  $k \cdot \omega$  the Euclidean inner product. For a fixed cocycle as we considered above, their result amounts to a reducibility result under a Diophantine condition (with respect to  $\omega$ ) on the so-called fibered rotation number  $\rho = \rho(A + F)$ , for a small analytic  $F$ , the smallness assumption depending on  $\rho$ . Rüssmann [12] later extended this result, under a more general arithmetic condition on  $\omega$  and  $\rho$  (this condition, weaker than the Diophantine condition, is slightly stronger than the so-called the Brjuno-Rüssmann condition). Moser and Pöschel [9] further extended the result to include some rational fibered rotation numbers, using a technique of resonance-cancellation, but the breakthrough came from Eliasson [6]: by sharpening this resonance-cancellation technique he obtained the reducibility for a set of full measure of fibered rotation number.

Since then, many works have been devoted to the reducibility of quasi-periodic cocycles, in different regularity classes and for cocycles taking values in different Lie algebras. In particular, many non-perturbative results have been obtained but they are restricted to two-dimensional frequencies  $\omega \in \mathbb{R}^2$ ; we should not try to review to state of the art as we will be interested only in perturbative results, but valid in any dimension  $\omega \in \mathbb{R}^d$ . More precisely, we will be interested in the interaction between the regularity of the cocycle and the arithmetic properties of the frequency vector  $\omega$ .

For smooth cocycles, the Diophantine condition on  $\omega$  is known to be sufficient, and it is not hard to see it is also necessary. The analytic case is more subtle. Chavaudret and Marmi [4] extended the result of Rüssmann to obtain reducibility under the Brjuno-Rüssmann condition; this sufficient arithmetic condition is not known to be optimal, but can be compared to a natural necessary condition (that we call the Rüssmann condition). The proof in [4] uses ideas of Rüssmann [13] and Pöschel [11], which deal with the corresponding results for respectively Hamiltonian systems and vector fields on the torus. Those results were later extended by Bounemoura and Féjoz [1], [2] for a more general class of systems with ultra-differentiable regularity. A particular case is the  $\alpha$ -Gevrey regularity, for a real parameter  $\alpha \geq 1$ , for which the analytic case is recovered by setting  $\alpha = 1$ ; an  $\alpha$ -Brjuno-Rüssmann condition is introduced in [1] (an equivalent condition was independently obtained in [10] for vector fields on the torus) and exactly as for  $\alpha = 1$ , this sufficient condition is shown to be comparable to the natural necessary condition. Unfortunately the results in [2] do not allow such a comparison in general as the sufficient arithmetic condition is affected by a technical assumption, and so the results are not as accurate as those obtained in the Gevrey case in [1].

The purpose of this article is to improve the results of [2] within the context of quasi-periodic cocycles: we will obtain a result of reducibility valid for a larger class of ultra-differentiable systems, with a better sufficient arithmetic condition in the sense that it can be compared to the natural necessary condition.

## 2. STATEMENT OF THE MAIN RESULTS

Let us recall the setting. We have  $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$  for an integer  $d \geq 1$  and we consider the cocycle

$$(2.1) \quad \begin{cases} x'(t) = (A + F(\theta(t)))x(t), \\ \theta'(t) = \omega, \end{cases}$$

where  $x \in \mathbb{R}^2$ ,  $\theta \in \mathbb{T}^d$  and  $A \in sl(2, \mathbb{R})$  is an elliptic matrix, or equivalently, fixing  $\theta(0) = 0 \in \mathbb{T}^d$ , we consider

$$x'(t) = (A + F(t\omega))x(t).$$

Such a cocycle will be simply denoted by  $(\omega, A+F)$ . It is said to be reducible if there exists a (fibered) conjugacy between  $(A+F)$  and a constant cocycle: there exist  $Y : \mathbb{T}^d \rightarrow GL(2, \mathbb{R})$  and an elliptic matrix  $B \in sl(2, \mathbb{R})$  such that

$$\partial_\omega Y = (A + F)Y - YB,$$

where  $\partial_\omega Y(\theta) = \partial_\theta Y(\theta) \cdot \omega$ . Since the matrices actually take values in  $sl(2, \mathbb{R})$ , a perhaps more natural definition would be to require that the conjugacy  $Y$  takes value in the corresponding Lie group  $SL(2, \mathbb{R})$ : it follows from the work in [3] that these two definitions are the same. In order to get such a reducibility, one need to impose regularity assumptions on  $F$  and an arithmetic condition on  $\omega$  (we will also impose a similar arithmetic condition on the fibered rotation number).

To quantify the regularity of  $F : \mathbb{T}^d \rightarrow sl(2, \mathbb{R})$ , we introduce a weight function

$$\Lambda : [1, +\infty) \rightarrow [1, +\infty)$$

which we assume is increasing and differentiable. Expanding a smooth function  $f \in C^\infty(\mathbb{T}^d, \mathbb{R})$  in Fourier series

$$f(\theta) = \sum_k \hat{f}(k) e^{2\pi i k \cdot \theta}$$

we will say it is  $\Lambda$ -ultra-differentiable if there exists  $r > 0$  such that

$$(2.2) \quad |f|_r = |f|_{\Lambda, r} := \sum_{k \in \mathbb{Z}^d} |\hat{f}(k)| e^{2\pi \Lambda(|k|)r} < \infty.$$

The ultra-differentiable weighted norm of  $f$  is then defined by (2.2), and we say  $f$  belongs to the ultra-differentiable function class  $U_{\Lambda, r}(\mathbb{T}^d, \mathbb{R})$ . This defines a Banach space. We will require the function  $\Lambda$  to be subadditive, namely

$$(S) \quad \Lambda(x + y) \leq \Lambda(x) + \Lambda(y), \quad x, y \geq 1.$$

This assumption turns  $U_{\Lambda, r}(\mathbb{T}^d, \mathbb{R})$  into a Banach algebra (for a proof of this elementary fact, see Appendix 8.1). Now for a matrix-valued function  $M : \mathbb{T}^d \rightarrow M_2(\mathbb{R})$ , one extends the definition of (2.2) in such a way that  $U_{\Lambda, r}(\mathbb{T}^d, M_2(\mathbb{R}))$  becomes a Banach algebra for the product of matrices. In the sequel, we will use the notation  $U_\Lambda = \bigcup_{r>0} U_{\Lambda, r}$  when convenient. Main examples of ultra-differentiable classes are the  $\alpha$ -Gevrey class associated to  $\Lambda_\alpha(v) = v^{\frac{1}{\alpha}}$  for  $\alpha \geq 1$  and the real analytic class for  $\Lambda_1(v) = v$ , but many more examples are readily available. In particular, the quasi-analytic class, i.e the class of functions which

are uniquely determined by the sequence of their derivatives at a point, corresponds to a function  $\Lambda$  satisfying

$$(2.3) \quad \int_1^\infty \frac{\Lambda(v)}{v^2} dv = +\infty$$

The parameter  $r$  can be called "ultra-differentiable parameter" if (2.2) holds, and it is essentially the radius of convergence for real-analytic functions.

Next we need to quantify the non-resonance condition on  $\omega \in \mathbb{R}^d$ . To do so, we introduce an approximating function

$$\Psi : [1, +\infty) \rightarrow [1, +\infty)$$

which we assume, without loss of generality, to be increasing and differentiable, and for which

$$(2.4) \quad \Psi(K) = \max\{ |2\pi k \cdot \omega|^{-1} \mid 0 < |k| \leq K \}, \quad K \in \mathbb{N}.$$

We also need to quantify the non-resonance condition on the fibered rotation number  $\rho = \rho(A + F)$ , a definition of which is recalled in Appendix 8.2. Without loss of generality, we use the same approximating function and requires that

$$(2.5) \quad |2\rho \pm 2\pi k \cdot \omega| \geq \frac{1}{\Psi(K)}, \quad 0 < |k| \leq K.$$

The approximating function  $\Psi$  will be assumed to satisfy the following arithmetic condition adapted to the weight  $\Lambda$ , that we call the  $\Lambda$ -Brjuno-Rüssmann condition

$$(\Lambda\text{-BR}) \quad \int_1^{+\infty} \frac{\Lambda'(v) \ln \Psi(v) dv}{\Lambda^2(v)} < \infty.$$

One easily check that the last condition is equivalent to

$$(2.6) \quad \int_1^{+\infty} \frac{\Psi'(v) dv}{\Psi(v) \Lambda(v)} < \infty.$$

In the Gevrey case  $\Lambda_\alpha(v) = v^{\frac{1}{\alpha}}$  (and thus in the analytic case when  $\alpha = 1$ ), the  $\Lambda_\alpha$ -Brjuno-Rüssmann condition is

$$(2.7) \quad \int_1^{+\infty} \frac{\ln \Psi(v) dv}{v^{1+\frac{1}{\alpha}}}$$

and one recovers the  $\alpha$ -Brjuno-Rüssmann condition introduced in [1] (for  $\alpha = 1$ , this is the Brjuno-Rüssmann condition as in [4]).

Now we can state the main theorem of this paper.

**Theorem 1.** *Assume that  $\Lambda$  satisfy (S) and  $\omega$  and  $\rho$  verify (2.4) and (2.5) with  $\Psi$  satisfying the  $\Lambda$ -Brjuno-Rüssmann condition ( $\Lambda$ -BR). Given any  $r > 0$  and any quasiperiodic cocycle  $(\omega, A + F)$  as in (2.1), with a non-zero elliptic matrix  $A \in sl(2, \mathbb{R})$  and  $F \in U_{\Lambda, r}(\mathbb{T}^d, sl(2, \mathbb{R}))$ , there exists  $\bar{\varepsilon}$  depending only on  $r, \Lambda, A, \omega$  such that if  $|F|_r \leq \bar{\varepsilon}$ , the cocycle  $(\omega, A + F)$  is reducible with a conjugacy  $Y \in U_{\Lambda, r/2}(\mathbb{T}^d, GL(2, \mathbb{R}))$  which satisfies  $|Y|_{\frac{r}{2}} \leq 2$  and  $|Y^{-1}|_{\frac{r}{2}} \leq 2$ .*

The  $\Lambda$ -Brjuno-Rüssmann condition ( $\Lambda$ -BR) is thus sufficient for the reducibility within the class  $U_\Lambda$ ; we do not know if the condition is necessary yet it implies the following  $\Lambda$ -Rüssmann condition

$$(\Lambda\text{-R}) \quad \lim_{v \rightarrow \infty} \frac{\log \Psi(v)}{\Lambda(v)} = 0$$

which is necessary as the next statement shows.

**Theorem 2.** *Assume that  $\Lambda$  satisfy (S),  $\omega$  verify (2.4) with  $\Psi$  not satisfying the  $\Lambda$ -Rüssmann condition ( $\Lambda$ -R) and  $\rho$  is arbitrary. Then there exists  $r > 0$  such that for all  $\varepsilon > 0$ , there exist a quasiperiodic cocycle  $(\omega, A + F)$  as in (2.1), with a non-zero elliptic matrix  $A \in sl(2, \mathbb{R})$  and  $F \in U_{\Lambda, r}(\mathbb{T}^d, sl(2, \mathbb{R}))$  satisfying  $|F|_r \leq \varepsilon$  which is not reducible by any continuous conjugacy  $Y : \mathbb{T}^d \rightarrow GL(2, \mathbb{R})$ .*

### 3. COROLLARIES AND APPLICATIONS

**3.1. A special case: quasi-analytic functions.** In the  $\alpha$ -Gevrey case when  $\Lambda_\alpha(v) = v^{\frac{1}{\alpha}}$ , in view of (2.7) the condition ( $\Lambda$ -BR) holds true for the approximating function  $\Psi(v) = e^{v^\beta}$  for any  $\beta < 1/\alpha$ ; in particular in the analytic case when  $\alpha = 1$  it holds true for any  $\beta < 1$ .

As a matter of fact, the latter also holds true for quasi-analytic functions, that is if  $\Lambda$  satisfies (2.3), then the condition ( $\Lambda$ -BR) holds true for  $\Psi(v) = e^{v^\beta}$  for any  $\beta < 1$ . Indeed, there exists

$$\frac{1 + \beta}{2} < \gamma < 1, \quad 0 < \delta < \frac{1 - \beta}{2}$$

and an unbounded sequence  $(v_n)$  such that  $1 \leq v_{n+1} - v_n \leq n^\delta$  and  $\Lambda(v_n) > v_n^\gamma$ . In particular  $v_n \geq n$  for all  $n$ . Thus for all  $n \geq 1$ , recalling that ( $\Lambda$ -BR) is equivalent to (2.6), one has

$$\int_1^{v_{n+1}} \frac{\Psi'(v) dv}{\Psi(v) \Lambda(v)} \leq \beta \sum_{k \leq n} \frac{v_k^{\beta-1}}{\Lambda(v_k)} (v_{k+1} - v_k) \leq \beta \sum_{k \leq n} \frac{1}{k^{1+\gamma-\beta-\delta}}$$

and this last sum converges as  $n \rightarrow +\infty$ .

**3.2. Regularity of the Lyapunov exponent.** Let  $\omega \in \mathbb{R}^d$  be fixed. For a matrix-valued function  $G$ , denote by  $L(G)$  the maximal Lyapunov exponent of the cocycle  $(\omega, G)$ .

**Corollary 1.** *Let  $(\omega, A + F)$  satisfy the assumptions of Theorem 1. Then for all  $A' \in U_\Lambda(\mathbb{T}^d, sl(2, \mathbb{R}))$ ,*

$$(3.1) \quad |L(A') - L(A + F)| \leq 4|A' - (A + F)|_0$$

*Proof.* Let  $Y$  be as in Theorem 1 and  $A_\infty$  the elliptic matrix such that

$$\partial_\omega Y = (A + F)Y - YA_\infty.$$

Let  $A' \in U_{\Lambda, r}(\mathbb{T}^d, M_2(\mathbb{R}))$ , then

$$\partial_\omega Y = A'Y - Y(A_\infty + Y^{-1}(A' - (A + F))Y).$$

Notice that the Lyapunov exponent of  $(\omega, A + F)$  is the same as for  $(\omega, A_\infty)$ , thus it is zero. Denoting by  $L$  the Lyapunov exponent, one has

$$\begin{aligned} |L(A') - L(A + F)| &= L(A') = L(A_\infty + Y^{-1}(A' - (A + F))Y) \\ &\leq \ln |A_\infty + Y^{-1}(A' - (A + F))Y|_0 \\ &\leq \ln(1 + |Y^{-1}(A' - (A + F))Y|_0) \\ &\leq |Y^{-1}(A' - (A + F))Y|_0 \end{aligned}$$

The bound on  $Y$  gives the estimate (3.1).  $\square$

Let us remark that in order to describe the regularity without any condition on the fibered rotation number, a statement about almost reducibility would be needed.

#### 4. PRELIMINARY REDUCTIONS AND CHOICE OF THE SEQUENCES OF PARAMETERS

Let us start with some preliminary lemmas. Recall that the spectrum of a non-zero elliptic matrix  $A \in sl(2, \mathbb{R})$  is of the form  $\text{Spec}(A) = \{\pm i\alpha\}$  for some real number  $\alpha > 0$  (such a number is well-defined up to a sign); this real number will be called the rotation number of  $A$  and denoted by  $\rho(A)$ . The following lemma gives us a real normal form for such an elliptic matrix and an estimate on the size of the transformation to normal form.

**Lemma 4.1.** *Given an elliptic matrix  $A \in sl(2, \mathbb{R})$  with  $\rho(A) = \alpha > 0$ , there exist a matrix  $P \in SL(2, \mathbb{R})$  such that*

$$PAP^{-1} = \alpha J, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$|P| \leq 2(|A|/\alpha)^{1/2}, \quad |P^{-1}| \leq 1.$$

For a proof, we refer to [8]. It will sometimes be useful to use a complex normal form in which the matrix is diagonal. To do this, we consider the matrix

$$(4.1) \quad M = \frac{1}{1=i} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \in U(2)$$

and we define the complex invertible matrix  $Q = MP$  where  $P$  is the matrix given by Lemma 4.1, so that

$$QAQ^{-1} = i\alpha R, \quad R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$|Q| \leq 2(|A|/\alpha)^{1/2}, \quad |Q^{-1}| \leq 1.$$

This lemma allows us to reduce the proof of Theorem 1 to the case where  $A = \alpha J$ ; indeed, it suffices to replace the smallness assumption on  $|F|_r$  by a smallness assumption on  $|PFP^{-1}|_r$  which is bounded by  $2(|A|/\alpha)^{1/2}|F_r|$ . In such a normal form, we have the following elementary lemma.

**Lemma 4.2.** *Assume that  $A = \alpha J$ . Then*

$$|\rho(A) - \rho(A + F)| \leq 4|F|_0.$$

For a proof, we refer to Appendix 8.2. Finally, a matrix  $A = \alpha J$  will remain elliptic under a small perturbation by a matrix in  $sl(2, \mathbb{R})$  but no longer in normal form. Yet Lemma 4.1 immediately implies the following.

**Lemma 4.3.** *Assume that  $A = \alpha J$  with  $\alpha > 0$  and  $B \in sl(2, \mathbb{R})$  such that  $|B| \leq \varepsilon$ . If  $\varepsilon \leq \alpha/4$ , then the matrix  $A + B$  is elliptic with  $\rho(A + B) = \beta$  satisfying*

$$\alpha/2 \leq \beta \leq \alpha + \alpha/2$$

and thus there exists  $P \in SL(2, \mathbb{R})$  such that  $P(A + B)P^{-1} = \beta J$  with

$$|P| \leq 4 \quad |P^{-1}| \leq 1.$$

Upon these preliminary reductions, we can define a sequence of parameters for the iterations. Let  $\rho(A) = \alpha$ ,  $\rho(A + F) = \rho$ ,  $\varepsilon = |F|_r$  and define for  $\nu \in \mathbb{N}$

$$\varepsilon_\nu = 4^{-\nu} \varepsilon.$$

We now choose  $N_0 \in \mathbb{N}$ , depending on  $r$  and the approximating function  $\Psi$ , sufficiently large so that

$$(4.2) \quad \int_{N_0}^{+\infty} \frac{\Lambda'(v) \ln \Psi(v) dv}{\Lambda^2(v)} \leq \frac{\pi}{6} r.$$

Then we set

$$N_\nu = \Psi^{-1}(2^\nu \Psi(N_0))$$

and observe that for all  $\nu \in \mathbb{N}$ , we have

$$(4.3) \quad \Psi(N_\nu) \varepsilon_\nu = 2^{-\nu} \Psi(N_0) \varepsilon_0 = 2^{-\nu} \Psi(N_0) \varepsilon.$$

and thus the sequence  $\Psi(N_\nu) \varepsilon_\nu$  is summable. We can define our threshold  $\varepsilon \leq \bar{\varepsilon}$  by the requirements that

$$\varepsilon \leq \alpha/4, \quad 2^8 \Psi(N_0) \varepsilon \leq 1.$$

One then easily check that

$$(4.4) \quad 2^8 \Psi(N_\nu) \varepsilon_\nu \leq 1$$

holds true for all  $\nu \in \mathbb{N}$ . Next we define another sequence  $\sigma_\nu > 0$  for  $\nu \in \mathbb{N}$  by

$$\sigma_\nu = \frac{3 \ln 2}{\pi \Lambda(N_\nu)}$$

so that for all  $\nu \in \mathbb{N}$ , we have the equality

$$(4.5) \quad 2^6 e^{-2\pi \Lambda(N_\nu) \sigma_\nu} = 1.$$

Finally we define recursively the sequence  $r_\nu$  by setting  $r_0 = r$  and  $r_{\nu+1} = r_\nu - \sigma_\nu$ ; we will see later, as a consequence of (4.2) and of our choice of  $\sigma_\nu$ , that this sequence is well-defined (in the sense that  $\sigma_\nu < r_\nu$ ),  $r_\mu \geq r/2$  and thus  $r_\nu$  converges to its infimum  $r^* \geq r/2$ .



## 5. THE ITERATION STEP

Given  $F : \mathbb{T}^d \rightarrow gl(2, \mathbb{R})$ , we define  $\text{tr}\langle F \rangle = \text{tr}(\hat{F}(0))$  where  $\hat{F}(0)$  is the average of  $F$  with respect to Lebesgue measure. We also define truncation operators  $T_N$  and  $\dot{T}_N$  on  $U_\Lambda(\mathbb{T}^d, gl(2, \mathbb{R}))$  as

$$(T_N F)(\theta) := \sum_{|k| \leq N} \hat{F}(k) e^{i2\pi k \cdot \theta}, \quad (\dot{T}_N F)(\theta) := \sum_{0 < |k| \leq N} \hat{F}(k) e^{i2\pi k \cdot \theta}.$$

In this section, for a fixed  $\nu \in \mathbb{N}$ , we consider a cocycle  $(\omega, A_\nu + F_\nu)$  which satisfy

$$(H_\nu) \quad \begin{cases} A_\nu = \alpha_\nu J, \\ |F_\nu|_{r_\nu} \leq \varepsilon_\nu, \quad \varepsilon_\nu \leq \alpha_\nu/4, \quad \text{tr}\langle F_\nu \rangle = 0, \\ \rho(A_\nu + F_\nu) = \rho. \end{cases}$$

We will conjugate this cocycle, by a transformation which is homotopic to the identity, to a cocycle  $(\omega, A_{\nu+1} + F_{\nu+1})$  which satisfy  $H_{\nu+1}$  together with estimates on such a transformation. First we have the following obvious lemma.

**Lemma 5.1.** *For any  $k \in \mathbb{Z}^d$  such that  $0 < |k| \leq N_\nu$ , we have*

$$|2\alpha_\nu \pm 2\pi k \cdot \omega| > \frac{1}{2\Psi(N_\nu)}.$$

*Proof.* Indeed we have  $\rho(A_\nu) = \alpha_\nu$  and  $\rho(A_\nu + F_\nu) = \rho$  hence

$$\begin{aligned} |2\alpha_\nu \pm 2\pi k \cdot \omega| &\geq |2\rho \mp 2\pi k \cdot \omega| - |2\rho(A_\nu + F_\nu) - 2\rho(A_\nu)| \\ &\geq \frac{1}{\Psi(N_\nu)} - 8\varepsilon_\nu \geq \frac{1}{2\Psi(N_\nu)} \end{aligned}$$

where we used Lemma 4.2 and the fact that  $16\Psi(N_\nu)\varepsilon_\nu \leq 1$ . □

Next we solve an approximate cohomological equation.

**Lemma 5.2.** *If  $G_\nu = \dot{T}_{N_\nu} F_\nu$ , there is a unique  $X_\nu$  such that  $X_\nu = \dot{T}_{N_\nu} X_\nu$  satisfying the equation*

$$(5.1) \quad \partial_\omega X_\nu = [A_\nu, X_\nu] + G_\nu$$

with the estimates

$$|X_\nu|_{r_\nu} \leq \Psi(N_\nu)\varepsilon_\nu, \quad |(I + X_\nu)^{-1}|_{r_\nu} \leq 2.$$

Moreover  $\text{tr}\langle X_\nu \rangle = 0$ .

*Proof.* Observe that conjugating the cocycle by the complex matrix  $M$  defined in (4.1), it is sufficient to prove the statement for  $A_\nu$  in complex normal form  $i\alpha R$ . Expanding  $F_\nu$  and  $X_\nu$  in Fourier series, the equation (5.1) yields

$$\sum_{0 < |k| \leq N_\nu} \partial_\omega \hat{X}_\nu(k) e^{2\pi i k \cdot \theta} = \sum_{0 < |k| \leq N_\nu} [A_\nu, \hat{X}_\nu(k)] e^{2\pi i k \cdot \theta} + \sum_{0 < |k| \leq N_\nu} \hat{F}_\nu(k) e^{2\pi i k \cdot \theta},$$

which is equivalent to

$$\partial_\omega \hat{X}_\nu(k) = [A_\nu, \hat{X}_\nu(k)] + \hat{F}_\nu(k), \quad 0 < |k| \leq N_\nu.$$

Since  $A_\nu$  is diagonal, the solution of the above equation is

$$\hat{X}_\nu(k) = L_k^{-1} \hat{F}_\nu(k),$$

where  $L_k$  is the operator defined by

$$L_k : sl(2, \mathbb{R}) \rightarrow sl(2, \mathbb{R}), \quad \tilde{X} \mapsto 2\pi i k \cdot \omega \tilde{X} - [\tilde{A}_\nu, \tilde{X}].$$

The spectrum of  $L_k$  is  $\{2\pi i k \cdot \omega \pm 2\alpha_\nu, 2\pi i k \cdot \omega\}$ . By Lemma 5.1 and (2.4), the operator  $L_k$  for  $0 < |k| \leq N_\nu$  is invertible with norm bounded by  $2\Psi(N_\nu)$  so

$$|\hat{X}_\nu(k)| = |L_k^{-1} \hat{F}_\nu(k)| \leq 2\Psi(N_\nu) |\hat{F}_\nu(k)|$$

and thus

$$|X_\nu|_{r_\nu} = \sum_{0 < |k| \leq N_\nu} |\hat{X}_\nu(k)| e^{2\pi \Lambda(k)r} \leq 2\Psi(N_\nu) \sum_{0 < |k| \leq N_\nu} |\hat{F}_\nu(k)| e^{2\pi \Lambda(k)r} \leq 2\Psi(N_\nu) \varepsilon_\nu.$$

Since  $4\Psi(N_\nu) \varepsilon_\nu \leq 1$  the estimate

$$|(I + X_\nu)^{-1}|_{r_\nu} \leq \frac{1}{1 - |X_\nu|_{r_\nu}} \leq 2$$

is obvious and so is  $\text{tr}\langle X_\nu \rangle = 0$  because  $\hat{X}_\nu(0) = 0$ . This completes the proof.  $\square$

We can finally state our main iterative proposition.

**Proposition 5.3.** *Let  $(\omega, A_\nu + F_\nu)$  be as in  $(H_\nu)$ . Then there exists a transformation  $Y_\nu$  homotopic to the identity such that*

$$\partial_\omega Y_\nu = (A_\nu + F_\nu)Y_\nu - Y_\nu(A_{\nu+1} + F_{\nu+1}),$$

with  $(\omega, A_{\nu+1} + F_{\nu+1})$  satisfying  $(H_{\nu+1})$  and such that

$$|Y_\nu - I|_{r_\nu} \leq 8\Psi(N_\nu) \varepsilon_\nu$$

*Proof.* The transformation  $Y_\nu$  will be the composition of a quasi-periodic linear transformation given by Lemma 5.2 and a constant transformation given by Lemma 4.3 to put back the constant elliptic part into normal form.

Let  $X_\nu$  be given by Lemma 5.2, and let  $Z_\nu = I + X_\nu$ . Since  $X_\nu$  solves

$$\partial_\omega X_\nu = [A_\nu, X_\nu] + G_\nu,$$

a computation leads to

$$\partial_\omega Z_\nu = (A_\nu + F_\nu)Z_\nu - Z_\nu(B_\nu + R_\nu),$$

with

$$B_\nu = A_\nu + \hat{F}_\nu(0), \quad R_\nu = (I + X_\nu)^{-1}[(F_\nu - \hat{F}_\nu(0) - G_\nu) + F_\nu X_\nu - X_\nu \hat{F}_\nu(0)].$$

We can estimate

$$\begin{aligned}
|R_\nu|_{r_\nu - \sigma_\nu} &= |(I + X_\nu)^{-1}[(F_\nu - \hat{F}_\nu(0) - G_\nu) + F_\nu X_\nu - X_\nu \hat{F}_\nu(0)]|_{r_\nu - \sigma_\nu} \\
&\leq 2|F_\nu - \hat{F}_\nu(0) - G_\nu|_{r_\nu - \sigma_\nu} + 2|F_\nu X_\nu|_{r_\nu - \sigma_\nu} + 2|X_\nu \hat{F}_\nu(0)|_{r_\nu - \sigma_\nu} \\
&\leq |F_\nu|_{r_\nu} 2e^{-2\pi\Lambda(N_\nu)\sigma_\nu} + 4|X_\nu|_r |F_\nu|_r \\
&\leq 2e^{-2\pi\Lambda(N_\nu)\sigma_\nu} \varepsilon_\nu + 8\Psi(N_\nu) \varepsilon_\nu \varepsilon_\nu \\
&\leq 2^{-5} \varepsilon_\nu + 2^{-5} \varepsilon_\nu
\end{aligned}$$

where we used (4.4) and (4.5) in the last inequality, and therefore

$$(5.2) \quad |R_\nu|_{r_\nu - \sigma_\nu} \leq 2^{-4} \varepsilon_\nu.$$

Let us now check that  $\text{tr}\langle R_\nu \rangle = 0$ . By the equality

$$\partial_\omega(I + X_\nu) = (A_\nu + F_\nu)(I + X_\nu) - (I + X_\nu)(B_\nu + R_\nu),$$

we know that

$$R_\nu = -(I + X_\nu)^{-1} \partial_\omega(I + X_\nu) + (I + X_\nu)^{-1} (A_\nu + F_\nu)(I + X_\nu) - B_\nu.$$

It follows from the assumptions  $\text{tr}A_\nu = \text{tr}\hat{F}_\nu(0) = \text{tr}B_\nu = 0$  that

$$\text{tr}\langle (I + X_\nu)^{-1} (A_\nu + F_\nu)(I + X_\nu) - B_\nu \rangle = 0.$$

On the other hand, using  $\text{tr}(AB) = \text{tr}(BA)$  we have

$$\begin{aligned}
&\text{tr}\langle (I + X_\nu)^{-1} \partial_\omega(I + X_\nu) \rangle \\
&= \text{tr}\langle \partial_\omega(I + X_\nu)(I + X_\nu)^{-1} \rangle \\
&= \text{tr}\langle (\partial_\omega X_\nu)(I + X_\nu)^{-1} \rangle \\
&= \text{tr}\langle \partial_\omega X_\nu - \frac{1}{2}((\partial_\omega X_\nu)X_\nu + X_\nu \partial_\omega X_\nu) \rangle + \frac{1}{3}((\partial_\omega X_\nu)X_\nu^2 + X_\nu(\partial_\omega X_\nu)X_\nu + (\partial_\omega X_\nu)X_\nu^2) + \dots \\
&= \text{tr}\langle \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \partial_\omega(X_\nu^{k+1}) \rangle \\
&= 0.
\end{aligned}$$

Thus  $\text{tr}\langle R_\nu \rangle = 0$ . Now we want to apply Lemma 4.3. Observe that  $\varepsilon_\nu \leq \alpha_\nu/4$ , therefore Lemma 4.3 gives that  $\rho(B_\nu) = \rho(A_\nu + \hat{F}_\nu(0)) = \alpha_{\nu+1}$  satisfy

$$\alpha_\nu/2 \leq \alpha_{\nu+1} \leq \alpha_\nu + \alpha_\nu/2$$

hence there exists  $P_\nu \in SL(2, \mathbb{R})$  such that  $P_\nu(B_\nu)P_\nu^{-1} = \alpha_{\nu+1}J$  with

$$(5.3) \quad |P_\nu| \leq 4 \quad |P_\nu^{-1}| \leq 1.$$

We can finally define

$$Y_\nu = P_\nu Z_\nu P_\nu^{-1}, \quad A_{\nu+1} = \alpha_{\nu+1}J, \quad F_{\nu+1} = P_\nu R_\nu P_\nu^{-1}$$

so that

$$\partial_\omega Y_\nu = (A_\nu + F_\nu)Y_\nu - Y_\nu(A_{\nu+1} + F_{\nu+1}).$$

Let us check that  $(H)_{\nu+1}$  is satisfied. By definition  $A_{\nu+1} = \alpha_{\nu+1}J$  and we know that

$$\varepsilon_{\nu+1} = \varepsilon_{\nu}/4 < \alpha_{\nu}/8 \leq \alpha_{\nu+1}/4.$$

Again by definition,  $F_{\nu+1} = P_{\nu}R_{\nu}P_{\nu}^{-1}$  and from the estimates (5.2) and (5.3) we have

$$|F_{\nu+1}|_{r_{\nu}-\sigma_{\nu}} \leq 2^{-4}4\varepsilon_{\nu} = \varepsilon_{\nu+1}$$

whereas  $\text{tr}\langle F_{\nu+1} \rangle = \text{tr}\langle R_{\nu} \rangle = 0$ . Then,  $Z_{\nu} = I + X_{\nu}$  is obviously homotopic to the identity and so is  $P_{\nu} \in SL(2, \mathbb{R})$  hence

$$\rho(A_{\nu+1} + F_{\nu+1}) = \rho(A_{\nu} + F_{\nu}) = \rho.$$

To conclude, from (5.3) and the estimates on  $X_{\nu}$  given by Lemma 5.2 we get

$$|Y_{\nu} - I|_{r_{\nu}} \leq 4|Z_{\nu} - I|_{r_{\nu}} \leq 4|X_{\nu}|_{r_{\nu}} \leq 8\Psi(N_{\nu})\varepsilon_{\nu}.$$

This concludes the proof.  $\square$

## 6. PROOF OF THEOREM 1

In this section we finally prove Theorem 1. Letting  $A_0 = A$  and  $F_0 = F$ , we can apply inductively Proposition 5.3 and for any  $\nu \in \mathbb{N}$ , if we define

$$Y^{\nu} = \prod_{\mu \leq \nu} Y_{\mu} \in U_{\Lambda, r_{\nu}}(\mathbb{T}^d, GL(2, \mathbb{R}))$$

we have

$$\partial_{\omega} Y^{\nu} = (A + F)Y^{\nu} - Y^{\nu}(A_{\nu+1} + F_{\nu+1}),$$

with

$$\rho(A_{\nu+1} + F_{\nu+1}) = \rho, \quad |F|_{r_{\nu+1}} \leq \varepsilon_{\nu+1}.$$

As the sequence  $\varepsilon_{\nu}$  converges to zero, the only thing that remains to be proved is that  $r_{\nu}$  converges to a non-zero limit and that  $Y^{\nu}$  converges. To do so, let us first observe that

$$\sum_{\nu \geq 1} \frac{1}{\Lambda(N_{\nu})} = \sum_{\nu \geq 1} \frac{1}{\Lambda(\Psi^{-1}(2^{\nu}\Psi(N_0)))} \leq \int_0^{+\infty} \frac{dx}{\Lambda(\Psi^{-1}(2^x\Psi(N_0)))}$$

and changing variables  $v = \Psi^{-1}(2^x\Psi(N_0))$ , this gives

$$\sum_{\nu \geq 1} \frac{1}{\Lambda(N_{\nu})} \leq \frac{1}{\ln 2} \int_{N_0}^{\infty} \frac{\Psi'(v)dv}{\Lambda(v)\Psi(v)}$$

and finally, by an integration by parts this yields

$$\sum_{\nu \geq 0} \frac{1}{\Lambda(N_{\nu})} \leq \frac{1}{\ln 2} \int_{N_0}^{\infty} \frac{\Lambda'(v) \ln \Psi(v) dv}{\Lambda^2(v)} \leq \frac{\pi}{6 \ln 2} r$$

where the last inequality follows from (4.2). By definition of  $\sigma_{\mu}$  this gives

$$\sum_{\nu \geq 0} \sigma_{\nu} = \frac{3 \ln 2}{\pi} \sum_{\nu \geq 0} \frac{1}{\Lambda(N_{\nu})} \leq r/2$$

and thus  $r_\nu \geq r/2$  converges to some  $r^* \geq r/2$ . To conclude,  $Y^\nu - I$  is easily seen to form a Cauchy sequence on the space  $U_{\Lambda, r/2}(\mathbb{T}^d, GL(2, \mathbb{R}))$  and thus  $Y^\nu$  converges to a limit  $Y$  which satisfies the bound

$$|Y - I|_{r/2} \leq 2 \sum_{\nu \geq 0} |Y_\nu - I| \leq 16 \sum_{\nu \geq 0} \Psi(N_\nu) \varepsilon_\nu \leq 32 \Psi(N_0) \varepsilon.$$

## 7. PROOF OF THEOREM 2

Let us start with the following lemma, which says that if  $\omega$  does not satisfy the  $\Lambda$ -Rüssmann condition ( $\Lambda$ -R), then one cannot solve the cohomological equation in general.

**Lemma 7.1.** *Assume  $\omega$  does not satisfy ( $\Lambda$ -R), that is*

$$\limsup_{v \rightarrow \infty} \frac{\log \Psi(v)}{\Lambda(v)} > 0.$$

*Then there exist  $r > 0$  such that for any  $\varepsilon \geq 0$  and any  $\rho \in \mathbb{R}$ , there exists a function  $u : \mathbb{T}^d \rightarrow \mathbb{R}$  for which*

$$\int_{\mathbb{T}^d} u(\theta) d\theta = \rho, \quad |u - \rho|_r \leq \varepsilon$$

*but such that the equation*

$$(E) \quad \omega \cdot \partial_\theta v(\theta) = u(\theta) - \rho$$

*has no continuous solution  $v : \mathbb{T}^d \rightarrow \mathbb{R}$ .*

*Proof.* Let  $r > 0$  be such that

$$\limsup_{v \rightarrow \infty} \frac{\log \Psi(v)}{\Lambda(v)} \geq 3\pi r$$

so that there exists a positive sequence  $v_j \rightarrow +\infty$  for which

$$\Psi(v_j)^{-1} \leq e^{-3\pi r \Lambda(v_j)}$$

By definition of  $\Psi$ , there exists infinitely many  $k_j \in \mathbb{Z}^d \setminus \{0\}$  (for which  $|k_j| = v_j$ ) and

$$|2\pi k_j \cdot \omega| \leq e^{-3\pi r \Lambda(|k_j|)}.$$

Let us define a constant

$$C = \sum_{j \in \mathbb{N}} e^{-\pi r \Lambda(|k_j|)} < +\infty$$

and a function

$$u(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{u}(k) e^{2\pi i k \cdot \theta},$$

by setting

$$\hat{u}(0) = \rho, \quad \hat{u}(k_j) = \varepsilon C^{-1} 2\pi k_j \cdot \omega$$

and all other Fourier coefficients equal to zero. By construction we do have

$$\int_{\mathbb{T}^d} u(\theta) d\theta = \rho, \quad |u - \rho|_r \leq \varepsilon.$$

Now a function  $v : \mathbb{T}^d \rightarrow \mathbb{R}$

$$v(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{v}(k) e^{2\pi i k \cdot \theta}$$

solves (E) if and only if

$$\hat{v}(k_j) = \frac{\hat{u}(k_j)}{2\pi i k_j \cdot \omega} = \varepsilon C^{-1}.$$

Clearly such Fourier coefficients do not define a function which is integrable, and therefore  $v$  cannot be continuous.  $\square$

To conclude the proof of Theorem 2, let  $u : \mathbb{T}^d \rightarrow \mathbb{R}$  be the function given by Lemma 7.1, and consider the  $sl(2, \mathbb{R})$  cocycle  $(\omega, A + F)$  defined by

$$A = \rho J, \quad F(\theta) = u(\theta)J - \rho J.$$

Its fibered rotation number is equal to  $\rho$  (see Appendix 8.2) and  $|F| \leq \varepsilon$ . Argue by contradiction that  $(\omega, A + F)$  is reducible. Since it takes values in  $so(2, \mathbb{R})$ , it follows from [3] that it is reducible by a transformation that takes values in  $SO(2, \mathbb{R})$ , therefore there exists  $v : \mathbb{T}^d \rightarrow \mathbb{R}$  and

$$Y(\theta) = \begin{pmatrix} \cos v(\theta) & \sin v(\theta) \\ -\sin v(\theta) & \cos v(\theta) \end{pmatrix} \in SO(2, \mathbb{R})$$

such that

$$\partial_\omega Y = MY - YB$$

for some constant matrix  $B = \beta J$ . But then necessarily  $\beta = \rho$  (that is  $B = A$ ) and the above equation is equivalent to (E) which has no continuous solution, which is a contradiction.

## 8. APPENDIX

### 8.1. Product estimates.

**Lemma 8.1.** *Suppose  $\Lambda$  satisfy the subadditivity condition (S). For any  $f, g \in U_{\Lambda, r}(\mathbb{T}^d, \mathbb{R})$ , we have  $fg \in U_{\Lambda, r}(\mathbb{T}^d, \mathbb{R})$  and*

$$|fg|_r \leq |f|_r |g|_r.$$

*Proof.* Expanding in Fourier series we have

$$f(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{f}(k) e^{2\pi i k \cdot \theta}, \quad g(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{g}(k) e^{2\pi i k \cdot \theta}$$

and

$$\begin{aligned} f(\theta)g(\theta) &= \left( \sum_{m \in \mathbb{Z}^d} \hat{f}(m) e^{2\pi i m \cdot \theta} \right) \left( \sum_{n \in \mathbb{Z}^d} \hat{g}(n) e^{2\pi i n \cdot \theta} \right) \\ &= \sum_{k \in \mathbb{Z}^d} \left( \sum_{m+n=k} \hat{f}(m) \hat{g}(n) \right) e^{2\pi i k \cdot \theta}. \end{aligned}$$

On the one hand

$$\begin{aligned}
|fg|_r &= \left| \sum_{k \in \mathbb{Z}^d} \left( \sum_{m+n=k} \hat{f}(m) \hat{g}(n) \right) e^{2\pi i k \cdot \theta} \right|_r \\
&= \sum_{k \in \mathbb{Z}^d} \left| \sum_{m+n=k} \hat{f}(m) \hat{g}(n) \right| e^{2\pi \Lambda(|k|)r} \\
&\leq \sum_{k \in \mathbb{Z}^d} \left( \sum_{m+n=k} |\hat{f}(m) \hat{g}(n)| e^{2\pi \Lambda(|k|)r} \right)
\end{aligned}$$

and on the other hand

$$\begin{aligned}
|f|_r |g|_r &= \left| \sum_{m \in \mathbb{Z}^d} \hat{f}(m) e^{2\pi i m \cdot \theta} \right|_r \left| \sum_{n \in \mathbb{Z}^d} \hat{g}(n) e^{2\pi i n \cdot \theta} \right|_r \\
&= \left( \sum_{m \in \mathbb{Z}^d} |\hat{f}(m)| e^{2\pi \Lambda(|m|)r} \right) \left( \sum_{n \in \mathbb{Z}^d} |\hat{g}(n)| e^{2\pi \Lambda(|n|)r} \right) \\
&= \sum_{k \in \mathbb{Z}^d} \sum_{m+n=k} |\hat{f}(m) \hat{g}(n)| e^{2\pi r(\Lambda(|m|) + \Lambda(|n|))}.
\end{aligned}$$

Since  $|k| = |m+n| \leq |m| + |n|$  we have  $\Lambda(|k|) \leq \Lambda(|m|) + \Lambda(|n|)$  by subadditivity and thus  $|f \cdot g|_r \leq |f|_r |g|_r$ .  $\square$

**8.2. Fibered rotation number.** Let us consider a quasi-periodic cocycle

$$x'(t) = M(t\omega)x(t),$$

where

$$M : \mathbb{T}^d \rightarrow sl(2, \mathbb{R}), \quad M(\theta) = \begin{pmatrix} a(\theta) & b(\theta) \\ c(\theta) & -a(\theta) \end{pmatrix}$$

with  $\omega \in \mathbb{R}^d$  non-resonant. Following Eliasson [6], we define the fibered rotation number  $\rho(M) \in \mathbb{R}$  as

$$\rho(M) = \lim_{t \rightarrow +\infty} \frac{\phi_M(t)}{t}$$

where  $\phi_M(t) = \phi(t)$  is any solution of the equation

$$\phi'(t) = 2a(t\omega) \cos \phi(t) \sin \phi(t) - (b(t\omega) + c(t\omega)) \cos^2 \phi(t) + b(t\omega).$$

A first special case is

$$M(\theta) = u(\theta)J, \quad u : \mathbb{T}^d \rightarrow \mathbb{R}$$

we then have

$$\rho(M) = \lim_{t \rightarrow +\infty} \frac{\phi_M(t)}{t} = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \phi'_M(s) ds = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t u(s\omega) ds = \int_{\mathbb{T}^d} u(\theta) d\theta$$

where the last equality follows from the unique ergodicity of the translation flow by  $\omega$ . Another special case is

$$M(\theta) = \alpha J + F(\theta), \quad \sup_{\theta \in \mathbb{T}^d} |F(\theta)| = \sup_{\theta \in \mathbb{T}^d} \left| \begin{pmatrix} f_1(\theta) & f_2(\theta) \\ f_3(\theta) & -f_1(\theta) \end{pmatrix} \right| \leq \varepsilon$$

from which one immediately obtains the estimate

$$|\rho(\alpha J) - \rho(M)| = |\alpha - \rho(M)| \leq \left| \frac{1}{t} \int_0^t \phi'_F(s) ds \right| \leq 4\varepsilon$$

which is nothing but Lemma 4.2.

## REFERENCES

- [1] Abed Bounemoura and Jacques Féjoz. KAM,  $\alpha$ -Gevrey regularity and the  $\alpha$ -Brjuno-Rüssmann condition. *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze*, XIX, 1-55, 2019.
- [2] Abed Bounemoura and Jacques Féjoz. Hamiltonian perturbation theory for ultra-differentiable functions. *Memoirs of the American Mathematical Society*, to appear.
- [3] Claire Chavaudret. Reducibility of quasiperiodic cocycles in Linear Lie groups. *Ergodic Theory and Dynamical Systems*, 31(3):741–769, 2011.
- [4] Claire Chavaudret and Stefano Marmi. Reducibility of cocycles under a Brjuno-Rüssmann arithmetical condition. *Journal of Modern Dynamics*, 6(1):59–78, 2012.
- [5] E.I. Dinaburg and Ya. G. Sinai. The one-dimensional Schrödinger equation with a quasiperiodic potential. *Functional Analysis and Its Applications*, 9(4):279–289, 1975.
- [6] L Håkan Eliasson. Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation. *Communications in mathematical physics*, 146(3):447–482, 1992.
- [7] Hai-Long Her and Jiangong You. Full measure reducibility for generic one-parameter family of quasi-periodic linear systems. *Journal of Dynamics and Differential Equations*, 20(4):831, 2008.
- [8] Xuanji Hou and Jiangong You. Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems. *Inventiones mathematicae*, 190(1):209–260, 2012.
- [9] Jürgen Moser and Jürgen Pöschel. An extension of a result by Dinaburg and Sinai on quasi-periodic potentials. *Commentarii Mathematici Helvetici*, 59(1):39–85, 1984.
- [10] João Lopes Dias and José Pedro Gaivão. Linearization of Gevrey flows on  $\mathbb{T}^d$  with a Brjuno type arithmetical condition *Journal of Differential Equations*, 267:7167–7212, 2019.
- [11] Jürgen Pöschel. KAM à la R. *Regular and Chaotic Dynamics*, 16(1-2):17–23, 2011.
- [12] Helmut Rüssmann. On the one-dimensional Schrödinger equation with a quasi-periodic potential. *Annals of the New York Academy of Sciences*, 357(1):90–107, 1980.
- [13] Helmut Rüssmann. KAM iteration with nearly infinitely small steps in dynamical systems of polynomial character. *Discrete and Continuous Dynamical Systems-S*, 3(4):683–718, 2010.

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