ON THE BETTI MAP ASSOCIATED TO ABELIAN LOGARITHMS

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ABSTRACT. (draft 5)

1. PROBLEM AND MAIN RESULTS.

1.1. Let A be a complex abelian variety of dimension g. We write Λ for the its *period lattice* $H_1(A(\mathbb{C}), \mathbb{Z})$, identified with the kernel of the exponential map $\exp_A : Lie\, A \to A$.

We fix a complex point σ of A, and let Λ_σ be the lattice formed by the pairs $(\ell \in Lie\, A, m \in \mathbb{Z})$ such that $exp_A(\ell) = m\sigma$. It sits in an extension

$$0 \to \Lambda \to \Lambda_\sigma \overset{(\ell,m)\mapsto m}{\to} \mathbb{Z}\sigma \to 0, \tag{1.1}$$

and we fix a splitting $\lambda : \mathbb{Z}\sigma \to \Lambda_\sigma$ (*i.e.* a branch of the "abelian logarithm"). On the other hand, let us consider the first projection $\Lambda_\sigma \overset{(\ell,m)\mapsto \ell}{\to} Lie\, A$ and its $\mathbb{R}$-linear extension $\Lambda_\sigma \otimes_{\mathbb{Z}} \mathbb{R} \to Lie\, A$. Since the composed map

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \to \Lambda_\sigma \otimes_{\mathbb{Z}} \mathbb{R} \to Lie\, A \tag{1.2}$$

is an isomorphism, one has a canonical projection $\Lambda_\sigma \otimes_{\mathbb{Z}} \mathbb{R} \to \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, and the image of $\lambda(\sigma)$ in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ can be expressed, in terms of a basis $\underline{\ell}$ of Λ, by $2g$ real coordinates: the *Betti coordinates*[1] of σ.

1.2. We are actually interested in the relative setting. Let S be a smooth connected complex algebraic variety.

Let $A \overset{f}{\to} S$ be an abelian scheme of relative dimension g. Its Lie algebra $Lie\, A$ is a rank g vector bundle on S. For our purpose, there will be no

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[1] the terminology seems due to D. Bertrand and refers to the fact that Λ_σ is nothing but the *Betti realization of the* 1*-motive* $[\mathbb{Z} \overset{1\mapsto\sigma}{\to} A]$ in the sense of [D2, X]. Note that the Betti coordinates depend not only on σ but also on the auxiliary choice of a splitting λ and of a basis of Λ.

loss of generality[2] in assuming, as we shall do, that S *is affine and that this vector bundle is trivial, i.e.* $\Gamma Lie\, A$ is a free $\Gamma\mathcal{O}_S$-module of rank g.

Let $\sigma : S \to A$ be a section of f. The above constructions extend to this relative setting as follows. Let $\tilde{S}$ be a universal covering of $S(\mathbb{C})$, with its canonical structure of complex analytic manifold. The kernel of $\exp_A$ is a locally constant sheaf on $S(\mathbb{C})$, which can be viewed as a constant lattice Λ on $\tilde{S}$. Similarly, one constructs an exact sequence of lattices on $\tilde{S}$

$$0 \to \Lambda \to \Lambda_\sigma \overset{(\ell,m)\mapsto m}{\to} \mathbb{Z}\sigma \to 0 \tag{1.3}$$

and fixes a splitting $\lambda : \mathbb{Z}\sigma \to \Lambda_\sigma$.

Let $\tilde{S}^{real}$ be $\tilde{S}$ viewed as a real-analytic manifold and let $\mathcal{O}^{\mathbb{R}}(\tilde{S}^{real})$ (*resp.* $\mathcal{O}(\tilde{S}^{real})$) be the ring of real-valued (*resp.* complex valued) real-analytic functions on $\tilde{S}$. The composed map

$$\Lambda \otimes_{\mathbb{Z}} \mathcal{O}^{\mathbb{R}}(\tilde{S}^{real}) \to \Lambda_\sigma \otimes_{\mathbb{Z}} \mathcal{O}^{\mathbb{R}}(\tilde{S}^{real}) \to \Gamma Lie\, A \otimes_{\Gamma O_S} \mathcal{O}(\tilde{S}^{real}) \tag{1.4}$$

is an isomorphism of free $\mathcal{O}^{\mathbb{R}}(\tilde{S}^{real})$-modules. One thus has a canonical projection $\Lambda_\sigma \otimes_{\mathbb{Z}} \mathcal{O}^{\mathbb{R}}(\tilde{S}^{real}) \to \Lambda \otimes_{\mathbb{Z}} \mathcal{O}^{\mathbb{R}}(\tilde{S}^{real})$, and the image of $\lambda(\sigma)$ in $\Lambda \otimes_{\mathbb{Z}} \mathcal{O}^{\mathbb{R}}(\tilde{S}^{real})$ can be expressed, in terms of a basis $\underline{\ell}$ of the lattice Λ, by a real-analytic map (the *Betti map*)

$$\beta = \beta(\sigma, \lambda, \underline{\ell}) : \quad \tilde{S} \to \mathbb{R}^{2g}. \tag{1.5}$$

1.3. We assume for convenience that $A \to S$ is *principally polarized*, which allows to consider to the *moduli map* $\mu : \ S \to \mathcal{A}_g$. We denote by $A_{\bar{\eta}}$ the geometric generic fiber, and we set

$$d = \dim \mu(S). \tag{1.6}$$

We are interested in the range of β. It is easy to see that the its dimension is $\leq \min(2d, 2g)$. *Under what condition is the range of* β *maximal, i.e. of dimension* $\min(2d, 2g)$*?*

This may not be the case for various trivial reasons (*e.g.* when the σ is a torsion section, or when $A \overset{f}{\to} S$ is the pull-back of an abelian scheme defined over a base of dimension $< d \leq g$).

1.4. Here is a sample of concrete results [not our final results, hopefully!]:

1.4.1. Theorem. *Assume that* $g \leq \min(3, d)$ *and that the section* σ *is not contained in a proper subgroup scheme of* A*. Then the range of* β *has real dimension* $2g$*. More precisely, the restriction of* β *to the inverse image of a dense Zariski-open subset of* S *is a submersion.*

[2] since S can be covered by such affine open subsets.

The proof relies on an analysis of the Kodaira-Spencer map.

1.4.2. Theorem. *Assume that σ is not contained in a proper subgroup scheme of A. Assume moreover that $d \leq \min(2, g)$. Then β has real image $2d$. More precisely, the restriction of β to the inverse image of a dense Zariski-open subset of S is an immersion.*

1.5. *[Motivation for the problem, and application of the results (if any)... General conjecture ....]*

2. AN INFINITESIMAL APPROACH, AND ITS HOLOMORPHIC TRANSLATION.

2.1. Our approach is infinitesimal. In fact, the problem we address is the following:

under what condition is β almost everywhere an immersion, resp. a submersion (according to whether $d \leq g$ or $g \leq d$) ?

2.1.1. Proposition. *For every $s \in \tilde{S}$, $\operatorname{Ker} \beta(s)$ is a complex vector subspace of $Lie\, A_s \cong \mathbb{R}^{2g}$. Moreover the fibers of β are complex-analytic subspaces of $\tilde{S}$, and all non-empty fibers have dimension $\geq 2d - r$, where r is the generic rank of β.*

Proof. - [cf. your prop 2.1 + 2.2]. □

2.2. We may first replace S by a finite etale covering and assume that for some $n \geq 3$, the n-torsion sections are defined over S, then replace it by its image in the moduli space $\mathcal{A}_{g,n}$ of principally polarized abelian varieties of dimension g with level n-structure. We thus consider the rank of a jacobian matrix of size $(2d, 2g)$, and for this purpose, we introduce the following notation:

$(\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g)$ is a symplectic basis of $\mathcal{H}^1_{dR}(A/S)$ with $\omega_1, \ldots, \omega_g \in \Omega_A$,

$(\gamma_1, \ldots, \gamma_g)$ is a symplectic basis of Λ,

$\Omega_1 := \left(\int_{\gamma_i} \omega_j\right)_{i,j=1,\ldots g}$, $\Omega_2 := \left(\int_{\gamma_{i+g}} \omega_j\right)_{i,j=1,\ldots g}$, $Z := \Omega_1 \cdot \Omega_2^{-1} = {}^tZ$ (which takes values in $\mathfrak{H}_g$), $L := \left(\int^{\sigma} \omega_j\right)_{j=1,\ldots g} \cdot \Omega_2^{-1}$.

Let r be $\min(d, g)$. In the next step, we transform the vanishing of a $2r$-$2r$-minor involving real-analytic functions by the vanishing of a r-r-minor of a related matrix involving only complex-analytic functions on $\tilde{S}$. In order to do this, we take advantage of the monodromy, using a theorem of linear independence of abelian logarithms with respect to periods [A1, th. 2] (*cf.* also [A2, §1]).

At the end, we come along with the following translation of the problem (under extra assumptions):

2.2.1. Proposition. *(with $S \subset \mathcal{A}_{g,n}$). Assume that σ is not contained in a proper sub-group scheme. Then for any $s \in \tilde{S}$, β is an immersion, resp. a submersion (according to whether $d \leq g$ or $g \leq d$) at s if and only if for some $(\lambda_1, \dots, \lambda_g) \in \mathbb{C}^g$, the matrix*

$$((\partial_i L)(s) + \sum_{k=1,\dots,g} \lambda_k (\partial_i Z_{kj})(s))_{i=1,\dots,d,\, j=1,\dots g} \tag{2.1}$$

has maximal rank (where the ∂_i form a basis of the tangent space $T_{\tilde{S}}(s)$).

This is already the case if for some $(\lambda_1, \dots, \lambda_g) \in \mathbb{C}^g$, the matrix

$$(\sum_{j=1,\dots,g} \lambda_k (\partial_i Z_{kj})(s))_{i=1,\dots,d,\, j=1,\dots g} \tag{2.2}$$

has maximal rank.

Proof. ... [your §3.5].

The second assertion follows from the first, putting an arbitrary coefficient λ_0 in front of $(\partial_i L)(s)$ and letting λ_0 tend to 0. □

Remark. For $d = g$, the negation of the conclusion leads to a special case of "Dieudonné's problem", which is notoriously difficult: describe vector spaces of singular matrices Z, *resp.* singular symmetric matrices, *cf. e.g.* [L][3].

3. Role of the Kodaira-Spencer map.

3.1. We know consider the following condition:

(3.1)

For every s in an dense open affine subset of S, the matrix (2.2) *has maximal rank.*

We shall translate it in terms of the Kodaira-Spencer map

$$\theta : T_S \otimes \Omega_A \to \Omega_A^\vee = \mathrm{Lie}\, A,$$

where Ω_A denotes the locally free $\mathcal{O}_S$-module of invariant 1-forms on A. We refer to [A3, 1.4, 2.1] for background about the Kodaira-Spencer map of a polarized abelian scheme A of relative dimension g over a smooth $\mathbb{C}$-scheme S. For any $\partial \in \Gamma T_S$, we denote by θ_∂ the corresponding map $\Omega_A \to \Omega_A^\vee$, which belongs to $\mathrm{Hom}_{\mathrm{sym}}(\Omega_A, \Omega_A^\vee)$. Let us recall that in terms of the symplectic basis $\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g$, the matrix of θ_∂ is given by the lower left quadrant T_∂ of the Gauss-Manin connection written in the form

$$\partial \begin{pmatrix} \Omega_1 & \mathrm{N}_1 \\ \Omega_2 & \mathrm{N}_2 \end{pmatrix} = \begin{pmatrix} \Omega_1 & \mathrm{N}_1 \\ \Omega_2 & \mathrm{N}_2 \end{pmatrix} \begin{pmatrix} R_\partial & S_\partial \\ T_\partial & U_\partial \end{pmatrix},$$

where $R_\partial, S_\partial, T_\partial, U_\partial \in M_g(\mathcal{O}(S))$. Moreover T_∂ is a symmetric matrix.

[3] we are grateful to M. Brion for this reference.

3.1.1. Lemma. *Assume that $d = g$. Then condition* (3.1) *is equivalent to:*

$$\exists \omega \in \Gamma \Omega_A,\ \forall \partial \in \Gamma T_S \setminus 0,\ \theta_\partial . \omega \neq 0. \tag{3.2}$$

Proof. One has

$$\partial Z = (\partial \Omega_1 - \Omega_1 \Omega_2^{-1} \partial \Omega_2) \Omega_2^{-1} = ((\Omega_1 R_\partial + N_1 T_\partial) - \Omega_1 \Omega_2^{-1} (\Omega_2 R_\partial + N_2 T_\partial)) \Omega_2^{-1}$$

$$= (N_1 - \Omega_1 \Omega_2^{-1} N_2) T_\partial \Omega_2^{-1} = -\Omega_2^{-1} ({}^t\Omega_1 N_2 - {}^t\Omega_2 N_1) T_\partial \Omega_2^{-1} = -2\pi i \Omega_2^{-1} T_\partial \Omega_2^{-1}.$$

If $\vec{\mu} = -2\pi i \vec{\lambda} \Omega_2^{-1}$, one has

$$\det(\sum_{k=1,\dots,g} \lambda_k (\partial_i Z_{kj})(s)) = \det(\sum_{k=1,\dots,g} \mu_k (T_{\partial_i})_{kj}(s)) \cdot (\det \Omega_2)^{-1}.$$

Setting $\omega = \sum \mu_k \omega_k$, one has $\sum \mu_k (T_{\partial_i})_{kj} = \theta_{\partial_i}(\omega) \cdot \eta_j$. Thus (3.1) is equivalent to: there is a dense open affine subset $S' \subset S$ such that

$$\forall s \in S'(\mathbb{C}), \exists \omega_s \in \Omega^1(A_s),\ \forall \partial_s \in T_{S,s} \setminus 0,\ (\theta(s)_{\partial_s})(\omega_s) \neq 0.$$

It is easy to see that this condition is equivalent in turn to (3.2) (for the direct implication, any ω lifting ω_s for some s will do). □

3.2. In [A3], a condition similar to (3.2) is studied (*cf.* [A3, lemma 1.4.5]), where the quantifiers are interchanged:

$$\forall \partial \in \Gamma T_S \setminus 0, \exists \omega \in \Gamma \Omega_A,\ \ \theta_\partial . \omega \neq 0. \tag{3.3}$$

3.2.1. Theorem. [A3, def. 4.2.1, th. 4.2.2] *In the restricted PE case, condition* (3.3) *is satisfied.* □

3.2.2. Lemma. *If $d = g$,* (3.2) $\Rightarrow$ (3.3).

Proof. Let $\theta^\omega : T_S \to \Omega_A^\vee$ the contraction of θ by $\omega \in \Omega_A$. Condition (3.2) is equivalent to the injectivity of θ^ω for some ω. Since $d = g$, this is equivalent to the generic surjectivity of θ^ω for some ω, which implies the generic surjectivity of θ, which is in turn equivalent to (3.3). □

3.2.3. Question. Does the converse of the lemma holds?

3.2.4. Remark. Let η be the generic point of S and $P(\Omega_{A_\eta})$ (*resp.* $P(T_\eta)$) be the projective space of lines on Ω_{A_η}) (*resp.* T_S). We denote by $\bar{\omega}$ (*resp.* $\bar{\partial}$) the class of ω in $P(\Omega_{A_\eta})$ (*resp.* ∂ in $P(T_\eta)$).

One could try to answer the question in general by explaining the "logical seesaw" between quantifiers from (3.3) to (3.2) through a "geometric seesaw" provided by the incidence variety $\Sigma \subset P(\Omega_{A_\eta}) \times P(T_\eta)$ formed by pairs $(\bar{\omega}, \bar{\partial})$ such that $\theta_\partial \cdot \omega = 0$, and let p_1, p_2 be the projections on the

factors. Let further $\breve{P}_h \subset P(T_\eta)$ be the locally closed subvariety formed by elements $\bar{\partial}$ such that $\theta_{\partial,\eta}$ has rank h. One has

$$p_2(\Sigma) \subset \coprod_{h=1}^{h=g-1} \breve{P}_h \subset P(T_\eta) = \coprod_{h=1}^{h=g} \breve{P}_h,$$

and

$$\dim p_2^{-1}\{\partial\} = g - 1 - h \text{ if } \partial \in \breve{P}_h.$$

By (3.3), $\breve{P}_g$ is non empty and in fact dense (the condition is open in ∂), hence $\dim p_2(\Sigma) \leq g - 2$.

The *key problem* is whether one has

$$\dim \breve{P}_{h-1} < \dim \breve{P}_h \tag{3.4}$$

(which is equivalent to $\dim \breve{P}_h \neq \emptyset$ since $\overline{\breve{P}_h} \supset \breve{P}_{h-1}$) for $e < h < g$, where e is the first index such that $\breve{P}_e \neq \emptyset$.

If so, then $\dim \breve{P}_h \leq h - 1$ and $\dim \Sigma \leq g - 2$, so p_1 is not dominant, and any $\bar{\omega}$ outside the image of p_1 will do.

We have only partial answers to 3.2.3.

3.2.5. Proposition. *If $d = g \leq 3$,* (3.3) $\Rightarrow$ (3.2)*.*

Proof. We use the previous remark: the case $g \leq 2$ is trivial ((3.4) is satisfied). For $g = 3$, $\dim \overline{\breve{P}}_2 = 1$, and it suffices to show that $\overline{\breve{P}}_1$ is of dimension 0 or empty. Let us identify $P(T_\eta)$ with a plane in the projective space of $S^2\mathrm{Lie}\, A_\eta$. Let $\Delta_h \subset P(S^2\mathrm{Lie}\, A_\eta)$ be the locus of matrices of rank h. Then Δ_1 may be identified with the Veronese surface $V \subset \mathbb{P}^5$, since its elements are the symmetric matrices of the form $\begin{pmatrix} x_0^2 & x_0x_1 & x_0x_2 \\ x_0x_1 & x_1^2 & x_1x_2 \\ x_0x_2 & x_1x_2 & x_2^2 \end{pmatrix}$, and $P(T_\eta) \cap V$ may be identified with $\overline{\breve{P}}_1$. It is enough to show that there any plane $P \subset \mathbb{P}^5$ such that $P \cap V$ is a curve is contained in $\overline{\Delta_2}$. Such a plane would be cut by three hyperplanes, which would correspond to three non collinear quadratic forms q_0, q_1, q_2 in x_0, x_1, x_2. If $P \cap V$ is a curve, then one could choose the q_i's of the form $\ell.x_i$, where ℓ is a linear form in x_0, x_1, x_2; by change of coordinates, ℓ may be taken to be x_0, and it is then clear that $P \subset \overline{\Delta_2}$. □

3.2.6. Proposition. *If $\mu(S)$ is a sufficiently general subvariety of dimension g of a special subvariety S_1 of $\mathcal{A}_g$ of restricted PEL type (cf.* [A3, 4.2.1]*), then* (3.3) $\Rightarrow$ (3.2)*.*

Proof. Let $D := \mathrm{End}\, A_\eta$ (assumed to be a division algebra), E a maximal subfield of D containing F (and chosen to be CM except if D is a totally real field) and E^+ the maximal totally real subfield of E. One has a decomposition $\Omega_A = \oplus_{\nu:E\to\mathbb{C}} \Omega_A \otimes_{E\otimes\mathbb{C},\nu} \mathbb{C}$. Fix $\omega \in \Omega_A$ with non-zero components ω_ν. Fix $s \in S(\mathbb{C})$ such that $\omega_{\nu,s}$ remains non-zero. With the notation of [A3, rem. 4.2.3 i)], the tangent space $T_{S_1,s}$ decomposes as $\oplus_{\nu:E\to\mathbb{C}} T_{S_1,s,\nu}$, and the universal Kodaira-Spencer map on the family of PEL type induces an isomorphism $T_{S_1,s,\nu} \cong S^2\Omega^\vee_{A,s,\nu}$ or $T_{S_1,s,\nu} \cong \otimes^2\Omega^\vee_{A,s,\nu}$ (according to whether D is of type I or II, or of type III or IV). In particular the contraction $(\theta_1^{\omega_\nu})_s : T_{S_1,s,\nu} \to \Omega^\vee_{A,s,\nu}$ is surjective, hence also $(\theta_1^{\omega})_s : T_{S_1,s} \to \Omega^\vee_{A,s}$. Since S is sufficiently general in S_1 of dimension g, then T_S is transverse to the kernel of θ_1^ω on a suitable dense open subset of S [details should be checked]. □

3.3. *A counterexample.* We indicate here what seems to be an example with smallest $g > 0$ of a subvariety (*resp.* a special subvariety) of dimension d of $\mathcal{A}_{g,n}$ with $d = g$, such that the restriction of the universal abelian scheme does not satisfy condition (3.3) (hence not (3.1); in this example the Betti map is nowhere a submersion). In fact $d = g = 5$ (*resp.* 6).

Let us first treat the modular example. Let E^+ be a real quadratic field and E a totally imaginary quadratic extension fo E^+. Let $A \to S$ be a principally polarized abelian scheme of relative dimension 6, with level $n \geq 3$ structure, such that $\mathrm{End}_S A = \mathcal{O}_E$, and with Shimura type $(r_\nu, s_\nu) = (1,2),(1,2)$ (for the two embeddings ν_1, ν_2 of E^+ in $\mathbb{C}$). According to [S], there exists such a family, with base S a special subvariety of dimension $\sum r_\nu \cdot s_\nu = 6$ of $\mathcal{A}_{6,n}$. By functoriality, θ commutes with the $\mathcal{O}_L$-action, hence respects the decomposition $\mathcal{H}^1_{dR}(A/S) = \oplus_{\rho:E\to\mathbb{C}} \mathcal{H}_\rho$. For every ρ, and general ∂, the restriction of the Kodaira-Spencer map θ_∂ to $(\Omega_A)_\rho$ has rank 1 (either the source or the target is of rank 1), hence θ_∂ has rank $4 < 6$.

The non modular example consists in taking a general hypersuface section in a modular example with $d = 6, g = 5$. Here E is a quadratic imaginary field, and $\mathrm{End}_S A = \mathcal{O}_E$ with Shimura type $(r,s) = (2,3)$. For each complex embedding ρ of E, the restriction of the Kodaira-Spencer map θ_∂ to $(\Omega_A)_\rho$ has rank 2 (either the source or the target is of rank 2), hence θ_∂ has rank $4 < 5$.

3.3.1. Remark. It would be interesting to go beyond, and settle (3.2) fro instance in the case of maximal monodromy Sp_{2g}. Let $A \to S$ be the jacobian of the universal hyperelliptic curve of genus $g > 0$. Here S is the affine space $\mathcal{M}_{0,2g+2} \cong (\mathbb{P}^1 \setminus \{0,1,\infty\})^{2g-1}$. By Torelli's theorem, one has $\dim\, \mu(S) = 2g - 1$. On the other hand, according to [M2, th. 5.6], $\mathrm{End} A_{\bar\eta} = \mathbb{Z}$ (in fact, it is knwon that the monodromy representation,

which is a specialization of the Burau representation of the braid group, has Zariski dense image in Sp_{2g}, *cf.* [aC]). Hence (3.2) whould imply that, after replacing $(\mathbb{P}^1 \setminus \{0,1,\infty\})^{2g-1}$ by a suitable dense Zariski-open affine subset, β is a submersion. [There are explicit calculations (without detail) by Bryant of θ in this case on MathOverflow (tag: hyperelliptic) which should settle the matter]

3.4. Let us turn to the proof of th. 1.4.1. [sketch] One may replace S by a finite etale covering and assume that A/S has a level n structure for some $n \geq 3$, then replace S by a smooth dense affine open subset of its image in $\mathcal{A}_{g,n}$. By assumption, $d = \dim \mu(S) \geq g$. Taking a sufficiently general locally closed affine subvariety $S' \subset S$ of dimension g, the restriction to S' of the abelian scheme and of the section satisfy the same assumption. Moreover, one checks that the condition $d \geq g$ implies that we are in the restricted PE case[4]. Remplacing S by S', one is in the situation $d = g$. One can then combine prop. 2.2.1, 3.2.5 and th. 3.2.1. [A similar deduction holds with prop. 3.2.6 in place of 3.2.5]. □

4. PROOF OF THEOREM 1.5.1.

[*your §4 and 5*].

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[4]here one needs a generalization of the notion of restricted PE to the case of an abelian scheme whose generic fiber is not necessarily simple. This does not pose problem.