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# ON THE BETTI MAP ASSOCIATED TO ABELIAN LOGARITHMS

Y. ANDRÉ, P. CORVAJA, U. ZANNIER

ABSTRACT. (draft 5)

## 1. PROBLEM AND MAIN RESULTS.

**1.1.** Let  $A$  be a complex abelian variety of dimension  $g$ . We write  $\Lambda$  for the its *period lattice*  $H_1(A(\mathbb{C}), \mathbb{Z})$ , identified with the kernel of the exponential map  $\exp_A : \text{Lie } A \rightarrow A$ .

We fix a complex point  $\sigma$  of  $A$ , and let  $\Lambda_\sigma$  be the lattice formed by the pairs  $(\ell \in \text{Lie } A, m \in \mathbb{Z})$  such that  $\exp_A(\ell) = m\sigma$ . It sits in an extension

$$(1.1) \quad 0 \rightarrow \Lambda \rightarrow \Lambda_\sigma \xrightarrow{(\ell, m) \mapsto m} \mathbb{Z}\sigma \rightarrow 0,$$

and we fix a splitting  $\lambda : \mathbb{Z}\sigma \rightarrow \Lambda_\sigma$  (i.e. a branch of the “abelian logarithm”). On the other hand, let us consider the first projection  $\Lambda_\sigma \xrightarrow{(\ell, m) \mapsto \ell} \text{Lie } A$  and its  $\mathbb{R}$ -linear extension  $\Lambda_\sigma \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \text{Lie } A$ . Since the composed map

$$(1.2) \quad \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \Lambda_\sigma \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \text{Lie } A$$

is an isomorphism, one has a canonical projection  $\Lambda_\sigma \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ , and the image of  $\lambda(\sigma)$  in  $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$  can be expressed, in terms of a basis  $\underline{\ell}$  of  $\Lambda$ , by  $2g$  real coordinates: the *Betti coordinates*<sup>1</sup> of  $\sigma$ .

**1.2.** We are actually interested in the relative setting. Let  $S$  be a smooth connected complex algebraic variety.

Let  $A \xrightarrow{f} S$  be an abelian scheme of relative dimension  $g$ . Its Lie algebra  $\text{Lie } A$  is a rank  $g$  vector bundle on  $S$ . For our purpose, there will be no

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<sup>1</sup>the terminology seems due to D. Bertrand and refers to the fact that  $\Lambda_\sigma$  is nothing but the *Betti realization of the 1-motive*  $[\mathbb{Z} \xrightarrow{1 \mapsto \sigma} A]$  in the sense of [D2, X]. Note that the Betti coordinates depend not only on  $\sigma$  but also on the auxiliary choice of a splitting  $\lambda$  and of a basis of  $\Lambda$ .

loss of generality<sup>2</sup> in assuming, as we shall do, that  $S$  is affine and that this vector bundle is trivial, i.e.  $\Gamma \text{Lie } A$  is a free  $\Gamma \mathcal{O}_S$ -module of rank  $g$ .

Let  $\sigma : S \rightarrow A$  be a section of  $f$ . The above constructions extend to this relative setting as follows. Let  $\tilde{S}$  be a universal covering of  $S(\mathbb{C})$ , with its canonical structure of complex analytic manifold. The kernel of  $\exp_A$  is a locally constant sheaf on  $S(\mathbb{C})$ , which can be viewed as a constant lattice  $\Lambda$  on  $\tilde{S}$ . Similarly, one constructs an exact sequence of lattices on  $\tilde{S}$

$$(1.3) \quad 0 \rightarrow \Lambda \rightarrow \Lambda_\sigma \xrightarrow{(\ell, m) \mapsto m} \mathbb{Z}\sigma \rightarrow 0$$

and fixes a splitting  $\lambda : \mathbb{Z}\sigma \rightarrow \Lambda_\sigma$ .

Let  $\tilde{S}^{real}$  be  $\tilde{S}$  viewed as a real-analytic manifold and let  $\mathcal{O}^{\mathbb{R}}(\tilde{S}^{real})$  (*resp.*  $\mathcal{O}(\tilde{S}^{real})$ ) be the ring of real-valued (*resp.* complex valued) real-analytic functions on  $\tilde{S}$ . The composed map

$$(1.4) \quad \Lambda \otimes_{\mathbb{Z}} \mathcal{O}^{\mathbb{R}}(\tilde{S}^{real}) \rightarrow \Lambda_\sigma \otimes_{\mathbb{Z}} \mathcal{O}^{\mathbb{R}}(\tilde{S}^{real}) \rightarrow \Gamma \text{Lie } A \otimes_{\Gamma \mathcal{O}_S} \mathcal{O}(\tilde{S}^{real})$$

is an isomorphism of free  $\mathcal{O}^{\mathbb{R}}(\tilde{S}^{real})$ -modules. One thus has a canonical projection  $\Lambda_\sigma \otimes_{\mathbb{Z}} \mathcal{O}^{\mathbb{R}}(\tilde{S}^{real}) \rightarrow \Lambda \otimes_{\mathbb{Z}} \mathcal{O}^{\mathbb{R}}(\tilde{S}^{real})$ , and the image of  $\lambda(\sigma)$  in  $\Lambda \otimes_{\mathbb{Z}} \mathcal{O}^{\mathbb{R}}(\tilde{S}^{real})$  can be expressed, in terms of a basis  $\underline{\ell}$  of the lattice  $\Lambda$ , by a real-analytic map (the *Betti map*)

$$(1.5) \quad \beta = \beta(\sigma, \lambda, \underline{\ell}) : \tilde{S} \rightarrow \mathbb{R}^{2g}.$$

**1.3.** We assume for convenience that  $A \rightarrow S$  is *principally polarized*, which allows to consider to the *moduli map*  $\mu : S \rightarrow \mathcal{A}_g$ . We denote by  $A_{\bar{\eta}}$  the geometric generic fiber, and we set

$$(1.6) \quad d = \dim \mu(S).$$

We are interested in the range of  $\beta$ . It is easy to see that the its dimension is  $\leq \min(2d, 2g)$ . *Under what condition is the range of  $\beta$  maximal, i.e. of dimension  $\min(2d, 2g)$ ?*

This may not be the case for various trivial reasons (*e.g.* when the  $\sigma$  is a torsion section, or when  $A \xrightarrow{f} S$  is the pull-back of an abelian scheme defined over a base of dimension  $< d \leq g$ ).

**1.4.** Here is a sample of concrete results [not our final results, hopefully!]:

**1.4.1. Theorem.** *Assume that  $g \leq \min(3, d)$  and that the section  $\sigma$  is not contained in a proper subgroup scheme of  $A$ . Then the range of  $\beta$  has real dimension  $2g$ . More precisely, the restriction of  $\beta$  to the inverse image of a dense Zariski-open subset of  $S$  is a submersion.*

<sup>2</sup>since  $S$  can be covered by such affine open subsets.

The proof relies on an analysis of the Kodaira-Spencer map.

**1.4.2. Theorem.** *Assume that  $\sigma$  is not contained in a proper subgroup scheme of  $A$ . Assume moreover that  $d \leq \min(2, g)$ . Then  $\beta$  has real image  $2d$ . More precisely, the restriction of  $\beta$  to the inverse image of a dense Zariski-open subset of  $S$  is an immersion.*

**1.5.** *[Motivation for the problem, and application of the results (if any)... General conjecture ....]*

## 2. AN INFINITESIMAL APPROACH, AND ITS HOLOMORPHIC TRANSLATION.

**2.1.** Our approach is infinitesimal. In fact, the problem we address is the following:

*under what condition is  $\beta$  almost everywhere an immersion, resp. a submersion (according to whether  $d \leq g$  or  $g \leq d$ ) ?*

**2.1.1. Proposition.** *For every  $s \in \tilde{S}$ ,  $\text{Ker } \beta(s)$  is a complex vector subspace of  $\text{Lie } A_s \cong \mathbb{R}^{2g}$ . Moreover the fibers of  $\beta$  are complex-analytic subspaces of  $\tilde{S}$ , and all non-empty fibers have dimension  $\geq 2d - r$ , where  $r$  is the generic rank of  $\beta$ .*

*Proof.* - [cf. your prop 2.1 + 2.2]. □

**2.2.** We may first replace  $S$  by a finite etale covering and assume that for some  $n \geq 3$ , the  $n$ -torsion sections are defined over  $S$ , then replace it by its image in the moduli space  $\mathcal{A}_{g,n}$  of principally polarized abelian varieties of dimension  $g$  with level  $n$ -structure. We thus consider the rank of a jacobian matrix of size  $(2d, 2g)$ , and for this purpose, we introduce the following notation:

$(\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$  is a symplectic basis of  $\mathcal{H}_{dR}^1(A/S)$  with  $\omega_1, \dots, \omega_g \in \Omega_A$ ,

$(\gamma_1, \dots, \gamma_g)$  is a symplectic basis of  $\Lambda$ ,

$\Omega_1 := \left( \int_{\gamma_i} \omega_j \right)_{i,j=1,\dots,g}$ ,  $\Omega_2 := \left( \int_{\gamma_{i+g}} \omega_j \right)_{i,j=1,\dots,g}$ ,  $Z := \Omega_1 \cdot \Omega_2^{-1} = {}^t Z$

(which takes values in  $\mathfrak{H}_g$ ),  $L := \left( \int^\sigma \omega_j \right)_{j=1,\dots,g} \cdot \Omega_2^{-1}$ .

Let  $r$  be  $\min(d, g)$ . In the next step, we transform the vanishing of a  $2r$ - $2r$ -minor involving real-analytic functions by the vanishing of a  $r$ - $r$ -minor of a related matrix involving only complex-analytic functions on  $\tilde{S}$ . In order to do this, we take advantage of the monodromy, using a theorem of linear independence of abelian logarithms with respect to periods [A1, th. 2] (cf. also [A2, §1]).

At the end, we come along with the following translation of the problem (under extra assumptions):

**2.2.1. Proposition.** (with  $S \subset \mathcal{A}_{g,n}$ ). Assume that  $\sigma$  is not contained in a proper sub-group scheme. Then for any  $s \in \tilde{S}$ ,  $\beta$  is an immersion, resp. a submersion (according to whether  $d \leq g$  or  $g \leq d$ ) at  $s$  if and only if for some  $(\lambda_1, \dots, \lambda_g) \in \mathbb{C}^g$ , the matrix

$$(2.1) \quad ((\partial_i L)(s) + \sum_{k=1, \dots, g} \lambda_k (\partial_i Z_{kj})(s))_{i=1, \dots, d, j=1, \dots, g}$$

has maximal rank (where the  $\partial_i$  form a basis of the tangent space  $T_{\tilde{S}}(s)$ ).

This is already the case if for some  $(\lambda_1, \dots, \lambda_g) \in \mathbb{C}^g$ , the matrix

$$(2.2) \quad \left( \sum_{j=1, \dots, g} \lambda_k (\partial_i Z_{kj})(s) \right)_{i=1, \dots, d, j=1, \dots, g}$$

has maximal rank.

*Proof.* ... [your §3.5].

The second assertion follows from the first, putting an arbitrary coefficient  $\lambda_0$  in front of  $(\partial_i L)(s)$  and letting  $\lambda_0$  tend to 0.  $\square$

*Remark.* For  $d = g$ , the negation of the conclusion leads to a special case of “Dieudonné’s problem”, which is notoriously difficult: describe vector spaces of singular matrices  $Z$ , resp. singular symmetric matrices, cf. e.g. [L]<sup>3</sup>.

### 3. ROLE OF THE KODAIRA-SPENCER MAP.

**3.1.** We now consider the following condition:

(3.1)

For every  $s$  in an dense open affine subset of  $S$ , the matrix (2.2) has maximal rank.

We shall translate it in terms of the Kodaira-Spencer map

$$\theta : T_S \otimes \Omega_A \rightarrow \Omega_A^\vee = \text{Lie } A,$$

where  $\Omega_A$  denotes the locally free  $\mathcal{O}_S$ -module of invariant 1-forms on  $A$ . We refer to [A3, 1.4, 2.1] for background about the Kodaira-Spencer map of a polarized abelian scheme  $A$  of relative dimension  $g$  over a smooth  $\mathbb{C}$ -scheme  $S$ . For any  $\partial \in \Gamma T_S$ , we denote by  $\theta_\partial$  the corresponding map  $\Omega_A \rightarrow \Omega_A^\vee$ , which belongs to  $\text{Hom}_{\text{sym}}(\Omega_A, \Omega_A^\vee)$ . Let us recall that in terms of the symplectic basis  $\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g$ , the matrix of  $\theta_\partial$  is given by the lower left quadrant  $T_\partial$  of the Gauss-Manin connection written in the form

$$\partial \begin{pmatrix} \Omega_1 & N_1 \\ \Omega_2 & N_2 \end{pmatrix} = \begin{pmatrix} \Omega_1 & N_1 \\ \Omega_2 & N_2 \end{pmatrix} \begin{pmatrix} R_\partial & S_\partial \\ T_\partial & U_\partial \end{pmatrix},$$

where  $R_\partial, S_\partial, T_\partial, U_\partial \in M_g(\mathcal{O}(S))$ . Moreover  $T_\partial$  is a symmetric matrix.

<sup>3</sup>we are grateful to M. Brion for this reference.

**3.1.1. Lemma.** *Assume that  $d = g$ . Then condition (3.1) is equivalent to:*

$$(3.2) \quad \exists \omega \in \Gamma \Omega_A, \forall \partial \in \Gamma T_S \setminus 0, \theta_{\partial} \cdot \omega \neq 0.$$

*Proof.* One has

$$\begin{aligned} \partial Z &= (\partial \Omega_1 - \Omega_1 \Omega_2^{-1} \partial \Omega_2) \Omega_2^{-1} = ((\Omega_1 R_{\partial} + N_1 T_{\partial}) - \Omega_1 \Omega_2^{-1} (\Omega_2 R_{\partial} + N_2 T_{\partial})) \Omega_2^{-1} \\ &= (N_1 - \Omega_1 \Omega_2^{-1} N_2) T_{\partial} \Omega_2^{-1} = -\Omega_2^{-1} ({}^t \Omega_1 N_2 - {}^t \Omega_2 N_1) T_{\partial} \Omega_2^{-1} = -2\pi i \Omega_2^{-1} T_{\partial} \Omega_2^{-1}. \end{aligned}$$

If  $\vec{\mu} = -2\pi i \vec{\lambda} \Omega_2^{-1}$ , one has

$$\det\left(\sum_{k=1, \dots, g} \lambda_k (\partial_i Z_{kj})(s)\right) = \det\left(\sum_{k=1, \dots, g} \mu_k (T_{\partial_i})_{kj}(s)\right) \cdot (\det \Omega_2)^{-1}.$$

Setting  $\omega = \sum \mu_k \omega_k$ , one has  $\sum \mu_k (T_{\partial_i})_{kj} = \theta_{\partial_i}(\omega) \cdot \eta_j$ . Thus (3.1) is equivalent to: there is a dense open affine subset  $S' \subset S$  such that

$$\forall s \in S'(\mathbb{C}), \exists \omega_s \in \Omega^1(A_s), \forall \partial_s \in T_{S,s} \setminus 0, (\theta(s)_{\partial_s})(\omega_s) \neq 0.$$

It is easy to see that this condition is equivalent in turn to (3.2) (for the direct implication, any  $\omega$  lifting  $\omega_s$  for some  $s$  will do).  $\square$

**3.2.** In [A3], a condition similar to (3.2) is studied (cf. [A3, lemma 1.4.5]), where the quantifiers are interchanged:

$$(3.3) \quad \forall \partial \in \Gamma T_S \setminus 0, \exists \omega \in \Gamma \Omega_A, \theta_{\partial} \cdot \omega \neq 0.$$

**3.2.1. Theorem.** [A3, def. 4.2.1, th. 4.2.2] *In the restricted PE case, condition (3.3) is satisfied.*  $\square$

**3.2.2. Lemma.** *If  $d = g$ , (3.2)  $\Rightarrow$  (3.3).*

*Proof.* Let  $\theta^{\omega} : T_S \rightarrow \Omega_A^{\vee}$  the contraction of  $\theta$  by  $\omega \in \Omega_A$ . Condition (3.2) is equivalent to the injectivity of  $\theta^{\omega}$  for some  $\omega$ . Since  $d = g$ , this is equivalent to the generic surjectivity of  $\theta^{\omega}$  for some  $\omega$ , which implies the generic surjectivity of  $\theta$ , which is in turn equivalent to (3.3).  $\square$

**3.2.3. Question.** Does the converse of the lemma holds?

**3.2.4. Remark.** Let  $\eta$  be the generic point of  $S$  and  $P(\Omega_{A_{\eta}})$  (resp.  $P(T_{\eta})$ ) be the projective space of lines on  $\Omega_{A_{\eta}}$  (resp.  $T_S$ ). We denote by  $\bar{\omega}$  (resp.  $\bar{\partial}$ ) the class of  $\omega$  in  $P(\Omega_{A_{\eta}})$  (resp.  $\partial$  in  $P(T_{\eta})$ ).

One could try to answer the question in general by explaining the ‘‘logical seesaw’’ between quantifiers from (3.3) to (3.2) through a ‘‘geometric seesaw’’ provided by the incidence variety  $\Sigma \subset P(\Omega_{A_{\eta}}) \times P(T_{\eta})$  formed by pairs  $(\bar{\omega}, \bar{\partial})$  such that  $\theta_{\partial} \cdot \omega = 0$ , and let  $p_1, p_2$  be the projections on the

factors. Let further  $\check{P}_h \subset P(T_\eta)$  be the locally closed subvariety formed by elements  $\bar{\partial}$  such that  $\theta_{\bar{\partial}, \eta}$  has rank  $h$ . One has

$$p_2(\Sigma) \subset \prod_{h=1}^{h=g-1} \check{P}_h \subset P(T_\eta) = \prod_{h=1}^{h=g} \check{P}_h,$$

and

$$\dim p_2^{-1}\{\bar{\partial}\} = g - 1 - h \text{ if } \bar{\partial} \in \check{P}_h.$$

By (3.3),  $\check{P}_g$  is non empty and in fact dense (the condition is open in  $\bar{\partial}$ ), hence  $\dim p_2(\Sigma) \leq g - 2$ .

The *key problem* is whether one has

$$(3.4) \quad \dim \check{P}_{h-1} < \dim \check{P}_h$$

(which is equivalent to  $\dim \check{P}_h \neq \emptyset$  since  $\overline{\check{P}_h} \supset \check{P}_{h-1}$ ) for  $e < h < g$ , where  $e$  is the first index such that  $\check{P}_e \neq \emptyset$ .

If so, then  $\dim \check{P}_h \leq h - 1$  and  $\dim \Sigma \leq g - 2$ , so  $p_1$  is not dominant, and any  $\bar{\omega}$  outside the image of  $p_1$  will do.

We have only partial answers to 3.2.3.

**3.2.5. Proposition.** *If  $d = g \leq 3$ , (3.3)  $\Rightarrow$  (3.2).*

*Proof.* We use the previous remark: the case  $g \leq 2$  is trivial ((3.4) is satisfied). For  $g = 3$ ,  $\dim \overline{\check{P}_2} = 1$ , and it suffices to show that  $\overline{\check{P}_1}$  is of dimension 0 or empty. Let us identify  $P(T_\eta)$  with a plane in the projective space of  $S^2 \text{Lie } A_\eta$ . Let  $\Delta_h \subset P(S^2 \text{Lie } A_\eta)$  be the locus of matrices of rank  $h$ . Then  $\Delta_1$  may be identified with the Veronese surface  $V \subset \mathbb{P}^5$ , since

its elements are the symmetric matrices of the form  $\begin{pmatrix} x_0^2 & x_0x_1 & x_0x_2 \\ x_0x_1 & x_1^2 & x_1x_2 \\ x_0x_2 & x_1x_2 & x_2^2 \end{pmatrix}$ ,

and  $P(T_\eta) \cap V$  may be identified with  $\overline{\check{P}_1}$ . It is enough to show that there any plane  $P \subset \mathbb{P}^5$  such that  $P \cap V$  is a curve is contained in  $\overline{\Delta_2}$ . Such a plane would be cut by three hyperplanes, which would correspond to three non collinear quadratic forms  $q_0, q_1, q_2$  in  $x_0, x_1, x_2$ . If  $P \cap V$  is a curve, then one could choose the  $q_i$ 's of the form  $\ell \cdot x_i$ , where  $\ell$  is a linear form in  $x_0, x_1, x_2$ ; by change of coordinates,  $\ell$  may be taken to be  $x_0$ , and it is then clear that  $P \subset \overline{\Delta_2}$ .  $\square$

**3.2.6. Proposition.** *If  $\mu(S)$  is a sufficiently general subvariety of dimension  $g$  of a special subvariety  $S_1$  of  $\mathcal{A}_g$  of restricted PEL type (cf. [A3, 4.2.1]), then (3.3)  $\Rightarrow$  (3.2).*

*Proof.* Let  $D := \text{End } A_\eta$  (assumed to be a division algebra),  $E$  a maximal subfield of  $D$  containing  $F$  (and chosen to be CM except if  $D$  is a totally real field) and  $E^+$  the maximal totally real subfield of  $E$ . One has a decomposition  $\Omega_A = \bigoplus_{\nu: E \rightarrow \mathbb{C}} \Omega_A \otimes_{E \otimes \mathbb{C}, \nu} \mathbb{C}$ . Fix  $\omega \in \Omega_A$  with non-zero components  $\omega_\nu$ . Fix  $s \in S(\mathbb{C})$  such that  $\omega_{\nu, s}$  remains non-zero. With the notation of [A3, rem. 4.2.3 i)], the tangent space  $T_{S_1, s}$  decomposes as  $\bigoplus_{\nu: E \rightarrow \mathbb{C}} T_{S_1, s, \nu}$ , and the universal Kodaira-Spencer map on the family of PEL type induces an isomorphism  $T_{S_1, s, \nu} \cong S^2 \Omega_{A, s, \nu}^\vee$  or  $T_{S_1, s, \nu} \cong \otimes^2 \Omega_{A, s, \nu}^\vee$  (according to whether  $D$  is of type I or II, or of type III or IV). In particular the contraction  $(\theta_1^{\omega_\nu})_s : T_{S_1, s, \nu} \rightarrow \Omega_{A, s, \nu}^\vee$  is surjective, hence also  $(\theta_1^\omega)_s : T_{S_1, s} \rightarrow \Omega_{A, s}^\vee$ . Since  $S$  is sufficiently general in  $S_1$  of dimension  $g$ , then  $T_S$  is transverse to the kernel of  $\theta_1^\omega$  on a suitable dense open subset of  $S$  [details should be checked].  $\square$

**3.3. A counterexample.** We indicate here what seems to be an example with smallest  $g > 0$  of a subvariety (*resp.* a special subvariety) of dimension  $d$  of  $\mathcal{A}_{g, n}$  with  $d = g$ , such that the restriction of the universal abelian scheme does not satisfy condition (3.3) (hence not (3.1); in this example the Betti map is nowhere a submersion). In fact  $d = g = 5$  (*resp.* 6).

Let us first treat the modular example. Let  $E^+$  be a real quadratic field and  $E$  a totally imaginary quadratic extension of  $E^+$ . Let  $A \rightarrow S$  be a principally polarized abelian scheme of relative dimension 6, with level  $n \geq 3$  structure, such that  $\text{End}_S A = \mathcal{O}_E$ , and with Shimura type  $(r_\nu, s_\nu) = (1, 2), (1, 2)$  (for the two embeddings  $\nu_1, \nu_2$  of  $E^+$  in  $\mathbb{C}$ ). According to [S], there exists such a family, with base  $S$  a special subvariety of dimension  $\sum r_\nu \cdot s_\nu = 6$  of  $\mathcal{A}_{6, n}$ . By functoriality,  $\theta$  commutes with the  $\mathcal{O}_L$ -action, hence respects the decomposition  $\mathcal{H}_{dR}^1(A/S) = \bigoplus_{\rho: E \rightarrow \mathbb{C}} \mathcal{H}_\rho$ . For every  $\rho$ , and general  $\partial$ , the restriction of the Kodaira-Spencer map  $\theta_\partial$  to  $(\Omega_A)_\rho$  has rank 1 (either the source or the target is of rank 1), hence  $\theta_\partial$  has rank  $4 < 6$ .

The non modular example consists in taking a general hypersurface section in a modular example with  $d = 6, g = 5$ . Here  $E$  is a quadratic imaginary field, and  $\text{End}_S A = \mathcal{O}_E$  with Shimura type  $(r, s) = (2, 3)$ . For each complex embedding  $\rho$  of  $E$ , the restriction of the Kodaira-Spencer map  $\theta_\partial$  to  $(\Omega_A)_\rho$  has rank 2 (either the source or the target is of rank 2), hence  $\theta_\partial$  has rank  $4 < 6$ .

**3.3.1. Remark.** It would be interesting to go beyond, and settle (3.2) for instance in the case of maximal monodromy  $Sp_{2g}$ . Let  $A \rightarrow S$  be the jacobian of the universal hyperelliptic curve of genus  $g > 0$ . Here  $S$  is the affine space  $\mathcal{M}_{0, 2g+2} \cong (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{2g-1}$ . By Torelli's theorem, one has  $\dim \mu(S) = 2g - 1$ . On the other hand, according to [M2, th. 5.6],  $\text{End } A_{\bar{\eta}} = \mathbb{Z}$  (in fact, it is known that the monodromy representation,



which is a specialization of the Burau representation of the braid group, has Zariski dense image in  $Sp_{2g}$ , cf. [aC]). Hence (3.2) would imply that, after replacing  $(\mathbb{P}^1 \setminus \{0, 1, \infty\})^{2g-1}$  by a suitable dense Zariski-open affine subset,  $\beta$  is a submersion. [There are explicit calculations (without detail) by Bryant of  $\theta$  in this case on MathOverflow (tag: hyperelliptic) which should settle the matter]

**3.4.** Let us turn to the proof of th. 1.4.1. [sketch] One may replace  $S$  by a finite étale covering and assume that  $A/S$  has a level  $n$  structure for some  $n \geq 3$ , then replace  $S$  by a smooth dense affine open subset of its image in  $\mathcal{A}_{g,n}$ . By assumption,  $d = \dim \mu(S) \geq g$ . Taking a sufficiently general locally closed affine subvariety  $S' \subset S$  of dimension  $g$ , the restriction to  $S'$  of the abelian scheme and of the section satisfy the same assumption. Moreover, one checks that the condition  $d \geq g$  implies that we are in the restricted PE case<sup>4</sup>. Replacing  $S$  by  $S'$ , one is in the situation  $d = g$ . One can then combine prop. 2.2.1, 3.2.5 and th. 3.2.1. [A similar deduction holds with prop. 3.2.6 in place of 3.2.5].  $\square$

#### 4. PROOF OF THEOREM 1.5.1.

[your §4 and 5].

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<sup>4</sup>here one needs a generalization of the notion of restricted PE to the case of an abelian scheme whose generic fiber is not necessarily simple. This does not pose problem.