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Purely dissipative solutions of Navier-Stokes equations for three-dimensional incompressible flows without wall

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Existing analytical solutions of Navier-Stokes equations (NSE) are rare. Starting from the helical decomposition, we derive analytically a series of purely dissipative solutions of NSE for three-dimensional incompressible flows without wall. The quasi-two-dimensional solutions are generalized Beltrami flows, and the three-dimensional solutions are found to be Beltrami flows. The two-dimensional Taylor-Green (TG) vortex and the Arnold-Beltrami-Childress (ABC) flows are our particular solutions. By choosing N different wave vectors at the same wave length, our solutions have $2N + 2$ degrees of freedom for the quasi-two-dimensional cases and $2N$ degrees of freedom for the three-dimensional cases, indicating that the solution space can be at any high dimensions.

Key words:

1. Introduction

Although the properties of the solutions of Navier-Stokes Equations (NSE) still need a lot of investigations, it is always interesting and important for us to search for the exact solutions of NSE. Various exact solutions of NSE have been already introduced in previous studies (see Wang 1991; Emanuel 2000), including a lot of famous laminar flows. In the following we will briefly revisit some of these exact solutions and emphasize the position of the present contribution. For consistency with the present contribution, we will restrict our discussion in three-dimensional incompressible flows.

(i) Classic exact solutions of three-dimensional NSE usually considers special wall-bounded flows with self-similarity or under special boundary conditions. They can be approximately classified into the following catalogs according to the type of boundary. *a)* Stationary parallel flows imply null velocity components perpendicular to the streamwise direction. Classic solutions include the Couette and Poiseuille flows in straight channels, and the Hagen-Poiseuille flow in pipes. *b)* The rotating Couette flow is an extension of the plane Couette flow to cylindrical coordinate. *c)* Non-stationary parallel flows consider the case that the velocity of boundary is not constant, such as the accelerating boundary (*i.e.*, the Rayleigh flow) and oscillating boundary. *d)* Other types of flows include the stagnation point flow, the Karman swirling flow, *etc.* Since the present contribution will not discuss the wall-bounded flows, we will not visit more about the related literature.

(ii) Another class of exact solutions of three-dimensional NSE considers the flows without

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wall. These flows are usually either generalized Beltrami flows, *i.e.*,

$$\nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = 0, \quad (1.1)$$

where \mathbf{u} is velocity and $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is vorticity, or more strictly, Beltrami flows, *i.e.*,

$$\boldsymbol{\omega} \times \mathbf{u} = 0. \quad (1.2)$$

Some exact solutions of these flows have been discussed in literature, for instance Hill (1894); Tsien (1943); Wang (1990*a,b*); Shi & Huang (1991); Shi *et al.* (1992); Fujimoto *et al.* (2015). Some of these solutions can be applied to wall-bounded cases, but the original solutions are solved in infinite space without wall. Also, almost all these solutions are either axisymmetric or planar flows. There are also several particular periodic solutions such as the two-dimensional Taylor-Green (TG) vortex (Sipp & Jacquin 1998) and the Arnold-Beltrami-Childress (ABC) flows (Dombre *et al.* 1986).

In the present contribution, we will introduce a series of new solutions of NSE for three-dimensional flows without wall. These solutions are combined by any finite number of wave vectors in Fourier space, therefore they are periodic, rather than axisymmetric or planar. We will show that the quasi-two-dimensional solutions are generalized Beltrami flows, while the three-dimensional solutions are Beltrami flows. The TG vortex and ABC flows are our particular solutions. The solutions are therefore purely dissipative. Analysis of free degrees will show that we can obtain non-trivial solution space at any high dimension.

2. Helical decomposition

In the present contribution we consider the three-dimensional velocity field without wall, and all velocity components are able to be represented in Fourier space. Following Constantin & Majda (1988); Waleffe (1992); Biferale *et al.* (2013), we can decompose an incompressible velocity field by using the helical decomposition. The velocity field $\mathbf{v}(\mathbf{x})$ in physical space is translated to $\mathbf{u}(\mathbf{k})$ in Fourier space with \mathbf{k} wave vector. Being divergence-free, $\mathbf{k} \cdot \mathbf{u}(\mathbf{k}) = 0$, each velocity component in Fourier space has only two degrees of freedom. Two orthogonal complex helical waves are chosen to be $\mathbf{h}^{\pm} = \hat{\boldsymbol{\omega}} \times \hat{\mathbf{k}} \pm i\hat{\boldsymbol{\omega}}$, with $i = \sqrt{-1}$ and $\hat{\boldsymbol{\omega}}$ unit vector. The vector $\hat{\boldsymbol{\omega}}$ can be chosen as $\hat{\boldsymbol{\omega}} = (\mathbf{z} \times \mathbf{k}) / \|\mathbf{z} \times \mathbf{k}\|$ with \mathbf{z} an arbitrary vector. We then have

$$\mathbf{u}(\mathbf{k}) = u^{+\mathbf{z}}(\mathbf{k})\mathbf{h}^{+\mathbf{z}}(\mathbf{k}) + u^{-\mathbf{z}}(\mathbf{k})\mathbf{h}^{-\mathbf{z}}(\mathbf{k}), \quad (2.1)$$

where $u^{+\mathbf{z}}$ and $u^{-\mathbf{z}}$ are complex numbers. The superscript \mathbf{z} is emphasized here since the decomposition depends on the selection of \mathbf{z} . In Waleffe (1992) \mathbf{z} is a constant vector, but in the present contribution we will allow the change of \mathbf{z} in order to obtain three-dimensional solutions. Constantin & Majda (1988) remarked that a single mode $u^{+\mathbf{z}}(\mathbf{k})$ or $u^{-\mathbf{z}}(\mathbf{k})$ corresponds to a Beltrami field, then any incompressible velocity field can be considered as a superposition of these Beltrami unit flows. Researchers, such as Waleffe (1992); Biferale *et al.* (2012, 2013); Zhu *et al.* (2019), used this decomposition in the treatment of turbulence to investigate the forward and backward energy backscatter. However, to our knowledge there is no discussion on combining these Beltrami unit flows to a complication NSE solution. In the present contribution we will deal with this problem to obtain a class of new solutions.

Under the helical decomposition, the NSE can be rewritten as

$$\left(\partial_t + \nu k^2\right) \mathbf{u}(\mathbf{k}) = \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} f(k, p, q) \sum_{s_k, s_p, s_q} s_{kpq}(s_k k + s_p p + s_q q)(s_p p - s_q q) \bar{u}^{s_p \mathbf{z}}(\mathbf{p}) \bar{u}^{s_q \mathbf{z}}(\mathbf{q}) \mathbf{h}^{s_k \mathbf{z}}(\mathbf{k}), \quad (2.2)$$

where ν is viscosity, $k = |\mathbf{k}|$ is the wave length, \mathbf{z} is always selected to be perpendicular to \mathbf{p} and \mathbf{q} with the same direction as $\mathbf{q} \times \mathbf{p}$ (for example, the unit vector $\mathbf{z} = \mathbf{q} \times \mathbf{p}$) such that

$f(k, p, q) = Q/(2kpq)$ is a geometrical factor with $Q = (2k^2p^2 + 2p^2q^2 + 2q^2k^2 - k^4 - p^4 - q^4)^{1/2}$, $s_{kpq} = s_k s_p s_q$, $\bar{\cdot}$ stands for complex conjugate, and $s_k, s_p, s_q = \pm$ denote different helical modes. If the right hand side of Eq. (2.2) is zero for any wave vector \mathbf{k} , we obtain purely dissipative solutions

$$\mathbf{u}(\mathbf{k}, t) = \mathbf{u}(\mathbf{k}, t = 0)e^{-\nu k^2 t}. \quad (2.3)$$

If \mathbf{p} and \mathbf{q} are in the same direction, we will have $f(k, p, q) = 0$, leading to vanishing right-hand side of Eq. (2.2). But for general cases it is complicated. In the present contribution we will therefore aim at finding a set of wave vector \mathcal{W} , such that $\forall \mathbf{p}, \mathbf{q} \in \mathcal{W}$,

$$\sum_{s_p, s_q} s_{kpq}(s_k k + s_p p + s_q q)(s_p p - s_q q) \bar{u}^{s_p}(\mathbf{p}) \bar{u}^{s_q}(\mathbf{q}) \equiv 0 \quad (2.4)$$

with $\mathbf{k} = -\mathbf{p} - \mathbf{q}$ and $s_k = \pm$. Transforming \mathcal{W} and the conjugation part to physical space will directly lead to solution (2.3).

We remark that these flows are always generalized Beltrami flows. In fact, substituting (2.4) into (2.2) and taking the curl, we simply obtain

$$\partial_t \boldsymbol{\omega} - \nu \nabla^2 \boldsymbol{\omega} = 0. \quad (2.5)$$

This is exactly the generalized Beltrami flow where $\nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = 0$ (see Wang 1991).

2.1. The case that $p = q$

Expanding Eq. (2.4) leads to

$$\begin{aligned} & (p - q)(s_k k + p + q) \bar{u}^{+z}(\mathbf{p}) \bar{u}^{+z}(\mathbf{q}) - (p + q)(s_k k + p - q) \bar{u}^{+z}(\mathbf{p}) \bar{u}^{-z}(\mathbf{q}) \\ & - (-p - q)(s_k k - p + q) \bar{u}^{-z}(\mathbf{p}) \bar{u}^{+z}(\mathbf{q}) + (-p + q)(s_k k - p - q) \bar{u}^{-z}(\mathbf{p}) \bar{u}^{-z}(\mathbf{q}) = 0. \end{aligned} \quad (2.6)$$

When $p = q$, this reduces to

$$(p + q) \bar{u}^{+z}(\mathbf{p}) \bar{u}^{-z}(\mathbf{q}) + (-p - q) \bar{u}^{-z}(\mathbf{p}) \bar{u}^{+z}(\mathbf{q}) = 0, \quad (2.7)$$

or equivalently

$$u^{+z}(\mathbf{p}) u^{-z}(\mathbf{q}) - u^{-z}(\mathbf{p}) u^{+z}(\mathbf{q}) = 0, \quad (2.8)$$

This means that we can define a two-dimensional complex vector at a three-dimensional wave vector $\mathbf{a}^z(\mathbf{k}) := (u^{+z}(\mathbf{k}), u^{-z}(\mathbf{k}))$. When $p = q$, Eq. (2.4) is then equivalent to the parallel condition between $\mathbf{a}^z(\mathbf{p})$ and $\mathbf{a}^z(\mathbf{q})$, denoted as

$$\mathbf{a}^z(\mathbf{p}) \parallel \mathbf{a}^z(\mathbf{q}). \quad (2.9)$$

Note that this parallel condition is only defined when z is perpendicular to \mathbf{p} and \mathbf{q} . For another vector $\boldsymbol{\xi}$ which is not perpendicular to both \mathbf{p} and \mathbf{q} , this condition will not be satisfied at the same time. The proof is as follows.

As mentioned in Waleffe (1992), defining $\boldsymbol{\lambda} = (\mathbf{p} \times \mathbf{q})/|\mathbf{p} \times \mathbf{q}|$ (which is a unit vector parallel to z), we have

$$\mathbf{h}^{\pm \boldsymbol{\xi}}(\mathbf{p}) = \hat{\mathbf{w}}^{\boldsymbol{\xi}}(\mathbf{p}) \times \hat{\mathbf{p}} \pm i \hat{\mathbf{w}}^{\boldsymbol{\xi}}(\mathbf{p}) = e^{i \pm \varphi_p^{\boldsymbol{\xi}}} [\boldsymbol{\lambda} \pm i \boldsymbol{\mu}(\mathbf{p})] \quad (2.10)$$

where $\boldsymbol{\mu}(\mathbf{p}) = \hat{\mathbf{p}} \times \boldsymbol{\lambda}$ is located in the plane of the triad $\{\mathbf{k}, \mathbf{p}, \mathbf{q}\}$, and $\varphi_p^{\boldsymbol{\xi}}$ depends only on the direction of $\boldsymbol{\xi}$. Thus we obtain

$$\mathbf{h}^{\pm \boldsymbol{\xi}}(\mathbf{p}) = e^{i \pm \varphi_p^{\boldsymbol{\xi}}} \mathbf{h}^{\pm \boldsymbol{\lambda}}(\mathbf{p}). \quad (2.11)$$

Particularly, after substituting $\boldsymbol{\xi} = \boldsymbol{\lambda}$ into Eq. (2.11), we obtain $\varphi_p^{\boldsymbol{\lambda}} = \pi$ and $\varphi_p^{-\boldsymbol{\lambda}} = 0$. As $\mathbf{u}(\mathbf{p})$ is invariant with the selection of the base of helical decomposition, $\mathbf{u}(\mathbf{p})$ can be rewritten under the new base $(\mathbf{h}^{\boldsymbol{\xi}}(\mathbf{p}), \mathbf{h}^{-\boldsymbol{\xi}}(\mathbf{p}))$ with

$$u^{+\boldsymbol{\xi}}(\mathbf{p}) = e^{-i \varphi_p^{\boldsymbol{\xi}}} u^{+\boldsymbol{\lambda}}(\mathbf{p}), \quad u^{-\boldsymbol{\xi}}(\mathbf{p}) = e^{i \varphi_p^{\boldsymbol{\xi}}} u^{-\boldsymbol{\lambda}}(\mathbf{p}), \quad (2.12)$$

and similarly for $\mathbf{u}(\mathbf{q})$. Then Eq. (2.8), which is satisfied when $\mathbf{z} = -\boldsymbol{\lambda}$, can be rewritten as

$$e^{i\varphi_p^\xi} u^{+\xi}(\mathbf{p}) e^{-i\varphi_q^\xi} u^{-\xi}(\mathbf{q}) - e^{-i\varphi_p^\xi} u^{-\xi}(\mathbf{p}) e^{i\varphi_q^\xi} u^{+\xi}(\mathbf{q}) = 0 \quad (2.13)$$

Clearly, if $\boldsymbol{\xi}$ is not parallel to $\boldsymbol{\lambda}$, the relation $e^{i(\varphi_p^\xi - \varphi_q^\xi)} = e^{-i(\varphi_p^\xi - \varphi_q^\xi)}$ is not satisfied, thus Eq. (2.13) can not be transformed into the form (2.8), which means the parallel condition is not valid for $\mathbf{a}^\xi(\mathbf{p})$ and $\mathbf{a}^\xi(\mathbf{q})$ when $\boldsymbol{\xi}$ is not perpendicular to both \mathbf{p} and \mathbf{q} .

2.2. The case that $p \neq q$

When $p \neq q$, we separate the left-hand side of Eq. (2.6) in two parts $P_1(s_k, p, q)$ and $P_2(k, p, q)$

$$\begin{aligned} P_1(s_k, p, q) &= (p^2 - q^2) s_k \left(\bar{u}^{+z}(\mathbf{p}) \bar{u}^{-z}(\mathbf{q}) - \bar{u}^{+z}(\mathbf{p}) \bar{u}^{-z}(\mathbf{q}) - \bar{u}^{-z}(\mathbf{p}) \bar{u}^{+z}(\mathbf{q}) + \bar{u}^{-z}(\mathbf{p}) \bar{u}^{+z}(\mathbf{q}) \right), \\ P_2(k, p, q) &= (p - q) k \left(\bar{u}^{+z}(\mathbf{p}) \bar{u}^{+z}(\mathbf{q}) - \bar{u}^{-z}(\mathbf{p}) \bar{u}^{-z}(\mathbf{q}) \right) + (p + q) (-k) \left(\bar{u}^{+z}(\mathbf{p}) \bar{u}^{-z}(\mathbf{q}) - \bar{u}^{-z}(\mathbf{p}) \bar{u}^{+z}(\mathbf{q}) \right). \end{aligned} \quad (2.14)$$

Equation (2.6) should be satisfied for $s_k = \pm 1$, which means

$$\begin{aligned} P_1(1, p, q) + P_2(k, p, q) &= 0, \\ P_1(-1, p, q) + P_2(k, p, q) &= 0. \end{aligned} \quad (2.15)$$

Due to the fact that $P_1(1, p, q) = -P_1(-1, p, q)$, Eq. (2.15) leads to

$$P_1(1, p, q) = P_2(k, p, q) = 0, \quad (2.16)$$

From Eqs. (2.14) and (2.16) we finally write

$$u^{+z}(\mathbf{p}) = u^{-z}(\mathbf{p}), \quad u^{+z}(\mathbf{q}) = u^{-z}(\mathbf{q}). \quad (2.17)$$

Since \mathbf{z} is perpendicular to the plane of \mathbf{p} , \mathbf{q} and $\hat{\mathbf{w}}(\mathbf{p}) = (\mathbf{z} \times \mathbf{p}) / |\mathbf{z} \times \mathbf{p}|$ is chosen in the helical decomposition, we have $\hat{\mathbf{w}}(\mathbf{p}) \times \hat{\mathbf{p}} = -\mathbf{z}$. From Eq. (2.17) we then obtain

$$\begin{aligned} \mathbf{u}(\mathbf{p}) &= u^{+z}(\mathbf{p}) \left(\mathbf{h}^{+z}(\mathbf{p}) + \mathbf{h}^{-z}(\mathbf{p}) \right) \\ &= u^{+z}(\mathbf{p}) (-2\mathbf{z}) \end{aligned} \quad (2.18)$$

and similarly for $\mathbf{u}(\mathbf{q})$. Performing the inverse Fourier transform for this system with two vectors as well as the conjugation leads to the velocity field

$$\begin{aligned} \mathbf{v}(\mathbf{x}) &= \mathbf{u}(\mathbf{p}) e^{i\mathbf{p}\mathbf{x}} + \mathbf{u}(\mathbf{q}) e^{i\mathbf{q}\mathbf{x}} + c.c. \\ &= \left(u^{+z}(\mathbf{p}) e^{i\mathbf{p}\mathbf{x}} + u^{+z}(\mathbf{q}) e^{i\mathbf{q}\mathbf{x}} + c.c. \right) (-2\mathbf{z}), \end{aligned} \quad (2.19)$$

which indicates that all velocities must be in the \mathbf{z} direction.

In addition, performing the similar analysis as in the case that $p = q$, we will not have $u^{+\xi}(\mathbf{p}) = u^{-\xi}(\mathbf{p})$ and $u^{+\xi}(\mathbf{p}) = u^{-\xi}(\mathbf{p})$ if $\boldsymbol{\xi}$ is not perpendicular to both \mathbf{p} and \mathbf{q} .

Equation (2.19) indicates that when $p \neq q$, purely dissipative solutions are trivial quasi-one-dimensional flows, which will not involve interesting results. Therefore, in the present contribution we will only consider the flow whose wave vectors are of the same magnitude.

3. Quasi-two-dimensional flows

In this section we will investigate the quasi-two-dimensional flows, which means that all wave vectors in \mathcal{W} are located in the same plane. In this case we can choose \mathbf{z} a constant vector perpendicular to this plane, therefore in this section we will omit the superscript \mathbf{z} in the equations. In this case we have the following lemma of transitivity.

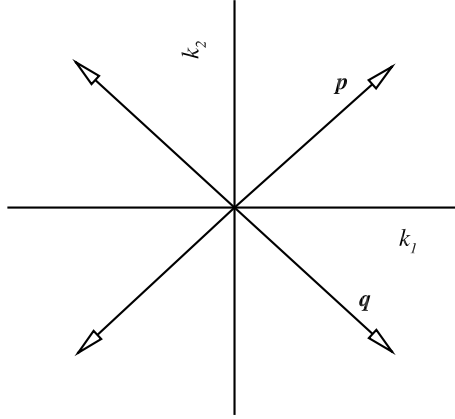


FIGURE 1. Sketch of the system with four wave vectors uniformly distributed in (k_1, k_2) plane of spectral space.

LEMMA 1. *If \mathbf{p} , \mathbf{q} and \mathbf{r} are located in the same plane, $\mathbf{a}(\mathbf{p})$, $\mathbf{a}(\mathbf{q})$ and $\mathbf{a}(\mathbf{r})$ are non-null complex vectors, and $\mathbf{a}(\mathbf{p}) \parallel \mathbf{a}(\mathbf{q})$, $\mathbf{a}(\mathbf{q}) \parallel \mathbf{a}(\mathbf{r})$, then $\mathbf{a}(\mathbf{p}) \parallel \mathbf{a}(\mathbf{r})$.*

Proof. We select a cartesian coordinate (k_1, k_2, k_3) such that \mathbf{p} , \mathbf{q} and \mathbf{r} are located in the (k_1, k_2) plane, and z is in the k_3 direction.

Supposing $\mathbf{a}(\mathbf{q}) \neq \mathbf{0}$, according to the parallel condition, $\mathbf{a}(\mathbf{p}) \parallel \mathbf{a}(\mathbf{q})$, $\mathbf{a}(\mathbf{q}) \parallel \mathbf{a}(\mathbf{r})$ leads to

$$\begin{aligned} \mathbf{a}(\mathbf{p}) &= \lambda_1 \mathbf{a}(\mathbf{q}), \\ \mathbf{a}(\mathbf{q}) &= \lambda_2 \mathbf{a}(\mathbf{r}), \end{aligned} \quad (3.1)$$

where $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ and then we have

$$\mathbf{a}(\mathbf{p}) = \lambda_1 \lambda_2 \mathbf{a}(\mathbf{r}) \quad (3.2)$$

which is equivalent to the parallel condition and we have directly $\mathbf{a}(\mathbf{p}) \parallel \mathbf{a}(\mathbf{r})$. \square

We can select $\mathcal{W} = \{\mathbf{p}_{(1)}, \mathbf{p}_{(2)}, \dots, \mathbf{p}_{(N)}\}$, where all wave vectors are located in the same plane (k_1, k_2) with the same wave length. Let $\mathbf{a}(\mathbf{p}_{(i)}) \parallel \mathbf{a}(\mathbf{p}_{(i+1)})$ for $i = 1, 2, \dots, N-1$. From Lemma 1 all these wave vectors are then parallel to each other. Therefore, Eq. (2.4) is always satisfied, leading to a purely dissipative solution (2.3). z is selected as $z = (0, 0, 1)$ in the (k_1, k_2, k_3) spectral space.

The above paragraphs describe a general method to generate purely dissipative quasi-two-dimensional velocity fields. In the following subsections, we will show several typical examples.

3.1. System with four wave vectors uniformly distributed

As shown in Fig.1, we start with a simple case by choosing $\mathcal{W} = \{\mathbf{p}, \mathbf{q}\}$ that has only two wave vectors with the same wavelength. Considering the conjugation set $\{-\mathbf{p}, -\mathbf{q}\}$, the whole system has four wave vectors. Specially, these four wave vectors are assumed to be uniformly distributed, *i.e.*,

$$\mathbf{p} = \begin{pmatrix} \frac{\sqrt{2}}{2} p \\ \frac{\sqrt{2}}{2} p \\ 0 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} \frac{\sqrt{2}}{2} p \\ -\frac{\sqrt{2}}{2} p \\ 0 \end{pmatrix}. \quad (3.3)$$

Constraints of these wave vectors include the parallel conditions

$$\mathbf{a}(\mathbf{p}) \parallel \mathbf{a}(\mathbf{q}), \quad \mathbf{a}(\mathbf{q}) \parallel \mathbf{a}(-\mathbf{p}), \quad \mathbf{a}(\mathbf{p}) \parallel \mathbf{a}(-\mathbf{p}) \quad (3.4)$$

and the conjugation conditions

$$\mathbf{a}(\mathbf{p}) = \bar{\mathbf{a}}(-\mathbf{p}), \quad \mathbf{a}(\mathbf{q}) = \bar{\mathbf{a}}(-\mathbf{q}). \quad (3.5)$$

These constraints lead to the general form of $\mathbf{a}(\mathbf{p})$ and $\mathbf{a}(\mathbf{q})$

$$\mathbf{a}(\mathbf{p}) = \begin{pmatrix} R_p \alpha_p + I_p \alpha_p i \\ R_p \beta_p + I_p \beta_p i \end{pmatrix}, \quad \mathbf{a}(\mathbf{q}) = \begin{pmatrix} R_q \alpha_p + I_q \alpha_p i \\ R_q \beta_p + I_q \beta_p i \end{pmatrix}, \quad (3.6)$$

where $(R_p, I_p, R_q, I_q, \alpha_p, \beta_p) \in \mathbb{R}^6$ are constants. Letting $\gamma_p = \alpha_p + \beta_p$ and $\sigma_p = \alpha_p - \beta_p$, we have

$$\mathbf{u}(\mathbf{p}) = \begin{pmatrix} \frac{\sqrt{2}}{2} I_p \sigma_p - \frac{\sqrt{2}}{2} R_p \sigma_p i \\ -\frac{\sqrt{2}}{2} I_p \sigma_p + \frac{\sqrt{2}}{2} R_p \sigma_p i \\ -R_p \gamma_p - I_p \gamma_p i \end{pmatrix}, \quad \mathbf{u}(\mathbf{q}) = \begin{pmatrix} -\frac{\sqrt{2}}{2} I_q \sigma_p + \frac{\sqrt{2}}{2} R_q \sigma_p i \\ -\frac{\sqrt{2}}{2} I_q \sigma_p + \frac{\sqrt{2}}{2} R_q \sigma_p i \\ -R_q \gamma_p - I_q \gamma_p i \end{pmatrix}, \quad (3.7)$$

with $(R_p, I_p, R_q, I_q, \gamma_p, \sigma_p) \in \mathbb{R}^6$ constants, and the velocity field in physical space is then

$$\begin{aligned} v_1(x_1, x_2, t = 0) &= \sqrt{2} I_p \sigma_p \cos\left(\frac{\sqrt{2}}{2} p x_1 + \frac{\sqrt{2}}{2} p x_2\right) + \sqrt{2} R_p \sigma_p \sin\left(\frac{\sqrt{2}}{2} p x_1 + \frac{\sqrt{2}}{2} p x_2\right) \\ &\quad - \sqrt{2} I_q \sigma_p \cos\left(\frac{\sqrt{2}}{2} p x_1 - \frac{\sqrt{2}}{2} p x_2\right) - \sqrt{2} R_q \sigma_p \sin\left(\frac{\sqrt{2}}{2} p x_1 - \frac{\sqrt{2}}{2} p x_2\right) \\ v_2(x_1, x_2, t = 0) &= -\sqrt{2} I_p \sigma_p \cos\left(\frac{\sqrt{2}}{2} p x_1 + \frac{\sqrt{2}}{2} p x_2\right) - \sqrt{2} R_p \sigma_p \sin\left(\frac{\sqrt{2}}{2} p x_1 + \frac{\sqrt{2}}{2} p x_2\right) \\ &\quad - \sqrt{2} I_q \sigma_p \cos\left(\frac{\sqrt{2}}{2} p x_1 - \frac{\sqrt{2}}{2} p x_2\right) - \sqrt{2} R_q \sigma_p \sin\left(\frac{\sqrt{2}}{2} p x_1 - \frac{\sqrt{2}}{2} p x_2\right) \\ v_3(x_1, x_2, t = 0) &= -2 R_p \gamma_p \cos\left(\frac{\sqrt{2}}{2} p x_1 + \frac{\sqrt{2}}{2} p x_2\right) + 2 I_p \gamma_p \sin\left(\frac{\sqrt{2}}{2} p x_1 + \frac{\sqrt{2}}{2} p x_2\right) \\ &\quad - 2 R_q \gamma_p \cos\left(\frac{\sqrt{2}}{2} p x_1 - \frac{\sqrt{2}}{2} p x_2\right) + 2 I_q \gamma_p \sin\left(\frac{\sqrt{2}}{2} p x_1 - \frac{\sqrt{2}}{2} p x_2\right) \end{aligned} \quad (3.8)$$

Starting from this general expression, when $p = \sqrt{2}$, $\gamma_p = 0$, $I_p = 0$, $I_q = 0$, $R_p = R_q$ and $2\sqrt{2}\sigma_p R_p = 1$, we obtain the well-known solution of two dimensional TG vortex with $\mathbf{v}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, t = 0)e^{-2\gamma t}$ where $\mathbf{v}(\mathbf{x}, t = 0)$ is

$$\begin{aligned} v_1(x_1, x_2, t = 0) &= \cos(x_1) \sin(x_2), \\ v_2(x_1, x_2, t = 0) &= -\sin(x_1) \cos(x_2), \\ v_3(x_1, x_2, t = 0) &= 0. \end{aligned} \quad (3.9)$$

3.2. System with six wave vectors uniformly distributed

The second example can be a system with six wave vectors of same wave length uniformly distributed in the plane (k_1, k_2) of spectral space (see Fig. 2 as a sketch), that is, $\mathcal{W} = \{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ with

$$\mathbf{p} = \begin{pmatrix} \frac{1}{2} p \\ \frac{\sqrt{3}}{2} p \\ 0 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} \frac{1}{2} p \\ -\frac{\sqrt{3}}{2} p \\ 0 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix} \quad (3.10)$$

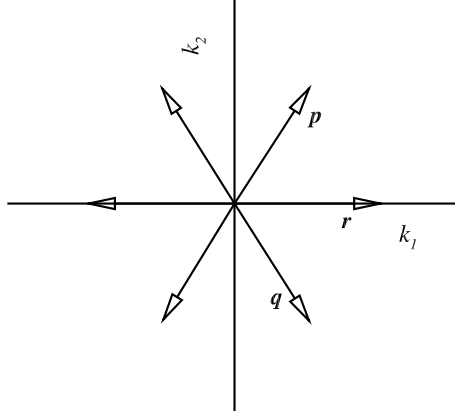


FIGURE 2. Sketch of the system with six wave vectors uniformly distributed in (k_1, k_2) plane of spectral space.

Similar to Sec. 3.1, for $\mathbf{a}(\mathbf{p})$, $\mathbf{a}(\mathbf{q})$ and $\mathbf{a}(\mathbf{r})$, the parallel conditions and conjugation conditions lead to

$$\mathbf{a}(\mathbf{p}) = \begin{pmatrix} R_p \alpha_p + I_p \alpha_p i \\ R_p \beta_p + I_p \beta_p i \end{pmatrix}, \quad \mathbf{a}(\mathbf{q}) = \begin{pmatrix} R_q \alpha_p + I_q \alpha_p i \\ R_q \beta_p + I_q \beta_p i \end{pmatrix}, \quad \mathbf{a}(\mathbf{r}) = \begin{pmatrix} R_r \alpha_p + I_r \alpha_p i \\ R_r \beta_p + I_r \beta_p i \end{pmatrix}, \quad (3.11)$$

where $(R_p, I_p, R_q, I_q, R_r, I_r, \alpha_p, \beta_p) \in \mathbb{R}^8$ are constants. Letting $\gamma_p = \alpha_p + \beta_p$ and $\sigma_p = \alpha_p - \beta_p$, we have

$$\mathbf{u}(\mathbf{p}) = \begin{pmatrix} \frac{\sqrt{3}}{2} I_p \sigma_p - \frac{\sqrt{3}}{2} R_p \sigma_p i \\ -\frac{1}{2} I_p \sigma_p + \frac{1}{2} R_p \sigma_p i \\ -R_p \gamma_p - I_p \gamma_p i \end{pmatrix}, \quad \mathbf{u}(\mathbf{q}) = \begin{pmatrix} -\frac{\sqrt{3}}{2} I_q \sigma_p + \frac{\sqrt{3}}{2} R_q \sigma_p i \\ -\frac{1}{2} I_q \sigma_p + \frac{1}{2} R_q \sigma_p i \\ -R_q \gamma_p - I_q \gamma_p i \end{pmatrix}, \quad \mathbf{u}(\mathbf{r}) = \begin{pmatrix} 0 \\ -I_r \sigma_p + R_r \sigma_p i \\ -R_r \gamma_p - I_r \gamma_p i \end{pmatrix}, \quad (3.12)$$

with $(R_p, I_p, R_q, I_q, R_r, I_r, \gamma_p, \sigma_p) \in \mathbb{R}^8$ constants, and the velocity field in physical space is then

$$\begin{aligned} v_1(x_1, x_2) &= \sqrt{3} I_p \sigma_p \cos\left(\frac{1}{2} p x_1 + \frac{\sqrt{3}}{2} p x_2\right) + \sqrt{3} R_p \sigma_p \sin\left(\frac{1}{2} p x_1 + \frac{\sqrt{3}}{2} p x_2\right) \\ &\quad - \sqrt{3} I_q \sigma_p \cos\left(\frac{1}{2} p x_1 - \frac{\sqrt{3}}{2} p x_2\right) - \sqrt{3} R_q \sigma_p \sin\left(\frac{1}{2} p x_1 - \frac{\sqrt{3}}{2} p x_2\right) \\ v_2(x_1, x_2) &= -I_p \sigma_p \cos\left(\frac{1}{2} p x_1 + \frac{\sqrt{3}}{2} p x_2\right) - R_p \sigma_p \sin\left(\frac{1}{2} p x_1 + \frac{\sqrt{3}}{2} p x_2\right) - 2I_r \sigma_p \cos(p x_1) \\ &\quad - I_q \sigma_p \cos\left(\frac{1}{2} p x_1 - \frac{\sqrt{3}}{2} p x_2\right) - R_q \sigma_p \sin\left(\frac{1}{2} p x_1 - \frac{\sqrt{3}}{2} p x_2\right) - 2R_r \sigma_p \sin(p x_1) \\ v_3(x_1, x_2) &= -2R_p \gamma_p \cos\left(\frac{1}{2} p x_1 + \frac{\sqrt{3}}{2} p x_2\right) + 2I_p \gamma_p \sin\left(\frac{1}{2} p x_1 + \frac{\sqrt{3}}{2} p x_2\right) - 2R_r \gamma_p \cos(p x_1) \\ &\quad - 2R_q \gamma_p \cos\left(\frac{1}{2} p x_1 - \frac{\sqrt{3}}{2} p x_2\right) + 2I_q \gamma_p \sin\left(\frac{1}{2} p x_1 - \frac{\sqrt{3}}{2} p x_2\right) + 2I_r \gamma_p \sin(p x_1) \end{aligned} \quad (3.13)$$

For visualization, we artificially select the values of the constants and show the velocity components in Fig. 3. Note that this velocity field is independent to the third coordinate x_3 , but the

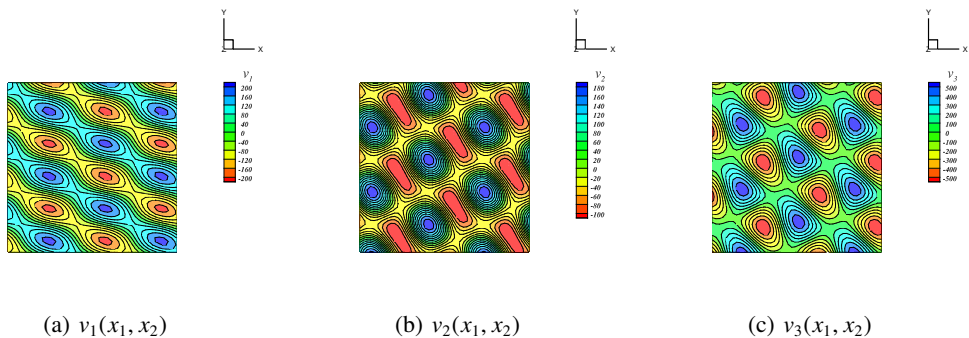


FIGURE 3. Velocity components in physical space of the system with six wave vectors uniformly distributed. As an example, constants are selected as $p = 1, R_p = 3, I_p = 14, R_q = 6, I_q = 2, R_r = 12, I_r = 3, \gamma_p = 10, \sigma_p = -6$ in Eq. (3.13). (a) $v_1(x_1, x_2)$, (b) $v_2(x_1, x_2)$, (c) $v_3(x_1, x_2)$.

velocity vector has the third component v_3 , so it is not the same as the traditional two-dimensional (planar) flow. We name it as quasi-two-dimensional flow in the present contribution. This flow belongs to a typical flow type (see Moffatt 1969) which was named as 3D2C by Zhu (2018).

As discussed in Sec. 2, we know that this solution is a generalized Beltrami flow. In order to show that it is generally not Beltrami flow, we can calculate analytically the vorticity $\boldsymbol{\omega}(\mathbf{x}, t) = \vec{\nabla} \times \mathbf{v}$ of the velocity field from Eq. (3.13). The isosurface of vorticity magnitude is shown in Fig. 4, where periodical straight vortex tubes are observed. When $\gamma_p \neq 0$ and $\sigma_p \neq 0$, we find

$$\begin{aligned}\omega_1(\mathbf{x}, t) &= \lambda_1 v_1(\mathbf{x}, t), \\ \omega_2(\mathbf{x}, t) &= \lambda_2 v_2(\mathbf{x}, t), \\ \omega_3(\mathbf{x}, t) &= \lambda_3 v_3(\mathbf{x}, t),\end{aligned}\tag{3.14}$$

where

$$\lambda_1 = \lambda_2 = p \frac{\gamma_p}{\sigma_p}, \quad \lambda_3 = p \frac{\sigma_p}{\gamma_p}.\tag{3.15}$$

Clearly, in the general case that $\gamma_p \neq \sigma_p$, we have $\lambda_1 = \lambda_2 \neq \lambda_3$, which indicates that $\boldsymbol{\omega} \times \mathbf{v} \neq 0$ and the flow is not a Beltrami flow.

3.3. System with six wave vectors non-uniformly distributed

The parallel condition in (2.8) is only dependent to the helical-decomposed velocity components, but is independent to the coordinates of wave vectors \mathbf{p} and \mathbf{q} . This means that the wave vectors can be artificially selected in the (k_1, k_2) plane in spectral space as long as there wave lengths are equal. In this subsection we then show the general cases of six-wave system. As presented in Fig. 5, we select $\mathcal{W} = \{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ with

$$\mathbf{p} = \begin{pmatrix} p \cos \theta_p \\ p \sin \theta_p \\ 0 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} p \cos \theta_q \\ p \sin \theta_q \\ 0 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} p \cos \theta_r \\ p \sin \theta_r \\ 0 \end{pmatrix}\tag{3.16}$$

As the same as Sec. 3.2, we have

$$\mathbf{a}(\mathbf{p}) = \begin{pmatrix} R_p \alpha_p + I_p \alpha_p i \\ R_p \beta_p + I_p \beta_p i \end{pmatrix}, \quad \mathbf{a}(\mathbf{q}) = \begin{pmatrix} R_q \alpha_p + I_q \alpha_p i \\ R_q \beta_p + I_q \beta_p i \end{pmatrix}, \quad \mathbf{a}(\mathbf{r}) = \begin{pmatrix} R_r \alpha_p + I_r \alpha_p i \\ R_r \beta_p + I_r \beta_p i \end{pmatrix},\tag{3.17}$$

where $(R_p, I_p, R_q, I_q, R_r, I_r, \alpha_p, \beta_p) \in \mathbb{R}^8$ are constants

Differences exist when substituting this to the helical decomposition. In the present general

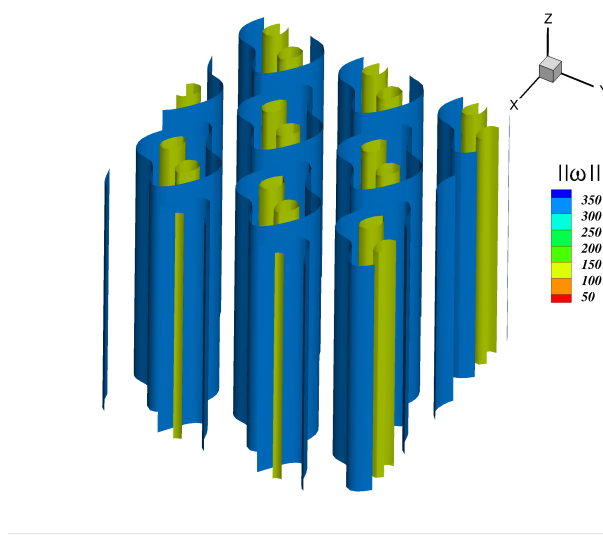


FIGURE 4. Isosurfaces of vorticity magnitude of the system with six wave vectors uniformly distributed. Parameters are the same as Fig. 3.

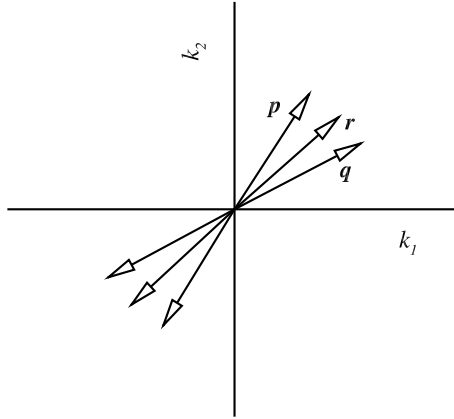


FIGURE 5. Sketch of the system with six wave vectors non-uniformly distributed in (k_1, k_2) plane of spectral space.

case, letting $\gamma_p = \alpha_p + \beta_p$ and $\sigma_p = \alpha_p - \beta_p$, we finally have

$$\mathbf{u}(\mathbf{p}) = \begin{pmatrix} I_p \sigma_p \sin \theta_p - R_p \sigma_p i \sin \theta_p \\ -I_p \sigma_p \cos \theta_p + R_p \sigma_p i \cos \theta_p \\ -R_p \gamma_p - I_p \gamma_p i \end{pmatrix}, \mathbf{u}(\mathbf{q}) = \begin{pmatrix} I_q \sigma_p \sin \theta_q - R_q \sigma_p i \sin \theta_q \\ -I_q \sigma_p \cos \theta_q + R_q \sigma_p i \cos \theta_q \\ -R_q \gamma_p - I_q \gamma_p i \end{pmatrix}, \quad (3.18)$$

$$\mathbf{u}(\mathbf{r}) = \begin{pmatrix} I_r \sigma_p \sin \theta_r - R_r \sigma_p i \sin \theta_r \\ -I_r \sigma_p \cos \theta_r + R_r \sigma_p i \cos \theta_r \\ -R_r \gamma_p - I_r \gamma_p i \end{pmatrix}$$

with $(\theta_p, \theta_q, \theta_r, R_p, I_p, R_q, I_q, R_r, I_r, \gamma_p, \sigma_p) \in \mathbb{R}^{11}$ constants, and the velocity field in physical

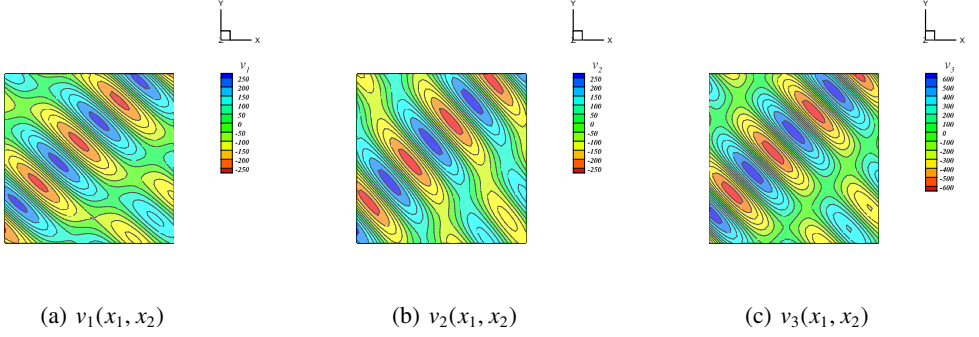


FIGURE 6. Velocity components in physical space of the system with six wave vectors non-uniformly distributed. As an example, constants are selected as $\theta_p = \frac{\pi}{3}, \theta_q = \frac{\pi}{6}, \theta_r = \frac{\pi}{4}, p = 1, R_p = 3, I_p = 14, R_q = 6, I_q = 2, R_r = 12, I_r = 3, \gamma_p = 10, \sigma_p = -6$ in Eq. (3.19). (a) $v_1(x_1, x_2)$, (b) $v_2(x_1, x_2)$, (c) $v_3(x_1, x_2)$.

space is then

$$\begin{aligned}
 v_1(x_1, x_2) &= 2I_p\sigma_p \sin(\theta_p) \cos(px_1 \cos \theta_p + px_2 \sin \theta_p) + 2R_p\sigma_p \sin(\theta_p) \sin(px_1 \cos \theta_p + px_2 \sin \theta_p) \\
 &\quad + 2I_q\sigma_p \sin(\theta_q) \cos(px_1 \cos \theta_q + px_2 \sin \theta_q) + 2R_q\sigma_p \sin(\theta_q) \sin(px_1 \cos \theta_q + px_2 \sin \theta_q) \\
 &\quad + 2I_r\sigma_p \sin(\theta_r) \cos(px_1 \cos \theta_r + px_2 \sin \theta_r) + 2R_r\sigma_p \sin(\theta_r) \sin(px_1 \cos \theta_r + px_2 \sin \theta_r) \\
 v_2(x_1, x_2) &= -2I_p\sigma_p \cos(\theta_p) \cos(px_1 \cos \theta_p + px_2 \sin \theta_p) - 2R_p\sigma_p \cos(\theta_p) \sin(px_1 \cos \theta_p + px_2 \sin \theta_p) \\
 &\quad - 2I_q\sigma_p \cos(\theta_q) \cos(px_1 \cos \theta_q + px_2 \sin \theta_q) - 2R_q\sigma_p \cos(\theta_q) \sin(px_1 \cos \theta_q + px_2 \sin \theta_q) \\
 &\quad - 2I_r\sigma_p \cos(\theta_r) \cos(px_1 \cos \theta_r + px_2 \sin \theta_r) - 2R_r\sigma_p \cos(\theta_r) \sin(px_1 \cos \theta_r + px_2 \sin \theta_r) \\
 v_3(x_1, x_2) &= -2R_p\gamma_p \cos(px_1 \cos \theta_p + px_2 \sin \theta_p) + 2I_p\gamma_p \sin(px_1 \cos \theta_p + px_2 \sin \theta_p) \\
 &\quad - 2R_q\gamma_p \cos(px_1 \cos \theta_q + px_2 \sin \theta_q) + 2I_q\gamma_p \sin(px_1 \cos \theta_q + px_2 \sin \theta_q) \\
 &\quad - 2R_r\gamma_p \cos(px_1 \cos \theta_r + px_2 \sin \theta_r) + 2I_r\gamma_p \sin(px_1 \cos \theta_r + px_2 \sin \theta_r)
 \end{aligned} \tag{3.19}$$

For visualization, we artificially select the values of the constants and show the velocity components in Fig. 6.

3.4. General form of quasi-two-dimensional flows with a finite number of wave vectors of the same wavelength

From the three examples above, we observe that if one more wave vector is added to \mathcal{W} of the system, two more independent variables will appear. In this subsection we then derive the general form for any finite number of wave vectors. Considering a system with $2N$ wave vectors in the (k_1, k_2) plane including $\mathcal{W} = \{\mathbf{p}_{(1)}, \mathbf{p}_{(2)}, \dots, \mathbf{p}_{(N)}\}$ and their conjugations, supposing $(\alpha, \beta) \in \mathbb{R}^2$, with $z = (0, 0, 1)$, we have the general form of $\mathbf{a}(\mathbf{p}_{(i)})$, $i = 1, 2, \dots, N$

$$\mathbf{a}(\mathbf{p}_{(i)}) = \begin{pmatrix} R_{p_{(i)}}\alpha + I_{p_{(i)}}\alpha i \\ R_{p_{(i)}}\beta + I_{p_{(i)}}\beta i \end{pmatrix}, \tag{3.20}$$

where $(R_{p_{(i)}}, I_{p_{(i)}}) \in \mathbb{R}^2$ are two independent variables for each $\mathbf{p}_{(i)}$. Letting $\gamma = \alpha + \beta$ and $\sigma = \alpha - \beta$, we have the general form of $\mathbf{u}(\mathbf{p}_{(i)})$, $i = 1, 2, \dots, N$

$$\mathbf{u}(\mathbf{p}_{(i)}) = \begin{pmatrix} I_{p_{(i)}}\sigma \sin \theta_{p_{(i)}} - R_{p_{(i)}}\sigma i \sin \theta_{p_{(i)}} \\ -I_{p_{(i)}}\sigma \cos \theta_{p_{(i)}} + R_{p_{(i)}}\sigma i \cos \theta_{p_{(i)}} \\ -R_{p_{(i)}}\gamma - I_{p_{(i)}}\gamma i \end{pmatrix} \tag{3.21}$$

Inverse Fourier transform then leads to the general form of the velocity field in physical space

$$\begin{aligned}
 \mathbf{v}(\mathbf{x}) &= \sum_{i=1}^N \mathbf{u}(\mathbf{p}_{(i)}) e^{i\mathbf{p}_{(i)}\mathbf{x}} \\
 v_1(\mathbf{x}) &= \sum_{i=1}^N 2I_{p_{(i)}} \sigma \sin(\theta_{p_{(i)}}) \cos(px_1 \cos \theta_{p_{(i)}} + px_2 \sin \theta_{p_{(i)}}) \\
 &\quad + \sum_{i=1}^N 2R_{p_{(i)}} \sigma \sin(\theta_{p_{(i)}}) \sin(px_1 \cos \theta_{p_{(i)}} + px_2 \sin \theta_{p_{(i)}}) \\
 v_2(\mathbf{x}) &= \sum_{i=1}^N (-2I_{p_{(i)}}) \sigma \cos(\theta_{p_{(i)}}) \cos(px_1 \cos \theta_{p_{(i)}} + px_2 \sin \theta_{p_{(i)}}) \\
 &\quad + \sum_{i=1}^N (-2R_{p_{(i)}}) \sigma \cos(\theta_{p_{(i)}}) \sin(px_1 \cos \theta_{p_{(i)}} + px_2 \sin \theta_{p_{(i)}}) \\
 v_3(\mathbf{x}) &= \sum_{i=1}^N (-2R_{p_{(i)}}) \gamma \cos(px_1 \cos \theta_{p_{(i)}} + px_2 \sin \theta_{p_{(i)}}) \\
 &\quad + \sum_{i=1}^N 2I_{p_{(i)}} \gamma \sin(px_1 \cos \theta_{p_{(i)}} + px_2 \sin \theta_{p_{(i)}})
 \end{aligned} \tag{3.22}$$

Where p is the wavelength for all wave vectors. There are in total $2N + 2$ independent variables in this expression of velocity field. As a result, for this quasi-two-dimensional system with $2N$ wave vectors of the same wavelength, we have in total $2N + 2$ degrees of freedom.

4. Three-dimensional flows

In the last section all wave vectors are located in the same plane, leading to a quasi-two-dimensional flow field which is independent to the x_3 direction. In the previous section, we will discuss the case of three-dimensional flows. As discussed in Sec. 2.1, for a system whose wave vectors have the same wave length, the condition of being a purely dissipative system is that for each two wave vectors $\mathbf{p}_{(i)}, \mathbf{p}_{(j)}$, choosing \mathbf{z} perpendicular to them, $\mathbf{a}^{\mathbf{z}}(\mathbf{p}_i)$ and $\mathbf{a}^{\mathbf{z}}(\mathbf{p}_j)$ should satisfy the parallel condition. Differing from the quasi-two-dimensional flows, here the vector \mathbf{z} can be different when considering different pair $(\mathbf{p}_{(i)}, \mathbf{p}_{(j)})$, and the transitivity Lemma 1 is not satisfied. This means that the parallelization of each pair $(\mathbf{p}_{(i)}, \mathbf{p}_{(j)})$ can be considered as an independent constraint. In the following subsections we then show that we can also obtain three-dimensional solutions, though the constraints are more complicated than the quasi-two-dimensional cases.

4.1. System with six wave vectors

We start from a simple example where we have six wave vectors located at the three axes in spectral space respectively. In this case we have $\mathcal{W} = \{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ with

$$\mathbf{p} = \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix}. \tag{4.1}$$

Figure 7 is a sketch for the wave vectors in spectral space. Visiting all wave vector pairs in this set, we find that the pair is in a plane of either (k_1, k_2) or (k_1, k_3) or (k_2, k_3) . The following paragraphs will then involve the parallel conditions of them respectively.

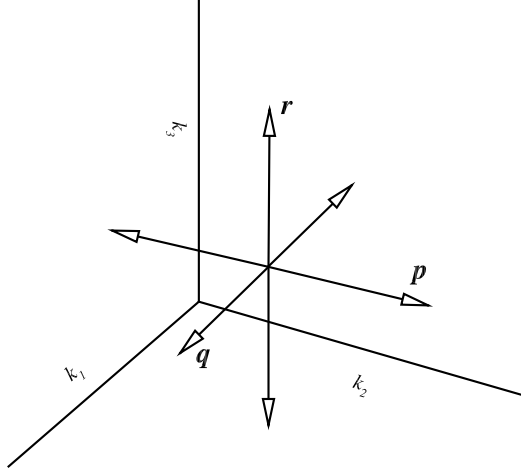


FIGURE 7. Sketch of the system with six wave vectors located at the axes of spectral space respectively.

(i) For the wave vector pair located in the plane (k_1, k_2) , *i.e.*, (\mathbf{p}, \mathbf{q}) as well as their conjugation, the orthogonal vector \mathbf{z}_{12} is chosen as $\mathbf{z}_{12} = (0, 0, 1)$. The parallel condition then implies that $\mathbf{a}^{\mathbf{z}_{12}}(\mathbf{p})$ and $\mathbf{a}^{\mathbf{z}_{12}}(\mathbf{q})$ should be at the following formula:

$$\mathbf{a}^{\mathbf{z}_{12}}(\mathbf{p}) = \begin{pmatrix} u^{+\mathbf{z}_{12}}(\mathbf{p}) \\ u^{-\mathbf{z}_{12}}(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} R_p \alpha_p + I_p \alpha_p i \\ R_p \beta_p + I_p \beta_p i \end{pmatrix}, \quad \mathbf{a}^{\mathbf{z}_{12}}(\mathbf{q}) = \begin{pmatrix} u^{+\mathbf{z}_{12}}(\mathbf{q}) \\ u^{-\mathbf{z}_{12}}(\mathbf{q}) \end{pmatrix} = \begin{pmatrix} R_q \alpha_p + I_q \alpha_p i \\ R_q \beta_p + I_q \beta_p i \end{pmatrix}, \quad (4.2)$$

with $(R_p, I_p, R_q, I_q, \alpha_p, \beta_p) \in \mathbb{R}^6$ constants.

(ii) For the wave vector pair located in the plane (k_1, k_3) , *i.e.*, (\mathbf{q}, \mathbf{r}) as well as their conjugation, the orthogonal vector \mathbf{z}_{13} is chosen as $\mathbf{z}_{13} = (0, 1, 0)$. The parallel condition then implies that $\mathbf{a}^{\mathbf{z}_{13}}(\mathbf{q})$ and $\mathbf{a}^{\mathbf{z}_{13}}(\mathbf{r})$ should be at the following formula:

$$\mathbf{a}^{\mathbf{z}_{13}}(\mathbf{q}) = \begin{pmatrix} u^{+\mathbf{z}_{13}}(\mathbf{q}) \\ u^{-\mathbf{z}_{13}}(\mathbf{q}) \end{pmatrix} = \begin{pmatrix} I_q \alpha_p - R_q \alpha_p i \\ -I_q \beta_p + R_q \beta_p i \end{pmatrix}, \quad \mathbf{a}^{\mathbf{z}_{13}}(\mathbf{r}) = \begin{pmatrix} u^{+\mathbf{z}_{13}}(\mathbf{r}) \\ u^{-\mathbf{z}_{13}}(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} -I_r \alpha_p - R_r \alpha_p i \\ I_r \beta_p + R_r \beta_p i \end{pmatrix}, \quad (4.3)$$

where $(R_r, I_r) \in \mathbb{R}^2$ are two more constants.

(iii) For the wave vector pair located in the plane (k_2, k_3) , *i.e.*, (\mathbf{p}, \mathbf{r}) as well as their conjugation, the orthogonal vector \mathbf{z}_{23} is chosen as $\mathbf{z}_{23} = (1, 0, 0)$. The parallel condition then implies that $\mathbf{a}^{\mathbf{z}_{23}}(\mathbf{p})$ and $\mathbf{a}^{\mathbf{z}_{23}}(\mathbf{r})$ should be at the following formula:

$$\mathbf{a}^{\mathbf{z}_{23}}(\mathbf{p}) = \begin{pmatrix} u^{+\mathbf{z}_{23}}(\mathbf{p}) \\ u^{-\mathbf{z}_{23}}(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} -I_p \alpha_p + R_p \alpha_p i \\ I_p \beta_p - R_p \beta_p i \end{pmatrix}, \quad \mathbf{a}^{\mathbf{z}_{23}}(\mathbf{r}) = \begin{pmatrix} u^{+\mathbf{z}_{23}}(\mathbf{r}) \\ u^{-\mathbf{z}_{23}}(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} -I_r \alpha_p + R_r \alpha_p i \\ -I_r \beta_p + R_r \beta_p i \end{pmatrix} \quad (4.4)$$

The above relations can guarantee that in each plane the parallel condition is satisfied. However, in three-dimensional cases the transitivity Lemma 1 is not satisfied, and more constraints should be considered. For example, the parallel condition $\mathbf{a}^{\mathbf{z}_{23}}(\mathbf{p}) \parallel \mathbf{a}^{\mathbf{z}_{23}}(\mathbf{r})$ leads to

$$u^{+\mathbf{z}_{23}}(\mathbf{p})u^{-\mathbf{z}_{23}}(\mathbf{r}) - u^{-\mathbf{z}_{23}}(\mathbf{p})u^{+\mathbf{z}_{23}}(\mathbf{r}) = 0, \quad (4.5)$$

which means

$$\alpha_p \beta_p = -\alpha_p \beta_p. \quad (4.6)$$

Equation (4.6) implies either $\alpha_p = 0$ or $\beta_p = 0$. It can be validated that when Eqs. (4.2), (4.3), (4.4) and (4.6) are satisfied, all parallel conditions in this system are automatically satisfied.

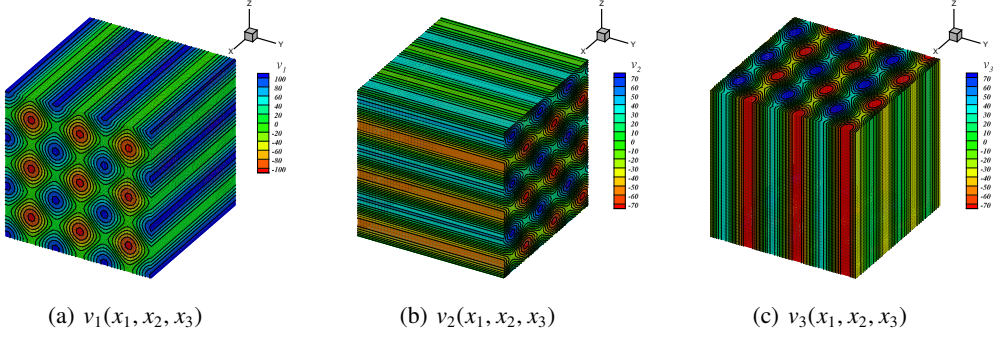


FIGURE 8. Velocity components in physical space of the system with six wave vectors. As an example, constants are selected as $p = 1, R_p = 6, I_p = 28, R_q = 12, I_q = 4, R_r = 24, I_r = 6$ in Eq. (4.8). (a) $v_1(x_1, x_2, x_3)$, (b) $v_2(x_1, x_2, x_3)$, (c) $v_3(x_1, x_2, x_3)$.

Choosing without loss of generality $\beta_p = 0$ and performing the inverse operation of helical decomposition, we have

$$\mathbf{u}(\mathbf{p}) = \begin{pmatrix} I_p \alpha_p - R_p \alpha_p i \\ 0 \\ -R_p \alpha_p - I_p \alpha_p i \end{pmatrix}, \quad \mathbf{u}(\mathbf{q}) = \begin{pmatrix} 0 \\ -I_q \alpha_p + R_q \alpha_p i \\ -R_q \alpha_p - I_q \alpha_p i \end{pmatrix}, \quad \mathbf{u}(\mathbf{r}) = \begin{pmatrix} R_r \alpha_p - I_r \alpha_p i \\ I_r \alpha_p + R_r \alpha_p i \\ 0 \end{pmatrix}, \quad (4.7)$$

Note that α_p appears in all terms which indicates that the choice of it has no influence on the form of \mathbf{u} and we can eliminate it. The system has only 6 degrees of freedom. Then the corresponding velocity field in physical space becomes

$$\mathbf{v}(x_1, x_2, x_3, t = 0) = \begin{pmatrix} 2I_p \cos(px_2) + 2R_p \sin(px_2) + 2R_r \cos(px_3) + 2I_r \sin(px_3) \\ -2I_q \cos(px_1) - 2R_q \sin(px_1) + 2I_r \cos(px_3) - 2R_r \sin(px_3) \\ -2R_p \cos(px_2) + 2I_p \sin(px_2) - 2R_q \cos(px_1) + 2I_q \sin(px_1) \end{pmatrix}. \quad (4.8)$$

We can also write this solution with defining another set of symbols

$$\begin{aligned} A &= 2I_r, & B &= -2R_q, & C &= 2I_p, \\ A' &= -2R_r, & B' &= 2I_q, & C' &= 2R_p, \end{aligned} \quad (4.9)$$

and obtain

$$\mathbf{v}(x_1, x_2, x_3, t = 0) = \begin{pmatrix} A \sin(px_3) + C \cos(px_2) \\ B \sin(px_1) + A \cos(px_3) \\ C \sin(px_2) + B \cos(px_1) \end{pmatrix} + \begin{pmatrix} -A' \cos(px_3) + C' \sin(px_2) \\ -B' \cos(px_1) + A' \sin(px_3) \\ -C' \cos(px_2) + B' \sin(px_1) \end{pmatrix} \quad (4.10)$$

The first part in the right-hand side is the ABC flows, while the second part is a symmetric form. When $A' = B' = C' = 0$ this system is simplified to the ABC flows.

It is possible to validate analytically that the vorticity of this flow field is always parallel to the velocity field, *i.e.*,

$$\boldsymbol{\omega}(\mathbf{x}, t) = \lambda \mathbf{v}(\mathbf{x}, t) \quad (4.11)$$

with λ constant. This means that the present solutions are Beltrami flows.

For visualization, we artificially select the values of the constants and show the velocity components in Fig. 8. The isosurfaces of vorticity magnitude is also shown in Fig.9.

4.2. System with eight wave vectors

In this subsection, we consider a more complicated problem with eight wave vectors. Six of them are located in the (k_1, k_2) plane, while the other two are at the k_3 axis (see Fig. 10 as a

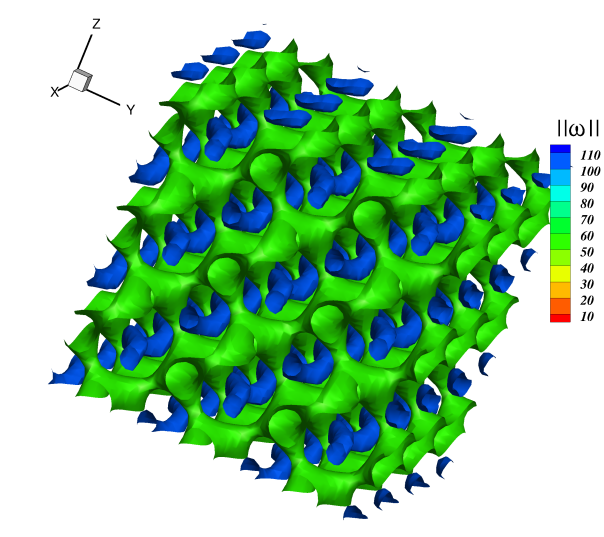


FIGURE 9. Isosurfaces of vorticity magnitude of the system with six wave vectors uniformly distributed. Parameters are the same as Fig. 8.

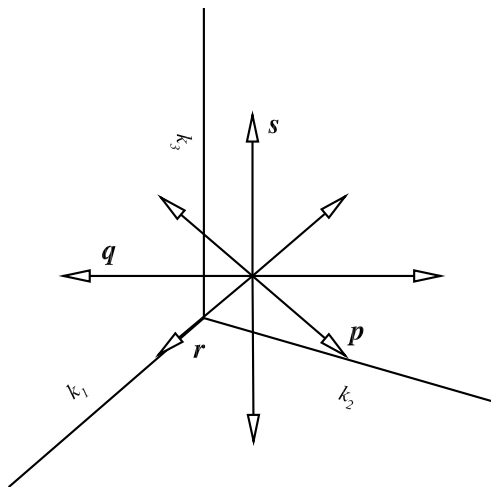


FIGURE 10. 8 wave vectors of same wavelength, 2 wave vectors overlap with k_3 axis in spectral space and the other 6 wave vectors uniformly distributed in k_1, k_2 plane

sketch). Specifically, the set $\mathcal{W} = \{p, q, r, s\}$ is

$$p = \begin{pmatrix} \frac{1}{2}p \\ \frac{\sqrt{3}}{2}p \\ 0 \end{pmatrix}, \quad q = \begin{pmatrix} \frac{1}{2}p \\ -\frac{\sqrt{3}}{2}p \\ 0 \end{pmatrix}, \quad r = \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix} \quad (4.12)$$

In this case, we have four different planes for different pairs of wave vectors. Using the same

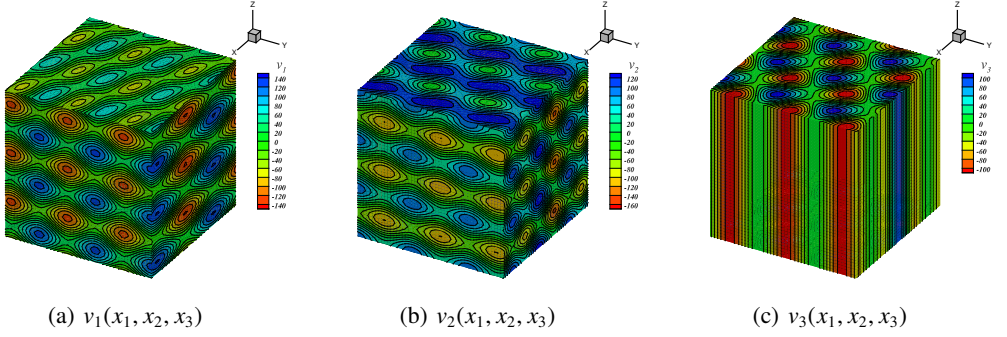


FIGURE 11. The cloud maps for components of velocity vector at $t = 0$ for the flow field in physical space. As an example, constants are select as $p = 1, R_p = 6, I_p = 28, R_q = 12, I_q = 4, R_r = 24, I_r = 6, R_s = 2, I_s = 38$ in Eq. (4.14). (a) $v_1(x_1, x_2, x_3)$, (b) $v_2(x_1, x_2, x_3)$, (c) $v_3(x_1, x_2, x_3)$

method as Sec. 4.1, finally we have

$$\begin{aligned}
 \mathbf{u}(\mathbf{p}) &= \begin{pmatrix} \frac{\sqrt{3}}{2}I_p - \frac{\sqrt{3}}{2}R_p i \\ -\frac{1}{2}I_p + \frac{1}{2}R_p i \\ -R_p - I_p i \end{pmatrix}, \mathbf{u}(\mathbf{q}) = \begin{pmatrix} -\frac{\sqrt{3}}{2}I_q + \frac{\sqrt{3}}{2}R_q i \\ -\frac{1}{2}I_q + \frac{1}{2}R_q i \\ -R_q - I_q i \end{pmatrix}, \\
 \mathbf{u}(\mathbf{r}) &= \begin{pmatrix} 0 \\ -I_r + R_r i \\ -R_r - I_r i \end{pmatrix}, \mathbf{u}(\mathbf{s}) = \begin{pmatrix} R_s - I_s i \\ I_s + R_s i \\ 0 \end{pmatrix},
 \end{aligned} \tag{4.13}$$

where $(R_p, I_p, R_q, I_q, R_r, I_r, R_s, I_s) \in \mathbb{R}^8$ are constants. The velocity components in physical space are

$$\begin{aligned}
 v_1(x_1, x_2, x_3, t = 0) &= \sqrt{3}I_p \cos\left(\frac{1}{2}px_1 + \frac{\sqrt{3}}{2}px_2\right) + \sqrt{3}R_p \sin\left(\frac{1}{2}px_1 + \frac{\sqrt{3}}{2}px_2\right) \\
 &\quad - \sqrt{3}I_q \cos\left(\frac{1}{2}px_1 - \frac{\sqrt{3}}{2}px_2\right) - \sqrt{3}R_q \sin\left(\frac{1}{2}px_1 - \frac{\sqrt{3}}{2}px_2\right) \\
 &\quad + 2R_s \cos(px_3) + 2I_s \sin(px_3) \\
 v_2(x_1, x_2, x_3, t = 0) &= -I_p \cos\left(\frac{1}{2}px_1 + \frac{\sqrt{3}}{2}px_2\right) - R_p \sin\left(\frac{1}{2}px_1 + \frac{\sqrt{3}}{2}px_2\right) \\
 &\quad - I_q \cos\left(\frac{1}{2}px_1 - \frac{\sqrt{3}}{2}px_2\right) - R_q \sin\left(\frac{1}{2}px_1 - \frac{\sqrt{3}}{2}px_2\right) \\
 &\quad - 2I_r \cos(px_1) - 2R_r \sin(px_1) + 2I_s \cos(px_3) - 2R_s \sin(px_3) \\
 v_3(x_1, x_2, x_3, t = 0) &= -2R_p \cos\left(\frac{1}{2}px_1 + \frac{\sqrt{3}}{2}px_2\right) + 2I_p \sin\left(\frac{1}{2}px_1 + \frac{\sqrt{3}}{2}px_2\right) \\
 &\quad - 2R_q \cos\left(\frac{1}{2}px_1 - \frac{\sqrt{3}}{2}px_2\right) + 2I_q \sin\left(\frac{1}{2}px_1 - \frac{\sqrt{3}}{2}px_2\right) \\
 &\quad - 2R_r \cos(px_1) + 2I_r \sin(px_1)
 \end{aligned} \tag{4.14}$$

For visualization, we artificially select the values of the constants and show the velocity components in Fig. 11. The isosurfaces of vorticity magnitude is also shown in Fig.12.

For this system, the analysis of vorticity vector gives the same property that indicates this flow also a Beltrami flow.

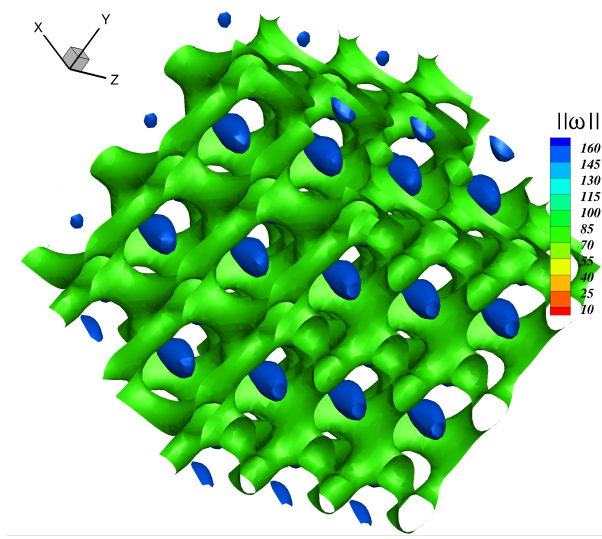


FIGURE 12. Isosurfaces of vorticity magnitude of the system with six wave vectors uniformly distributed. Parameters are the same as Fig. 11.

4.3. General form of three-dimensional flows with a finite number of wave vectors of the same wavelength

From the two examples above, we have the same observation as in the quasi-two-dimensional case that if one more wave vector is added to \mathcal{W} of the system, two more independent variables will appear. In this subsection we then derive the general form for any finite number of wave vectors three dimensional spectral space. Considering a system with $2N$ wave vectors in spectral space including $\mathcal{W} = \{\mathbf{p}_{(1)}, \mathbf{p}_{(2)}, \dots, \mathbf{p}_{(N)}\}$ and their conjugations, we need to search a general form of $\mathbf{u}(\mathbf{p}_{(i)})$, $i = 1, 2, \dots, N$. The parallel conditions lead to the conclusion that for every pair of wave vectors $(\mathbf{p}_{(i)}, \mathbf{p}_{(j)})$, $i, j = 1, 2, \dots, N$ and $i \neq j$

$$\mathbf{a}^{z_{ij}}(\mathbf{p}_{(i)}) = \begin{pmatrix} u^{+z_{ij}}(\mathbf{p}_{(i)}) \\ 0 \end{pmatrix}, \quad \mathbf{a}^{z_{ij}}(\mathbf{p}_{(j)}) = \begin{pmatrix} u^{+z_{ij}}(\mathbf{p}_{(j)}) \\ 0 \end{pmatrix}, \quad (4.15)$$

or

$$\mathbf{a}^{z_{ij}}(\mathbf{p}_{(i)}) = \begin{pmatrix} 0 \\ u^{-z_{ij}}(\mathbf{p}_{(i)}) \end{pmatrix}, \quad \mathbf{a}^{z_{ij}}(\mathbf{p}_{(j)}) = \begin{pmatrix} 0 \\ u^{-z_{ij}}(\mathbf{p}_{(j)}) \end{pmatrix}, \quad (4.16)$$

where $z_{ij} = \mathbf{p}_{(j)} \times \mathbf{p}_{(i)}$. From Eqs. (2.10) and (2.12), for another vector ξ , as long as ξ is not in the same direction with $\mathbf{p}_{(i)}$, the new coordinates become

$$\mathbf{a}^{\xi}(\mathbf{p}_{(i)}) = \begin{pmatrix} e^{-i\varphi_{\mathbf{p}_{(i)}}^{\xi}} u^{+z_{ij}}(\mathbf{p}_{(i)}) \\ 0 \end{pmatrix} \quad (4.17)$$

or

$$\mathbf{a}^{\xi}(\mathbf{p}_{(i)}) = \begin{pmatrix} 0 \\ e^{i\varphi_{\mathbf{p}_{(i)}}^{\xi}} u^{-z_{ij}}(\mathbf{p}_{(i)}) \end{pmatrix}. \quad (4.18)$$

This relation is valid for all wave vectors in \mathcal{W} . As a result, we can choose one vector $\mathbf{z} = (0, 0, 1)$ for all wave vectors except for $\mathbf{p} = (0, 0, p)$. If $\mathbf{p} = (0, 0, p) \in \mathcal{W}$, we denote the

coordinates of other wave vectors as

$$\mathbf{p}_{(i)} = \begin{pmatrix} k_{1_{p(i)}} p \\ k_{2_{p(i)}} p \\ k_{3_{p(i)}} p \end{pmatrix}, \forall i = 1, 2, \dots, N-1, \quad (4.19)$$

where p is the wavelength, $k_{1_{p(i)}}^2 + k_{2_{p(i)}}^2 + k_{3_{p(i)}}^2 = 1$ and $k_{1_{p(i)}}^2 + k_{2_{p(i)}}^2 \neq 0$. For $i = N$, we write

$$\mathbf{p}_{(N)} = \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix}. \quad (4.20)$$

With these notations, we can have the general form of $\mathbf{a}^z(\mathbf{p}_{(i)})$, $i = 1, 2, \dots, N-1$

$$\mathbf{a}^z(\mathbf{p}_{(i)}) = \begin{pmatrix} R_{p(i)} + I_{p(i)} i \\ 0 \end{pmatrix} \quad (4.21)$$

or

$$\mathbf{a}^z(\mathbf{p}_{(i)}) = \begin{pmatrix} 0 \\ R_{p(i)} + I_{p(i)} i \end{pmatrix}, \quad (4.22)$$

where $(R_{p(i)}, I_{p(i)}) \in \mathbb{R}^2$ are two independent variables for each wave vector. For the case in which we use the general form of $\mathbf{a}^z(\mathbf{p}_{(i)})$, $i = 1, 2, \dots, N-1$ in Eq. (4.21), we have $\mathbf{u}(\mathbf{p}_{(i)})$, $i = 1, 2, \dots, N-1$

$$\begin{aligned} \mathbf{h}^{+z}(\mathbf{p}_{(i)}) &= \begin{pmatrix} k_{1_{p(i)}} k_{2_{p(i)}} - k_{2_{p(i)}} i \\ k_{2_{p(i)}} k_{3_{p(i)}} + k_{1_{p(i)}} i \\ -k_{1_{p(i)}}^2 - k_{2_{p(i)}}^2 \end{pmatrix}, \\ \mathbf{u}(\mathbf{p}_{(i)}) &= \begin{pmatrix} (R_{p(i)} k_{1_{p(i)}} k_{2_{p(i)}} + I_{p(i)} k_{2_{p(i)}}) + (I_{p(i)} k_{1_{p(i)}} k_{2_{p(i)}} - R_{p(i)} k_{2_{p(i)}}) i \\ (R_{p(i)} k_{2_{p(i)}} k_{3_{p(i)}} - I_{p(i)} k_{1_{p(i)}}) + (I_{p(i)} k_{2_{p(i)}} k_{3_{p(i)}} + R_{p(i)} k_{1_{p(i)}}) i \\ -R_{p(i)} (k_{1_{p(i)}}^2 + k_{2_{p(i)}}^2) - I_{p(i)} (k_{1_{p(i)}}^2 + k_{2_{p(i)}}^2) i \end{pmatrix}. \end{aligned} \quad (4.23)$$

For $i = N$, we have

$$\mathbf{u}(\mathbf{p}_{(N)}) = \begin{pmatrix} R_{p(N)} - I_{p(N)} i \\ I_{p(N)} + R_{p(N)} i \\ 0 \end{pmatrix} \quad (4.24)$$

Inverse Fourier transform then leads to the general form of velocity field in physical space for

this case

$$\begin{aligned}
v_1(\mathbf{x}, t = 0) &= \sum_{i=1}^{N-1} 2(R_{p(i)} k_{1_{p(i)}} k_{2_{p(i)}} + I_{p(i)} k_{2_{p(i)}}) \cos(k_{1_{p(i)}} x_1 + k_{2_{p(i)}} x_2 + k_{3_{p(i)}} x_3) + 2R_{p(N)} \cos(px_3) \\
&\quad + \sum_{i=1}^{N-1} (-2)(I_{p(i)} k_{1_{p(i)}} k_{2_{p(i)}} - R_{p(i)} k_{2_{p(i)}}) \sin(k_{1_{p(i)}} x_1 + k_{2_{p(i)}} x_2 + k_{3_{p(i)}} x_3) + 2I_{p(N)} \sin(px_3), \\
v_2(\mathbf{x}, t = 0) &= \sum_{i=1}^{N-1} 2(R_{p(i)} k_{2_{p(i)}} k_{3_{p(i)}} - I_{p(i)} k_{1_{p(i)}}) \cos(k_{1_{p(i)}} x_1 + k_{2_{p(i)}} x_2 + k_{3_{p(i)}} x_3) + 2I_{p(N)} \cos(px_3) \\
&\quad + \sum_{i=1}^{N-1} (-2)(I_{p(i)} k_{2_{p(i)}} k_{3_{p(i)}} + R_{p(i)} k_{1_{p(i)}}) \sin(k_{1_{p(i)}} x_1 + k_{2_{p(i)}} x_2 + k_{3_{p(i)}} x_3) - 2R_{p(N)} \sin(px_3), \\
v_3(\mathbf{x}, t = 0) &= \sum_{i=1}^{N-1} (-2)R_{p(i)} (k_{1_{p(i)}}^2 + k_{2_{p(i)}}^2) \cos(k_{1_{p(i)}} x_1 + k_{2_{p(i)}} x_2 + k_{3_{p(i)}} x_3) \\
&\quad + \sum_{i=1}^{N-1} 2I_{p(i)} (k_{1_{p(i)}}^2 + k_{2_{p(i)}}^2) \sin(k_{1_{p(i)}} x_1 + k_{2_{p(i)}} x_2 + k_{3_{p(i)}} x_3).
\end{aligned} \tag{4.25}$$

If $\mathbf{p} = (0, 0, p) \notin \mathcal{W}$, we just eliminate the components associated with $\cos(px_3)$ and $\sin(px_3)$. We can also use the spherical coordinates to represent $k_{1_{p(i)}}$, $k_{2_{p(i)}}$ and $k_{3_{p(i)}}$ and introduce the angles in Eq. (4.25) similar as the method in Sec. 3.4. We have $k_{1_{p(i)}} = \sin \theta_{p(i)} \cos \phi_{p(i)}$, $k_{2_{p(i)}} = \sin \theta_{p(i)} \sin \phi_{p(i)}$, $k_{3_{p(i)}} = \cos \theta_{p(i)}$.

For the other case in which the general form of $\alpha^z(\mathbf{p}_{(i)})$, $i = 1, 2, \dots, N-1$ is Eq. (4.22), we can perform a similar derivation and obtain another general form of velocity similar to the one presented.

From these general results for three-dimensional flows, we can conclude on the degrees of freedom that for a three dimensional system with N wave vectors of the same wavelength, we have $2N$ degrees of freedom.

5. Concluding remarks

Starting from the helical decomposition, we derive analytically a series of purely dissipative solutions of NSE for three-dimensional incompressible flows without wall. The quasi-two-dimensional solutions are generalized Beltrami flows, and the three-dimensional solutions are, according to present examples, Beltrami flows. The two-dimensional TG vortex and the ABC flows are our particular solutions. Although not rigorously proved, we guess that these solutions might be general non-trivial solutions for the cases that flow is three-dimensional, incompressible, and can be presented via a summation of a finite number of discrete Fourier wave modes. This is suggested as an open question.

The present solutions are complementations to existing analytical solutions of NSE. In fact, our solutions can be expanded to any high degree of freedom, indicating that the solution space can be at any high dimensions. For the stabilities of these solutions, we have numerically tested some examples and found they are all instable. Rigorous proves are required in the future studies.

This result might also involve new understandings about the generation and energy transfer of turbulence. Imaging two different flow fields: the first flow is one of our solutions, while the second flow changes the phases at one or more wave vectors, such at the parallel condition (2.8) is not always satisfied. Clearly, these two flows have exactly the same energy distribution, but their behaviors of energy transfer are different: the first flow does not have any multi-scale energy

exchange, but the second flow will exchange energy to smaller scales. In fact, we illustrated in Secs. 2.1 and 2.2 that if a system contains different wave lengths, energy will (non-trivially) transfer, but for a system with only one wave length, energy transfer does not always happen. To quantitatively measure the multi-scale energy transfer, a quantity based on the parallel condition (2.8) might be appropriate. For example, we could define the complex angle between $\mathbf{a}^z(\mathbf{p})$ and $\mathbf{a}^z(\mathbf{q})$ at a fixed wave length to estimate whether this scale is likely to transfer energy to other scales. We expect that these ideas might inspire future studies.

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