Dissipative boundary conditions and entropic solutions in dynamical perfect plasticity

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Abstract. We prove the well–posedness of a dynamical perfect plasticity model under general assumptions on the stress constraint set and on the reference configuration. The problem is studied by combining both calculus of variations and hyperbolic methods. The hyperbolic point of view enables one to derive a class of dissipative boundary conditions, somehow intermediate between homogeneous Dirichlet and Neumann ones. By using variational methods, we show the existence and uniqueness of solutions. Then we establish the equivalence between the original variational solutions and generalized entropic–dissipative ones, derived from a weak hyperbolic formulation for initial–boundary value Friedrichs' systems with convex constraints.

Contents

1. Introduction

In this paper we study the problem of small–strain dynamical perfect plasticity, under general constitutive assumptions, exploiting both the theory of constrained initial–boundary value Friedrichs' systems and techniques from the calculus of variations. Perfect plasticity is a classical theory in continuum mechanics (see e.g. [26, 32]), developed to predict the appearance of permanent deformations in solids as the result of critical internal stresses. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set representing the reference configuration of an elasto–plastic body, $u : \Omega \times (0,T) \to \mathbb{R}^n$ be a kinematically admissible displacement field, and $\sigma : \Omega \times (0,T) \to \mathbb{M}^n_{\rm sym}$ be a statically admissible Cauchy stress tensor. In a dynamical framework, given an external body load $f \colon \Omega \times (0,T) \to \mathbb{R}^n$, u and σ satisfy the equation of motion

$$\ddot{u} - \operatorname{div} \sigma = f \quad \text{in } \Omega \times (0,T)\,, \tag{1.1a}$$

where $\ddot{u}$, the second partial derivative of u with respect to time, is the acceleration. In small strain perfect plasticity, the linearized strain tensor $\mathrm{E}u := (\mathrm{D}u + \mathrm{D}u^T)/2$ decomposes as

$$\mathrm{E}u = e + p\,, \tag{1.1b}$$

where e and $p : \Omega \times (0,T) \to \mathbb{M}^n_{\rm sym}$ are the elastic and the plastic strains, corresponding to reversible and irreversible deformations, respectively. The *Hooke's law*

$$\sigma := \mathbf{A}e = \lambda(\operatorname{tr} e)\mathrm{I}_n + 2\mu e \tag{1.1c}$$

gives the expression of the Cauchy stress as a linear mapping of the elastic strain, for some fourth order symmetric and isotropic elasticity tensor $\mathbf{A}$, whose expression involves the *Lamé coefficients* λ, μ satisfying the ellipticity conditions $\mu > 0$ and $2\mu + n\lambda > 0$. In perfect plasticity, the Cauchy stress tensor is constrained to stay in a fixed closed and convex set $\mathbf{K} \subset \mathbb{M}^n_{\rm sym}$ containing the origin as an interior point

$$\sigma \in \mathbf{K}\,. \tag{1.1d}$$

When σ lies inside the interior of $\mathbf{K}$, the material behaves elastically and no additional inelastic strains are created ($\dot{p} = 0$). On the contrary, when σ reaches the boundary of $\mathbf{K}$ a plastic flow may develop in such a way that a non–trivial permanent plastic strain p may remain after unloading. The evolution of p is described by the *Prandtl-Reuss law* $\dot{p} \in N_{\mathbf{K}}(\sigma)$, where $N_{\mathbf{K}}(\sigma)$ is the normal cone to $\mathbf{K}$ at σ, or equivalently by the *Hill's principle of maximum plastic work*

$$\sigma : \dot{p} = H(\dot{p}) \,, \tag{1.1e}$$

with $H(q) = \sup_{\tau \in \mathbf{K}} \tau : q$ the support function of $\mathbf{K}$.

The problem of dynamical perfect plasticity consists thus in finding a displacement $u : \Omega \times (0,T) \to \mathbb{R}^n$, an elastic strain $e : \Omega \times (0,T) \to \mathbb{M}^n_{\rm sym}$, a plastic strain $p : \Omega \times (0,T) \to \mathbb{M}^n_{\rm sym}$, and a Cauchy stress $\sigma : \Omega \times (0,T) \to \mathbb{M}^n_{\rm sym}$ such that (1.1a)–(1.1e) hold in $\Omega \times (0,T)$, starting from initial conditions

$$(u(0), \dot{u}(0), e(0), p(0)) = (u_0, v_0, e_0, p_0), \tag{1.1f}$$

and with suitable boundary conditions on $\partial\Omega \times (0,T)$. In the quasistatic setting, i.e., when the equation of motion (1.1a) is replaced by the equilibrium equation $-\mathrm{div}\,\sigma = f$ in $\Omega \times (0,T)$, the problem has been studied in [40] and later revisited in [15] in the framework of rate independent processes. A similar static problem, also known as Hencky plasticity, has been investigated in [42, 5]. In the dynamical case, existence and uniqueness of solutions to this problem has been proven in the case of Dirichlet boundary conditions on u in [6] when $\mathbf{K}$ is cylindrical and bounded in the direction $\mathbb{M}^n_D$ of deviatoric matrices (typically the Von Mises or Tresca models), and in [10] for a general $\mathbf{K}$. The proof in [6, 10] relies on the approximation through visco–plastic regularized models. Another approach introduced in [17] is based on the minimization of time–space convex functionals (called WIDE for weighted–inertia–dissipation–energy) involving a small parameter in an exponentially time decreasing weight, and by passing to the parameter limit. We remark that in this last reference, the authors also consider the case of mixed Dirichlet and homogeneous–Neumann boundary conditions but with a cylindrical elasticity set as in [6], bounded in the direction of $\mathbb{M}^n_D$.

The present contribution focuses on an approach to dynamic perfect plasticity based on entropic formulations of hyperbolic conservation laws. A simplified anti–plane model has been studied in [8]. We extend here the result of [8] to the general vectorial case of (1.1), by removing the antiplane shear displacement assumption, by considering a general convex set $\mathbf{K}$, and by working in general Lipschitz reference configurations Ω. In the sequel, we describe the abstract approach, and point out the mathematical issues arising in the general elasto–plastic setting.

The starting point is to express (1.1a), (1.1b), (1.1c) and (1.1f) as a non–stationary Friedrichs' system

$$\begin{cases} \partial_t U + \displaystyle\sum_{i=1}^{n} A_i \, \partial_i U + P = F \,, \\ U(0) = U_0 \,, \end{cases} \tag{1.2a}$$

where each component of $U : \Omega \times (0,T) \to \mathbb{R}^d$, with $d = n + \frac{n(n+1)}{2}$ is a suitable explicit linear combination of the n components of $\dot{u}$ and of the $\frac{n(n+1)}{2}$ components of the $n \times n$ symmetric matrix σ (and similarly for U_0). The vector $P : \Omega \times (0,T) \to \mathbb{R}^d$ is a residual term (usually not present in Friedrichs systems) coming from the plastic strain rate $\dot{p}$, the source term $F : \Omega \times (0,T) \to \mathbb{R}^d$ comes from the body loads f, and the $d \times d$ symmetric matrices $A_1, \dots, A_n$ are only depending explicitly on the Lamé coefficients. The plasticity conditions (1.1d), (1.1e) then correspond to

$$U \in \mathbf{C}, \qquad P \cdot (U - \xi) \geq 0 \quad \text{for every } \xi \in \mathbf{C} \,, \tag{1.2b}$$

for a suitable closed and convex set $\mathbf{C} \subset \mathbb{R}^d$ containing $0_{\mathbb{R}^d}$ as an interior point. Therefore, (1.1) can be formally recast into (1.2), that we interpret as a Friedrichs' system with a convex constraint.

In the classical case of elastodynamics, we have $P = 0$ and this formulation of the generalized wave equation describing the equation of motion in the framework of Friedrichs' systems has been introduced and studied in [27]. Besides elastodynamics, many problems can be formulated in terms of Friedrichs' systems, for instance advection–reaction equations, advection–diffusion–reaction equations, the wave equation, the linearized Euler equations, the Maxwell equations in the so-called elliptic regime, and several other equations in mathematical physics (see e.g. [12, 2]).

When the problem is set in the whole space $\mathbb{R}^n$, a notion of entropic solutions of Friedrichs' systems with convex constraints has been introduced in [20] and characterized in [9] as the unique

limit of regularized solutions by means of a constraint penalization and a vanishing diffusion. This theory turns out to be well defined in Lebesgue type spaces. However, problems in continuum mechanics are often formulated in bounded domains. For that reason, it is important to extend the previous approach to initial–boundary value problems. The difficulty is that the boundary conditions must be compatible with the evolution, especially in the hyperbolic framework where the initial conditions are propagated at finite speed through the characteristics and, possibly, up to the boundary. The starting point of our analysis (which was also the point of view of [35, 8]) are so-called dissipative boundary conditions introduced in [24]. This notion of boundary conditions was then used in [35] to formulate an L^2-theory for initial–boundary value Friedrichs' system with an intrinsic formulation of boundary conditions, in the spirit of the L^∞-theory of boundary value scalar conservation laws [33, 36] (see also the abstract approach to Friedrichs' systems in Hilbert spaces of [22, 21], and [1] on the relations between different types of boundary conditions). The advantage of this so-called entropic–dissipative formulation is that it does not involve any notion of trace of the solution on the boundary, which is meaningful when one is looking for solutions in Lebesgue type spaces.

This paper completes the analysis of a notion of *entropic–dissipative solution* for initial–boundary value Friedrichs' systems with a convex constraint, obtained by a combination of [35] and [20], in the case of dynamic perfect plasticity. Such solutions are functions $U \in L^2(\Omega \times (0,T); \mathbf{C})$ such that for all constant vectors $\xi \in \mathbf{C}$ and all test functions $\varphi \in C_c^\infty(\mathbb{R}^n \times (-\infty, T))$ with $\varphi \geq 0$,

$$\int_0^T \int_\Omega |U-\xi|^2 \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t + \sum_{i=1}^n \int_0^T \int_\Omega A_i(U-\xi)\cdot(U-\xi)\partial_i\varphi \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_\Omega |U_0-\xi|^2\varphi(0)\,\mathrm{d}x + 2\int_0^T\int_\Omega F\cdot(U-\xi)\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\partial\Omega} M\xi^+\cdot\xi^+\varphi\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t \geq 0. \quad (1.3a)$$

In the expression above, for $x \in \partial\Omega$, we denote by $\nu(x)$ the unit outer normal to $\partial\Omega$ at x, $A_\nu(x) := \sum_{i=1}^n \nu_i(x) A_i$ and $M(x) \in \mathbb{M}^d_{\mathrm{sym}}$ satisfies

$$\begin{cases} M(x) \text{ is nonnegative,} \\ \operatorname{Ker} A_\nu(x) \subset \operatorname{Ker} M(x)\,, \\ \mathbb{R}^d = \operatorname{Ker}\big(A_\nu(x) - M(x)\big) + \operatorname{Ker}\big(A_\nu(x) + M(x)\big)\,, \end{cases} \quad (1.3b)$$

and ξ^+ is the projection of ξ onto $\operatorname{Ker}\big(A_\nu(x)+M(x)\big) \cap \operatorname{Im} A_\nu$. The formulation (1.3a) is a combination of the unconstrained initial–boundary value problem introduced in [35, Definition 1], and the constrained Cauchy problem in the full space introduced in [20, Definition 2]. It has two main features: first, the fact the test vectors ξ belong to the convex set $\mathbf{C}$ entirely contains the information of the flow rule; and second, the fact that the boundary condition is expressed through a boundary term not involving the trace of the solution which allows for a Lebesgue spaces type theory. The formulation (1.3a) is referred to as an *entropic formulation* since it involves functions $U \mapsto |U-\xi|^2$ which are analogous to the Kružkov entropies in [30] for scalar conservations laws, and to as a *dissipative formulation* by analogy of the loss of kinetic energy for Euler equations in [31] since, in the case $F = 0$, the boundary condition implies the time decrease of $\|U(t)\|_{L^2(\Omega)}$.

In [24] the admissible boundary conditions in a "strong sense" are given by

$$(A_\nu - M)U = 0 \quad \text{on } \partial\Omega \times (0,T)\,, \quad (1.4)$$

for a matrix M satisfying (1.3b). In the case where A_ν is invertible, as in plasticity, the difference between [24] and [35] on admissible matrices M is that in the latter case, M has to be symmetric since only its symmetric part is involved in the formulation (1.3a).

In the present case, the characterization of all admissible M in the sense of (1.3b) requires a delicate abstract algebraic treatment. This is detailed in Paragraph 3.2.3 in dimension $n = 3$, but it may be extended to any dimension. It turns out that, for a given A_ν, the matrix M is determined by a $n \times n$ positive definite matrix S, in terms of which we reformulate (1.4) as

$$S\dot{u} + \sigma\nu = 0 \quad \text{on } \partial\Omega \times (0,T)\,. \quad (1.5)$$

Thanks to this characterization of the admissible boundary conditions, we can formulate (1.3a) directly for the pair $(\dot{u}, \sigma)$ (instead of U), to specialize the notion of *entropic–dissipative solution* to the present dynamic elasto–plasticity model (1.2) (see Definition 5.1). The boundary condition

(1.5) now formally complements the problem (1.1). Unfortunately, the convex constraint (1.1d) might be incompatible with (1.5) because $S\dot{u}$ may not belong to $K\nu$ on some part of $\partial\Omega$. Motivated by this observation, we consider an elasto–visco–plastic approximation, where in correspondence to a viscosity parameter $\varepsilon > 0$, we add a damping term by replacing the stress σ by $\sigma_\varepsilon + \varepsilon \mathrm{E}\dot{u}_\varepsilon$ in (1.1a) and (1.5), following a Kelvin-Voigt rheology, and the convex constraint (1.1d), (1.1e) is penalized through a Perzyna visco–plastic approximation

$$\dot{p}_\varepsilon = \frac{\sigma_\varepsilon - \mathrm{P}_{\mathbf{K}}(\sigma_\varepsilon)}{\varepsilon}\,.$$

In the spirit of [35] and [8], we study the asymptotics of these approximating solutions (whose existence and uniqueness is standard, see e.g. Proposition 4.6) as $\varepsilon \to 0$. Our main results are twofold:

(1) In Theorem 4.5, we prove the existence and uniqueness of a *variational solution to* (1.1) supplemented with the *relaxed boundary conditions*

$$P_{-\mathbf{K}\nu}(S\dot{u}) + \sigma\nu = 0 \quad \text{on } \partial\Omega \times (0,T)\,,$$

which is defined in Definition 4.2;
(2) In Theorems 5.2 and 5.3, we establish the equivalence between variational and entropic–dissipative solutions.

Although the outcomes of our analysis are similar to those in the simplified setting of [8], we develop here completely different tools due to the lack of counterparts for the regularity results employed in [8], specific to the anti–plane case, and of the setting therein. In particular, this allowed us both to simplify the argument and to avoid further regularity assumptions on Ω and $\mathbf{K}$, as we briefly discuss below.

One of the main features of plasticity models is the linear growth of the plastic dissipation (the support function H of the elasticity set $\mathbf{K}$), which leads to strain concentration and to a possibly singular measure plastic strain p. From a functional analytic point of view, kinematically admissible displacements fields turn out to belong to the space BD of functions of bounded deformation introduced in [38] for the study of perfect plasticity models (see also [41]). This pathological functional setting prevents to easily give a sense to the flow rule (1.1e) since the stress σ (which is usually square integrable as in classical elasticity) is not in duality with the plastic strain p which is a singular measure. It is although possible to define a generalized stress–strain duality $[\sigma : \dot{p}]$ as in [29, 23, 10] in Ω, but the lack of control of the boundary terms prevents to extend this definition to $\overline{\Omega}$. A remedy to this obstacle in variational approaches to evolution problems is to express the flow rule as an energy–dissipation balance. It thus leads to study the limit of the energy–dissipation balance for approximated evolutions. In the present model, the relaxation of boundary conditions follows from the interplay between the stress constraint (1.1d), (1.1e) and the unrelaxed boundary conditions (1.5) for the viscous approximations.

Concerning the lower energy–dissipation inequality, the difficulty is related to the lack of lower semicontinuity of the term

$$\int_0^t \int_\Omega H(\dot{p}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\partial\Omega} S\dot{u}_\varepsilon \cdot \dot{u}_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s$$

with respect to the weak convergence in the energy space. Using the approximate boundary condition, it can be rewritten as

$$\int_0^t \int_\Omega H(\dot{p}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_0^t \int_{\partial\Omega} S\dot{u}_\varepsilon \cdot \dot{u}_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s + \frac{1}{2}\int_0^t \int_{\partial\Omega} S^{-1}(\sigma_\varepsilon\nu)\cdot(\sigma_\varepsilon\nu)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s + o(1). \quad (1.6)$$

Here, the approximate displacement $u_\varepsilon(t) \overset{*}{\rightharpoonup} u(t)$ weakly* in $BD(\Omega)$ with a trace which is bounded in $L^2(\partial\Omega;\mathbb{R}^n)$. Unfortunately, the trace operator in $BD(\Omega)$ is not continuous with respect to the weak* convergence in that space (see [7]) so that $u_\varepsilon(t) \rightharpoonup w(t)$ weakly in $L^2(\partial\Omega;\mathbb{R}^n)$ for some $w(t)$ which differs from the trace of $u(t)$. Passing to the lower limit in (1.6), owing to standard lower

semicontinuity results for convex functionals of measures, yields a lower bound of the form

$$\int_0^t \int_\Omega H(\dot p)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\partial\Omega} H((\dot w-\dot u)\odot\nu)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s \\ + \frac12\int_0^t\int_{\partial\Omega} S\dot w\cdot\dot w\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s + \frac12\int_0^t\int_{\partial\Omega} S^{-1}(\sigma\nu)\cdot(\sigma\nu)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s,$$

where the first integral should be interpreted as a convex functional of a measure. This lower bound suggests to define the following infimal convolution boundary function

$$\psi(x,z) = \inf_{z'\in\mathbb{R}^n}\left\{\frac12 S(x)z'\cdot z' + H((z'-z)\odot\nu(x))\right\}, \tag{1.7}$$

which leads to the lower bound

$$\int_0^t \int_\Omega H(\dot p)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\partial\Omega} \psi(x,\dot u)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s + \frac12\int_0^t\int_{\partial\Omega} S^{-1}(\sigma\nu)\cdot(\sigma\nu)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s \\ \le \liminf_{\varepsilon\to0}\left\{\int_0^t\int_\Omega H(\dot p_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\partial\Omega} S\dot u_\varepsilon\cdot\dot u_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\right\}. \tag{1.8}$$

In particular, this lower semicontinuity result improves the analogue result [8, Proposition 5.1] in the scalar case, where the $\mathcal{C}^1$ regularity of the boundary was needed to use the optimal constant in the trace inequality in BV (see [4]). The lower bound (1.8) is used to infer the lower inequality in the limit energy–dissipation balance (4.5).

As for the upper energy–dissipation inequality, the formal argument leading to the missing energy inequality is very simple and essentially rests on both convexity inequalities:

$$H(\dot p(t))\ge\sigma(t):\dot p(t)\text{ in }\Omega,\quad \psi(x,\dot u(t)) + \frac12 S^{-1}(\sigma(t)\nu)\cdot(\sigma(t)\nu)\ge-(\sigma(t)\nu)\cdot\dot u(t)\text{ on }\partial\Omega$$

together with an integration by part formula. In our case, the first inequality can be proven to be satisfied in the sense of measures, while the validity of the second one depends on whether $\sigma(t)\nu\in K\nu$ on $\partial\Omega$. This property is known to be true in the case where $\mathbf{K}$ is cylindrical and bounded in the direction of deviatoric matrices (see [23, Theorem 3.5]). The generalization to any arbitrary convex set $\mathbf{K}$ depends on the possibility to approximate the space of admissible stresses

$$\left\{\sigma\in L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}})\colon\ \operatorname{div}\sigma\in L^2(\Omega;\mathbb{R}^n),\ \sigma\nu\in L^2(\partial\Omega;\mathbb{R}^n)\right\}$$

by smooth functions. The main difficulty is related to the fact that the normal trace $\sigma\nu$ of $\sigma\in L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}})$ with $\operatorname{div}\sigma\in L^2(\Omega;\mathbb{R}^n)$ is canonically defined as an element of $H^{-\frac12}(\partial\Omega;\mathbb{R}^n)$, so that any approximation will have a normal trace strongly converging in that space instead of $L^2(\partial\Omega;\mathbb{R}^n)$ as required. The desired approximation ensuring all properties is known from [14] when σ is vector–valued (instead of matrix-valued), where it is proven the density of smooth functions in the space $\{z\in L^2(\Omega;\mathbb{R}^n)\colon \operatorname{div} z\in L^2(\Omega),\ z\cdot\nu\in L^2(\partial\Omega)\}$, which naturally arises in the study of Maxwell's equations. The proof relies on an abstract regularity result [28] for nonhomogeneous Neumann problem which falls outside the scope of standard elliptic regularity. Unfortunately, up to our knowledge, there is no analogue of that result for the symmetric matrix-valued case. Even assuming the validity of both convexity formulas, we then need to be able to apply a generalized integration by parts formula, which is the crucial tool in [8] (see [3]). In our case, again, the validity of such an integration by parts formula would again equire to prove a density result as above. We overcome the lack of integration by parts formula by proving an integral version of the formula (1.7) to obtain $\int_{\partial\Omega}\psi(x,\dot u(t))\,\mathrm{d}\mathcal{H}^{n-1}$ by global minimization over a set of smooth functions w. Then we extend the measure $\dot p(t)$, defined only on Ω, up to the boundary by a boundary term of the form $(w-\dot u(t))\odot\nu\mathcal{H}^{n-1}\llcorner\partial\Omega$ where w is an arbitrary smooth function. We are now in a situation very similar to plasticity models with a Dirichlet boundary condition given by w. We can in particular use the results in [10], more precisely the generalized convexity inequality $H(\dot p(t))\ge[\sigma(t):\dot p(t)]$ in the sense of measures in $\overline{\Omega}$, and the generalized integration by parts formula for fixed w proven therein. The desired upper inequality is then obtained by minimizing with respect to w.

A last issue to overcome is the uniqueness of variational solutions. Again the formal argument rests on taking the difference of the equations of motion associated to two different solutions, multiply by the difference of the velocities and integrate by parts. It usually leads to a comparison

principle between solutions, known as Kato inequality in the context of hyperbolic equations. Once again, we are stucked by the nonvalidity of the generalized integration by parts formula. We use here a variational argument based on the strict convexity of the total energy, which, to our knowledge, was never used in such an evolutionary context.

Finally, we show that both notions of entropic–dissipative solutions and variational solutions are actually identical. The implication that variational solutions are entropic–dissipative ones is quite easy and relies on the visco–plastic approximation. For fixed ε, the formal operations leading to (1.3a) can be fully justified and then we pass to the limit as $\varepsilon \to 0$. The converse implication is more involved and is inspired from [8] except for the derivation of the relaxed boundary condition. In [8], a further C^2 regularity property of Ω is used to infer that any vector field $z \in L^\infty(\Omega;\mathbb{R}^n)$ with $\operatorname{div} z \in L^2(\Omega)$ has a normal trace $z\cdot\nu \in L^\infty(\partial\Omega)$ which can be recovered as the weak* limit in L^∞ of $z\cdot\nu(x)$ as the point $x\in\Omega$ tends to the boundary (see [13, Theorem 2.2]). This results strongly uses the fact that z belongs to $L^\infty(\Omega;\mathbb{R}^n)$, which is not the case anymore in our (matrix-valued) situation since $\sigma(t)\in\mathbf{K}$ and $\mathbf{K}$ is not bounded. In order to recover the relaxed boundary conditions, we use the entropic inequality (1.3a) to ensure that the normal trace $\sigma(t)\nu$ in actually a bounded measure on $\partial\Omega$ (instead of just a distribution in $H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^n)$), which allows us to localize the inequality on $\partial\Omega$. Then, since the term $\sigma(t)\nu$ is multiplied by an arbitrary vector, and all other terms in this inequality are absolutely continuous with respect to $\mathcal{H}^{n-1}\llcorner\partial\Omega$, we infer that the singular part of the measure $\sigma(t)\nu$ vanishes, and we obtain so that $\sigma\nu\in L^2(\partial\Omega;\mathbb{R}^n)$. Finally the relaxed boundary condition is obtained as a consequence of the energy–dissipation balance and the uniqueness result proved for variational solutions. This new approach allows us to weaken the C^2 regularity hypothesis done in [8] into a Lipschitz regularity.

The paper is organized as follows. In Section 2, we introduce the main notation and recall basic facts used throughout this work. Section 3 is devoted to describe the model of small strain dynamic perfect plasticity and its reformulations as a constrained Friedrichs' system. We introduce the notion of entropic solution and derive all admissible dissipative conditions. In Section 4, we prove the existence and uniqueness of variational solutions to the dynamic perfectly plastic model endowed with the dissipative boundary condition. Finally, in Section 5 we prove that both notions of variational and entropic–dissipative solutions are equivalent, and, as a byproduct, the well posedness of the entropic–dissipative formuation.

2. Notation and preliminaries

2.1. **Linear algebra.** If a and $b\in\mathbb{R}^n$, we write $a\cdot b := \sum_{i=1}^n a_i b_i$ for the Euclidean scalar product, and we denote the corresponding norm by $|a| := \sqrt{a\cdot a}$.

For any $m,n\in\mathbb{N}$, we denote by $\mathbb{M}^{m\times n}$ the space of $m\times n$ matrices, and by $\mathbb{M}^n_{\rm sym}$, $\mathbb{M}^n_D$, $\mathbb{M}^{n,+}_{\rm sym}$, and $\mathbb{M}^n_{\rm skew}$ the spaces of $n\times n$ matrices which are symmetric, symmetric deviatoric, symmetric positive definite, and skew symmetric, respectively. Given a matrix $A\in\mathbb{M}^{m\times n}$, we denote by $\operatorname{Ker} A$ its kernel, by $\operatorname{Im} A$ its range, and by $\operatorname{rk} A$ its rank. If $m=n$, then $\det A$ stands for the determinant of $A\in\mathbb{M}^n$ and $\operatorname{tr} A$ denotes its trace. The space $\mathbb{M}^n$ is endowed with the Frobenius scalar product $A:B := \operatorname{tr}(A^TB)$ and with the corresponding Frobenius norm $|A| := \sqrt{A:A}$. The identity and zero matrices in $\mathbb{M}^n$ are denoted by I_n and $\mathbf{0}_n$. Finally, if $a\in\mathbb{R}^m$ and $b\in\mathbb{R}^n$, we denote their tensor product by $a\otimes b := ab^T\in\mathbb{M}^{m\times n}$. If $m=n$, the symmetric tensor product between a and $b\in\mathbb{R}^n$ is defined by $a\odot b = (a\otimes b + b\otimes a)/2\in\mathbb{M}^n_{\rm sym}$.

2.2. **Convex analysis.** We recall several definitions and basic facts from convex analysis (see [37]). Let $(V,\langle\cdot,\cdot\rangle)$ be an Euclidean space $\psi: V\to[0,+\infty]$ be a proper function (i.e. not identically $+\infty$). The convex conjugate of ψ is defined as

$$\psi^*(q) := \sup_{z\in V}\big\{\langle q,z\rangle - \psi(z)\big\} \qquad \text{for all } q\in V,$$

which is a convex and lower semicontinuous function. In particular, if $C\subset V$ is a convex set, we define the indicator function I_C of C as $\mathrm{I}_C := 0$ in C and $+\infty$ otherwise. The convex conjugate $(\mathrm{I}_C)^*$ of I_C is called the support function of C.

If $\phi_1,\phi_2: V\to[0,+\infty]$ are proper convex functions, then the infimal convolution of ϕ_1 and ϕ_2 is defined as

$$(\phi_1\,\square\,\phi_2)(z) := \inf_{z'\in V}\big\{\phi_1(z-z')+\phi_2(z')\big\},$$

which turns out to be a convex function. It can be shown that

$$\phi_1 \,\square\, \phi_2 = (\phi_1^* + \phi_2^*)^* .$$

2.3. **Measures.** The Lebesgue measure in $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$, and the $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}^{n-1}$. If $X \subset \mathbb{R}^n$ is a Borel set and Y is an Euclidean space, we denote by $\mathcal{M}_b(X;Y)$ the space of Y-valued bounded Radon measures in X endowed with the norm $\|\mu\| := |\mu|(X)$, where $|\mu|$ is the variation of the measure μ. If $Y = \mathbb{R}$ we simply write $\mathcal{M}_b(X)$ instead of $\mathcal{M}_b(X;\mathbb{R})$, and we denote by $\mathcal{M}_b^+(X)$ the cone of all nonnegative bounded Radon measures.

If the relative topology of X is locally compact, by Riesz representation theorem, $\mathcal{M}_b(X;Y)$ can be identified with the dual space of $C_0(X;Y)$, the space of continuous functions $\varphi : X \to Y$ such that $\{|\varphi| \geq \varepsilon\}$ is compact for every $\varepsilon > 0$. The weak* topology of $\mathcal{M}_b(X;Y)$ is defined using this duality.

Let $\mu \in \mathcal{M}_b(X;Y)$ and $f : Y \to [0,+\infty]$ be a convex, positively one-homogeneous function. Using the theory of convex functions of measures developed in [25, 18, 19], we introduce the nonnegative Radon measure $f(\mu) \in \mathcal{M}_b^+(X)$, defined for every Borel set $A \subset X$ by

$$f(\mu)(A) := \sup \left\{ \sum_{i=1}^m f(\mu(A_i)) \right\}$$

where the supremum is taken over all $m \in \mathbb{N}$ and all finite Borel partitions $\{A_1, \dots, A_m\}$ of A. In addition, it can be established that

$$f(\mu) = f\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}\right) |\mu| ,$$

where $\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}$ stands for the Radon-Nikodým derivative of μ with respect to $|\mu|$.

2.4. **Functional spaces.** We use standard notation for Lebesgue spaces (L^p) and Sobolev spaces ($W^{s,p}$ and $H^s = W^{s,2}$).

The space of functions of bounded deformation is defined by

$$BD(\Omega) = \{u \in L^1(\Omega;\mathbb{R}^n) : \mathrm{E}u \in \mathcal{M}_b(\Omega;\mathbb{M}^n_{\mathrm{sym}})\} ,$$

where $\mathrm{E}u := (Du + Du^T)/2$ stands for the distributional symmetric gradient of u. We recall (see [41, 7]) that, if Ω has Lipschitz boundary, every function $u \in BD(\Omega)$ admits a trace, still denoted u, in $L^1(\partial\Omega;\mathbb{R}^n)$ such that the integration by parts formula holds: for all $\varphi \in C^1(\overline{\Omega};\mathbb{M}^n_{\mathrm{sym}})$,

$$\int_{\partial\Omega} u \cdot (\varphi\nu)\, \mathrm{d}\mathcal{H}^{n-1} = \int_\Omega \operatorname{div}\varphi \cdot u \,\mathrm{d}x + \int_\Omega \varphi : \mathrm{dE}u .$$

Note that the trace operator is continuous with respect to the strong convergence of $BD(\Omega)$ but not with respect to the weak* convergence in $BD(\Omega)$.

Let us define

$$H(\operatorname{div},\Omega) = \{\sigma \in L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}}) : \operatorname{div}\sigma \in L^2(\Omega;\mathbb{R}^n)\} .$$

If Ω has Lipschitz boundary, for any $\sigma \in H(\operatorname{div},\Omega)$ we can define the normal trace $\sigma\nu$ as an element of $H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^n)$ (cf. e.g. [41, Theorem 1.2, Chapter 1]) by setting

$$\langle \sigma\nu, \psi\rangle_{\partial\Omega} := \int_\Omega \psi \cdot \operatorname{div}\sigma \,\mathrm{d}x + \int_\Omega \sigma : \mathrm{E}\psi \,\mathrm{d}x . \tag{2.1}$$

for every $\psi \in H^1(\Omega;\mathbb{R}^n)$.

Following [29, 23, 10], we define a generalized notion of stress/strain duality as follows.

Definition 2.1. Let $\sigma \in H(\operatorname{div},\Omega)$ and $p \in \mathcal{M}_b(\overline{\Omega};\mathbb{M}^n_{\mathrm{sym}})$ be such that $\mathrm{E}u = e + p$ in Ω and $p = (w-u)\odot\nu\mathcal{H}^{n-1}$ on $\partial\Omega$ for some $u \in BD(\Omega)\cap L^2(\Omega;\mathbb{R}^n)$, $e \in L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}})$ and $w \in H^1(\Omega;\mathbb{R}^n)$. We define the first order distribution $[\sigma : p] \in \mathcal{D}'(\mathbb{R}^n)$ by

$$\langle [\sigma : p], \varphi\rangle := -\int_\Omega \varphi\sigma : (e - \mathrm{E}w)\,\mathrm{d}x - \int_\Omega (u-w)\cdot\operatorname{div}\sigma\,\varphi\,\mathrm{d}x - \int_\Omega \sigma : \big((u-w)\odot\nabla\varphi\big)\,\mathrm{d}x$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$.

In the sequel we will be interested in stresses $\sigma \in H(\mathrm{div},\Omega)$ taking values in a given closed and convex subset $\mathbf{K}$ of $\mathbb{M}^n_{sym}$ containing 0 in its interior. We define the set of all plastically admissible stresses by

$$\mathcal{K}(\Omega) := \{\sigma \in H(\mathrm{div},\Omega)\colon\ \sigma(x) \in \mathbf{K} \text{ for a.e. } x \in \Omega\}$$

which defines a closed and convex subset of the space $H(\mathrm{div},\Omega)$.

If Ω has a Lipschitz boundary, $\sigma \in \mathcal{K}(\Omega)$ and $H(p)$ is a finite measure, where H is the support function of $\mathbf{K}$, then, using an approximation result for σ by smooth functions (see [15, Lemma 2.3]) as well as the integration by parts formula in $BD(\Omega)$ (see [7, Theorem 3.2]), we can show as in [10, Section 2] that $[\sigma\colon p]$ may be extended to a bounded Radon measure in $\overline{\Omega}$, i.e. $[\sigma\colon p] \in \mathcal{M}_b(\overline{\Omega};\mathbb{M}^n_{\mathrm{sym}})$, with

$$H(p) \geq [\sigma\colon p] \quad \text{in } \mathcal{M}^+_b(\overline{\Omega})\,. \tag{2.2}$$

Note also that if $\varphi \in C^1_c(\Omega)$, thanks to the integration by parts formula in $H^1(\Omega;\mathbb{R}^n)$ the expression of duality reduces to

$$\langle [\sigma\colon p], \varphi\rangle = -\int_\Omega \varphi\sigma\colon e\,\mathrm{d}x - \int_\Omega u\cdot \mathrm{div}\,\sigma\,\varphi\,\mathrm{d}x - \int_\Omega \sigma\colon (u\odot\nabla\varphi)\,\mathrm{d}x\,. \tag{2.3}$$

3. Description of the model

3.1. **Small strain dynamic perfect plasticity.** Let us consider a bounded open set $\Omega \subset \mathbb{R}^n$ (in dimension $n = 2$ or 3) which stands for the reference configuration of an elasto-plastic body. We will work in the framework of small strain elasto-plasticity where the natural kinematic variable is the displacement field $u : \Omega \times (0,T) \to \mathbb{R}^n$ (or the velocity field $v := \dot{u}$). In small strain elasto-plasticity, the linearized strain tensor $\mathrm{E}u := (\mathrm{D}u + \mathrm{D}u^T)/2$ decomposes additively in the following form:

$$\mathrm{E}u = e + p,$$

where $e : \Omega \times (0,T) \to \mathbb{M}^n_{\mathrm{sym}}$ is the elastic strain and $p : \Omega \times (0,T) \to \mathbb{M}^n_{\mathrm{sym}}$ the plastic strain. The elastic strain is related to the Cauchy stress tensor $\sigma : \Omega \times (0,T) \to \mathbb{M}^n_{\mathrm{sym}}$ by means of Hooke's law $\sigma := \mathbf{A}e$, where $\mathbf{A}$ is the symmetric fourth order elasticity tensor. In the sequel we will assume that $\mathbf{A}$ is isotropic, which means that

$$\mathbf{A}e = \lambda(\mathrm{tr}\,e)\mathrm{I}_n + 2\mu e\,,$$

where $\mu > 0$ and $2\mu + n\lambda > 0$. In a dynamical framework and in the presence of an external body load $f : \Omega \times (0,T) \to \mathbb{R}^n$, the equation of motion writes

$$\ddot{u} - \mathrm{div}\,\sigma = f \quad \text{in } \Omega \times (0,T)\,.$$

Perfect plasticity is characterized by the existence of a fixed closed and convex subset $\mathbf{K}$ of $\mathbb{M}^n_{\mathrm{sym}}$ with non empty interior, in which the Cauchy stress tensor is constrained to stay:

$$\sigma \in \mathbf{K}\,.$$

If σ lies inside the interior of $\mathbf{K}$, the material behaves elastically, so that unloading will bring back the body into its initial configuration ($\dot{p} = 0$). On the other hand, if σ reaches the boundary of $\mathbf{K}$ (called the yield surface), a plastic flow may develop, so that, after unloading, a non-trivial permanent plastic strain p will remain. Its evolution is described by means of the flow rule and is expressed with the Prandtl-Reuss law

$$\dot{p} \in N_{\mathbf{K}}(\sigma)\,,$$

where $N_{\mathbf{K}}(\sigma) = \partial \mathrm{I}_{\mathbf{K}}(\sigma)$ is the normal cone to $\mathbf{K}$ at σ. Note that the flow rule can be equivalently written as

$$\sigma\colon \dot{p} = H(\dot{p})\,,$$

where $H(q) = \sup_{\tau\in\mathbf{K}} \tau\colon q$ is the support function of $\mathbf{K}$, which expresses Hill's principle of maximum plastic work.

In summary, the problem of dynamical perfect plasticity consists in finding a displacement field $u\colon \Omega\times(0,T) \to \mathbb{R}^n$, a Cauchy stress $\sigma\colon \Omega\times(0,T) \to \mathbb{M}^n_{\mathrm{sym}}$, an elastic strain $e\colon \Omega\times(0,T) \to \mathbb{M}^n_{\mathrm{sym}}$

and a plastic strain $p\colon \Omega \times (0,T) \to \mathbb{M}^n_{\rm sym}$ such that in $\Omega \times (0,T)$ the following hold:

$$\begin{cases} \ddot{u} - \operatorname{div} \sigma = f\,; \\ \mathrm{E}u = e + p\,; \\ \sigma = \mathbf{A}e \in \mathbf{K}\,; \\ \sigma : \dot{p} = H(\dot{p})\,. \end{cases} \tag{3.1}$$

To formulate the Cauchy problem, we add initial conditions of the form

$$(u(0), \dot{u}(0), e(0), p(0)) = (u_0, v_0, e_0, p_0)\,. \tag{3.2}$$

Eventually, in order to close the initial-boundary value problem, we have also to add suitable boundary conditions, whose derivation will be the object of the following subsections.

3.2. **Entropic formulation of the model.** The object of this subsection is to show that, formally, the system of dynamic elasto–plasticity can be reformulated in a framework similar to initial–boundary value conservations laws, provided the boundary conditions are compatible with the hyperbolic structure of the problem. We will first recast (3.1) as a constrained Friedrichs' system as in [20]. Then, defining a suitable notion of so-called dissipative boundary conditions introduced in [35] and taking its origin in the seminal paper [24], we will in turn reformulate the problem as infinitely many nonlinear inequalities involving entropies as for conservations laws. The advantage of this new formulation is that, as in [36], it involves a boundary term containing the information of the boundary condition without appealing to any notion of trace of the solution on the boundary. In particular, it allows one to define a weaker notion of solutions in a (Lebesgue) functional space which is larger than the energy space provided by variational methods.

3.2.1. *Dynamic perfect plasticity as a constrained Friedrichs' system.* We first perform formal operations on the system (3.1) to write it in a fashion similar to that of linear, symmetric hyperbolic systems called Friedrichs' systems. In this paragraph we actually show that the first three conditions in (3.1) correspond to consider a "constrained Friedrichs' system" of the form

$$\partial_t U + \sum_{i=1}^{n} A_i \partial_i U + P = F\,,$$

where each line of the unknown $U : \Omega \times (0,T) \to \mathbb{R}^d$ ($d = n + \frac{n(n+1)}{2}$) is a linear combination of the n components of the velocity $v = \dot{u}$ and of the $\frac{n(n+1)}{2}$ components of the (symmetric) Cauchy stress σ (see also [27] in the case elastodynamics). For $i \in \{1, \dots, n\}$, the matrices $A_i \in \mathbb{M}^d_{\rm sym}$ are constants matrices whose coefficients only depend on the Lamé coefficients λ and μ, and the vectors P and F result from the plastic strain $\dot{p}$ and the source term f, respectively. For simplicity, we will perform our computations only in the case $n = 3$.

Let us define the following (column) vectors representing, respectively, the velocity, the Cauchy stress and the linearized strain:

$$\begin{cases} V := (\dot{u}_1, \dot{u}_2, \dot{u}_3)^T \in \mathbb{R}^3\,, \\ \Sigma := (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23})^T \in \mathbb{R}^6\,, \\ Y := (\partial_1 \dot{u}_1, \partial_2 \dot{u}_2, \partial_3 \dot{u}_3, \partial_1 \dot{u}_2 + \partial_2 \dot{u}_1, \partial_1 \dot{u}_3 + \partial_3 \dot{u}_1, \partial_2 \dot{u}_3 + \partial_3 \dot{u}_2)^T \in \mathbb{R}^6\,. \end{cases} \tag{3.3}$$

We also introduce the vectors corresponding to the initial velocity and the initial stress:

$$V_0 = ((v_0)_1, (v_0)_2, (v_0)_3)^T \in \mathbb{R}^3, \quad \Sigma_0 := ((\sigma_0)_{11}, (\sigma_0)_{22}, (\sigma_0)_{33}, (\sigma_0)_{12}, (\sigma_0)_{13}, (\sigma_0)_{23})^T \in \mathbb{R}^6.$$

First of all, we have that

$$Y = -\sum_{i=1}^{3} (B_i)^T \partial_i V\,, \tag{3.4}$$

where the matrices B_1, B_2 and $B_3 \in \mathbb{M}^{3\times 6}$ are given by

$$B_1 = -\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$B_2 = -\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$B_3 = -\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Employing such matrices B_i we express the equation of motion $\ddot{u} - \mathrm{div}\sigma = f$ as

$$\partial_t V + \sum_{i=1}^{3} B_i \partial_i \Sigma = f\,, \tag{3.5}$$

and the constitutive law $\dot{\sigma} = \mathbf{A}\dot{e} = \lambda\,(\mathrm{div}\,\dot{u})\,\mathrm{I}_3 + 2\mu \mathrm{E}\dot{u} - \lambda\,(\mathrm{tr}\,\dot{p})\,\mathrm{I}_3 - 2\mu\dot{p}$ as

$$\partial_t \Sigma + CQ = CY\,, \tag{3.6}$$

where

$$Q := (\dot{p}_{11}, \dot{p}_{22}, \dot{p}_{33}, 2\dot{p}_{12}, 2\dot{p}_{13}, 2\dot{p}_{23})^T \in \mathbb{R}^6\,,$$

and

$$C := \begin{pmatrix} \tilde{C} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mu\,\mathrm{I}_3 \end{pmatrix} \in \mathbb{M}^{6,+}_{\mathrm{sym}}\,, \qquad \tilde{C} := \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda \\ \lambda & \lambda + 2\mu & \lambda \\ \lambda & \lambda & \lambda + 2\mu \end{pmatrix} \in \mathbb{M}^{3,+}_{\mathrm{sym}}\,.$$

Collecting (3.4), (3.5) and (3.6), we obtain

$$\widetilde{A}_0 \partial_t \widetilde{U} + \sum_{i=1}^{3} \widetilde{A}_i \partial_i \widetilde{U} + \widetilde{Q} = F, \quad \widetilde{U} \in \widetilde{\mathbf{C}} \tag{3.7}$$

where

$$\widetilde{U} := \begin{pmatrix} V \\ \Sigma \end{pmatrix} \in \mathbb{R}^9\,, \quad \widetilde{A}_0 := \begin{pmatrix} \mathrm{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \tilde{C}^{-1} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \frac{1}{\mu}\,\mathrm{I}_3 \end{pmatrix} \in \mathbb{M}^{9,+}_{\mathrm{sym}}\,, \quad \widetilde{A}_i := \begin{pmatrix} \mathbf{0}_3 & B_i \\ B_i^T & \mathbf{0}_6 \end{pmatrix} \in \mathbb{M}^{9,+}_{\mathrm{sym}}\,, \tag{3.8}$$

$$\widetilde{Q} = \begin{pmatrix} 0_{\mathbb{R}^3} \\ Q \end{pmatrix}, \quad F = \begin{pmatrix} f \\ 0_{\mathbb{R}^6} \end{pmatrix},$$

and $\widetilde{\mathbf{C}} \subset \mathbb{R}^9$ is the set of all vectors $\widetilde{U} = (V, \Sigma)^T \in \mathbb{R}^9$ such that

$$V \in \mathbb{R}^3, \quad \begin{pmatrix} \Sigma_1 & \Sigma_4 & \Sigma_5 \\ \Sigma_4 & \Sigma_2 & \Sigma_6 \\ \Sigma_5 & \Sigma_6 & \Sigma_3 \end{pmatrix} \in \mathbf{K}\,,$$

which is a closed and convex set containing $0_{\mathbb{R}^9}$ in its interior by the properties of $\mathbf{K}$.

In order to get rid of the matrix $\widetilde{A}_0$ in front of the time derivative in (3.7), we set $A_i := \widetilde{A}_0^{-\frac{1}{2}} \widetilde{A}_i \widetilde{A}_0^{-\frac{1}{2}}$ for all $i \in \{1, 2, 3\}$, and

$$U := \widetilde{A}_0^{\frac{1}{2}} \widetilde{U}, \quad P := \widetilde{A}_0^{-\frac{1}{2}} \begin{pmatrix} 0_{\mathbb{R}^3} \\ Q \end{pmatrix}, \quad U_0 = \widetilde{A}_0^{\frac{1}{2}} \begin{pmatrix} V_0 \\ \Sigma_0 \end{pmatrix}.$$

Using that $F = \widetilde{A}_0^{-\frac{1}{2}} F$, we can recast (3.7) in the form

$$\begin{cases} \partial_t U + \displaystyle\sum_{i=1}^{3} A_i\, \partial_i U + P = F\,, \\ U \in \mathbf{C}\,, \\ U(0) = U_0\,, \end{cases} \tag{3.9}$$

where $\mathbf{C} = \widetilde{A}_0^{\frac{1}{2}} \widetilde{\mathbf{C}}$, which is a closed and convex subset of $\mathbb{R}^9$ containing $0_{\mathbb{R}^9}$ in its interior. Note that from the expression of $\widetilde{A}_0$, we have that

$$\widetilde{A}_0^{\pm\frac{1}{2}} = \begin{pmatrix} \mathrm{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \tilde{C}^{\mp\frac{1}{2}} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mu^{\mp\frac{1}{2}}\,\mathrm{I}_3 \end{pmatrix} \tag{3.10}$$

and, for all $i \in \{1,2,3\}$, denoting by $B_i = (B_i'|B_i'')$ with B_i' and $B_i'' \in \mathbb{M}^3$, then the matrices A_i may be expressed as

$$A_i = \begin{pmatrix} \mathbf{0}_3 & B_i' \widetilde{C}^{\frac{1}{2}} & \sqrt{\mu} B_i'' \\ \widetilde{C}^{\frac{1}{2}} (B_i')^T & \mathbf{0}_3 & \mathbf{0}_3 \\ \sqrt{\mu} (B_i'')^T & \mathbf{0}_3 & \mathbf{0}_3 \end{pmatrix}.$$

Remark 3.1. We observe that $\tilde{C}$ may be diagonalized as

$$\tilde{C} = RDR^T\,,$$

where $R \in SO(3)$ and D are given by

$$R = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \qquad D = \operatorname{diag}(2\mu, 2\mu, 2\mu + 3\lambda)\,.$$

Thus, for any power $\gamma \in \mathbb{R}$, a direct computation gives that

$$\widetilde{C}^\gamma = RD^\gamma R^T = \begin{pmatrix} \alpha_\gamma & \beta_\gamma & \beta_\gamma \\ \beta_\gamma & \alpha_\gamma & \beta_\gamma \\ \beta_\gamma & \beta_\gamma & \alpha_\gamma \end{pmatrix},$$

for

$$\alpha_\gamma = \frac{2}{3}(2\mu)^\gamma + \frac{1}{3}(2\mu + 3\lambda)^\gamma\,, \qquad \beta_\gamma = -\frac{1}{3}(2\mu)^\gamma + \frac{1}{3}(2\mu + 3\lambda)^\gamma\,.$$

We now employ the fourth condition in (3.1) to infer an inequality useful in the subsequent analysis. We show that

$$P \cdot (U - \xi) \geq 0 \quad \text{for every } \xi \in \mathbf{C}\,. \tag{3.11}$$

In fact, writing $\xi = \widetilde{A}_0^{\frac{1}{2}} \widetilde{\xi}$ for some $\widetilde{\xi} = (z, \mathcal{T})^T \in \widetilde{\mathbf{C}}$ with

$$z \in \mathbb{R}^3, \quad \tau := \begin{pmatrix} \mathcal{T}_1 & \mathcal{T}_4 & \mathcal{T}_5 \\ \mathcal{T}_4 & \mathcal{T}_2 & \mathcal{T}_6 \\ \mathcal{T}_5 & \mathcal{T}_6 & \mathcal{T}_3 \end{pmatrix} \in \mathbf{K},$$

we get, using the symmetry of $\widetilde{A}_0^{\frac{1}{2}}$, that $P \cdot (U - \xi) = (\widetilde{A}_0^{-\frac{1}{2}} \widetilde{Q}) \cdot (\widetilde{A}_0^{\frac{1}{2}} \widetilde{U} - \widetilde{A}_0^{\frac{1}{2}} \widetilde{\xi}) = \widetilde{Q} \cdot (\widetilde{U} - \widetilde{\xi}) = Q \cdot (\Sigma - \mathcal{T})$. Then, by the fourth condition in (3.1), we have that $\sigma \colon p \geq \tau \colon p$, and since $\tau \in \mathbf{K}$, we get that $Q \cdot (\Sigma - \mathcal{T}) = \dot{p} \colon (\sigma - \tau) \geq 0$, so (3.11) holds true.

3.2.2. *Entropic formulation.* In the spirit of entropic formulations for conservations laws, we are going to reformulate (3.1), or (3.9) and (3.11), as infinitely many nonlinear inequalities involving entropies well adapted to the system. At this stage we need to impose suitable admissible boundary conditions following an approach introduced in [35, 8] and in the pioneering work [24]. We are formally interested in dissipative boundary conditions of the form

$$(A_\nu - M)U = 0 \quad \text{on } \partial\Omega \times (0,T)\,, \tag{3.12}$$

where (denoting by $\nu(x)$ the unit outer normal to $\partial\Omega$ at $x \in \partial\Omega$)

$$A_\nu(x) := \sum_{i=1}^3 \nu_i(x) A_i \in \mathbb{M}^9_{\mathrm{sym}}$$

and, for each $x \in \partial\Omega$, $M(x) \in \mathbb{M}^9_{\mathrm{sym}}$ satisfies

$$\begin{cases} M(x) \text{ is nonnegative,} \\ \operatorname{Ker} A_\nu(x) \subset \operatorname{Ker} M(x)\,, \\ \mathbb{R}^9 = \operatorname{Ker}\big(A_\nu(x) - M(x)\big) + \operatorname{Ker}\big(A_\nu(x) + M(x)\big)\,. \end{cases} \tag{3.13}$$

Taking the scalar product of the first equation in (3.9) with U yields

$$\frac{1}{2}\partial_t |U|^2 + \frac{1}{2}\sum_{i=1}^3 \partial_i (A_i U \cdot U) + P \cdot U = F \cdot U \tag{3.14}$$

while, for every constant vector $\xi \in \mathbf{C}$, taking the scalar product of (3.9) with ξ leads to

$$\partial_t(U\cdot\xi)+\sum_{i=1}^3\partial_i(A_iU\cdot\xi)+P\cdot\xi=F\cdot\xi. \tag{3.15}$$

Substracting (3.14) and (3.15), and using (3.11), leads to

$$\partial_t|U-\xi|^2+\sum_{i=1}^3\partial_i\big(A_i(U-\xi)\cdot(U-\xi)\big)\le 2F\cdot(U-\xi).$$

We now multiply the above inequality by a test function $\varphi\in C_c^\infty(\mathbb{R}^3\times\mathbb{R})$ with $\varphi\ge 0$, and integrate by parts to get that

$$\begin{aligned}&\int_{\mathbb{R}^+}\int_\Omega|U-\xi|^2\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t+\sum_{i=1}^3\int_{\mathbb{R}^+}\int_\Omega A_i(U-\xi)\cdot(U-\xi)\partial_i\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega|U_0-\xi|^2\varphi(0)\,\mathrm{d}x\\&\quad+2\int_{\mathbb{R}^+}\int_\Omega F\cdot(U-\xi)\varphi\,\mathrm{d}x\,\mathrm{d}t-\int_{\mathbb{R}^+}\int_{\partial\Omega}A_\nu(U-\xi)\cdot(U-\xi)\varphi\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t\ge 0\,.\end{aligned} \tag{3.16}$$

According to [35, Lemma 1], we have that

$$\mathbb{R}^9=\operatorname{Ker}A_\nu\oplus\big(\operatorname{Ker}(A_\nu-M)\cap\operatorname{Im}A_\nu\big)\oplus\big(\operatorname{Ker}(A_\nu+M)\cap\operatorname{Im}A_\nu\big).$$

For each $\xi\in\mathbb{R}^9$, we denote by $\xi^\pm$ the projection of ξ onto $\operatorname{Ker}(A_\nu\pm M)\cap\operatorname{Im}A_\nu$. Using the (strong) boundary condition (3.12), we have that $U\in\operatorname{Ker}(A_\nu-M)$, or still $U^+=0$. The algebraic conditions (3.13) together with [35, Lemma 1] thus yield

$$\begin{aligned}A_\nu(U-\xi)\cdot(U-\xi)&=-M(U-\xi)^+\cdot(U-\xi)^++M(U-\xi)^-\cdot(U-\xi)^-\\&=-M\xi^+\cdot\xi^++M(U-\xi)^-\cdot(U-\xi)^-\ge-M\xi^+\cdot\xi^+.\end{aligned}$$

Inserting in (3.16), we get that for all constant vectors $\xi\in\mathbf{C}$ and all test functions $\varphi\in C_c^\infty(\mathbb{R}^3\times\mathbb{R})$ with $\varphi\ge 0$, it holds that

$$\begin{aligned}&\int_{\mathbb{R}^+}\int_\Omega|U-\xi|^2\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t+\sum_{i=1}^3\int_{\mathbb{R}^+}\int_\Omega A_i(U-\xi)\cdot(U-\xi)\partial_i\varphi\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_\Omega|U_0-\xi|^2\varphi(0)\,\mathrm{d}x+2\int_{\mathbb{R}^+}\int_\Omega F\cdot(U-\xi)\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}^+}\int_{\partial\Omega}M\xi^+\cdot\xi^+\varphi\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t\ge 0\,.\end{aligned} \tag{3.17}$$

The previous family of inequalities defines a notion of entropic solutions $U\in L^2(\Omega\times\mathbb{R}^+;\mathbf{C})$ to the dynamical elasto–plastic problem in the spirit of constrained Friedrichs' systems in the whole space as in [20]. Note that in the presence of a boundary, our formulation is meaningful within a L^2 theory of Friedrichs' systems (as suggested by (3.17)) since the trace of U on the boundary $\partial\Omega\times\mathbb{R}^+$ is not involved in this definition (see also [36, 33] for an L^∞-theory of initial–boundary value conservation laws).

3.2.3. *Characterization of all admissible boundary matrices.* In order to derive the "dissipative" boundary conditions involved in our model, we need to characterize all admissible boundary matrices M. Since the problem is local, let $\nu\in\mathbb{S}^2$ be fixed, and consider the matrix A_ν given by

$$A_\nu=\sum_{i=1}^3\nu_iA_i.$$

A direct computation shows that

$$A_\nu=\begin{pmatrix}\mathbf{0}_3&A'_\nu&A''_\nu\\(A'_\nu)^T&\mathbf{0}_3&\mathbf{0}_3\\(A''_\nu)^T&\mathbf{0}_3&\mathbf{0}_3\end{pmatrix} \tag{3.18}$$

where, according to Remark 3.1,

$$A'_\nu=B'_\nu\widetilde{C}^{\frac12}=-\begin{pmatrix}\nu_1\,\alpha&\nu_1\,\beta&\nu_1\,\beta\\\nu_2\,\beta&\nu_2\,\alpha&\nu_2\,\beta\\\nu_3\,\beta&\nu_3\,\beta&\nu_3\,\alpha\end{pmatrix}\in\mathbb{M}^3\,,\quad A''_\nu=\sqrt{\mu}B''_\nu=-\sqrt{\mu}\begin{pmatrix}\nu_2&\nu_3&0\\\nu_1&0&\nu_3\\0&\nu_1&\nu_2\end{pmatrix}\in\mathbb{M}^3. \tag{3.19}$$

with

$$B'_\nu = \sum_{i=1}^{3} B'_i \nu_i = -\begin{pmatrix} \nu_1 & 0 & 0 \\ 0 & \nu_2 & 0 \\ 0 & 0 & \nu_3 \end{pmatrix}, \quad B''_\nu = \sum_{i=1}^{3} B''_i \nu_i = -\begin{pmatrix} \nu_2 & \nu_3 & 0 \\ \nu_1 & 0 & \nu_3 \\ 0 & \nu_1 & \nu_2 \end{pmatrix}$$

and

$$\alpha = \alpha_{\frac{1}{2}} = \frac{2}{3}\sqrt{2\mu} + \frac{1}{3}\sqrt{2\mu+3\lambda}\,, \qquad \beta = \beta_{\frac{1}{2}} = -\frac{1}{3}\sqrt{2\mu} + \frac{1}{3}\sqrt{2\mu+3\lambda}\,.$$

Let us start by proving some dimensional properties of A_ν.

Lemma 3.2. *For all $\nu \in \mathbb{S}^2$, we have*

$$\operatorname{rk}(A'_\nu | A''_\nu) = 3 \tag{3.20}$$

and

$$\operatorname{rk} A_\nu = 6\,, \qquad \dim \operatorname{Ker} A_\nu = 3. \tag{3.21}$$

Proof. We first note that

$$\det(A'_\nu) = -\nu_1\nu_2\nu_3(2\mu)\sqrt{3\lambda+2\mu}\,. \tag{3.22}$$

Then, (3.22) gives (3.20) if $\nu_i \neq 0$ for all $i \in \{1,2,3\}$. Otherwise, assume that $\nu_i = 0$ for some $i \in \{1,2,3\}$. Note that since $\nu \in \mathbb{S}^2$, then ν has at least one nonzero component. Since $\mu > 0$ and $3\lambda + 2\mu > 0$, then we have $\alpha \neq \beta$, $\alpha^2 \neq \beta^2$. Thus the nonzero lines of A'_ν are linearly independent, while the lines of A''_ν corresponding to the zero lines of A'_ν are linearly independent as well, which implies (3.20) also in that case.

Finally, (3.21) is a consequence of the structure of the matrix A_ν in (3.18) and the Rank Theorem. □

According to (3.20), in correspondence to $\nu \in \mathbb{S}^2$ there exists a matrix $C_\nu \in \mathbb{M}^9_{\rm sym}$ such that the fourth, fifth and sixth columns of $C_\nu A_\nu$ are linearly independent. We denote by

$$\widehat{A}_\nu := C_\nu A_\nu C_\nu^T = \begin{pmatrix} \mathbf{0}_3 & \widehat{A}'_\nu & \widehat{A}''_\nu \\ (\widehat{A}'_\nu)^T & \mathbf{0}_3 & \mathbf{0}_3 \\ (\widehat{A}''_\nu)^T & \mathbf{0}_3 & \mathbf{0}_3 \end{pmatrix}, \tag{3.23}$$

with $\widehat{A}'_\nu$, and $\widehat{A}''_\nu \in \mathbb{M}^3$, which operates a suitable change of both lines and columns of A_ν in such a way that

$$\operatorname{rk} \widehat{A}'_\nu = 3\,. \tag{3.24}$$

Notice that we may assume that $C_\nu = \mathrm{I}_9$ (i.e. $A_\nu = \widehat{A}_\nu$), if and only if $\nu_i \neq 0$ for all $i \in \{1,2,3\}$.

Remark 3.3. Since C_ν^T and C_ν^{-T} are permutation matrices which leave unchanged the first three components of a vector, we deduce that for any $U \in \mathbb{R}^9$,

$$(C_\nu^T U)_i = U_i = (C_\nu^{-T} U)_i \quad \text{for all } i \in \{1,2,3\}.$$

Let us now pass to the characterization of the admissible boundary matrices, fixing $x \in \partial\Omega$ in (3.13), so that we are given $M \in \mathbb{M}^9_{\rm sym}$ satisfying

$$\begin{cases} M \text{ is nonnegative,} \\ \operatorname{Ker} A_\nu \subset \operatorname{Ker} M\,, \\ \mathbb{R}^9 = \operatorname{Ker}(A_\nu - M) + \operatorname{Ker}(A_\nu + M)\,. \end{cases} \tag{3.25}$$

We write

$$M = \begin{pmatrix} M_1 & M_2 & M'_2 \\ (M_2)^T & M_3 & M'_3 \\ (M'_2)^T & (M'_3)^T & M''_3 \end{pmatrix}, \tag{3.26}$$

with M_1, M_3, $M''_3 \in \mathbb{M}^3_{\rm sym}$, M_2, M'_2 and $M'_3 \in \mathbb{M}^3$. Moreover, let

$$\widehat{M} := C_\nu M C_\nu^T = \begin{pmatrix} \widehat{M}_1 & \widehat{M}_2 & \widehat{M}'_2 \\ (\widehat{M}_2)^T & \widehat{M}_3 & \widehat{M}'_3 \\ (\widehat{M}'_2)^T & (\widehat{M}'_3)^T & \widehat{M}''_3 \end{pmatrix}, \tag{3.27}$$

with $\widehat{M}_1$, $\widehat{M}_3$, $\widehat{M}''_3 \in \mathbb{M}^3_{\rm sym}$, $\widehat{M}_2$, $\widehat{M}'_2$ and $\widehat{M}'_3 \in \mathbb{M}^3$. Notice that $\widehat{M}_1 = M_1$, by the properties of C_ν.

We have the following characterization of all admissible boundary matrices M.

Theorem 3.4. *Let $\nu \in \mathbb{S}^2$ and A_ν, $\widehat{A}_\nu$ be the corresponding matrices defined by* (3.18) *and* (3.23). *A matrix $M \in \mathbb{M}^9_{\rm sym}$ satisfies* (3.25)*, if and only if (recall* (3.27)*)*

$$\widehat{M}_1,\ \widehat{M}_3 \text{ are positive definite,} \tag{3.28a}$$

$$(\widehat{A}'_\nu)^T\,\widehat{M}_1^{-1}\widehat{M}_2 \in \mathbb{M}^3_{\rm skew},\quad \widehat{A}'_\nu\,\widehat{M}_3^{-1}(\widehat{M}_2)^T \in \mathbb{M}^3_{\rm skew}, \tag{3.28b}$$

$$\widehat{M}_3 = (\widehat{A}'_\nu)^T\,\widehat{M}_1^{-1}\widehat{A}'_\nu + (\widehat{M}_2)^T\widehat{M}_1^{-1}\widehat{M}_2,\quad \widehat{M}_1 = \widehat{A}'_\nu\,\widehat{M}_3^{-1}(\widehat{A}'_\nu)^T + \widehat{M}_2\widehat{M}_3^{-1}(\widehat{M}_2)^T, \tag{3.28c}$$

$$\widehat{M}'_2 = \widehat{M}_2(\widehat{A}'_\nu)^{-1}\widehat{A}''_\nu,\quad \widehat{M}'_3 = \widehat{M}_3(\widehat{A}'_\nu)^{-1}\widehat{A}''_\nu,\quad \widehat{M}''_3 = (\widehat{A}''_\nu)^T(\widehat{A}'_\nu)^{-T}\widehat{M}_3(\widehat{A}'_\nu)^{-1}\widehat{A}''_\nu, \tag{3.28d}$$

and

$$\operatorname{rk}(A_\nu + M) = \operatorname{rk}(A_\nu - M) = 3\,. \tag{3.28e}$$

We now subdivide the proof into several technical lemmas.

Lemma 3.5. *We have*

$$\operatorname{Ker} A_\nu = \operatorname{Ker}(A_\nu + M) \cap \operatorname{Ker}(A_\nu - M)\,. \tag{3.29}$$

In addition,

$$\operatorname{Ker}(A'_\nu|A''_\nu) \subset \operatorname{Ker}(M_2|M'_2) \cap \operatorname{Ker}(M_3|M'_3) \cap \operatorname{Ker}\left((M'_3)^T|M''_3\right). \tag{3.30}$$

Proof. As for (3.29), $\operatorname{Ker} A_\nu \subset \operatorname{Ker} M$ readily gives that $\operatorname{Ker} A_\nu \subset \operatorname{Ker}(A_\nu + M) \cap \operatorname{Ker}(A_\nu - M)$. On the other hand, if $U \in \operatorname{Ker}(A_\nu + M) \cap \operatorname{Ker}(A_\nu - M)$, then $(A_\nu + M)U = (A_\nu - M)U = 0$, so summing up both equalities yields $A_\nu\,U = 0$, hence $U \in \operatorname{Ker} A_\nu$.

Finally, we observe that if $U \in \operatorname{Ker} A_\nu$, then

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \in \operatorname{Ker} \begin{pmatrix} (A'_\nu)^T \\ (A''_\nu)^T \end{pmatrix}$$

which implies that $U_1 = U_2 = U_3 = 0$ according to (3.20) and the Rank Theorem. Thus $\operatorname{Ker} A_\nu = \{\mathbf{0}_3\} \times \operatorname{Ker}(A'_\nu|A''_\nu)$ and thanks to the decomposition (3.26) of the matrix M, the inclusion (3.30) follows from the second condition in (3.25). □

Remark 3.6. Notice that (3.28) hold with A'_ν, A''_ν, M_1, M_2, M_3 in place of $\widehat{A}'_\nu$, $\widehat{A}''_\nu$, $\widehat{M}_1$, $\widehat{M}_2$, $\widehat{M}_3$ if and only if $\nu_i \neq 0$ for all $i \in \{1,2,3\}$. Moreover, the admissibility conditions (3.25) are invariant if both $\widehat{A}_\nu$ and $\widehat{M}$ replace A_ν and M. This allows us, in order to ease the reading, to argue for a fixed $\nu \in \mathbb{S}^2$ with $\nu_i \neq 0$ for all $i \in \{1,2,3\}$, as we do in the following. The case with general ν is readily obtained by replacing A'_ν, A''_ν, M_1, M_2, M'_2, M_3, M'_3, M''_3 by $\widehat{A}'_\nu$, $\widehat{A}''_\nu$, $\widehat{M}_1$, $\widehat{M}_2$, $\widehat{M}'_2$, $\widehat{M}_3$, $\widehat{M}'_3$, $\widehat{M}''_3$.

From now on we assume that $\nu_i \neq 0$ for all $i \in \{1,2,3\}$. Since, by (3.24), $\operatorname{rk} A'_\nu = 3$, there holds

$$\operatorname{Ker} A_\nu = \operatorname{Vect}\left\{ \begin{pmatrix} 0_{\mathbb{R}^3} \\ (A'_\nu)^{-1}(A''_\nu)^{(j)} \\ -e_j \end{pmatrix} \right\}_{1\le j\le 3}, \tag{3.31}$$

where $E^{(j)}$ denotes the j-th column of a matrix E and e_j is the j-th vector of the canonical basis of $\mathbb{R}^3$. Using that $\operatorname{Ker} A_\nu \subset \operatorname{Ker} M$, we get that

$$M'_2 = M_2(A'_\nu)^{-1}A''_\nu,\quad M'_3 = M_3(A'_\nu)^{-1}A''_\nu,\quad M''_3 = (M'_3)^T(A'_\nu)^{-1}A''_\nu\,. \tag{3.32}$$

As a consequence, the matrix M will be entirely determined by the submatrices M_1, M_2 and M_3.

Lemma 3.7. *Let $\nu \in \mathbb{S}^2$ be such that $\nu_i \neq 0$ for all $i \in \{1,2,3\}$. The matrices M_1 and M_3 are nonnegative,*

$$\operatorname{rk} M_1 = \operatorname{rk} M_3 = 3, \tag{3.33}$$

and

$$\operatorname{rk}(A_\nu + M) = \operatorname{rk}(A_\nu - M) = 3\,. \tag{3.34}$$

Proof. By the first condition in (3.25), since M is non negative, we have that M_1 and M_3 are non negative. Next we only prove that $\operatorname{rk} M_3 = 3$ since the argument for M_1 is similar. Assume by contradiction that $\operatorname{rk} M_3 = m < 3$.

Step 1. Let us show that

$$\operatorname{rk}\begin{pmatrix} M_2 \\ M_3 \end{pmatrix} = m\,. \tag{3.35}$$

We first notice that

$$\operatorname{Ker}\begin{pmatrix} M_2 \\ M_3 \end{pmatrix} = \operatorname{Ker} M_2 \cap \operatorname{Ker} M_3 \subset \operatorname{Ker} M_3\,.$$

On the other hand, if $x \in \operatorname{Ker} M_3 \setminus \operatorname{Ker} M_2$, then for any $c \in \mathbb{R}$

$$M\begin{pmatrix} M_2 x \\ c\,x \\ 0 \end{pmatrix} \cdot \begin{pmatrix} M_2 x \\ c\,x \\ 0 \end{pmatrix} = M_1(M_2 x)\cdot M_2 x + 2c|M_2 x|^2\,,$$

which tends to $-\infty$ as $c \to -\infty$, in contradiction with the nonnegativity of M. This implies that $\operatorname{Ker} M_3 \subset \operatorname{Ker} M_2$, hence

$$\operatorname{Ker} M_3 = \operatorname{Ker} M_2 \cap \operatorname{Ker} M_3 = \operatorname{Ker}\begin{pmatrix} M_2 \\ M_3 \end{pmatrix}, \tag{3.36}$$

and thus, the Rank Theorem gives (3.35).

Step 2. Let us prove that

$$\operatorname{rk}\begin{pmatrix} M_2 \pm A'_\nu \\ M_3 \end{pmatrix} = 3\,. \tag{3.37}$$

Indeed, assume that there exists $x \in \mathbb{R}^3$ with $x \neq 0$ and

$$x \in \operatorname{Ker}\begin{pmatrix} M_2 \pm A'_\nu \\ M_3 \end{pmatrix}.$$

In particular, using (3.36),

$$x \in \operatorname{Ker} M_3 = \operatorname{Ker}\begin{pmatrix} M_2 \\ M_3 \end{pmatrix},$$

hence $x \in \operatorname{Ker} A'_\nu$, which is impossible since A'_ν is invertible. This yields

$$\operatorname{Ker}\begin{pmatrix} M_2 \pm A'_\nu \\ M_3 \end{pmatrix} = \{0\},$$

and thus (3.37) holds according to the Rank Theorem.

Step 3. Let us show that

$$\operatorname{rk}(A_\nu \pm M) \geq 6 - m\,. \tag{3.38}$$

We denote by

$$A_\pm := \begin{pmatrix} M_1 & M_2 \pm A'_\nu \\ (M_2 \pm A'_\nu)^T & M_3 \end{pmatrix} \in \mathbb{M}^6_{\mathrm{sym}},$$

and we observe that $\operatorname{rk}(A_\nu \pm M) \geq \operatorname{rk} A_\pm$. By (3.37), the last three lines of $A_\pm$ are linearly independent. Moreover, since $\operatorname{rk} M_3 = m$, using again (3.37) we get that at least $3 - m$ lines of $M_2 \pm A'_\nu$ are not contained in $\operatorname{Im} M_3$. We denote their indices by $j_1, \dots, j_{3-m} \in \{1, 2, 3\}$. Since the lines with indices $j_1, \dots, j_{3-m}$ of $M_2 \pm A'_\nu$ are not a linear combination of those of M_3, *a fortiori* the lines with indices $j_1, \dots, j_{3-m}$ of $A_\pm$ are not a linear combination of the last three lines of $A_\pm$. We deduce that the lines of $A_\pm$ with indices $j_1, \dots, j_{3-m}$, 4, 5 and 6 are linearly independent, and thus that $\operatorname{rk} A_\pm \geq 3 + (3 - m)$.

Step 4. We now conclude the proof of (3.33). By Lemma 3.5, we have that

$$\dim\Big(\operatorname{Ker}(A_\nu + M) \cap \operatorname{Ker}(A_\nu - M)\Big) = \dim \operatorname{Ker} A_\nu = 3\,,$$

while (3.38) together with the Rank Theorem imply that

$$\dim(\operatorname{Ker}(A_\nu \pm M)) = 9 - \operatorname{rk}(A_\nu \pm M) \leq 3 + m\,.$$

Then, we deduce that

$$\begin{aligned} \dim(\operatorname{Ker}(A_\nu + M) + \operatorname{Ker}(A_\nu - M)) &= \dim(\operatorname{Ker}(A_\nu + M)) + \dim(\operatorname{Ker}(A_\nu - M)) \\ &\quad -\dim(\operatorname{Ker}(A_\nu + M) \cap \operatorname{Ker}(A_\nu - M)) \\ &\leq 2(3+m) - 3 < 9\,, \end{aligned}$$

since $m < 3$, which is against the third condition in (3.25). This gives $m = 3$, and then completes the proof of (3.33) for M_3. The same holds for M_1.

Step 5. From the previous step, we infer that

$$\begin{cases} \dim(\mathrm{Ker}\,(A_\nu \pm M)) \le 6, \\ \dim(\mathrm{Ker}\,(A_\nu + M)) + \dim\,(\mathrm{Ker}\,(A_\nu - M)) - \dim\,(\mathrm{Ker}\,(A_\nu + M) \cap \mathrm{Ker}\,(A_\nu - M)) = 9, \\ \dim\big(\mathrm{Ker}\,(A_\nu + M) \cap \mathrm{Ker}\,(A_\nu - M)\big) = 3. \end{cases}$$

Thus, we deduce that $\dim(\mathrm{Ker}\,(A_\nu+M))+\dim\,(\mathrm{Ker}\,(A_\nu-M)) = 12$, hence $\dim\,(\mathrm{Ker}\,(A_\nu \pm M)) = 6$, leading to (3.34) owing to the Rank Theorem. □

Lemma 3.8. *Let $\nu \in \mathbb{S}^2$ be such that $\nu_i \neq 0$ for all $i \in \{1,2,3\}$. Then, $(A'_\nu)^T\, M_1^{-1} M_2 \in \mathbb{M}^3_{\mathrm{skew}}$, $A'_\nu\, M_3^{-1}(M_2)^T \in \mathbb{M}^3_{\mathrm{skew}}$ and we have*

$$\begin{aligned} M_3 &= (A'_\nu)^T\, M_1^{-1} A'_\nu + (M_2)^T M_1^{-1} M_2, \\ M_1 &= A'_\nu\, M_3^{-1}(A'_\nu)^T + M_2 M_3^{-1}(M_2)^T. \end{aligned}$$

Proof. Since $\mathrm{rk}\,(A_\nu \pm M) = 3$ and M_1 is invertible, then each column of $A_\nu \pm M$ is generated by the first three columns of $A_\nu \pm M$. Thus

$$\begin{pmatrix} M_2 \pm A'_\nu \\ M_3 \end{pmatrix} = \begin{pmatrix} M_1 \\ (M_2 \pm A'_\nu)^T \end{pmatrix} M_1^{-1}(M_2 \pm A'_\nu),$$

and we obtain that

$$\begin{aligned} M_3 &= (M_2 \pm A'_\nu)^T M_1^{-1}(M_2 \pm A'_\nu) \\ &= (A'_\nu)^T\, M_1^{-1} A'_\nu + (M_2)^T M_1^{-1} M_2 \pm (M_2)^T M_1^{-1} A'_\nu \pm (A'_\nu)^T\, M_1^{-1} M_2. \end{aligned}$$

Note that in particular

$$\mathrm{rk}\,(M_2 \pm A'_\nu) = 3. \tag{3.39}$$

Similarly, using that M_3 is invertible, we have

$$M_1 = A'_\nu\, M_3^{-1}(A'_\nu)^T + M_2 M_3^{-1}(M_2)^T \pm M_2 M_3^{-1}(A'_\nu)^T \pm A'_\nu\, M_3^{-1}(M_2)^T.$$

These conditions are equivalent to those in the statement of the lemma. □

We are now in position to complete the proof of Theorem 3.4.

Proof of Theorem 3.4. Let us start by proving the necessary condition. We consider a matrix $M \in \mathbb{M}^9_{\mathrm{sym}}$ satisfying (3.25). By Lemma 3.7, the matrices M_1 and M_3 are nonnegative and invertible so that (3.28a) holds. Next (3.28b) and (3.28c) have been proved in Lemma 3.8, while (3.28d) is a consequence of (3.32). Finally, according to Lemma 3.7, conditions (3.28e) are satisfied.

We now prove the sufficient condition. By (3.28d) and the expression (3.31) of $\mathrm{Ker}\,A_\nu$, we have that the second condition $\mathrm{Ker}\,A_\nu \subset \mathrm{Ker}\,M$ of (3.25) is satisfied, from which we deduce that

$$\mathrm{Ker}\,(A_\nu + M) \cap \mathrm{Ker}\,(A_\nu - M) = \mathrm{Ker}\,A_\nu.$$

Since by (3.28e), we have $\mathrm{rk}\,(A_\nu \pm M) = 3$, then $\dim \mathrm{Ker}\,(A_\nu \pm M) = 6$, and

$$\begin{aligned} \dim\big(\mathrm{Ker}\,(A_\nu + M) + \mathrm{Ker}\,(A_\nu - M)\big) &= \dim\big(\mathrm{Ker}\,(A_\nu + M)\big) + \dim\big(\mathrm{Ker}\,(A_\nu - M)\big) \\ &\quad -\dim\big(\mathrm{Ker}\,(A_\nu + M) \cap \mathrm{Ker}\,(A_\nu - M)\big) = 9\,, \end{aligned}$$

so that the last condition $\mathrm{Ker}\,(A_\nu + M) + \mathrm{Ker}\,(A_\nu - M) = \mathbb{R}^9$ in (3.25) holds. It lasts to show that M is non-negative. We write any $v \in \mathbb{R}^9$ as $v = (v_1, v_2, v_3)$, with v_1, v_2, and $v_3 \in \mathbb{R}^3$. With this notation, and using (3.28d), we have that

$$\begin{aligned} Mv \cdot v = M_1 v_1 \cdot v_1 + M_3 v_2 \cdot v_2 + \big(M_3(A'_\nu)^{-1} A''_\nu v_3\big) \cdot \big((A'_\nu)^{-1} A''_\nu v_3\big) \\ + 2M_2 v_2 \cdot v_1 + 2\big(M_2(A'_\nu)^{-1} A''_\nu v_3\big) \cdot v_1 + 2\big(M_3(A'_\nu)^{-1} A''_\nu v_3\big) \cdot v_2\,. \end{aligned}$$

According to (3.28c), we thus deduce that

$$\begin{aligned} Mv\cdot v &= M_1v_1\cdot v_1+(M_1^{-1}A'_\nu v_2)\cdot(A'_\nu v_2)+(M_1^{-1}M_2v_2)\cdot(M_2v_2)\\ &\quad+\big(M_1^{-1}A''_\nu v_3\big)\cdot\big(A''_\nu v_3\big)+\big(M_1^{-1}M_2(A'_\nu)^{-1}A''_\nu v_3\big)\cdot\big(M_2(A'_\nu)^{-1}A''_\nu v_3\big)\\ &\quad+2(M_1^{-1}A''_\nu v_3)\cdot(A'_\nu v_2)+2(M_1^{-1}M_2(A'_\nu)^{-1}A''_\nu v_3)\cdot(M_2v_2)\\ &\quad+2M_2v_2\cdot v_1+2(M_2(A'_\nu)^{-1}A''_\nu v_3)\cdot v_1\\ &= \big(M_1^{-1}(M_1v_1+M_2v_2)\big)\cdot(M_1v_1+M_2v_2)\\ &\quad+\big(M_1^{-1}A'_\nu(v_2+(A'_\nu)^{-1}A''_\nu v_3)\big)\cdot\big(A'_\nu(v_2+(A'_\nu)^{-1}A''_\nu v_3)\big)\\ &\quad+2(M_1^{-1}M_2(A'_\nu)^{-1}A''_\nu v_3)\cdot(M_1v_1+M_2v_2)\\ &\quad+(M_1^{-1}M_2(A'_\nu)^{-1}A''_\nu v_3)\cdot(M_2(A'_\nu)^{-1}A''_\nu v_3)\\ &= \big(M_1^{-1}(M_1v_1+M_2v_2+M_2(A'_\nu)^{-1}A''_\nu v_3)\big)\cdot(M_1v_1+M_2v_2+M_2(A'_\nu)^{-1}A''_\nu v_3)\\ &\quad+\big(M_1^{-1}A'_\nu(v_2+(A'_\nu)^{-1}A''_\nu v_3)\big)\cdot(A'_\nu(v_2+(A'_\nu)^{-1}A''_\nu v_3))\geq 0\,, \end{aligned}$$

where we have used that M_1^{-1} is nonnegative in the last inequality. Therefore $M\in\mathbb{M}^n_{\rm sym}$ is non negative, and then it satisfies all conditions in (3.25). □

3.2.4. *Derivation of all dissipative boundary conditions.* We are now in the position to write explicitly all admissible dissipative boundary conditions. Let us introduce the following notation.

Definition 3.9. Given a matrix $\sigma\in\mathbb{M}^3$ (non necessarily symmetric), we associate to σ the vector

$$\sigma_{\rm pr}=(\sigma_{11},\sigma_{22},\sigma_{33},\sigma_{12},\sigma_{13},\sigma_{23})^T\in\mathbb{R}^6.$$

Remark 3.10. Let $\sigma\in\mathbb{M}^3$ and $\sigma_{\rm pr}\in\mathbb{R}^6$ be the associated vector given by Definition 3.9. We further decompose $\sigma_{\rm pr}$ as $\sigma_{\rm pr}=(\sigma'_{\rm pr},\sigma''_{\rm pr})^T$ where $\sigma'_{\rm pr}=(\sigma_{11},\sigma_{22},\sigma_{33})^T\in\mathbb{R}^3$ and $\sigma''_{\rm pr}=(\sigma_{12},\sigma_{13},\sigma_{23})^T\in\mathbb{R}^3$. An immediate computation shows that

$$B'_\nu\sigma'_{\rm pr}+B''_\nu\sigma''_{\rm pr}=-\sigma_{\rm sym}\nu \tag{3.40}$$

where $\sigma_{\rm sym}\in\mathbb{M}^3_{\rm sym}$ is defined by

$$\sigma_{\rm sym}:=\begin{pmatrix}\sigma_{11}&\sigma_{12}&\sigma_{13}\\ \sigma_{12}&\sigma_{22}&\sigma_{23}\\ \sigma_{13}&\sigma_{23}&\sigma_{33}\end{pmatrix}.$$

Note that $\sigma_{\rm sym}$ differs from the symmetric part of σ defined by $(\sigma+\sigma^T)/2$. In addition, if σ is already symmetric, then $\sigma_{\rm sym}=\sigma$.

Proposition 3.11. *Let $\nu\in\mathbb{S}^2$ and $A_\nu\in\mathbb{M}^9_{\rm sym}$ be the matrix given by* (3.18)*. Then $M\in\mathbb{M}^9_{\rm sym}$ is a boundary matrix satisfying properties* (3.25) *if and only if there exist matrices $S_1\in\mathbb{M}^{3,+}_{\rm sym}$, $S_2\in\mathbb{M}^3_{\rm skew}$ such that both $S_1\pm S_2$ are invertible and the following property holds: for any $(v,\sigma)\in\mathbb{R}^3\times\mathbb{M}^3_{\rm sym}$, defining*

$$U:=\widetilde{A}_0^{\frac12}\begin{pmatrix}v\\ \sigma_{\rm pr}\end{pmatrix}\in\mathbb{R}^9,$$

then

$$(A_\nu\pm M)U=0\quad\Longleftrightarrow\quad(S_1\pm S_2)v\mp\sigma\nu=0\,. \tag{3.41}$$

Proof. According to (3.10), it holds that

$$U=\begin{pmatrix}v\\ \widetilde{C}^{-\frac12}\sigma'_{\rm pr}\\ \mu^{-\frac12}\sigma''_{\rm pr}\end{pmatrix},$$

where $\sigma'=(\sigma_{11},\sigma_{22},\sigma_{33})^T\in\mathbb{R}^3$ and $\sigma''=(\sigma_{12},\sigma_{13},\sigma_{23})^T\in\mathbb{R}^3$ so that $\sigma_{\rm pr}=(\sigma'_{\rm pr},\sigma''_{\rm pr})^T$.

Step 1. We claim that (recall (3.27))

$$(A_\nu\pm M)U=0\quad\Longleftrightarrow\quad\widehat{A}'_\nu\widehat{M}_3^{-1}(\widehat{A}'_\nu\pm\widehat{M}_2)^Tv\mp\sigma\nu=0\,. \tag{3.42}$$

We prove (3.42) first in the case where $\nu_i \neq 0$ for all $i \in \{1,2,3\}$ (where A'_ν, A''_ν, M_1, M_2, M_3 coincide with $\widehat{A}'_\nu$, $\widehat{A}''_\nu$, $\widehat{M}_1$, $\widehat{M}_2$, $\widehat{M}_3$). Using (3.18) and (3.26), we have

$$A_\nu \pm M = \begin{pmatrix} \pm M_1 & A'_\nu \pm M_2 & A''_\nu \pm M'_2 \\ (A'_\nu \pm M_2)^T & \pm M_3 & \pm M'_3 \\ (A''_\nu \pm M'_2)^T & \pm (M'_3)^T & \pm M''_3 \end{pmatrix}.$$

By Theorem 3.4, we have $\mathrm{rk}\,(A_\nu \pm M) = 3$ and $\mathrm{rk}\,(M_3) = 3$. Therefore $\mathrm{Im}\,(A_\nu \pm M)$ is generated by the lines 4, 5 and 6 of $A_\nu \pm M$. Thus, $(A_\nu \pm M)U = 0$ if and only if

$$(A'_\nu \pm M_2)^T v \pm M_3 \widetilde{C}^{-\frac{1}{2}} \sigma'_{\rm pr} \pm \mu^{-\frac{1}{2}} M'_3 \sigma''_{\rm pr} = 0,$$

or still, using condition (3.28d) of Theorem 3.4,

$$(A'_\nu \pm M_2)^T v \pm M_3 \widetilde{C}^{-\frac{1}{2}} \sigma'_{\rm pr} \pm \mu^{-\frac{1}{2}} M_3 (A'_\nu)^{-1} A''_\nu \sigma''_{\rm pr} = 0.$$

Since M_3 is invertible, using (3.19) this last equation is again equivalent to

$$A'_\nu M_3^{-1} (A'_\nu \pm M_2)^T v \pm B'_\nu \sigma'_{\rm pr} \pm B''_\nu \sigma''_{\rm pr} = 0\,,$$

and then, by (3.40), to

$$A'_\nu M_3^{-1} (A'_\nu \pm M_2)^T v \mp \sigma\nu = 0.$$

Let us now address the case where ν has at least a null component. Recalling C_ν introduced in (3.23), we get that

$$\begin{aligned} (A_\nu \pm M)U = 0 \quad &\Longleftrightarrow \quad C_\nu (A_\nu \pm M) C_\nu^T C_\nu^{-T} U = 0 \\ &\Longleftrightarrow \quad (\widehat{A}_\nu \pm \widehat{M})\widehat{U} = 0 \end{aligned} \tag{3.43}$$

where $\widehat{U} = C_\nu^{-T} U$. Since $\mathrm{rk}\,(\widehat{A}_\nu \pm \widehat{M}) = 3$ and

$$\mathrm{rk} \begin{pmatrix} \widehat{M}_2 \pm \widehat{A}'_\nu \\ \widehat{M}_3 \end{pmatrix} = 3\,,$$

then the condition (3.43) is equivalent to

$$\big((\widehat{A}'_\nu \pm \widehat{M}_2)^T | \pm \widehat{M}_3| \pm \widehat{M}'_3\big)\widehat{U} = 0\,,$$

or still, multiplying on the left by $\widehat{A}'_\nu \widehat{M}_3^{-1}$ and using (3.28d), to

$$\big(\widehat{A}'_\nu \widehat{M}_3^{-1} (\widehat{A}'_\nu \pm \widehat{M}_2)^T | \pm \widehat{A}'_\nu | \pm \widehat{A}''_\nu\big)\widehat{U} = 0\,. \tag{3.44}$$

According to Remark 3.3, we have that $\widehat{U}_i = U_i$ for all $i \in \{1,2,3\}$, while using that $\widehat{A}_\nu \widehat{U} = C_\nu A_\nu U$ gives

$$\widehat{A}'_\nu \begin{pmatrix} \widehat{U}_4 \\ \widehat{U}_5 \\ \widehat{U}_6 \end{pmatrix} + \widehat{A}''_\nu \begin{pmatrix} \widehat{U}_7 \\ \widehat{U}_8 \\ \widehat{U}_9 \end{pmatrix} = A'_\nu \begin{pmatrix} U_4 \\ U_5 \\ U_6 \end{pmatrix} + A''_\nu \begin{pmatrix} U_7 \\ U_8 \\ U_9 \end{pmatrix}.$$

Hence, (3.44) is equivalent to

$$\Big(\widehat{A}'_\nu \widehat{M}_3^{-1} (\widehat{A}'_\nu \pm \widehat{M}_2)^T | \pm A'_\nu | \pm A''_\nu\Big) U = 0$$

and arguing as above, we get that

$$\Big(\widehat{A}'_\nu \widehat{M}_3^{-1} (\widehat{A}'_\nu \pm \widehat{M}_2)^T | \pm A'_\nu | \pm A''_\nu\Big) U = \widehat{A}'_\nu \widehat{M}_3^{-1} (\widehat{A}'_\nu \pm \widehat{M}_2)^T v \mp \sigma\nu\,,$$

so that (3.42) is proven.

Step 2. Given M satisfying (3.25), let us define $S_1 := \widehat{A}'_\nu \widehat{M}_3^{-1} (\widehat{A}'_\nu)^T$ and $S_2 := \widehat{A}'_\nu \widehat{M}_3^{-1} (\widehat{M}_2)^T$. According to (3.28b), the matrix S_2 is skew symmetric. Moreover, since $\widehat{A}'_\nu$ is invertible and $\widetilde{M}_3$ is positive definite, we deduce that S_1 is positive definite. Finally, using (3.39), we have that $S_1 \pm S_2 = \widehat{A}'_\nu \widehat{M}_3^{-1} (\widehat{A}'_\nu \pm \widehat{M}_2)^T \in \mathbb{M}^3$ is invertible.

Conversely, given $S_1 \in \mathbb{M}^{3,+}_{\rm sym}$ and $S_2 \in \mathbb{M}^3_{\rm skew}$, then we define

$$\widehat{M}_3 := (\widehat{A}'_\nu)^T (S_1)^{-1} \widehat{A}'_\nu\,, \quad \widehat{M}_2 := \widehat{M}_3 (\widehat{A}'_\nu)^{-T} S_2\,,$$
$$\widehat{M}_1 := \widehat{A}'_\nu\, \widehat{M}_3^{-1} (\widehat{A}'_\nu)^T + \widehat{M}_2 \widehat{M}_3^{-1} (\widehat{M}_2)^T,$$
$$\widehat{M}'_2 := \widehat{M}_2 (\widehat{A}'_\nu)^{-1} \widehat{A}''_\nu, \quad \widehat{M}'_3 := \widehat{M}_3 (\widehat{A}'_\nu)^{-1} \widehat{A}''_\nu, \quad \widehat{M}''_3 := (\widehat{A}''_\nu)^T (\widehat{A}'_\nu)^{-T} \widehat{M}_3 (\widehat{A}'_\nu)^{-1} \widehat{A}''_\nu\,,$$

and let M and $\widehat{M} \in \mathbb{M}^9_{\rm sym}$ be defined by (3.26) and (3.27). From Theorem 3.4, it follows that M satisfies (3.25). With this correspondence between M and (S_1, S_2), (3.42) guarantees that (3.41) holds true, and this concludes the proof. □

3.2.5. *Final formulation of the models.* We complement the dynamical system of perfect plasticity (3.1)–(3.2) with dissipative boundary conditions of the form

$$S\dot{u} + \sigma\nu = 0 \quad \text{ on } \partial\Omega \times (0,T)\,, \tag{3.45}$$

for some $S \in \mathbb{M}^3$ an invertible matrix whose symmetric part $S_1 := (S + S^T)/2 \in \mathbb{M}^{3,+}_{\rm sym}$ is positive definite. Indeed, setting $S_2 := -(S - S^T)/2 \in \mathbb{M}^3_{\rm skew}$, then $S_1 - S_2 = S$, $S_1 + S_2 = S^T$ and (3.41) with the minus sign provides the correspondence between boundary conditions of different type. Therefore, the full dynamical system of perfect plasticity reads as

$$\begin{cases} \ddot{u} - \operatorname{div}\sigma = f & \text{in } \Omega \times (0,T) \\ \mathrm{E}u = e + p & \text{in } \Omega \times (0,T) \\ \sigma = \mathbf{A}e \in \mathbf{K} & \text{in } \Omega \times (0,T) \\ \sigma : \dot{p} = H(\dot{p}) & \text{in } \Omega \times (0,T) \\ S\dot{u} + \sigma\nu = 0 & \text{on } \partial\Omega \times (0,T) \\ (u(0), \dot{u}(0), e(0), p(0)) = (u_0, v_0, e_0, p_0) & \text{in } \Omega. \end{cases} \tag{3.46}$$

The final part of this section is devoted to recast the entropic formulation from (3.17), according to the correspondence between U and $(\dot{u}, \sigma)$ given in (3.3). We thus fix $\xi \in \mathbb{R}^9$ of the form

$$\xi = \widetilde{A}_0^{\frac{1}{2}} \begin{pmatrix} z \\ \tau_{\rm pr} \end{pmatrix}$$

for some $(z, \tau) \in \mathbb{R}^3 \times \mathbb{M}^3_{\rm sym}$.

The following remark gathers some further properties of the matrices A_i and A_ν.

Remark 3.12. We observe three facts, following from direct computations. First, for all $X \in \mathbb{R}^3$, we have that

$$\begin{cases} |U - \xi|^2 = |\dot{u} - z|^2 + \mathbf{A}^{-1}(\sigma - \tau) : (\sigma - \tau)\,, \\ \displaystyle\sum_{i=1}^3 X_i A_i (U - \xi) \cdot (U - \xi) = -2\big((\sigma - \tau)X\big) \cdot (\dot{u} - z)\,. \end{cases} \tag{3.47}$$

Second, since

$$\mathbf{A}(w \odot \nu)\nu = \Big(\mu \mathrm{I}_3 + (\mu + \lambda)(\nu \otimes \nu)\Big) w \quad \text{for every } w \in \mathbb{R}^3\,,$$

and $\mu \mathrm{I}_3 + (\mu + \lambda)(\nu \otimes \nu)$ is invertible when $\mu > 0$ and $3\lambda + 2\mu > 0$, then there exists a linear map $\Phi \colon \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\Phi(\eta) := \Big(\mu \mathrm{I}_3 + (\mu + \lambda)(\nu \otimes \nu)\Big)^{-1} \eta \quad \text{ satisfies } \quad \mathbf{A}(\Phi(\eta) \odot \nu)\nu = \eta \text{ for any } \eta \in \mathbb{R}^3\,.$$

Third, for any vector $\theta \in \mathbb{R}^9$ of the form $\theta = \widetilde{A}_0^{\frac{1}{2}}(\theta_1|\theta_2)^T$ with $\theta_1 \in \mathbb{R}^3$, $\theta_2 \in \mathbb{R}^6$, we have by (3.40) and (3.8),

$$A_\nu \theta = \widetilde{A}_0^{-\frac{1}{2}} \Big(\sum_{i=1}^3 \widetilde{A}_i \nu_i \Big) (\theta_1|\theta_2)^T = \widetilde{A}_0^{\frac{1}{2}} \Big(-(\theta_2)_{\rm sym}\nu | \big(\mathbf{A}(\theta_1 \odot \nu)\big)_{\rm pr} \Big)^T,$$

and we obtain that

$$\operatorname{Im} A_\nu = \Big\{ \widetilde{A}_0^{\frac{1}{2}} \Big(w_1 | \big(\mathbf{A}(w_2 \odot \nu)\big)_{\rm pr} \Big)^T : w_1,\, w_2 \in \mathbb{R}^3 \Big\}\,.$$

We now have at our disposal all the elements to determine the projections ξ_0 (resp. $\xi^\pm$) of ξ onto $\operatorname{Ker} A_\nu$ (resp. $\operatorname{Ker}(A_\nu \pm M) \cap \operatorname{Im} A_\nu$). Let us assume that S is a symmetric matrix, which is the case where the variational approach can be developed, see Remark 4.1 below. Then, in view of Remark 3.10 and of Proposition 3.11, we obtain that

$$\xi_0 = \frac{1}{2} \widetilde{A}_0^{\frac{1}{2}} \begin{pmatrix} 0_{\mathbb{R}^3} \\ \big(\tau - \mathbf{A}(\Phi(\tau\nu) \odot \nu)\big)_{\rm pr} \end{pmatrix}, \qquad \xi^\pm = \frac{1}{2} \widetilde{A}_0^{\frac{1}{2}} \begin{pmatrix} z \pm S^{-1}\tau\nu \\ \big(\mathbf{A}(\Phi(\tau\nu \pm Sz) \odot \nu)\big)_{\rm pr} \end{pmatrix}.$$

This finally gives, with (3.47), that

$$M\xi^{\pm}\cdot\xi^{\pm}=\pm\frac{1}{2}S^{-1}(\tau\nu\pm Sz)\cdot(\tau\nu\pm Sz)\,. \tag{3.48}$$

Therefore, setting $v:=\dot u$ and using (3.47)–(3.48), (3.17) is equivalent to the following family of inequalities: for all constant vectors $(z,\tau)\in\mathbb{R}^3\times\mathbf{K}$ and all test functions $\varphi\in C_c^\infty(\mathbb{R}^3\times\mathbb{R})$ with $\varphi\geq 0$ it holds that

$$\begin{aligned}\int_{\mathbb{R}^+}\int_\Omega\left(|v-z|^2+\mathbf{A}^{-1}(\sigma-\tau):(\sigma-\tau)\right)\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t-2\int_{\mathbb{R}^+}\int_\Omega(\sigma-\tau):((v-z)\odot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t\\+\int_\Omega\left(|v_0-z|^2+\mathbf{A}^{-1}(\sigma_0-\tau):(\sigma_0-\tau)\right)\varphi(0)\,\mathrm{d}x\\+2\int_{\mathbb{R}^+}\int_\Omega f\cdot(v-z)\varphi\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{2}\int_{\mathbb{R}^+}\int_{\partial\Omega}S^{-1}(\tau\nu+Sz)\cdot(\tau\nu+Sz)\varphi\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t\geq 0\,.\end{aligned}$$

4. Variational solutions

4.1. **The elastoplastic model.** In this section, we study the model (3.46), corresponding to (3.1)–(3.2) complemented with the dissipative boundary conditions (3.45). Our analysis rests on a variational approach, starting from a visco–elasto–plastic model, for which well-posedness is easily established. Then we study the limit of these viscous solutions as the viscosity parameter becomes vanishingly small. We derive a unique evolution satisfying (3.1) (in a suitable sense), and a relaxed version of the boundary conditions (3.45). Indeed the stress contraint $\sigma\in\mathbf{K}$ and the boundary condition $\sigma\nu+S\dot u=0$ are incompatible since formally $\sigma\nu$ belongs to $\mathbf{K}\nu$ while $\dot u$ is free. Thus the boundary condition will have to accommodate the constraint by projecting $S\dot u$ onto $-\mathbf{K}\nu$. From a mathematical point of view, this is related to the lack of continuity of the trace operator in $BD(\Omega)$ with respect to weak* convergence in that space (see [41, 7]), as well as the lack of lower semicontinuity of the energy which has to be relaxed, i.e., replaced by its lower semicontinuous envelope. The relaxation procedure will make appear a new boundary energy related to the previous projection property.

Remark 4.1. Note that our variational approach is based on the minimization of an energy functional at the discrete time level, see Subsection 4.2. This so-called incremental problem involves various energies (the kinetic and elastic energies, and the plastic dissipation) which are convex, except a boundary energy of the form

$$u\mapsto\int_{\partial\Omega}Su\cdot u\,\mathrm{d}\mathcal{H}^{n-1}$$

which is related to our choice of boundary conditions. Since the variational structure is lost while passing to the limit as the time-step tends to zero, it is necessary the minimality to be equivalent to the Euler-Lagrange equation (in order not to lose information). As usual, this property depends on the convexity of the energy functional and it rests on the symmetry of the matrix S. For that reason, we will assume the matrix S to be symmetric (although the algebraic conditions derived in Proposition 3.11 allow one to include matrices S with a nontrivial skew symmetric part), in hypothesis (H_4) below. Our variational method does not permit to consider the more general situation of non symmetric matrices S.

Let us now describe precisely the various hypotheses needed.

(H_1) **Reference configuration.** Let $\Omega\subset\mathbb{R}^n$ is a bounded Lipschitz open set which stands for the reference configuration of a linearly elastic and perfectly plastic body.

(H_2) **Elasticity properties.** We suppose that the material is isotropic, which means that the Hooke's tensor involved in the constitutive law is expressed by

$$\mathbf{A}e=\lambda(\operatorname{tr}e)\,\mathrm{I}_n+2\mu e\quad\text{ for all }e\in\mathbb{M}^n_{\mathrm{sym}},$$

where λ and μ are the Lamé coefficients satisfying $\mu>0$ and $2\mu+n\lambda>0$ (ensuring the ellipticity of the tensor $\mathbf{A}$). It generates a quadratic form given by

$$Q(e):=\frac{1}{2}\mathbf{A}e:e=\frac{\lambda}{2}(\operatorname{tr}e)^2+\mu|e|^2\quad\text{ for all }e\in\mathbb{M}^n_{\mathrm{sym}}.$$

If $e \in L^2(\Omega;\mathbb{R}^n)$, we define the elastic energy by

$$\mathcal{Q}(e) := \int_\Omega Q(e)\,\mathrm{d}x.$$

(H_3) **Plastic properties.** We suppose that the stress is constrained to stay inside a fixed closed and convex set $\mathbf{K} \subset \mathbb{M}^n_{\mathrm{sym}}$ containing 0 in its interior. In particular, there exists $r > 0$ such that

$$\{\tau \in \mathbb{M}^n_{\mathrm{sym}} :\ |\tau| \le r\} \subset \mathbf{K}. \tag{4.1}$$

The support function $H : \mathbb{M}^n_{\mathrm{sym}} \to [0,+\infty]$ of $\mathbf{K}$ is defined by

$$H(q) := \sup_{\sigma\in\mathbf{K}} \sigma : q \quad \text{ for all } q \in \mathbb{M}^n_{\mathrm{sym}}.$$

Note that according to (4.1), we get that

$$H(q) \ge r|q| \quad \text{ for all } q \in \mathbb{M}^n_{\mathrm{sym}}. \tag{4.2}$$

If $p \in \mathcal{M}_b(\Omega;\mathbb{M}^n_{\mathrm{sym}})$, we denote the convex function of measure $H(p)$ by

$$H(p) := H\left(\frac{\mathrm{d}p}{\mathrm{d}|p|}\right)|p|,$$

and we define the plastic dissipation by

$$\mathcal{H}(p) := \int_\Omega H\left(\frac{\mathrm{d}p}{\mathrm{d}|p|}\right)\mathrm{d}|p|.$$

(H_4) **Dissipative boundary condition.** We suppose that the dissipative boundary condition is expressed by means of a boundary matrix

$$S \in L^\infty(\partial\Omega;\mathbb{M}^{n,+}_{\mathrm{sym}}) \tag{4.3}$$

satisfying the following condition: there exists $c > 0$ such that for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$ and all $z \in \mathbb{R}^n$

$$S(x)z \cdot z \ge c|z|^2\,. \tag{4.4}$$

(H_5) **External forces.** We assume that the body is subjected to external body loads

$$f \in H^1(0,T;L^2(\Omega;\mathbb{R}^n))\,.$$

(H_6) **Initial conditions.** Let $u_0 \in H^1(\Omega;\mathbb{R}^n)$, $v_0 \in H^2(\Omega;\mathbb{R}^n)$, $e_0 \in L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}})$, $p_0 \in L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}})$ and $\sigma_0 := \mathbf{A}e_0 \in \mathcal{K}(\Omega)$ be such that

$$\begin{cases} \mathrm{E}u_0 = e_0 + p_0 \quad \text{a.e. in } \Omega\,, \\ Sv_0 + \sigma_0\nu = 0 \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega\,. \end{cases}$$

Definition 4.2. Assume that (H_1)–(H_4) are satisfied. A triple (u,e,p) is a *variational solution to the dynamic elasto-plastic model* associated to the initial data $(u_0,v_0,e_0,p_0) \in L^2(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}}) \times L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}})$ and the source term $f \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))$ if

$$\begin{cases} u \in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)) \cap C^{0,1}([0,T];BD(\Omega))\,, \\ e \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}}))\,, \\ p \in C^{0,1}([0,T];\mathcal{M}_b(\Omega;\mathbb{M}^n_{\mathrm{sym}}))\,, \end{cases}$$

$$\sigma := \mathbf{A}e \in L^\infty(0,T;H(\mathrm{div},\Omega)), \quad \sigma\nu \in L^\infty(0,T;L^2(\partial\Omega;\mathbb{R}^n)),$$

and it satisfies the following properties:

1. *The initial conditions*:
$$u(0) = u_0, \quad \dot{u}(0) = v_0, \quad e(0) = e_0, \quad p(0) = p_0;$$
2. *The additive decomposition*: for all $t \in [0,T]$,
$$\mathrm{E}u(t) = e(t) + p(t) \quad \text{in } \mathcal{M}_b(\Omega;\mathbb{M}^n_{\mathrm{sym}})\,;$$
3. *The equation of motion:*
$$\ddot{u} - \mathrm{div}\,\sigma = f \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^n))\,;$$

4. *The relaxed dissipative boundary conditions*:
$$\mathrm{P}_{-\mathbf{K}\nu}(S\dot{u}) + \sigma\nu = 0 \quad \text{in } L^2(0,T;L^2(\partial\Omega;\mathbb{R}^n))\,;$$
5. *The stress constraint*: for every $t \in [0,T]$,
$$\sigma(t) \in \mathbf{K} \quad \text{a.e. in } \Omega\,;$$
6. *The flow rule*: for every $t \in [0,T]$,
$$H(\dot{p}(t)) = [\sigma(t) : \dot{p}(t)] \quad \text{in } \mathcal{M}_b(\Omega)\,.$$
7. *The energy balance*: for every $t \in [0,T]$,
$$\begin{aligned}&\frac{1}{2}\int_\Omega |\dot{u}(t)|^2\,\mathrm{d}x + \mathcal{Q}(e(t)) + \int_0^t \mathcal{H}(\dot{p}(s))\,\mathrm{d}s + \int_0^t\int_{\partial\Omega}\psi(x,\dot{u})\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\\ &\quad + \frac{1}{2}\int_0^t\int_{\partial\Omega} S^{-1}(\sigma\nu)\cdot(\sigma\nu)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s = \frac{1}{2}\int_\Omega |v_0|^2\,\mathrm{d}x + \mathcal{Q}(e_0) + \int_0^t\int_\Omega f\cdot\dot{u}\,\mathrm{d}x\,\mathrm{d}s\,,\end{aligned} \tag{4.5}$$
where $\psi\colon \partial\Omega\times\mathbb{R}^n \to \mathbb{R}^n$ is defined, for $\mathcal{H}^{n-1}$-a.e. $x\in\partial\Omega$ and all $z\in\mathbb{R}^n$, by
$$\psi(x,z) = \inf_{z'\in\mathbb{R}^n}\left\{\frac{1}{2}S(x)z'\cdot z' + H((z'-z)\odot\nu(x))\right\}. \tag{4.6}$$

Remark 4.3. In the relaxed dissipative boundary conditions at point 4. above, $\mathrm{P}_{-\mathbf{K}\nu(x)}$ stands for the orthogonal projection in $\mathbb{R}^n$ onto the closed and convex set $-\mathbf{K}\nu(x)$ in $\mathbb{R}^n$ with respect to the scalar product $\langle u,v\rangle_{S^{-1}(x)} := S^{-1}(x)u\cdot v$. It precisely states that $\sigma\nu\in\mathbf{K}\nu$ a.e. on $\partial\Omega\times(0,T)$ and that $S\dot{u}$ has to be projected onto $-\mathbf{K}\nu$ to let the boundary conditions accommodate the stress constraint.

Remark 4.4. If the convex set $\mathbf{K}$ is of the form
$$\mathbf{K} = \left\{\tau\in\mathbb{M}^n_{\mathrm{sym}} :\ \tau_D := \tau - \frac{\mathrm{tr}\,\sigma}{n}\mathrm{I_n}\in K\right\}, \tag{4.7}$$
where K is a compact and convex set in $\mathbb{M}^n_D$ containing 0 in its interior (as e.g. for Von Mises or Tresca models), then the plastic strain turns out to be a deviatoric measure, i.e.,
$$p\in C^{0,1}([0,T];\mathcal{M}_b(\Omega;\mathbb{M}^n_D))$$
and the normal component of the dissipative boundary condition reads as
$$(S^{-1}\sigma\nu)\cdot\nu + \dot{u}\cdot\nu = 0 \quad \text{in } L^2(0,T;L^2(\partial\Omega;\mathbb{R}^n)).$$
In other words, since the stress constraint only acts on the deviatoric part of the Cauchy stress, then only shearing stresses are responsible of plastification. It follows that only the tangential component of the boundary condition (the one involving the boundary shearing stress) needs to be relaxed, and the normal component of the boundary condition (involving the boundary hydrostatic pressure) is not affected by the stress constraint.

The present section is devoted to the proof of the following existence and uniqueness result.

Theorem 4.5. *Under the assumptions (H_1)–(H_6), there exists a unique variational solution to the elasto–plastic problem associated to the initial data (u_0,v_0,e_0,p_0) and the source term f according to Definition 4.2.*

The result is obtained starting from a elasto–visco–plastic approximation, introduced in Subsection 4.2. Subsection 4.3 collects some preliminary tools, in view of the existence and uniqueness proof, performed in Subsection 4.4.

4.2. **The visco-elastoplastic model.** The visco-elastoplastic regularization consists in adding a visco–elastic diffusion term of Kelvin–Voigt type in the constitutive law, and a visco–plastic term of Perzyna type in the plastic flow rule penalizing the stress constraint. This type of models has been considered

- in [39, 10, 16] for variational models as in (3.1) but with different boundary conditions;
- in [9, 20] for constrained Friedrichs' systems in the whole space;
- in [8] for a simplified antiplane elastoplastic model with dissipative boundary conditions.

One deduces the following existence and uniqueness result arguing exactly as done for the antiplane simplified case of [8, 10].

Proposition 4.6. *Assume that (H_1)–(H_6) are satisfied. For each $\varepsilon > 0$, we define $g_\varepsilon := \varepsilon E v_0\, \nu \in L^2(\partial\Omega;\mathbb{R}^n)$. Then, there exists a unique triple $(u_\varepsilon, e_\varepsilon, p_\varepsilon)$ with*

$$\begin{cases} u_\varepsilon \in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)) \cap H^2(0,T;H^1(\Omega;\mathbb{R}^n))\,, \\ e_\varepsilon \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{\rm sym}))\,, \\ p_\varepsilon \in H^1(0,T;L^2(\Omega;\mathbb{M}^n_{\rm sym}))\,, \end{cases}$$

which satisfies the following properties:

1. The initial conditions*:*
$$u_\varepsilon(0) = u_0, \quad \dot u_\varepsilon(0) = v_0, \quad e_\varepsilon(0) = e_0, \quad p_\varepsilon(0) = p_0;$$
2. The additive decomposition*:*
$$\mathrm{E}u_\varepsilon = e_\varepsilon + p_\varepsilon \quad \textit{a.e. in } \Omega\times(0,T)\,;$$
3. The constitutive law*:*
$$\sigma_\varepsilon = \mathbf{A}e_\varepsilon \quad \textit{a.e. in } \Omega\times(0,T)\,;$$
4. The equation of motion*:*
$$\ddot u_\varepsilon - \mathrm{div}\,(\sigma_\varepsilon + \varepsilon \mathrm{E}\dot u_\varepsilon) = f \quad \textit{in } L^2(0,T;L^2(\Omega;\mathbb{R}^n))\,;$$
5. The dissipative boundary conditions*:*
$$S\dot u_\varepsilon + (\sigma_\varepsilon + \varepsilon \mathrm{E}\dot u_\varepsilon)\nu = g_\varepsilon \quad \textit{in } L^2(0,T;L^2(\partial\Omega;\mathbb{R}^n))\,;$$
6. The visco–plastic flow rule*:*
$$\dot p_\varepsilon = \frac{\sigma_\varepsilon - \mathrm{P}_{\mathbf{K}}(\sigma_\varepsilon)}{\varepsilon} \quad \textit{a.e. in } \Omega\times(0,T)\,;$$
7. The energy balance*: for every $t\in[0,T]$,*

$$\begin{aligned} &\frac12\int_\Omega |\dot u_\varepsilon(t)|^2\,\mathrm{d}x + \mathcal{Q}(e_\varepsilon(t)) + \int_0^t\int_\Omega H(\dot p_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad + \int_0^t\int_{\partial\Omega} S\dot u_\varepsilon\cdot\dot u_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s + \varepsilon\int_0^t\int_\Omega |\mathrm{E}\dot u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s + \varepsilon\int_0^t\int_\Omega |\dot p_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad\qquad = \frac12\int_\Omega |v_0|^2\,\mathrm{d}x + \mathcal{Q}(e_0) + \int_0^t\int_\Omega f\cdot\dot u_\varepsilon\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\partial\Omega} g_\varepsilon\cdot\dot u_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\,. \end{aligned} \tag{4.8}$$

Moreover, the following uniform estimate holds

$$\begin{aligned} &\|\ddot u_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} + \|\dot e_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^n_{\rm sym}))} \\ &\qquad + \varepsilon\|\mathrm{E}\ddot u_\varepsilon\|^2_{L^2(0,T;L^2(\Omega;\mathbb{M}^n_{\rm sym}))} + \int_0^T\!\!\int_{\partial\Omega} S\ddot u_\varepsilon\cdot\ddot u_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\mathrm{d}t \le C, \end{aligned} \tag{4.9}$$

for some constant $C>0$ independent of ε and S.

We do not give a proof of this result which closely follows the arguments in [34, Subsection 3.4.1], to which we refer for details with just some formal modifications to pass from the antiplane case setting to the present general one (see also [10] in the vectorial case but with different boundary conditions). We just point out the general method. The starting point consists in defining suitable time discrete approximating solutions. For any $N\in\mathbb{N}$, large enough, we consider a discretisation $\{t_i\}_{i=0}^N := \{i\frac{T}{N}\}_{i=0}^N$ of the time interval $[0,T]$, and approximate the values of (u,e,p) at the nodes of the discretisation as follows. Setting $\delta := \frac{T}{N}$ the time step, we start by the initial values (u_0,e_0,p_0), $(u_1,e_1,p_1) := (u_0,e_0,p_0) + \delta(v_0,0,\mathrm{E}u_0)$ and define by induction, for $i\ge 2$,

$$(u_i,e_i,p_i)\in X := \{(v,\eta,q)\in H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_{\rm sym})\times L^2(\Omega;\mathbb{M}^n_{\rm sym}):\ \mathrm{E}v = \eta + q \text{ in } \Omega\}$$

as the unique minimizer over X of

$$(v,\eta,q)\mapsto \mathcal{Q}(\eta)+\int_\Omega H(q-p_{i-1})\,\mathrm{d}x+\frac{1}{2\delta^2}\int_\Omega (v-2u_{i-1}+u_{i-2})^2\,\mathrm{d}x$$
$$+\frac{\varepsilon}{2\delta}\int_\Omega \big(|\mathrm{E}v-\mathrm{E}u_{i-1}|^2+|q-p_{i-1}|^2\big)\,\mathrm{d}x+\frac{1}{2\delta}\int_{\partial\Omega} S(v-u_{i-1})\cdot(v-u_{i-1})\,\mathrm{d}\mathcal{H}^{n-1}$$
$$-\int_\Omega f(t_{i-1})\cdot v\,\mathrm{d}x-\int_{\partial\Omega} g_\varepsilon\cdot v\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

We use the discrete solutions $\{(u_i,e_i,p_i)\}_{i=0}^N$ to construct N-dependent quadratic interpolations $(u_N(t),e_N(t),p_N(t))$, which can be shown to satisfy suitable *a priori* estimates and then converge in a weak sense as $N\to+\infty$ towards the weak notion of evolution. Then it is possible to prove uniqueness, and *a posteriori* regularity estimates that guarantees the properties stated in Proposition 4.6.

We notice that the matrix S has to be symmetric in order to guarantee the minimality of the incremental minimization problems above to be equivalent to the corresponding Euler–Lagrange equation, which is in turn a key tool, e.g. in the derivation of the *a priori* estimates.

4.3. **Some technical results.** In the sequel, $\nu(x)$ denotes the outward unit normal to $\partial\Omega$ at $x\in\partial\Omega$, which is defined $\mathcal{H}^{n-1}$-almost everywhere in $\partial\Omega$ by the Lipschitz regularity of $\partial\Omega$. Let us define the functions f and $h:\partial\Omega\times\mathbb{R}^n\to\mathbb{R}$ by setting, for $\mathcal{H}^{n-1}$-a.e. $x\in\partial\Omega$ and all $z\in\mathbb{R}^n$,

$$f(x,z):=\frac{1}{2}S(x)z\cdot z$$

and

$$h(x,z):=H(-z\odot\nu(x))=\sup_{\sigma\in\mathbf{K}} -\sigma:(z\odot\nu(x))=\sup_{\sigma\in\mathbf{K}} -(\sigma\nu(x))\cdot z=(\mathrm{I}_{-\mathbf{K}\nu(x)})^*(z).$$

We next remark that $\psi:\partial\Omega\times\mathbb{R}^n\to\mathbb{R}$, introduced in (4.6), is the inf-convolution (with respect to the second variable) of the functions $f(x,\bullet)$ and $h(x,\bullet)$ for $\mathcal{H}^{n-1}$-a.e. $x\in\partial\Omega$.

Remark 4.7. By standard properties of the inf-convolution and of the convex conjugate (see e.g. [37, Theorem 16.4]), we have

$$\psi(x,\bullet)=f(x,\bullet)\,\square\, h(x,\bullet)=\Big(f^*(x,\bullet)+h^*(x,\bullet)\Big)^*.$$

As a consequence, for $\mathcal{H}^{n-1}$-a.e. $x\in\partial\Omega$ and all $z\in\mathbb{R}^n$,

$$\begin{aligned}\psi(x,z) &= \sup_{q\in-\mathbf{K}\nu(x)}\left\{z\cdot q-\frac{1}{2}S^{-1}(x)q\cdot q\right\}\\ &= \frac{1}{2}S(x)z\cdot z-\frac{1}{2}\inf_{q\in-\mathbf{K}\nu(x)} S^{-1}(x)\big(q-S(x)z\big)\cdot\big(q-S(x)z\big)\\ &= \frac{1}{2}S(x)z\cdot z-\frac{1}{2}\|\mathrm{P}_{-\mathbf{K}\nu(x)}\big(S(x)z\big)-S(x)z\|^2_{S^{-1}(x)}\,,\end{aligned}$$

where $\mathrm{P}_{-\mathbf{K}\nu(x)}$ stands for the orthogonal projection in $\mathbb{R}^n$ onto the closed and convex set $-\mathbf{K}\nu(x)$ in $\mathbb{R}^n$ with respect to the Euclidean scalar product $\langle u,v\rangle_{S^{-1}(x)}:=S^{-1}(x)u\cdot v$. It follows that, for $\mathcal{H}^{n-1}$-a.e. $x\in\partial\Omega$, the function $\psi(x,\cdot)$ is of class $C^{1,1}$ in $\mathbb{R}^n$ because

$$D_z\psi(x,z)=S(x)z-\big(S(x)z-P_{-\mathbf{K}\nu(x)}(S(x)z)\big)=P_{-\mathbf{K}\nu(x)}(S(x)z),\tag{4.10}$$

and the projection is Lipschitz continuous.

In the following lemma, we prove an explicit expression for the element realizing the minimum value in the infimal convolution defining the function ψ.

Lemma 4.8. *For $\mathcal{H}^{n-1}$-a.e. $x\in\partial\Omega$ and all $z\in\mathbb{R}^n$, there exists a unique $\bar z=\bar z(x,z)\in\mathbb{R}^n$ such that*

$$\psi(x,z)=\frac{1}{2}S(x)(z-\bar z)\cdot(z-\bar z)+h(x,\bar z).$$

In addition, we have $z-\bar z=S^{-1}(x)D_z\psi(x,z)$.

Proof. Since the function $z' \mapsto \frac{1}{2}S(x)(z-z')\cdot(z-z')+h(x,z')$ is continuous, coercive and strictly convex, it admits a unique minimum point $\bar{z}\in\mathbb{R}^n$ which thus satisfies

$$\psi(x,z)=\frac{1}{2}S(x)(z-\bar{z})\cdot(z-\bar{z})+h(x,\bar{z}).$$

Let $t>0$ and $\xi\in\mathbb{R}^n$. Since $\psi(x,z-t\xi)\le\frac{1}{2}S(x)(z-t\xi-\bar{z})\cdot(z-t\xi-\bar{z})+h(x,\bar{z})$ then

$$\begin{aligned}\frac{\psi(x,z)-\psi(x,z-t\xi)}{t} &\ge \frac{S(x)(z-\bar{z})\cdot(z-\bar{z})-S(x)(z-t\xi-\bar{z})\cdot(z-t\xi-\bar{z})}{2t}\\ &= S(x)(z-\bar{z})\cdot\xi-\frac{t}{2}S(x)\xi\cdot\xi.\end{aligned}$$

Passing to the limit as $t\to 0$ and using that $\psi(x,\bullet)$ is differentiable, we get that $D_z\psi(x,z)\cdot\xi\ge S(x)(z-\bar{z})\cdot\xi$, and since ξ is arbitrary, $D_z\psi(x,z)=S(x)(z-\bar{z})$, which completes the proof. □

From the previous lemma, we deduce the following measurable selection result when the argument of ψ is not anymore a constant but a function defined on $\partial\Omega$.

Lemma 4.9. *For a given $u\in L^2(\partial\Omega;\mathbb{R}^n)$, there exists a unique function $w\in L^2(\partial\Omega;\mathbb{R}^n)$ such that, for $\mathcal{H}^{n-1}$-a.e. $x\in\partial\Omega$,*

$$\psi(x,u(x))=\frac{1}{2}S(x)w(x)\cdot w(x)+H\big((w(x)-u(x))\odot\nu(x)\big).$$

Proof. For $\mathcal{H}^{n-1}$-a.e. $x\in\partial\Omega$, let $w(x):=S^{-1}(x)D_z\psi(x,u(x))$. Since $(x,z)\mapsto D_z\psi(x,z)$ is a Carathéodory function, we deduce that the function $w:\partial\Omega\to\mathbb{R}^n$ is $\mathcal{H}^{n-1}$-measurable. In addition, according to Lemma 4.8, we have

$$\psi(x,u(x))=\frac{1}{2}S(x)w(x)\cdot w(x)+H\big((w(x)-u(x))\odot\nu(x)\big).$$

Finally, since H is nonnegative and vanishes in 0, we have that for $\mathcal{H}^{n-1}$-a.e. $x\in\partial\Omega$,

$$\frac{c}{2}|w(x)|^2\le\frac{1}{2}S(x)w(x)\cdot w(x)\le\psi(x,u(x))\le\frac{1}{2}S(x)u(x)\cdot u(x)$$

using (4.4), which shows that $w\in L^2(\partial\Omega;\mathbb{R}^n)$ since $u\in L^2(\partial\Omega;\mathbb{R}^n)$ and by (4.3). □

We now prove that the integral functional associated to the density ψ can be globally expressed as an infimum.

Lemma 4.10. *For a given $u\in L^2(\partial\Omega;\mathbb{R}^n)$, it holds that*

$$\int_{\partial\Omega}\psi(x,u)\,\mathrm{d}\mathcal{H}^{n-1}=\inf_{z\in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)}\left\{\int_{\partial\Omega}\left(\frac{1}{2}Sz\cdot z+H\big((z-u)\odot\nu\big)\right)\mathrm{d}\mathcal{H}^{n-1}\right\}.$$

Proof. According to Lemma 4.9, we have

$$\int_{\partial\Omega}\psi(x,u)\,\mathrm{d}\mathcal{H}^{n-1}=\inf_{z\in L^2(\partial\Omega;\mathbb{R}^n)}\left\{\int_{\partial\Omega}\left(\frac{1}{2}Sz\cdot z+H\big((z-u)\odot\nu\big)\right)\mathrm{d}\mathcal{H}^{n-1}\right\}.$$

By density of $C(\partial\Omega;\mathbb{R}^n)$ in $L^2(\partial\Omega;\mathbb{R}^n)$ and Tietze's extension Theorem, for any $z\in L^2(\partial\Omega;\mathbb{R}^n)$ and $\eta>0$ there exists a function $\tilde{z}\in C_c(\mathbb{R}^n;\mathbb{R}^n)$ such that $\|z-\tilde{z}\|_{L^2(\partial\Omega;\mathbb{R}^n)}\le\eta/2$. Then approximating by convolution, there exists $\hat{z}\in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ such that $\|\hat{z}-z\|_{L^2(\partial\Omega;\mathbb{R}^n)}\le\eta$. The conclusion then follows from the continuity of the functional

$$z\in L^2(\partial\Omega;\mathbb{R}^n)\mapsto\int_{\partial\Omega}\left(\frac{1}{2}Sz\cdot z+H\big((z-u)\odot\nu\big)\right)\mathrm{d}\mathcal{H}^{n-1}$$

with respect to the strong $L^2(\partial\Omega;\mathbb{R}^n)$-convergence. □

In the following result we establish a convexity inequality which would be easily obtained if we had at our disposal a generalized Green formula, and if we a priori knew that $\sigma\nu\in\mathbf{K}\nu$ when $\sigma\in\mathcal{K}(\Omega)$. This result will be essential to obtain the upper energy–dissipation inequality in Section 4.4.3.

Proposition 4.11. *Let $(u,e,p)\in (BD(\Omega)\cap L^2(\Omega;\mathbb{R}^n))\times L^2(\Omega;\mathbb{M}^n_{\rm sym})\times\mathcal{M}_b(\Omega;\mathbb{M}^n_{\rm sym})$ be such that $\mathcal{H}(p)<+\infty$, and let $\sigma\in\mathcal{K}(\Omega)$ with $\sigma\nu\in L^2(\partial\Omega;\mathbb{R}^n)$. Then for all $\varphi\in C^1_c(\mathbb{R}^n)$ with $\varphi\geq 0$, we have*

$$\int_\Omega \varphi\,\mathrm{d}H(p)+\int_{\partial\Omega}\varphi\psi(x,u)\,\mathrm{d}\mathcal{H}^{n-1}+\frac12\int_{\partial\Omega}\varphi S^{-1}(\sigma\nu)\cdot(\sigma\nu)\,\mathrm{d}\mathcal{H}^{n-1}$$
$$\geq -\int_\Omega \varphi\sigma:e\,\mathrm{d}x-\int_\Omega\varphi u\cdot\operatorname{div}\sigma\,\mathrm{d}x-\int_\Omega\sigma\colon(u\odot\nabla\varphi)\,\mathrm{d}x.$$

Proof. Let $z\in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$, and let us define an extension $q\in\mathcal{M}_b(\overline{\Omega};\mathbb{M}^n_{\rm sym})$ of p by setting

$$q:=p\llcorner\Omega+(z-u)\odot\nu\mathcal{H}^{n-1}\llcorner\partial\Omega.$$

Then from (2.2) and the definition of the stress–strain duality, for all $\varphi\in C^1(\mathbb{R}^n)$ with $\varphi\geq 0$, we have that

$$\int_\Omega\varphi\,\mathrm{d}H(p)+\int_{\partial\Omega}\varphi\left(\frac12 Sz\cdot z+H((z-u)\odot\nu)\right)\mathrm{d}\mathcal{H}^{n-1}$$
$$=\int_{\overline{\Omega}}\varphi\,\mathrm{d}H(q)+\frac12\int_{\partial\Omega}\varphi Sz\cdot z\,\mathrm{d}\mathcal{H}^{n-1}\geq\langle[\sigma\colon q],\varphi\rangle+\frac12\int_{\partial\Omega}\varphi Sz\cdot z\,\mathrm{d}\mathcal{H}^{n-1}$$
$$=-\int_\Omega\varphi\sigma\colon(e-Ez)\,\mathrm{d}x-\int_\Omega(u-z)\cdot\operatorname{div}\sigma\,\varphi\,\mathrm{d}x-\int_\Omega\sigma\colon\big((u-z)\odot\nabla\varphi\big)\,\mathrm{d}x+\frac12\int_{\partial\Omega}\varphi Sz\cdot z\,\mathrm{d}\mathcal{H}^{n-1}.$$

Integrating by parts the terms involving z yields

$$\int_\Omega\varphi\,\mathrm{d}H(p)+\int_{\partial\Omega}\varphi\left(\frac12 Sz\cdot z+H((z-u)\odot\nu)\right)\mathrm{d}\mathcal{H}^{n-1}$$
$$\geq-\int_\Omega\varphi\sigma:e\,\mathrm{d}x-\int_\Omega\varphi u\cdot\operatorname{div}\sigma\,\mathrm{d}x-\int_\Omega\sigma\colon(u\odot\nabla\varphi)\,\mathrm{d}x+\int_{\partial\Omega}\varphi\left((\sigma\nu)\cdot z+\frac12 Sz\cdot z\right)\mathrm{d}\mathcal{H}^{n-1}$$
$$\geq-\int_\Omega\varphi\sigma:e\,\mathrm{d}x-\int_\Omega\varphi u\cdot\operatorname{div}\sigma\,\mathrm{d}x-\int_\Omega\sigma\colon(u\odot\nabla\varphi)\,\mathrm{d}x-\frac12\int_{\partial\Omega}\varphi S^{-1}(\sigma\nu)\cdot(\sigma\nu)\,\mathrm{d}\mathcal{H}^{n-1},$$

where we used that $(\sigma\nu)\cdot z+\frac12 Sz\cdot z\geq-\frac12 S^{-1}(\sigma\nu)\cdot(\sigma\nu)$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$. Passing to the infimum in $z\in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$ in the left hand side of the previous inequality, and using Lemma 4.10 leads to the desired inequality. □

Our last tool is the following lower bound for the dissipation energy along sequences converging in a natural topology associated to the energy space. This lower bound estimate will be the main tool employed in the lower energy–dissipation inequality in Section 4.4.3.

Proposition 4.12. *Let $(u_k,e_k,p_k)_{k\in\mathbb{N}}$ be a sequence in $BD(\Omega)\times L^2(\Omega;\mathbb{M}^n_{\rm sym})\times\mathcal{M}_b(\Omega;\mathbb{M}^n_{\rm sym})$ such that $\mathrm{E}u_k=e_k+p_k$ in $\mathcal{M}_b(\Omega;\mathbb{M}^n_{\rm sym})$, and*

$$\begin{cases} u_k\overset{*}{\rightharpoonup}u & \text{weakly* in } BD(\Omega),\\ e_k\rightharpoonup e & \text{weakly in } L^2(\Omega;\mathbb{M}^n_{\rm sym}),\\ p_k\overset{*}{\rightharpoonup}p & \text{weakly* in } \mathcal{M}_b(\Omega;\mathbb{M}^n_{\rm sym}),\\ u_k\rightharpoonup w & \text{weakly in } L^2(\partial\Omega;\mathbb{R}^n), \end{cases}$$

for some $(u,e,p)\in BD(\Omega)\times L^2(\Omega;\mathbb{M}^n_{\rm sym})\times\mathcal{M}_b(\Omega;\mathbb{M}^n_{\rm sym})$ and $w\in L^2(\partial\Omega;\mathbb{R}^n)$. Then

$$p_k\overset{*}{\rightharpoonup}p+(w-u)\odot\nu\mathcal{H}^{n-1}\llcorner\partial\Omega\quad\text{weakly* in }\mathcal{M}_b(\overline{\Omega};\mathbb{M}^n_{\rm sym})\tag{4.11}$$

and

$$\mathcal{H}(p)+\int_{\partial\Omega}\psi(x,u)\,\mathrm{d}\mathcal{H}^{n-1}\leq\liminf_{k\to\infty}\left\{\mathcal{H}(p_k)+\frac12\int_{\partial\Omega}Su_k\cdot u_k\,\mathrm{d}\mathcal{H}^{n-1}\right\}.\tag{4.12}$$

Proof. We first establish (4.11). By Green's formula in $BD(\Omega)$ (see [7, Theorem 3.2]), we have that for any $\varphi\in C^1(\mathbb{R}^n;\mathbb{M}^n_{\rm sym})$,

$$\int_\Omega\varphi\colon\mathrm{d}(\mathrm{E}u_k-\mathrm{E}u)=-\int_\Omega\operatorname{div}\varphi\cdot(u_k-u)\,\mathrm{d}x+\int_{\partial\Omega}\varphi\colon(u_k-u)\odot\nu\,\mathrm{d}\mathcal{H}^{n-1},$$

so that, using the convergences in the hypotheses,

$$\lim_{k\to\infty}\int_\Omega \varphi\colon \mathrm{d}(\mathrm{E}u_k - \mathrm{E}u) = \int_{\partial\Omega} \varphi\colon (w-u)\odot\nu\,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{4.13}$$

By a density argument, the same convergence holds also for every $\varphi \in C(\overline{\Omega};\mathbb{M}^n_{\mathrm{sym}})$. Since $e_k \rightharpoonup e$ in $L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}})$, then using that $p_k = \mathrm{E}u_k - e_k$ and $p = \mathrm{E}u - e$, and by (4.13), we get that

$$\lim_{k\to\infty}\int_\Omega \varphi\colon \mathrm{d}(p_k - p) = \int_{\partial\Omega} \varphi\colon (w-u)\odot\nu\,\mathrm{d}\mathcal{H}^{n-1} \quad \text{for every } \varphi \in C(\overline{\Omega};\mathbb{M}^n_{\mathrm{sym}})\,,$$

which gives (4.11).

We now prove (4.12). By (4.11) and Reshetnyak's lower semicontinuity Theorem we obtain that

$$\begin{aligned}\liminf_{k\to\infty}&\left\{\mathcal{H}(p_k) + \frac12\int_{\partial\Omega} Su_k\cdot u_k\,\mathrm{d}\mathcal{H}^{n-1}\right\}\\ &\geq \mathcal{H}(p) + \int_{\partial\Omega} H\big((w-u)\odot\nu\big)\,\mathrm{d}\mathcal{H}^{n-1} + \frac12\int_{\partial\Omega} Sw\cdot w\,\mathrm{d}\mathcal{H}^{n-1} \geq \mathcal{H}(p) + \int_{\partial\Omega}\psi(x,u)\,\mathrm{d}\mathcal{H}^{n-1}\,,\end{aligned}$$

by definition of ψ. □

Remark 4.13. In the case of a cylindrical convex set $\mathbf{K}$ of the form (4.7), since the plastic strain is deviatoric then the condition

$$p_k \overset{*}{\rightharpoonup} p \quad \text{weakly* in } \mathcal{M}_b(\Omega;\mathbb{M}^n_D)$$

implies that $\mathrm{tr}\,\big((w-u)\odot\nu\big) = (w-u)\cdot\nu = 0$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$.

4.4. **Proof of Theorem 4.5.** In this subsection we prove Theorem 4.5. Using the energy balance as well as the estimate (4.9), we deduce weak convergence properties of the various fields. This enables one to pass to the limit in the linear relations (the initial condition, the additive decomposition, the constitutive law and the equation of motion) as well as in the convex relation (the stress constraint). Then we improve these weak convergences into strong ones (in particular for the trace of the displacement) thanks to which we can pass to the lower limit in the localized energy balance. This gives a first energy inequality which turns out to be an equality. Specializing this energy balance to test functions compactly supported in Ω we obtain the measure theoretic flow rule. Finally we derive the relaxed dissipative boundary condition thanks to a convexity argument by proving that the limit of the traces of $(u_\varepsilon)_{\varepsilon>0}$ is the solution of the minimization problem in the infimal convolution defining $\psi(x,\dot u(t,x))$. Eventually we prove the uniqueness of the solution.

4.4.1. *Weak compactness.* For any $\varepsilon>0$, let $(u_\varepsilon, e_\varepsilon, p_\varepsilon)$ be the solution of the elasto–visco–plastic approximation given by Proposition 4.6. Thanks to estimates (4.9) in Proposition 4.6, there exist functions $u \in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^n))$, $e \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}}))$, $\sigma \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}}))$ and $w \in H^2(0,T;L^2(\partial\Omega;\mathbb{R}^n))$ such that, up to a subsequence,

$$\begin{cases} u_\varepsilon \overset{*}{\rightharpoonup} u & \text{weakly* in } W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^n))\,,\\ e_\varepsilon \overset{*}{\rightharpoonup} e & \text{weakly* in } W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}}))\,,\\ \sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma & \text{weakly* in } W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}}))\,,\\ u_\varepsilon \rightharpoonup w & \text{weakly in } H^2(0,T;L^2(\partial\Omega;\mathbb{R}^n))\,,\end{cases} \tag{4.14}$$

and, by Ascoli-Arzelà Theorem, for every $t\in[0,T]$,

$$\begin{cases} u_\varepsilon(t)\rightharpoonup u(t), \quad \dot u_\varepsilon(t)\rightharpoonup \dot u(t) & \text{weakly in } L^2(\Omega;\mathbb{R}^n)\,,\\ e_\varepsilon(t)\rightharpoonup e(t), \quad \sigma_\varepsilon(t)\rightharpoonup \sigma(t) & \text{weakly in } L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}})\,,\\ u_\varepsilon(t)\rightharpoonup w(t), \quad \dot u_\varepsilon(t)\rightharpoonup \dot w(t) & \text{weakly in } L^2(\partial\Omega;\mathbb{R}^n)\,.\end{cases} \tag{4.15}$$

By the energy balance (4.8) between two arbitrary times $0\leq t_1<t_2\leq T$ we get that

$$\begin{aligned}\int_{t_1}^{t_2}\mathcal{H}(\dot p_\varepsilon(s))\,\mathrm{d}s \leq \frac12\Big(\|\dot u_\varepsilon(t_1)\|^2_{L^2(\Omega;\mathbb{R}^n)} - \|\dot u_\varepsilon(t_2)\|^2_{L^2(\Omega;\mathbb{R}^n)}\Big) + \big(\mathcal{Q}(e_\varepsilon(t_1)) - \mathcal{Q}(e_\varepsilon(t_2))\big)\\ + \int_{t_1}^{t_2}\int_\Omega f\cdot\dot u_\varepsilon\,\mathrm{d}x\,\mathrm{d}s + \int_{t_1}^{t_2}\int_{\partial\Omega} g_\varepsilon\cdot\dot u_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\,.\end{aligned}$$

According to the bounds (4.9), the right hand side above can be controled by $C(t_2-t_1)$, for some constant $C>0$ independent of ε and t_1, t_2. On the other hand, by Jensen's inequality and (4.2), we get that

$$\|p_\varepsilon(t_2)-p_\varepsilon(t_1)\|_{L^1(\Omega;\mathbb{M}^n_{\rm sym})}\le C(t_2-t_1)\,,$$

so, by Ascoli-Arzelà Theorem, there is $p\in C^{0,1}([0,T];\mathcal{M}_b(\Omega;\mathbb{M}^n_{\rm sym}))$ such that, up to a further subsequence (independent of t), for every $t\in[0,T]$

$$p_\varepsilon(t)\overset{*}{\rightharpoonup}p(t)\quad\text{weakly* in }\mathcal{M}_b(\Omega;\mathbb{M}^n_{\rm sym})\,. \tag{4.16}$$

By the additive decomposition $\mathrm{E}u_\varepsilon(t)=e_\varepsilon(t)+p_\varepsilon(t)$, and the convergences in (4.15) and (4.16), we have that $u\in C^{0,1}([0,T];BD(\Omega))$, and for every $t\in[0,T]$

$$u_\varepsilon(t)\overset{*}{\rightharpoonup}u(t)\quad\text{weakly* in }BD(\Omega)\,.$$

According to the previously established weak convergences, we can pass to the limit in the initial conditions

$$u(0)=u_0,\quad \dot u(0)=v_0,\quad e(0)=e_0,\quad p(0)=p_0,$$

the additive decomposition,

$$\mathrm{E}u(t)=e(t)+p(t)\quad\text{in }\mathcal{M}_b(\Omega;\mathbb{M}^n_{\rm sym})\,,$$

the constitutive law

$$\sigma=\mathbf{A}e\quad\text{a.e. in }\Omega\times(0,T)\,,$$

and the equation of motion

$$\ddot u-\operatorname{div}\sigma=f\quad\text{in }L^2(0,T;L^2(\Omega;\mathbb{R}^n))\,,$$

where we used that $\varepsilon\mathrm{E}\dot u_\varepsilon\to0$ strongly in $L^2(0,T;L^2(\Omega;\mathbb{M}^n_{\rm sym}))$ in the last relation. Note that $\sigma_\varepsilon+\varepsilon\mathrm{E}\dot u_\varepsilon\rightharpoonup\sigma$ weakly in $L^2(0,T;L^2(\Omega;\mathbb{M}^n_{\rm sym}))$ and $\operatorname{div}(\sigma_\varepsilon+\varepsilon\mathrm{E}\dot u_\varepsilon)=\ddot u_\varepsilon+f\rightharpoonup\ddot u+f=\operatorname{div}\sigma$ weakly in $L^2(0,T;L^2(\Omega;\mathbb{R}^n))$ so that $\sigma_\varepsilon+\varepsilon\mathrm{E}\dot u_\varepsilon\rightharpoonup\sigma$ weakly in $L^2(0,T;H(\operatorname{div},\Omega))$. Thus $(\sigma_\varepsilon+\varepsilon\mathrm{E}\dot u_\varepsilon)\nu\rightharpoonup\sigma\nu$ weakly in $L^2(0,T;H^{-\frac12}(\partial\Omega;\mathbb{R}^n))$, and since $\big((\sigma_\varepsilon+\varepsilon\mathrm{E}\dot u_\varepsilon)\nu\big)_{\varepsilon>0}$ is bounded in $L^2(0,T;L^2(\partial\Omega;\mathbb{R}^n))$ by the energy balance, we have that $(\sigma_\varepsilon+\varepsilon\mathrm{E}\dot u_\varepsilon)\nu\rightharpoonup\sigma\nu$ weakly in $L^2(0,T;L^2(\partial\Omega;\mathbb{R}^n))$. In particular, passing to the limit in the boundary condition, we obtain that

$$\sigma\nu+S\dot w=0\quad\text{in }L^2(0,T;L^2(\partial\Omega;\mathbb{R}^n)) \tag{4.17}$$

which will be made more precise later.

It is also possible to pass to the limit in the stress constraint by setting $\tilde\sigma_\varepsilon:=\mathrm{P}_{\mathbf{K}}(\sigma_\varepsilon)$, where $\mathrm{P}_{\mathbf{K}}$ is the orthogonal projection in $\mathbb{M}^n_{\rm sym}$ onto the closed and convex set $\mathbf{K}$ with respect to the canonical scalar product. Since $\mathrm{P}_{\mathbf{K}}$ is 1-Lipschitz and $0\in\mathbf{K}$, we infer that $(\tilde\sigma_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(0,T;L^2(\Omega;\mathbb{M}^n_{\rm sym}))$, like $(\sigma_\varepsilon)_{\varepsilon>0}$. Therefore, up to a subsequence, $\tilde\sigma_\varepsilon\rightharpoonup\tilde\sigma$ weakly in $L^2(0,T;L^2(\Omega;\mathbb{M}^n_{\rm sym}))$, for some $\tilde\sigma\in L^2(0,T;L^2(\Omega;\mathbb{M}^n_{\rm sym}))$ with $\tilde\sigma\in\mathbf{K}$ a.e. in $\Omega\times(0,T)$. By the energy balance (4.8) together with the visco–plastic flow rule, we have that $\sigma_\varepsilon-\tilde\sigma_\varepsilon=\varepsilon\dot p_\varepsilon\to0$ strongly in $L^2(0,T;L^2(\Omega;\mathbb{M}^n_{\rm sym}))$. As a consequence $\sigma=\tilde\sigma$ a.e. in $\Omega\times(0,T)$ which shows the validy of the stress constraint

$$\sigma\in\mathbf{K}\quad\text{a.e. in }\Omega\times(0,T).$$

4.4.2. *Strong compactness.* We next improve the weak convergences into strong convergences. Indeed, substracting the equations of motion, we have

$$\ddot u_\varepsilon-\ddot u-\operatorname{div}(\sigma_\varepsilon-\sigma+\varepsilon\mathrm{E}\dot u_\varepsilon)=0\quad\text{in }L^2(0,T;L^2(\Omega;\mathbb{R}^n))\,.$$

Taking $\dot u_\varepsilon\mathbf{1}_{[0,t]}\in L^2(0,T;H^1(\Omega;\mathbb{R}^n))$ as test function, we get that

$$\begin{aligned}\int_0^t\int_\Omega(\ddot u_\varepsilon-\ddot u)\cdot\dot u_\varepsilon\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_\Omega(\sigma_\varepsilon-\sigma):\mathrm{E}\dot u_\varepsilon\,\mathrm{d}x\,\mathrm{d}s+\varepsilon\int_0^t\int_\Omega|\mathrm{E}\dot u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s\\ =\int_0^t\int_{\partial\Omega}((\sigma_\varepsilon+\varepsilon\mathrm{E}\dot u_\varepsilon)\nu-\sigma\nu)\cdot\dot u_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\\ =-\int_0^t\int_{\partial\Omega}S(\dot u_\varepsilon-\dot w)\cdot\dot u_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s+\int_0^t\int_{\partial\Omega}g_\varepsilon\cdot\dot u_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\,,\end{aligned}$$

where we used that $S\dot{u}_\varepsilon + (\sigma_\varepsilon + \varepsilon \mathrm{E}\dot{u}_\varepsilon)\nu = 0$ and $S\dot{w} + \sigma\nu = 0$ in $L^2(0,T;L^2(\partial\Omega;\mathbb{R}^n))$. According to the additive decomposition, we have that

$$\int_0^t \int_\Omega (\sigma_\varepsilon - \sigma) : \mathrm{E}\dot{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}s = \int_0^t \int_\Omega (\sigma_\varepsilon - \sigma) : \dot{e}_\varepsilon \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega (\sigma_\varepsilon - \sigma) : \dot{p}_\varepsilon \,\mathrm{d}x\,\mathrm{d}s.$$

Since $\sigma \in \mathbf{K}$ a.e. in $\Omega \times (0,T)$, the visco–plastic flow rule implies that

$$\int_0^t \int_\Omega (\sigma_\varepsilon - \sigma) : \dot{p}_\varepsilon \,\mathrm{d}x\,\mathrm{d}s \geq 0\,.$$

Thus

$$\begin{aligned}\int_0^t \int_\Omega (\ddot{u}_\varepsilon - \ddot{u}) \cdot (\dot{u}_\varepsilon - \dot{u}) \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega (\sigma_\varepsilon - \sigma) : (\dot{e}_\varepsilon - \dot{e}) \,\mathrm{d}x\,\mathrm{d}s + \varepsilon \int_0^t \int_\Omega |\mathrm{E}\dot{u}_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}s \\ + \int_0^t \int_{\partial\Omega} S(\dot{u}_\varepsilon - \dot{w}) \cdot (\dot{u}_\varepsilon - \dot{w}) \,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s \leq - \int_0^t \int_\Omega (\ddot{u}_\varepsilon - \ddot{u}) \cdot \dot{u} \,\mathrm{d}x\,\mathrm{d}s \\ - \int_0^t \int_\Omega (\sigma_\varepsilon - \sigma) : \dot{e} \,\mathrm{d}x\,\mathrm{d}s - \int_0^t \int_{\partial\Omega} S(\dot{u}_\varepsilon - \dot{w}) \cdot \dot{w} \,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s + \int_0^t \int_{\partial\Omega} g_\varepsilon \cdot \dot{u}_\varepsilon \,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s.\end{aligned}$$

The weak convergences (4.14) imply that the right hand side of the previous inequality tends to zero as $\varepsilon \to 0$. Thus, since

$$\int_0^t \int_\Omega (\ddot{u}_\varepsilon - \ddot{u}) \cdot (\dot{u}_\varepsilon - \dot{u}) \,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\|\dot{u}_\varepsilon(t) - \dot{u}(t)\|^2_{L^2(\Omega;\mathbb{R}^n)},$$

$$\int_0^t \int_\Omega (\sigma_\varepsilon - \sigma) : (\dot{e}_\varepsilon - \dot{e}) \,\mathrm{d}x\,\mathrm{d}s = \mathcal{Q}(e_\varepsilon(t) - e(t)),$$

and, by (4.4),

$$\int_0^t \int_{\partial\Omega} S(\dot{u}_\varepsilon - \dot{w}) \cdot (\dot{u}_\varepsilon - \dot{w}) \,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s \geq c \int_0^t \int_{\partial\Omega} |\dot{u}_\varepsilon - \dot{w}|^2 \,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s$$

we get the following strong convergences, for all $t \in [0,T]$,

$$\begin{cases} \dot{u}_\varepsilon(t) \to \dot{u}(t) & \text{strongly in } L^2(\Omega;\mathbb{R}^n), \\ \sigma_\varepsilon(t) \to \sigma(t) & \text{strongly in } L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}}), \\ \dot{u}_\varepsilon \to \dot{w} & \text{strongly in } L^2(0,T;L^2(\partial\Omega;\mathbb{R}^n)). \end{cases}$$

In addition, the estimates (4.9) and Lebesgue's dominated convergence theorem yield

$$\begin{cases} \dot{u}_\varepsilon \to \dot{u} & \text{strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^n)), \\ \sigma_\varepsilon \to \sigma & \text{strongly in } L^2(0,T;L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}})), \end{cases}$$

According to the boundary condition at ε fixed, we deduce that the sequence $\big((\sigma_\varepsilon + \varepsilon \mathrm{E}\dot{u}_\varepsilon)\nu\big)_{\varepsilon>0}$ is strongly converging in $L^2(0,T;L^2(\partial\Omega;\mathbb{R}^n))$, and thus,

$$(\sigma_\varepsilon + \varepsilon \mathrm{E}u_\varepsilon)\nu \to \sigma\nu \quad \text{strongly in } L^2(0,T;L^2(\partial\Omega;\mathbb{R}^n)).$$

Remark 4.14. In the case of a cylindrical convex set $\mathbf{K}$ of the form (4.7), since the plastic strain is deviatoric then the convergence (4.16) has to be replaced by

$$p_\varepsilon(t) \overset{*}{\rightharpoonup} p(t) \quad \text{weakly* in } \mathcal{M}_b(\Omega;\mathbb{M}^n_D)$$

for all $t \in [0,T]$. Thus, Remark 4.13 ensures that $u(t)\cdot\nu = w(t)\cdot\nu$ on $\partial\Omega$, hence $\dot{w}(t)\cdot\nu = \dot{u}(t)\cdot\nu$ on $\partial\Omega$ for a.e. $t \in (0,T)$. Taking the normal component of the boundary condition (4.17) yields

$$\dot{u}\cdot\nu + (S^{-1}\sigma\nu)\cdot\nu = 0 \quad \text{in } L^2(0,T;L^2(\partial\Omega;\mathbb{R}^n)).$$

4.4.3. *Energy balance.* We first localize the energy balance (4.8) for fixed ε. To this end, we consider a test function $\varphi \in C_c^\infty(\mathbb{R}^n)$ and we multiply the equation of motion by $\varphi \dot{u}_\varepsilon$. After integration by parts in space and time, we obtain for all $t \in [0,T]$,

$$\begin{aligned}\frac{1}{2}\int_\Omega \varphi|\dot{u}_\varepsilon(t)|^2\,\mathrm{d}x + \int_\Omega \varphi\, Q(e_\varepsilon(t))\,\mathrm{d}x + \int_0^t\int_\Omega \varphi\, H(\dot{p}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\partial\Omega} \varphi\, S\dot{u}_\varepsilon\cdot\dot{u}_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\\ + \varepsilon\int_0^t\int_\Omega \varphi|\mathrm{E}\dot{u}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s + \varepsilon\int_0^t\int_\Omega \varphi|\dot{p}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \sigma_\varepsilon : (\nabla\varphi\odot\dot{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s\\ = \frac{1}{2}\int_\Omega \varphi|v_0|^2\,\mathrm{d}x + \int_\Omega \varphi\, Q(e_0)\,\mathrm{d}x + \int_0^t\int_\Omega \varphi f\cdot\dot{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\partial\Omega}\varphi g_\varepsilon\cdot\dot{u}_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\,. \quad (4.18)\end{aligned}$$

If further $\varphi \geq 0$, we write the energy balance (4.18) as

$$\begin{aligned}\frac{1}{2}\int_\Omega \varphi|\dot{u}_\varepsilon(t)|^2\,\mathrm{d}x + \int_\Omega \varphi\, Q(e_\varepsilon(t))\,\mathrm{d}x + \int_0^t\int_\Omega \varphi H(\dot{p}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\partial\Omega} \varphi\, S\dot{u}_\varepsilon\cdot\dot{u}_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\\ + \int_0^t\int_\Omega \sigma_\varepsilon : (\nabla\varphi\odot\dot{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s\\ \leq \frac{1}{2}\int_\Omega \varphi|v_0|^2\,\mathrm{d}x + \int_\Omega \varphi\, Q(e_0)\,\mathrm{d}x + \int_0^t\int_\Omega \varphi f\cdot\dot{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\partial\Omega}\varphi g_\varepsilon\cdot\dot{u}_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\,.\end{aligned}$$

According to the strong convergences of the velocity and the stress, we have

$$\frac{1}{2}\int_\Omega \varphi|\dot{u}_\varepsilon(t)|^2\,\mathrm{d}x + \int_\Omega \varphi\, Q(e_\varepsilon(t))\,\mathrm{d}x \to \frac{1}{2}\int_\Omega \varphi|\dot{u}(t)|^2\,\mathrm{d}x + \int_\Omega \varphi\, Q(e(t))\,\mathrm{d}x,$$

$$\int_0^t\int_\Omega \sigma_\varepsilon : (\nabla\varphi\odot\dot{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s \to \int_0^t\int_\Omega \sigma : (\nabla\varphi\odot\dot{u})\,\mathrm{d}x\,\mathrm{d}s$$

and

$$\int_0^t\int_\Omega \varphi f\cdot\dot{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\partial\Omega}\varphi g_\varepsilon\cdot\dot{u}_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s \to \int_0^t\int_\Omega \varphi f\cdot\dot{u}\,\mathrm{d}x\,\mathrm{d}s.$$

Using the boundary condition and the strong convergences of the boundary velocity and the normal stress, we get

$$\begin{aligned}\int_0^t\int_{\partial\Omega} \varphi\, S\dot{u}_\varepsilon\cdot\dot{u}_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s &= \frac{1}{2}\int_0^t\int_{\partial\Omega} \varphi\, S\dot{u}_\varepsilon\cdot\dot{u}_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\\ &+ \frac{1}{2}\int_0^t\int_{\partial\Omega} \varphi\, S^{-1}\big((\sigma_\varepsilon+\varepsilon\mathrm{E}\dot{u}_\varepsilon)\nu - g_\varepsilon\big)\cdot\big((\sigma_\varepsilon+\varepsilon\mathrm{E}\dot{u}_\varepsilon)\nu - g_\varepsilon\big)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\\ &\to \frac{1}{2}\int_0^t\int_{\partial\Omega} \varphi\, S\dot{w}\cdot\dot{w}\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s + \frac{1}{2}\int_0^t\int_{\partial\Omega} \varphi\, S^{-1}(\sigma\nu)\cdot(\sigma\nu)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s.\end{aligned}$$

Let $0 = t_0 \leq t_1 \leq \cdots \leq t_N = t$ be a partition of $[0,t]$. Since, for all $0 \leq i \leq N$, we have $p_\varepsilon(t_i) \overset{*}{\rightharpoonup} p(t_i)\llcorner\Omega + (w(t_i)-u(t_i))\odot\nu\mathcal{H}^{n-1}\llcorner\partial\Omega$ weakly* in $\mathcal{M}_b(\overline{\Omega};\mathbb{M}^n_{\mathrm{sym}})$, using Jensen's inequality and Reshetnyak's lower semicontinuity Theorem, we get that

$$\begin{aligned}\liminf_{\varepsilon\to0}\int_0^t\int_\Omega \varphi H(\dot{p}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s &\geq \sum_{i=1}^N \liminf_{\varepsilon\to0}\int_\Omega \varphi H(p_\varepsilon(t_i)-p_\varepsilon(t_{i-1}))\,\mathrm{d}x\\ &\geq \sum_{i=1}^N\Big\{\int_\Omega \varphi\,\mathrm{d}H(p(t_i)-p(t_{i-1}))\\ &+ \int_{\partial\Omega}\varphi H\big(((w(t_i)-u(t_i))-(w(t_{i-1})-u(t_{i-1}))\odot\nu\big)\,\mathrm{d}\mathcal{H}^{n-1}\Big\}.\end{aligned}$$

Passing to the supremum with respect to all partitions and using [15, Theorem 7.1], we get that

$$\liminf_{\varepsilon\to0}\int_0^t\int_\Omega \varphi H(\dot{p}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s \geq \int_0^t\int_\Omega \varphi\,\mathrm{d}H(\dot{p}(s))\,\mathrm{d}s + \int_0^t\int_{\partial\Omega}\varphi H\big((\dot{w}(s)-\dot{u}(s))\odot\nu\big)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\,.$$

Gathering all previous convergences and using the definition of ψ, we deduce that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \varphi|\dot u(t)|^2\,\mathrm{d}x + \int_\Omega \varphi\, Q(e(t))\,\mathrm{d}x + \int_0^t\int_\Omega \varphi\,\mathrm{d}H(\dot p(s))\,\mathrm{d}s\\
&+\int_0^t\int_\Omega \sigma : (\nabla\varphi\odot\dot u)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\partial\Omega}\varphi\psi(x,\dot u)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s + \frac12\int_0^t\int_{\partial\Omega}\varphi S^{-1}(\sigma\nu)\cdot(\sigma\nu)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\\
&\qquad\le \frac{1}{2}\int_\Omega \varphi|\dot u(t)|^2\,\mathrm{d}x + \int_\Omega \varphi\, Q(e(t))\,\mathrm{d}x + \int_0^t\int_\Omega \varphi\,\mathrm{d}H(\dot p(s))\,\mathrm{d}s\\
&\qquad+\int_0^t\int_\Omega \sigma : (\nabla\varphi\odot\dot u)\,\mathrm{d}x\,\mathrm{d}s + \frac12\int_0^t\int_{\partial\Omega}\varphi S^{-1}(\sigma\nu)\cdot(\sigma\nu)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\\
&\qquad+\int_0^t\int_{\partial\Omega}\varphi H\big((\dot w-\dot u)\odot\nu\big)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s + \frac12\int_0^t\int_{\partial\Omega}\varphi\, S\dot w\cdot\dot w\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\\
&\qquad\le \frac{1}{2}\int_\Omega \varphi|v_0|^2\,\mathrm{d}x + \int_\Omega \varphi\, Q(e_0)\,\mathrm{d}x + \int_0^t\int_\Omega \varphi f\cdot\dot u\,\mathrm{d}x\,\mathrm{d}s\,. \quad (4.19)
\end{aligned}$$

We now prove the opposite inequality. According to Proposition 4.11 and using the equation of motion, we have that

$$\begin{aligned}
&\int_\Omega \varphi\,\mathrm{d}H(\dot p(t)) + \int_{\partial\Omega}\varphi\psi(x,\dot u(t))\,\mathrm{d}\mathcal{H}^{n-1} + \frac12\int_{\partial\Omega}\varphi S^{-1}(\sigma(t)\nu)\cdot(\sigma(t)\nu)\,\mathrm{d}\mathcal{H}^{n-1}\\
&\quad\ge -\int_\Omega \varphi\sigma(t) : \dot e(t)\,\mathrm{d}x - \int_\Omega \varphi\dot u(t)\cdot\ddot u(t)\,\mathrm{d}x + \int_\Omega \varphi\dot u(t)\cdot f(t)\,\mathrm{d}x - \int_\Omega \sigma(t)\colon(\dot u(t)\odot\nabla\varphi)\,\mathrm{d}x.
\end{aligned}$$

Integrating in time between 0 and t and integrating by parts, leads to the remaining energy inequality

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \varphi|\dot u(t)|^2\,\mathrm{d}x + \int_\Omega \varphi\, Q(e(t))\,\mathrm{d}x + \int_0^t\int_\Omega \varphi\,\mathrm{d}H(\dot p(s))\,\mathrm{d}s\\
&+\int_0^t\int_\Omega \sigma : (\nabla\varphi\odot\dot u)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\partial\Omega}\varphi\psi(x,\dot u)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s + \frac12\int_0^t\int_{\partial\Omega}\varphi S^{-1}(\sigma\nu)\cdot(\sigma\nu)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\\
&\qquad\ge \frac{1}{2}\int_\Omega \varphi|v_0|^2\,\mathrm{d}x + \int_\Omega \varphi\, Q(e_0)\,\mathrm{d}x + \int_0^t\int_\Omega \varphi f\cdot\dot u\,\mathrm{d}x\,\mathrm{d}s\,. \quad (4.20)
\end{aligned}$$

Gathering both energy inequalities (4.19) and (4.20) leads to the energy balance: for all $t\in[0,T]$ and all $\varphi\in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \varphi|\dot u(t)|^2\,\mathrm{d}x + \int_\Omega \varphi\, Q(e(t))\,\mathrm{d}x + \int_0^t\int_\Omega \varphi\,\mathrm{d}H(\dot p(s))\,\mathrm{d}s\\
&+\int_0^t\int_\Omega \sigma : (\nabla\varphi\odot\dot u)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\partial\Omega}\varphi\psi(x,\dot u)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s + \frac12\int_0^t\int_{\partial\Omega}\varphi S^{-1}(\sigma\nu)\cdot(\sigma\nu)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\\
&\qquad= \frac{1}{2}\int_\Omega \varphi|v_0|^2\,\mathrm{d}x + \int_\Omega \varphi\, Q(e_0)\,\mathrm{d}x + \int_0^t\int_\Omega \varphi f\cdot\dot u\,\mathrm{d}x\,\mathrm{d}s\,. \quad (4.21)
\end{aligned}$$

4.4.4. *Flow rule and relaxed boundary condition.* Derivating with respect to time the energy balance (4.21), we get for a.e. $t\in[0,T]$ and all $\varphi\in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}
\int_\Omega \varphi\,\mathrm{d}H(\dot p(t)) + \int_{\partial\Omega}\varphi\psi(x,\dot u(t))\,\mathrm{d}\mathcal{H}^{n-1} &= -\int_\Omega \varphi\sigma(t) : \dot e(t)\,\mathrm{d}x - \int_\Omega \varphi\dot u(t)\cdot\operatorname{div}\sigma(t)\,\mathrm{d}x\\
&\quad - \int_\Omega \sigma(t)\colon(\dot u(t)\odot\nabla\varphi)\,\mathrm{d}x - \frac12\int_{\partial\Omega}\varphi S^{-1}(\sigma(t)\nu)\cdot(\sigma(t)\nu)\,\mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}$$

Taking in particular a test function $\varphi\in C_c^\infty(\Omega)$ with compact support in Ω,

$$\begin{aligned}
\int_\Omega \varphi\,\mathrm{d}H(\dot p(t)) &= -\int_\Omega \varphi\sigma(t) : \dot e(t)\,\mathrm{d}x - \int_\Omega \varphi\dot u(t)\cdot\operatorname{div}\sigma(t)\,\mathrm{d}x - \int_\Omega \sigma(t)\colon(\dot u(t)\odot\nabla\varphi)\,\mathrm{d}x\\
&= \langle[\sigma(t) : \dot p(t)],\varphi\rangle,
\end{aligned}$$

hence we get the measure theoretic flow rule

$$H(\dot p(t)) = [\sigma(t) : \dot p(t)] \quad \text{ in } \mathcal{M}_b(\Omega).$$

Taking $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi = 1$ in $\overline{\Omega}$, we get from the energy balance (4.21) that

$$\frac12\int_\Omega |\dot u(t)|^2\,\mathrm{d}x + \mathcal{Q}(e(t)) + \int_0^t \mathcal{H}(\dot p(s))\,\mathrm{d}s + \int_0^t\int_{\partial\Omega}\psi(x,\dot u)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s \\ + \frac12\int_0^t\int_{\partial\Omega} S^{-1}(\sigma\nu)\cdot(\sigma\nu)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s = \frac12\int_\Omega |v_0|^2\,\mathrm{d}x + \mathcal{Q}(e_0) + \int_0^t\int_\Omega f\cdot\dot u\,\mathrm{d}x\,\mathrm{d}s\,,$$

and also from (4.19) that

$$\int_0^t\int_{\partial\Omega}\psi(x,\dot u)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s = \int_0^t\int_{\partial\Omega} H\big((\dot w-\dot u)\odot\nu\big)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s + \frac12\int_0^t\int_{\partial\Omega} S\dot w\cdot\dot w\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s.$$

As a consequence for a.e. $(x,t)\in\partial\Omega\times(0,T)$, we have

$$\psi(x,\dot u(x,t)) = H\big((\dot w(x,t)-\dot u(x,t))\odot\nu(x)\big) + \frac12 S(x)\dot w(x,t)\cdot\dot w(x,t)$$

which implies, in view of Lemma 4.8 and (4.10), that

$$\dot w(x,t) = S^{-1}(x)D_z\psi(x,\dot u(x,t)) = S^{-1}(x)\mathrm{P}_{-\mathbf{K}\nu(x)}(S(x)\dot u(x,t)).$$

Inserting this expression into (4.17) leads to the desired boundary condition

$$\sigma\nu + \mathrm{P}_{-\mathbf{K}\nu}(S\dot u) = 0 \quad \text{ in } L^2(0,T;L^2(\partial\Omega;\mathbb{R}^n)).$$

4.4.5. *Uniqueness.* Let (u_1,e_1,σ_1,p_1) and (u_2,e_2,σ_2,p_2) be two variational solutions of the elasto–plastic model, according to Definition 4.2, associated to the same initial data (u_0,v_0,e_0,p_0), the same source term f and the same boundary matrix S. Let us define

$$u := \frac{u_1+u_2}{2},\quad e := \frac{e_1+e_2}{2},\quad p := \frac{p_1+p_2}{2},\quad \sigma := \frac{\sigma_1+\sigma_2}{2}.$$

Then, by linearity and convexity, we have

$$u(0)=u_0,\quad \dot u(0)=v_0,\quad e(0)=e_0,\quad p(0)=p_0, \tag{4.22}$$

for all $t\in[0,T]$,

$$\mathrm{E}u(t) = e(t)+p(t) \quad \text{in } \mathcal{M}_b(\Omega;\mathbb{M}^n_{\mathrm{sym}}),$$
$$\sigma = \mathbf{A}e\in\mathbf{K} \quad \text{a.e. in } \Omega\times(0,T),$$

and

$$\ddot u - \operatorname{div}\sigma = f \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^n)).$$

For $i=1,2$ and all $t\in[0,T]$, we define

$$\mathcal{E}_t(u_i,e_i,p_i) := \frac12\int_\Omega|\dot u_i(t)|^2\,\mathrm{d}x + \mathcal{Q}(e_i(t)) + \int_0^t\mathcal{H}(\dot p_i(s))\,\mathrm{d}s - \int_0^t\int_\Omega f\cdot\dot u_i\,\mathrm{d}x\,\mathrm{d}s \\ + \int_0^t\int_{\partial\Omega}\psi(x,\dot u_i)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s + \frac12\int_0^t\int_{\partial\Omega} S^{-1}(\sigma_i\nu)\cdot(\sigma_i\nu)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s,$$

and similarly for $\mathcal{E}_t(u,e,p)$. By convexity of H and ψ, we have for all $t\in[0,T]$

$$\int_\Omega\left|\frac{u_1(t)-u_2(t)}{2}\right|^2\mathrm{d}x + \mathcal{Q}\left(\frac{e_1(t)-e_2(t)}{2}\right) + \mathcal{E}_t(u,e,p) \\ \le \frac12\mathcal{E}_t(u_1,e_1,p_1) + \frac12\mathcal{E}_t(u_2,e_2,p_2) = \frac12\int_\Omega|v_0|^2\,\mathrm{d}x + \mathcal{Q}(e_0)\,. \tag{4.23}$$

According to Proposition 4.11 together with the equation of motion, we have that for a.e. $t\in(0,T)$,

$$\mathcal{H}(\dot p(t)) + \int_{\partial\Omega}\psi(x,\dot u(t))\,\mathrm{d}\mathcal{H}^{n-1} + \frac12\int_{\partial\Omega} S^{-1}(\sigma(t)\nu)\cdot(\sigma(t)\nu)\,\mathrm{d}\mathcal{H}^{n-1} \\ \ge -\int_\Omega\sigma(t):\dot e(t)\,\mathrm{d}x - \int_\Omega\ddot u(t)\cdot\dot u(t)\,\mathrm{d}x + \int_\Omega f(t)\cdot\dot u(t)\,\mathrm{d}x.$$

We next integrate by parts with respect to time and use the initial conditions (4.22) to obtain that

$$\mathcal{E}_t(u,e,p) \geq \frac{1}{2}\int_\Omega |v_0|^2 \,\mathrm{d}x + \mathcal{Q}(e_0). \tag{4.24}$$

Gathering (4.23) and (4.24) implies that $u_1 = u_2$ and $e_1 = e_2$, hence $p_1 = p_2$ as well.

Remark 4.15. Let us observe that our proof of uniqueness does not explicitely use the dissipative boundary condition, but rather its weak formulation through the energy balance. As a consequence, we proved the uniqueness of solutions among all triples (u,e,p) satisfying all requirements in Definition 4.2 except the relaxed boundary condition.

5. Entropic-dissipative solutions

Following the discussion of Subsection 3.2, we introduce the following generalized notion of solutions to the dynamic elasto–plasticity model.

Definition 5.1. Assume that hypotheses (H_1)–(H_4) are satisfied. A pair

$$(v,\sigma) \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n\times\mathbf{K}))$$

defines an *entropic–dissipative solution to the dynamic elasto–plasticity model* associated to the initial data $(v_0,\sigma_0)\in L^2(\Omega;\mathbb{R}^n\times\mathbf{K})$ and the source term $f\in L^2(0,T;L^2(\Omega;\mathbb{R}^n))$ if

$$\begin{aligned}\int_0^T\int_\Omega \left(|v-z|^2+\mathbf{A}^{-1}(\sigma-\tau):(\sigma-\tau)\right)\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t - 2\int_0^T\int_\Omega(\sigma-\tau):((v-z)\odot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t\\ +\int_\Omega\varphi(0)\left(|v_0-z|^2+\mathbf{A}^{-1}(\sigma_0-\tau):(\sigma_0-\tau)\right)\mathrm{d}x\\ +2\int_0^T\int_\Omega f\cdot(v-z)\varphi\,\mathrm{d}x\,\mathrm{d}t+\frac12\int_0^T\int_{\partial\Omega}S^{-1}(\tau\nu+Sz)\cdot(\tau\nu+Sz)\varphi\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t\geq 0,\end{aligned}$$

for all constant vectors $(z,\tau)\in\mathbb{R}^n\times\mathbf{K}$ and all test functions $\varphi\in C_c^\infty(\mathbb{R}^n\times[0,T))$ with $\varphi\geq 0$.

We intend to show that both notions of variational and entropic–dissipative solutions actually coincide. Using the elasto–visco–plastic approximation, we start by proving that variational solutions are entropic–dissipative ones.

Theorem 5.2. *Assuming (H_1)–(H_6), let (u,e,p) be the unique variational solution (obtained in Theorem 4.5) to the dynamic elasto–plastic model associated to the initial data (u_0,v_0,e_0,p_0) and the source term f according to Definition 4.2. Then $(v=\dot u,\sigma=\mathbf{A}e)$ is an entropic–dissipative solution associated to the initial data $(v_0,\sigma_0=\mathbf{A}e_0)$ and the source term f according to Definition 5.1.*

Proof. We multiply the equation of motion $\ddot u_\varepsilon - \operatorname{div}(\sigma_\varepsilon+\varepsilon\mathrm{E}\dot u_\varepsilon-\tau)=f$ by $\varphi(\dot u_\varepsilon - z)$ where $\varphi\in C_c^\infty(\mathbb{R}^n\times[0,T))$ with $\varphi\geq 0$, and integrate by parts in space and time. Then

$$\begin{aligned}&\int_0^T\int_\Omega\varphi f\cdot(\dot u_\varepsilon-z)\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\partial\Omega}(\sigma_\varepsilon+\varepsilon\mathrm{E}\dot u_\varepsilon-\tau)\nu\cdot(\dot u_\varepsilon-z)\varphi\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t\\ &=\int_0^T\int_\Omega\left(\varphi\ddot u_\varepsilon\cdot(\dot u_\varepsilon-z)+\varphi(\sigma_\varepsilon+\varepsilon\mathrm{E}\dot u_\varepsilon-\tau):\mathrm{E}\dot u_\varepsilon+(\sigma_\varepsilon+\varepsilon\mathrm{E}\dot u_\varepsilon-\tau):((\dot u_\varepsilon-z)\odot\nabla\varphi)\right)\mathrm{d}x\,\mathrm{d}t\\ &=\frac12\int_0^T\int_\Omega\varphi\partial_t\left(|\dot u_\varepsilon-z|^2+\mathbf{A}(e_\varepsilon-\mathbf{A}^{-1}\tau):(e_\varepsilon-\mathbf{A}^{-1}\tau)\right)\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega\varphi(\sigma_\varepsilon-\tau):\dot p_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\varepsilon\int_0^T\int_\Omega|\mathrm{E}\dot u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega(\varepsilon\mathrm{E}\dot u_\varepsilon+(\sigma_\varepsilon-\tau)):((\dot u_\varepsilon-z)\odot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

Since $\tau \in \mathbf{K}$, the flow rule implies that $(\sigma_\varepsilon - \tau) : \dot{p}_\varepsilon \geq 0$ a.e. in $\Omega \times (0,T)$. Hence, integrating by parts the first integral in the right–hand-side yields

$$\begin{aligned}
&\int_0^T \int_\Omega \varphi f \cdot (\dot{u}_\varepsilon - z)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\partial\Omega} (\sigma_\varepsilon + \varepsilon \mathrm{E}\dot{u}_\varepsilon - \tau)\nu \cdot (\dot{u}_\varepsilon - z)\varphi\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t \\
&\geq -\frac{1}{2}\int_0^T \int_\Omega \partial_t\varphi \left(|\dot{u}_\varepsilon - z|^2 + \mathbf{A}(e_\varepsilon - \mathbf{A}^{-1}\tau) : (e_\varepsilon - \mathbf{A}^{-1}\tau)\right)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \frac{1}{2}\int_\Omega \varphi(0)\left(|v_0 - z|^2 + \mathbf{A}(e_0 - \mathbf{A}^{-1}\tau) : (e_0 - \mathbf{A}^{-1}\tau)\right)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega (\varepsilon \mathrm{E}\dot{u}_\varepsilon + (\sigma_\varepsilon - \tau)) : ((\dot{u}_\varepsilon - z) \odot \nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ and using the previously established convergence leads to

$$\begin{aligned}
2\int_0^T \int_\Omega \varphi f \cdot (\dot{u} - z)\,\mathrm{d}x\,\mathrm{d}t + 2\int_0^T \int_{\partial\Omega} \varphi(\sigma - \tau)\nu \cdot \left(S^{-1}D_z\psi(\,\bullet\,, \dot{u}) - z\right)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t& \\
+ \int_0^T \int_\Omega \partial_t\varphi \left(|\dot{u} - z|^2 + \mathbf{A}^{-1}(\sigma - \tau) : (\sigma - \tau)\right)\,\mathrm{d}x\,\mathrm{d}t& \\
+ \int_\Omega \varphi(0)\left(|v_0 - z|^2 + \mathbf{A}^{-1}(\sigma_0 - \tau) : (\sigma_0 - \tau)\right)\,\mathrm{d}x\,\mathrm{d}t& \\
- 2\int_0^T \int_\Omega (\sigma - \tau) : ((\dot{u} - z) \odot \nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t \geq 0\,.&
\end{aligned}$$

Using the parallelogram identity, we have

$$\begin{aligned}
&2(\sigma - \tau)\nu \cdot \left(S^{-1}D_z\psi(\,\bullet\,, \dot{u}) - z\right) = 2(\sigma - \tau)\nu \cdot S^{-1}\left(D_z\psi(\,\bullet\,, \dot{u}) - Sz\right) \\
&= 2\left\|\frac{(\sigma - \tau)\nu}{2} + \frac{D_z\psi(\,\bullet\,, \dot{u}) - Sz}{2}\right\|_{S^{-1}}^2 - 2\left\|\frac{(\sigma - \tau)\nu}{2} - \frac{D_z\psi(\,\bullet\,, \dot{u}) - Sz}{2}\right\|_{S^{-1}}^2 \\
&\leq 2\left\|\frac{(\sigma - \tau)\nu}{2} + \frac{D_z\psi(\,\bullet\,, \dot{u}) - Sz}{2}\right\|_{S^{-1}}^2 = \frac{1}{2}\left\|\tau\nu + Sz\right\|_{S^{-1}}^2 = \frac{1}{2}S^{-1}(\tau\nu + Sz) \cdot (\tau\nu + Sz)\,,
\end{aligned}$$

where we used the boundary condition $\sigma\nu + D_z\psi(\,\bullet\,, \dot{u}) = 0$ on $\partial\Omega \times (0,T)$. We deduce that

$$\begin{aligned}
2\int_0^T \int_\Omega \varphi f \cdot (\dot{u} - z)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^T \int_{\partial\Omega} \varphi S^{-1}(\tau\nu + Sz) \cdot (\tau\nu + Sz)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t& \\
+ \int_0^T \int_\Omega \partial_t\varphi \left(|\dot{u} - z|^2 + \mathbf{A}^{-1}(\sigma - \tau) : (\sigma - \tau)\right)\,\mathrm{d}x\,\mathrm{d}t& \\
+ \int_\Omega \varphi(0)\left(|v_0 - z|^2 + \mathbf{A}^{-1}(\sigma_0 - \tau) : (\sigma_0 - \tau)\right)\,\mathrm{d}x\,\mathrm{d}t& \\
- 2\int_0^T \int_\Omega (\sigma - \tau) : ((\dot{u} - z) \odot \nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t \geq 0\,,&
\end{aligned}$$

which completes the proof of the Theorem. □

We now prove the converse implication, namely that any entropic–dissipative solution generates a variational solution to the dynamic elasto–plastic model.

Theorem 5.3. *Assuming (H_1)–(H_5), let $(v,\sigma) \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n \times \mathbf{K}))$ be an entropic–dissipative solution to the dynamic elasto–plasticity model associated to the initial data $(v_0,\sigma_0) \in H^2(\Omega;\mathbb{R}^n) \times \mathcal{K}(\Omega)$ with $Sv_0 + \sigma_0\nu = 0$ on $\partial\Omega$, and the source term $f \in H^1(0,T;L^2(\Omega;\mathbb{R}^n))$. Then, for any initial displacement $u_0 \in H^1(\Omega;\mathbb{R}^n)$, there exists a triple (u,e,p), with $v = \dot{u}$, $\sigma = \mathbf{A}e$, which is a variational solution to the dynamic elasto–plastic model associated to the initial data $(u_0, v_0, e_0, p_0 = \mathrm{E}u_0 - e_0)$ and the source term f.*

Proof. We divide the proof in several steps, the four first ones closely following the proof of [8, Proposition 7.2].

Step 1: Initial conditions for v and σ. By a general property of dissipative formulations of initial-boundary value problems proven in [35, Lemma 3], the initial condition is satisfied in the essential limit sense, i.e.,

$$\lim_{t\to 0}\lim_{\alpha\to 0}\frac{1}{\alpha}\int_{t-\alpha}^{t}\int_\Omega |\sigma(x,s)-\sigma_0(x)|^2\,\mathrm{d}x\,\mathrm{d}s=0,\quad \lim_{t\to 0}\lim_{\alpha\to 0}\frac{1}{\alpha}\int_{t-\alpha}^{t}\int_\Omega |v(x,s)-v_0(x)|^2\,\mathrm{d}x\,\mathrm{d}s=0\,.$$

Note that in [35], the authors consider unconstrained Friedrichs systems. However, a careful inspection of the proof of [35, Lemma 3] reveals that this result still holds in the constrained case. Together with the property that $\sigma\in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}}))$ and $v\in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n))$, that gives $(\sigma(t),v(t))\to(\sigma(0),v(0))$ in $L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}})\times L^2(\Omega;\mathbb{R}^n)$ as $t\to 0$, the identities above imply that $(\sigma(0),v(0))=(\sigma_0,v_0)$.

Step 2: Definition of (u,e,p) and preliminary properties. For all $t\in[0,T]$, we set

$$u(t):=u_0+\int_0^t v(s)\,\mathrm{d}s$$

as a Bochner integral in $L^2(\Omega;\mathbb{R}^n)$. In view of the Lipschitz regularity of v, we get that $u\in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^n))$. By construction $u(0)=u_0$ and, by Step 1, $\dot u(0)=v_0$. Let us define

$$e:=\mathbf{A}^{-1}\sigma\ \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{\mathrm{sym}}))\,,\qquad p:=\mathrm{E}u-e\in W^{1,\infty}(0,T;H^{-1}(\Omega;\mathbb{M}^n_{\mathrm{sym}}))\,. \tag{5.1}$$

Integrating by parts with respect to time the entropic–dissipative inequality in Definition 5.1, we get that for any $\varphi\in C^1_c(\mathbb{R}^n\times[0,T))$ with $\varphi\ge 0$ and any $(z,\tau)\in\mathbb{R}^n\times\mathbf{K}$

$$\begin{aligned}-2\int_0^T\int_\Omega(\ddot u\cdot(\dot u-z)+\dot e\colon(\sigma-\tau))\,\varphi\,\mathrm{d}x\,\mathrm{d}t-2\int_0^T\int_\Omega(\sigma-\tau):((\dot u-z)\odot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t\\ +2\int_0^T\int_\Omega f\cdot(\dot u-z)\varphi\,\mathrm{d}x\,\mathrm{d}t+\frac12\int_0^T\int_{\partial\Omega}S^{-1}(\tau\nu+Sz)\cdot(\tau\nu+Sz)\varphi\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t\ge 0\,.\end{aligned}$$

Factorizing in z and τ, and performing a further integration by parts, we get that

$$\begin{aligned}&-\int_0^T\int_\Omega\Big((\ddot u\cdot\dot u+\dot e\colon\sigma-f\cdot\dot u)\,\varphi+\sigma:(\nabla\varphi\odot\dot u)\Big)\,\mathrm{d}x\,\mathrm{d}t+z\cdot\int_0^T\int_\Omega(\ddot u\varphi+\sigma\,\nabla\varphi-f\,\varphi)\,\mathrm{d}x\,\mathrm{d}t\\ &+\tau:\int_0^T\int_\Omega(\dot e\varphi+\dot u\odot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t+\frac14\int_0^T\int_{\partial\Omega}S^{-1}(\tau\nu-Sz)\cdot(\tau\nu-Sz)\varphi\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t\ge 0\,.\end{aligned}\tag{5.2}$$

With the choice $(z,\tau)=0$ we obtain that

$$-\int_0^T\int_\Omega\Big((\ddot u\cdot\dot u+\dot e\colon\sigma-f\cdot\dot u)\,\varphi+\sigma:(\nabla\varphi\odot\dot u)\Big)\,\mathrm{d}x\ \mathrm{d}t\ge 0$$

for any $\varphi\in C^1_c(\mathbb{R}^n\times[0,T))$ with $\varphi\ge 0$, and localizing in time this gives for a.e. $t\in(0,T)$,

$$\langle\mu(t),\phi\rangle:=-\int_\Omega\Big((\ddot u(t)\cdot\dot u(t)+\dot e(t)\colon\sigma(t)-f(t)\cdot\dot u(t))\,\phi+\sigma(t):(\nabla\phi\odot\dot u(t))\Big)\,\mathrm{d}x\ge 0 \tag{5.3}$$

for any $\phi\in C^1_c(\mathbb{R}^n)$ with $\phi\ge 0$. We deduce that for a.e. $t\in(0,T)$ the (nonnegative) distribution $\mu(t)$ uniquely extends to a nonnegative measure, still denoted by $\mu(t)$, compactly supported in $\overline\Omega$. Moreover, a density argument and Fubini's Theorem imply that the function $t\mapsto\langle\mu(t),\phi\rangle$ is measurable for any $\phi\in C_c(\mathbb{R}^n)$, so that $\mu\colon t\mapsto\mu(t)$ is weak*-measurable in $\mathcal{M}_b(\mathbb{R}^n)$. Eventually, since $\mu(t)$ has compact support in $\overline\Omega$, we can evaluate its mass by taking $\phi\equiv 1$, which gives

$$\mu(t)(\mathbb{R}^n)=\langle\mu(t),1\rangle=-\int_\Omega\big(\ddot u(t)\cdot\dot u(t)+\dot e(t)\colon\sigma(t)-f(t)\cdot\dot u(t)\big)\,\mathrm{d}x\,.$$

By the regularity properties on u, σ, and f, we obtain $\operatorname{ess\,sup}_{t\in(0,T)}\mu(t)(\mathbb{R}^n)<+\infty$, so that

$$\mu\in L^\infty_{w^*}(0,T;\mathcal{M}^+_b(\mathbb{R}^n))\,. \tag{5.4}$$

Step 3: Equation of motion. Choosing $\tau = 0$, $z \in \mathbb{R}^n$ and $\varphi \in C^1_c(\Omega\times(0,T))$ with $\varphi \geq 0$ arbitrarily in (5.2), we get that

$$\int_0^T \int_\Omega (\ddot{u}\varphi + \sigma\,\nabla\varphi - f\,\varphi)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

which implies the validity of the equation of motion

$$\ddot{u} - \operatorname{div}\sigma = f \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^n))\,,$$

and, in particular, $\sigma \in L^\infty(0,T;H(\operatorname{div};\Omega))$. Recalling (2.1), we can define the normal trace $\sigma\nu$ as a element of $L^\infty(0,T;H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^n))$ and, for every $z\in\mathbb{R}^n$ and $\varphi\in C^1_c(\mathbb{R}^n\times[0,T))$

$$z\cdot\int_0^T\int_\Omega (\ddot{u}\,\varphi + \sigma\,\nabla\varphi - f\,\varphi)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle \sigma\nu, z\,\varphi\rangle_{\partial\Omega}\,\mathrm{d}t\,.$$

Inserting the above identity in (5.2) yields

$$\begin{aligned}&-\int_0^T\int_\Omega \Big((\ddot{u}\cdot\dot{u} + \dot{e}\colon\sigma - f\cdot\dot{u})\,\varphi + \sigma : (\nabla\varphi\odot\dot{u})\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\langle\sigma\nu, z\varphi\rangle_{\partial\Omega}\,\mathrm{d}t\\ &+\tau:\int_0^T\int_\Omega(\dot{e}\varphi + \dot{u}\odot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t + \frac14\int_0^T\int_{\partial\Omega} S^{-1}(\tau\nu - Sz)\cdot(\tau\nu - Sz)\varphi\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t \geq 0\,, \quad (5.5)\end{aligned}$$

for any $\varphi\in C^1_c(\mathbb{R}^n\times[0,T))$ with $\varphi\geq 0$ and any $(z,\tau)\in\mathbb{R}^n\times\mathbf{K}$.

Step 4: Flow rule. With the choice $z=0$ in (5.5), and recalling the expression (5.3) of $\mu(t)$, we have that

$$-\tau:\int_0^T\int_\Omega(\dot{e}\varphi + \dot{u}\odot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t \leq \int_0^T\int_\Omega \varphi\,\mathrm{d}\mu(t)\,\mathrm{d}t\,,$$

for any $\tau\in\mathbf{K}$ and any $\varphi\in C^1_c(\Omega\times(0,T))$ with $\varphi\geq 0$. By (5.1) we can recast the above estimate into

$$\int_0^T \langle\tau:\dot{p}(t),\varphi(t)\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\,\mathrm{d}t \leq \int_0^T\int_\Omega\varphi\,\mathrm{d}\mu(t)\,\mathrm{d}t \quad \text{for any } \varphi\in C^1_c(\Omega\times[0,T))\,,\ \varphi\geq 0\,,$$

that may be localized in time, since $\dot{p}\in L^\infty(0,T;H^{-1}(\Omega;\mathbb{M}^n_{\mathrm{sym}}))$, to get, for a.e. $t\in(0,T)$,

$$\langle\tau:\dot{p}(t),\phi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} \leq \int_\Omega \phi\,\mathrm{d}\mu(t) \quad \text{for any } \phi\in C^1_c(\Omega)\,,\ \phi\geq 0. \quad (5.6)$$

This last property implies that, for a.e. $t\in(0,T)$ and all $\tau\in\mathbf{K}$, the distribution $\mu(t)-\tau:\dot{p}(t)$ is nonnegative. Thus, since $\mu(t)$ is a measure, we infer that the distribution $\dot{p}(t)$ uniquely extends to a measure in Ω. Moreover, since $p\in W^{1,\infty}(0,T;H^{-1}(\Omega;\mathbb{M}^n_{\mathrm{sym}}))$, the function $t\mapsto\langle\dot{p}(t),\phi\rangle$ is measurable for all $\phi\in C^1_c(\Omega)$, and by density this holds true also for all $\phi\in C_c(\Omega)$. Therefore $t\mapsto\dot{p}(t)$ is weak* measurable in $\mathcal{M}_b(\Omega;\mathbb{M}^n_{\mathrm{sym}})$. In addition, for any Borel set $A\subset\Omega$, approximating the characteristic function $\mathbf{1}_A$ by $C^1_c(\Omega)$ functions, we deduce from (5.6) that for a.e. $t\in(0,T)$

$$\tau\colon\dot{p}(t)(A)\leq\mu(t)(A) \quad \text{for every Borel set } A\subset\Omega \text{ and every } \ \tau\in\mathbf{K}$$

Maximizing with respect to all $\tau\in\mathbf{K}$ yields by definition of H,

$$H\big(\dot{p}(t)(A)\big)\leq\mu(t)(A) \quad \text{for every Borel set } A\subset\Omega.$$

By the definition of a convex function of measures (cf. e.g. [25]), this leads to

$$H(\dot{p}(t))\leq\mu(t) \quad \text{in } \mathcal{M}^+_b(\Omega)\,. \quad (5.7)$$

Moreover, using (4.2), for a.e. $t\in(0,T)$, we get that $0\leq r|\dot{p}(t)|\leq\mu(t)$ in $\mathcal{M}^+_b(\Omega)$, which, together with (5.4), implies that

$$\dot{p}\in L^\infty_{w*}(0,T;\mathcal{M}_b(\Omega;\mathbb{M}^n_{\mathrm{sym}}))\,. \quad (5.8)$$

From this last property, we deduce that

$$p\in C^{0,1}([0,T];\mathcal{M}_b(\Omega;\mathbb{M}^n_{\mathrm{sym}}))\,,\quad u\in C^{0,1}([0,T];BD(\Omega))\,.$$

Indeed, for all $\phi\in C_c(\Omega)$ the function $t\mapsto\langle p(t),\phi\rangle$ belongs to $W^{1,\infty}(0,T;\mathbb{M}^n_{\mathrm{sym}})$ and $\frac{\mathrm{d}}{\mathrm{d}t}\langle p(t),\phi\rangle = \langle\dot{p}(t),\phi\rangle$ for a.e. $t\in(0,T)$. By (5.8) we infer that

$$|\langle p(t_2)-p(t_1),\phi\rangle|\leq\sup_{t\in(0,T)}|\dot{p}(t)|(\Omega)\;\|\phi\|_{L^\infty(\Omega;\mathbb{M}^n_{\mathrm{sym}})}(t_2-t_1) \quad \text{for all } 0\leq t_1\leq t_2\leq T\,,$$

and the previous inequality gives $p \in C^{0,1}([0,T];\mathcal{M}_b(\Omega;\mathbb{M}^n_{\rm sym}))$, while $u \in C^{0,1}([0,T];BD(\Omega))$ follows from (5.1).

Thus, for a.e. $t \in (0,T)$, we have the additive decomposition $\mathrm{E}\dot u(t) = \dot e(t) + \dot p(t)$ with $\dot u(t) \in BD(\Omega) \cap L^2(\Omega;\mathbb{R}^n)$, $\dot e(t) \in L^2(\Omega;\mathbb{M}^n_{\rm sym})$, $\dot p(t) \in \mathcal{M}_b(\Omega;\mathbb{M}^n_{\rm sym})$, and $\sigma(t) \in \mathcal{K}(\Omega)$. Moreover, using (5.7), we get that $H(\dot p(t))$ is a finite measure. Thus we satisfy the conditions that guarantee (2.2) and (2.3), which in turn give $[\sigma(t)\colon \dot p(t)] \in \mathcal{M}_b(\Omega)$ for a.e. $t \in (0,T)$ and

$$\mu(t) \llcorner \Omega = [\sigma(t)\colon \dot p(t)] \llcorner \Omega \le H(\dot p(t)) \quad \text{in } \mathcal{M}_b^+(\Omega) \ \text{ for a.e. } t \in (0,T)\,.$$

Together with (5.7), the estimate above gives that

$$\mu(t) \llcorner \Omega = [\sigma(t)\colon \dot p(t)] \llcorner \Omega = H(\dot p(t)) \quad \text{in } \mathcal{M}_b(\Omega) \ \text{ for a.e. } t \in (0,T)\,. \tag{5.9}$$

Step 5: Integrability property of the normal stress. Putting the previous information into (5.5) and using the integration by parts formula in BD yields

$$\begin{aligned}&\int_0^T \int_\Omega \varphi \,\mathrm{d}H(\dot p(t))\,\mathrm{d}t + \int_0^T \int_{\partial\Omega} \varphi \,\mathrm{d}\mu(t)\,\mathrm{d}t + \int_0^T \langle \sigma\nu, z\varphi\rangle_{\partial\Omega}\,\mathrm{d}t + \int_0^T \int_{\partial\Omega} (\tau\nu)\cdot \dot u\,\varphi\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t\\ &- \int_0^T \langle \tau : \dot p(t), \varphi(t)\rangle\,\mathrm{d}t + \frac14 \int_0^T \int_{\partial\Omega} S^{-1}(\tau\nu - Sz)\cdot(\tau\nu - Sz)\varphi\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t \ge 0\,,\end{aligned}$$

for any $\varphi \in C^1_c(\mathbb{R}^n \times (0,T))$ with $\varphi \ge 0$ and any $(z,\tau) \in \mathbb{R}^n \times \mathbf{K}$. Localizing in time, we get that for a.e. $t \in (0,T)$

$$\begin{aligned}&\int_\Omega \phi\,\mathrm{d}H(\dot p(t)) + \int_{\partial\Omega} \phi\,\mathrm{d}\mu(t) + \langle \sigma(t)\nu, z\phi\rangle_{\partial\Omega} - \langle \tau : \dot p(t), \phi\rangle\\ &\qquad + \int_{\partial\Omega} (\tau\nu)\cdot \dot u(t)\,\phi\,\mathrm{d}\mathcal{H}^{n-1} + \frac14 \int_{\partial\Omega} S^{-1}(\tau\nu - Sz)\cdot(\tau\nu - Sz)\,\phi\,\mathrm{d}\mathcal{H}^{n-1} \ge 0\,,\end{aligned}$$

for any $\phi \in C^1_c(\mathbb{R}^n)$ with $\phi \ge 0$ and any $(z,\tau) \in \mathbb{R}^n \times \mathbf{K}$. As a consequence of the previous inequality, we deduce that, for a.e. $t \in (0,T)$, the normal trace $\sigma(t)\nu$ is a measure in $\mathcal{M}_b(\partial\Omega;\mathbb{R}^n)$ so that the previous inequality can be localized on $\partial\Omega$. We thus get that for a.e. $t \in (0,T)$,

$$\begin{aligned}&\mu(t) \llcorner \partial\Omega + z\cdot(\sigma(t)\nu)\\ &\qquad \ge \left(-(\tau\nu)\cdot\dot u(t) - \frac14 S^{-1}(\tau\nu - Sz)\cdot(\tau\nu - Sz)\right)\mathcal{H}^{n-1} \llcorner \partial\Omega \quad \text{in } \mathcal{M}_b^+(\partial\Omega)\end{aligned} \tag{5.10}$$

for any $(z,\tau) \in \mathbb{R}^n \times \mathbf{K}$. Using the Besicovitch decomposition Theorem we can split $\sigma(t)\nu$ as $\sigma(t)\nu = \sigma_t \mathcal{H}^{n-1} \llcorner \partial\Omega + \sigma_t^s$ for some $\sigma_t \in L^1(\partial\Omega;\mathbb{R}^n)$ and some singular measure $\sigma_t^s \in \mathcal{M}_b(\partial\Omega;\mathbb{R}^n)$ with respect to $\mathcal{H}^{n-1} \llcorner \partial\Omega$. Similarly, we can decompose $\mu(t) \llcorner \partial\Omega$ as $\mu(t) \llcorner \partial\Omega = \mu_t^a + \mu_t^s$ for some nonnegative measures μ_t^a and μ_t^s which are, respectively, absolutely continuous and singular with respect to $\mathcal{H}^{n-1} \llcorner \partial\Omega$. According to (5.10), we have that

$$\mu(t) \llcorner \partial\Omega \ge \mu_t^s \ge -z\cdot\sigma_t^s \quad \text{in } \mathcal{M}_b^+(\partial\Omega)$$

so that if $A \subset \partial\Omega$ is a Borel set such that $\sigma_t^s(A) \ne 0$, then the previous inequality leads to a contradiction by sending z to infinity since $\mu(t)$ is a finite measure. Therefore, the singular part σ_t^s of the measure $\sigma(t)\nu$ vanishes, and $\sigma(t)\nu$ is absolutely continuous with respect to $\mathcal{H}^{n-1} \llcorner \partial\Omega$. Identifying the measure $\sigma(t)\nu$ with its density σ_t and going back to (5.10) yields

$$\begin{aligned}&\mu(t) \llcorner \partial\Omega + z\cdot(\sigma(t)\nu)\mathcal{H}^{n-1} \llcorner \partial\Omega\\ &\qquad \ge \left(-(\tau\nu)\cdot\dot u(t) - \frac14 S^{-1}(\tau\nu - Sz)\cdot(\tau\nu - Sz)\right)\mathcal{H}^{n-1} \llcorner \partial\Omega \quad \text{in } \mathcal{M}_b^+(\partial\Omega).\end{aligned}$$

A classical measure theoreric argument (see e.g. [11, Proposition 1.6]) implies that one can pass to the supremum in $\tau \in \mathbf{K}$ under the integral sign, hence

$$\begin{aligned}&\mu(t) \llcorner \partial\Omega + z\cdot(\sigma(t)\nu)\mathcal{H}^{n-1} \llcorner \partial\Omega\\ &\qquad \ge \left[\sup_{\tau\in\mathbf{K}} \left(-(\tau\nu)\cdot\dot u(t) - \frac14 S^{-1}(\tau\nu - Sz)\cdot(\tau\nu - Sz)\right)\right]\mathcal{H}^{n-1} \llcorner \partial\Omega \quad \text{in } \mathcal{M}_b^+(\partial\Omega).\end{aligned} \tag{5.11}$$

Using standard convex analysis arguments, the previous supremum may be expressed, for a.e. $(x,t)\in\partial\Omega\times(0,T)$, as

$$\begin{aligned}\sup_{\tau\in\mathbf{K}}\left(-(\tau\nu(x))\cdot\dot u(x,t)-\frac14 S^{-1}(x)(\tau\nu(x)-S(x)z)\cdot(\tau\nu(x)-S(x)z)\right)\\ =\sup_{q\in-\mathbf{K}\nu(x)}\left(q\cdot\dot u(x,t)-\frac14 S^{-1}(x)(S(x)z+q)\cdot(S(x)z+q)\right)\\ =\left(\frac14 S^{-1}(x)(S(x)z+\bullet)\cdot(S(x)z+\bullet)+\mathrm{I}_{-\mathbf{K}\nu(x)}\right)^*(\dot u(x,t))\\ =\left(\frac14 S^{-1}(x)(S(x)z+\bullet)\cdot(S(x)z+\bullet)\right)^*\square\,(\mathrm{I}_{-\mathbf{K}\nu(x)})^*(\dot u(x,t))\,.\end{aligned}$$

Recalling that $(\mathrm{I}_{-\mathbf{K}\nu(x)})^*=h(x,\bullet)$, and computing the convex conjugate

$$\begin{aligned}\left(\frac14 S^{-1}(x)(S(x)z+\bullet)\cdot(S(x)z+\bullet)\right)^*(q)&=\frac12 S(x)q\cdot q+\frac12 S(x)(q-z)\cdot(q-z)-\frac12 S(x)z\cdot z\\&\geq\frac12 S(x)q\cdot q-\frac12 S(x)z\cdot z\end{aligned}$$

we obtain that

$$\begin{aligned}\sup_{\tau\in\mathbf{K}}\left(-(\tau\nu(x))\cdot\dot u(x,t)-\frac14 S^{-1}(x)(\tau\nu(x)-S(x)z)\cdot(\tau\nu(x)-S(x)z)\right)\\ \geq f(x,\bullet)\square\,h(x,\bullet)(\dot u(x,t))-\frac12 S(x)z\cdot z=\psi(x,\dot u(x,t))-\frac12 S(x)z\cdot z\,.\end{aligned}$$

Inserting this last inequality into (5.11), we infer that for a.e. $t\in(0,T)$

$$\int_{\partial\Omega}\phi\,\mathrm{d}\mu(t)\geq\int_{\partial\Omega}\phi\left(-z\cdot(\sigma(t)\nu)-\frac12 Sz\cdot z\right)\mathrm{d}\mathcal{H}^{n-1}+\int_{\partial\Omega}\phi\,\psi(x,\dot u(t))\,\mathrm{d}\mathcal{H}^{n-1}$$

for all $\phi\in C(\partial\Omega)$ with $\phi\geq0$. Using again [11, Proposition 1.6], we can pass to the supremum in $z\in\mathbb{R}^n$ under the integral sign which leads to

$$\int_{\partial\Omega}\phi\,\mathrm{d}\mu(t)\geq\frac12\int_{\partial\Omega}\phi S^{-1}(\sigma(t)\nu)\cdot(\sigma(t)\nu)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\partial\Omega}\phi\,\psi(x,\dot u(t))\,\mathrm{d}\mathcal{H}^{n-1}.\tag{5.12}$$

Employing the coercivity property (4.4) of S and (5.4), we deduce that

$$\sigma\nu\in L^\infty(0,T;L^2(\partial\Omega;\mathbb{R}^n)).$$

Step 6: The boundary condition. Using (5.12), (5.3) and the flow rule (5.9) yields, for a.e. $t\in(0,T)$,

$$\begin{aligned}\frac12\int_{\partial\Omega}S^{-1}(\sigma(t)\nu)\cdot(\sigma(t)\nu)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\partial\Omega}\psi(x,\dot u(t))\,\mathrm{d}\mathcal{H}^{n-1}\leq\mu(t)(\partial\Omega)=\mu(t)(\mathbb{R}^n)-\mu(t)(\Omega)\\ =-\int_\Omega(\ddot u(t)\cdot\dot u(t)+\dot e(t)\colon\sigma(t)-f(t)\cdot\dot u(t))\,\mathrm{d}x-\mathcal{H}(\dot p(t)).\end{aligned}$$

Integrating by parts with respect to time we deduce that for all $0\leq t\leq T$

$$\begin{aligned}&\frac12\int_\Omega|\dot u(t)|^2\,\mathrm{d}x+\mathcal{Q}(e(t))+\int_0^t\mathcal{H}(\dot p(s))\,\mathrm{d}s+\int_0^t\int_{\partial\Omega}\psi(x,\dot u)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\\&+\frac12\int_0^t\int_{\partial\Omega}S^{-1}(\sigma\nu)\cdot(\sigma\nu)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\leq\frac12\int_\Omega|v_0|^2\,\mathrm{d}x+\mathcal{Q}(e_0)+\int_0^t\int_\Omega f\cdot\dot u\,\mathrm{d}x\,\mathrm{d}s\,.\end{aligned}$$

Using Proposition 4.11 and arguing word for word as in the second part of Subsection 4.4.3, we obtain the other energy inequality

$$\begin{aligned}&\frac12\int_\Omega|\dot u(t)|^2\,\mathrm{d}x+\mathcal{Q}(e(t))+\int_0^t\mathcal{H}(\dot p(s))\,\mathrm{d}s+\int_0^t\int_{\partial\Omega}\psi(x,\dot u)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\\&+\frac12\int_0^t\int_{\partial\Omega}S^{-1}(\sigma\nu)\cdot(\sigma\nu)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s\geq\frac12\int_\Omega|v_0|^2\,\mathrm{d}x+\mathcal{Q}(e_0)\,\mathrm{d}x+\int_0^t\int_\Omega f\cdot\dot u\,\mathrm{d}x\,\mathrm{d}s\,.\end{aligned}$$

To summarize, we have proved that the triple (u, e, p) satisfies

$$\begin{cases} u \in W^{2,\infty}(0,T; L^2(\Omega;\mathbb{R}^n)) \cap C^{0,1}([0,T]; BD(\Omega))\,, \\ e \in W^{1,\infty}(0,T; L^2(\Omega;\mathbb{M}^n_{\rm sym}))\,, \\ p \in C^{0,1}([0,T]; \mathcal{M}_b(\Omega;\mathbb{M}^n_{\rm sym}))\,, \end{cases}$$

$$\sigma := \mathbf{A}e \in L^\infty(0,T; H(\mathrm{div},\Omega)), \quad \sigma\nu \in L^\infty(0,T; L^2(\partial\Omega;\mathbb{R}^n))$$

and there hold, at the moment, all the conditions in Definition 4.2 except condition 4.

Step 7: Dissipative boundary conditions. From Remark 4.15, it follows that (u, e, p) is the unique variational solution the dynamic elasto–plastic model associated to the initial data (u_0, v_0, e_0, p_0) and the source term f. In particular the dissipative boundary condition

$$\sigma\nu + \mathrm{P}_{-\mathbf{K}\nu}(S\dot{u}) = 0 \quad \text{ in } L^2(0,T; L^2(\partial\Omega;\mathbb{R}^n))$$

is satisfied, and this completes the proof of the Theorem. □

By Theorems 4.5, 5.2 and 5.3, it turns out that both notions of variational and entropic–dissipative solutions coincide. We thus deduce the following well–posedness result for entropic–dissipative solutions.

Corollary 5.4. *Assuming (H_1)–(H_6), there exists a unique entropic–dissipative solution (v, σ) associated to the initial data (v_0, σ_0) and the source term f according to Definition 5.1.*

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