A brief introduction to the scaling limits and effective equations in kinetic theory

M Pulvirenti, Sergio Simonella

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Abstract These lecture notes provide the material for a short introductory course on effective equations for classical particle systems. They concern the basic equations in kinetic theory, written by Boltzmann and Landau, describing rarefied gases and weakly interacting plasmas respectively. These equations can be derived formally, under suitable scaling limits, taking classical particle systems as a starting point. A rigorous proof of this limiting procedure is difficult and still largely open. We discuss some mathematical problems arising in this context.

1 The foundations of kinetic theory

Many interesting systems in physics and applied sciences consist of a large number of identical components so that they are difficult to analyze from a mathematical point of view. On the other hand, quite often, we are not interested in a detailed description of the system but rather in its collective behaviour. Therefore, it is necessary to look for all procedures leading to simplified models, retaining the interesting features of the original system, cutting away redundant information. This is the methodology of statistical mechanics and of kinetic theory. Here we want to outline the limiting procedure leading from the microscopic description of a large particle system (based

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on the fundamental laws like the Newton or Schrödinger equations) to the more practical picture dictated by kinetic theory.

Although the methods of kinetic theory are frequently applied to a large variety of complex systems (consisting of a huge number of individuals), we will discuss only models arising in physics and more precisely in classical mechanics. The starting point is a system of N identical particles in the physical space. A microscopic state of the system is a sequence $z_1, \cdots, z_N$ where $z_i = (x_i, v_i)$ denotes position and velocity of the i-th particle. The equations of motion are given by Newton's laws of dynamics.

We are interested in a situation where N is very large (for instance, a cubic centimeter of a rarefied gas contains approximately 10^{19} molecules). The knowledge of the microscopic states becomes useless, and we turn to a statistical description. We introduce a probability measure $W_0^N(Z_N)dZ_N$ (absolutely continuous with respect to the Lebesgue measure), defined on the phase space $\mathbb{R}^{3N} \times \mathbb{R}^{3N}$, where

$$Z_N = (z_1, \cdots, z_N) = (x_i, v_i, \cdots, x_N, v_N) \, .$$

W_0^N assigns the same statistical weight to two different vectors Z_N and Z'_N differing only for the order of particles, i.e., identifying the same physical configuration.

The time-evolved measure is defined by

$$W^N(Z_N, t) = W_0^N(\Phi^{-t}(Z_N)) \, . \tag{1}$$

Here $\Phi^t(Z_N)$ denotes the dynamical, measure-preserving flow constructed by solving the equations of motion.

We can establish a partial differential equation, called the Liouville equation, describing the evolution of the measure (1). However, this equation is also not tractable in practice. To have an efficient reduced description, one can focus on the time evolution for the probability distribution of a *given* particle (say particle 1), all the particles being identical. To this end, we define the j-particle marginals

$$f_j^N(Z_j, t) := \int_{\mathbb{R}^{3N} \times \mathbb{R}^{3N}} dz_{j+1} \cdots dz_N W^N(Z_j, z_{j+1}, \cdots, z_N, t) \, , \qquad j = 1, \cdots, N \, , \tag{2}$$

and we look for an equation describing the evolution of f_1^N. We deduce, in most of the physically relevant situations, an evolution equation of the form

$$\partial_t f_1^N = -v \cdot \nabla f_1^N + Q \, . \tag{3}$$

The first term in the right-hand side is due to the free transport of particles, while the term Q should describe the interaction of particle 1 with the rest of the system.

We face a big difficulty. Since the interaction is binary, Q will depend on f_2^N, namely the two-particle marginal. In other words, (3) is still useless: to know f_1^N we need to know f_2^N, and to know f_2^N we need to know f_3^N, and so on. We handle a hierarchy of equations, called BBGKY hierarchy (from the names of the physicists Bogolyubov, Born, Green, Kirkwood, Yvon).

Here enters the property called *propagation of chaos*, that is,

$$f_2^N(x_1, v_1, x_2, v_2, t) \simeq f_1^N(x_1, v_1, t) f_1^N(x_2, v_2, t). \tag{4}$$

Accepting (4), Q becomes an operator acting on f_1^N and (3) is a closed equation. We have thus replaced a huge ordinary differential system by a single PDE. The price we pay is that (3) is nonlinear.

The equality in Eq. (4) is certainly false, since it expresses the statistical independence of particle 1 and particle 2 which, even if assumed at time 0, cannot hold at later times. Indeed, the dynamics creates correlations. Nevertheless, one can hope to recover this property in some *asymptotic* situation described by a suitable scaling limit. This is what happens in two different physical contexts: the low-density and the weak-coupling limits, yielding two different kinetic equations, namely the Boltzmann and the Landau equations, respectively. The passage from hamiltonian mechanics to this kinetic description is actually very delicate. As we shall see later on, we go from a deterministic time-reversible system to an irreversible equation.

A different scaling procedure is the so-called mean-field limit. This leads to the Vlasov equation, which has still a time-reversible, hamiltonian nature. It is a sort of continuum limit and hence much simpler than the previous two. Some challenging and interesting problems concerning the mean-field limit are anyway still open, but we shall not discuss them in this note.

2 Low-density limit and Boltzmann equation

Ludwig Boltzmann established an evolution equation to describe the behaviour of a rarefied gas in 1872, starting from the mathematical model of elastic balls and using mechanical and statistical considerations [3]. The importance of this equation is twofold. On one side, it provides (as well as the hydrodynamical equations) a reduced description of the microscopic world. On the other, it is also an important tool for applications, especially for dilute fluids when the hydrodynamical equations fail to hold.

According to the general paradigm of kinetic theory, the starting point of Boltzmann's analysis is to renounce to study the gas in terms of the detailed motion of the molecules of the full system. It is preferable to investigate a function $f(x, v)$, the probability density of a given particle, where x and v denote its position and velocity. Or, following the original approach proposed by Boltzmann, $f(x, v)dxdv$ is to rather be interpreted as the fraction of molecules happening to be in the cell of the phase space of size $dxdv$ around (x, v). The two quantities are not exactly the same, but they are asymptotically equivalent (when the number of particles diverges) if a *law of large numbers* holds.

Boltzmann considered a gas as microscopically described by a system of elastic (hard) balls, colliding according to the laws of classical mechanics. In this case, the *Boltzmann equation* for the one-particle distribution function reads

$$(\partial_t + v \cdot \nabla_x) f = Q_B(f, f) \tag{5}$$

where Q_B, the collision operator, is defined by

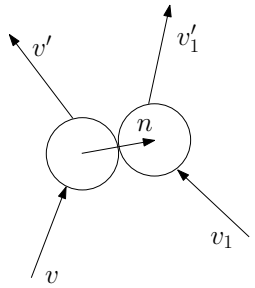
$$Q_B(f,f)(x,v) := \int_{\mathbb{R}^3} dv_1 \int_{S^2_+} dn\,(v-v_1)\cdot n\,[f(x,v')f(x,v_1')-f(x,v)f(x,v_1)]\,, \quad (6)$$

with

$$\begin{aligned} v' &= v - n[n\cdot(v-v_1)] \\ v_1' &= v_1 + n[n\cdot(v-v_1)] \end{aligned} \quad (7)$$

and n a unit vector (impact vector) varying in $S^2_+ = \{n \in S^2 \mid n\cdot(v-v_1)\geq 0\}$.

Note that v' and v_1' are the outgoing velocities after a collision of two elastic balls with incoming velocities v and v_1 and centers x and $x+\varepsilon n$, with ε the diameter of the spheres. The collision takes place if $n\cdot(v-v_1)>0$. Formulas (7) are consequences of the conservation of energy and momenta. Note that ε does not enter (5) as a parameter.



As a fundamental feature of (5), one has the formal conservation (in time) of the five quantities

$$\int dx \int dv\, f(x,v,t)v^\alpha \quad (8)$$

with $\alpha = 0,1,2$, expressing conservation of probability, momentum and energy, respectively. From now on, we shall often abbreviate $\int = \int_{\mathbb{R}^3}$.

Moreover, Boltzmann introduced the (kinetic) entropy defined by

$$H(f) = \int dx \int dv\, f\log f(x,v) \quad (9)$$

and proved the famous H theorem asserting the decrease of $H(f(t))$ along the solutions of (5).

Finally, in the case of bounded domains or homogeneous solutions ($f = f(v,t)$ independent of x), the distribution defined for some $\beta>0$, $\rho>0$ and $u\in\mathbb{R}^3$ by

$$M(v) = \frac{\rho}{(2\pi/\beta)^{3/2}} e^{-\beta/2|v-u|^2}, \quad (10)$$

called Maxwellian distribution, is stationary for the evolution given by (5). In addition, M minimizes H among all distributions with given total mass ρ, mean velocity u and mean energy. The parameter β is interpreted as the inverse temperature.

In conclusion, Boltzmann was able to introduce an evolution equation with the remarkable properties of expressing mass, momentum and energy conservation and also the tendency to thermal equilibrium. In this way, he tried to conciliate Newton's laws with the second principle of thermodynamics.

The H Theorem is apparently in contrast with the laws of mechanics, which are time-reversible. This fact caused skepticism among the scientific community, and the work of Boltzmann was attacked repeatedly. We refer the reader to the monograph by C. Cercignani [5], which is a beautiful compromise between historical account and scientific divulgation, to have a faithful idea of the debate at the time.

To formally derive (5), let us consider a system of N identical hard spheres of diameter ε and unitary mass, interacting by means of the collision law (7). We denote by ε the diameter of the particles which, for the moment, is fixed and not necessarily small.

The phase space Γ_N of the system is the subset of $\mathbb{R}^{3N} \times \mathbb{R}^{3N}$ fulfilling the hard-core condition, namely $|x_i - x_j| \geq \varepsilon$ for $i \neq j$. The dynamical flow $Z_N \to \Phi^t(Z_N)$ is defined as the free flow, i.e., $Z_N \to \Phi^t(Z_N) = (x_1 + v_1 t, v_1, \cdots, x_N + v_N t, v_N)$ up to the first impact time (when $|x_i - x_j| = \varepsilon$); then an instantaneous collision takes place according to the law (7), and the flow goes on up to the next collision instant.

The well-posedness of the hard-sphere dynamics is not obvious, due to the occurrence of multiple collisions or to the a priori possibility that collision times accumulate at a finite limiting time. However, such pathologies cannot occur outside a set of initial conditions Z_N of vanishing measure. Indeed following [1] (see also [6]), the flow $Z_N \to \Phi^t(Z_N)$ can be defined for all $t \in \mathbb{R}$ almost everywhere with respect to the Lebesgue measure, which is enough for what will follow (even the proof of this result is not relevant in the following, so that we omit further details).

Given a probability measure with density W_0^N on Γ_N, thanks to the invariance of the Lebesgue measure under the above evolution, we define the time-evolved measure as the measure with density given by (1). Notice that this density is now restricted to Γ_N, however we can, equivalently and at any time, extend W^N to zero outside Γ_N and work with densities "with holes" in $\mathbb{R}^{3N} \times \mathbb{R}^{3N}$.

We recall that we consider probability distributions W^N which are initially (hence at any positive time) symmetric in the exchange of the particles. The probability density of j particles is then given by the j-particle marginal (2).

Note also that here $\Gamma_N, \Phi^t, W^N, f_j^N \cdots$ should exhibit a double dependence on N and ε. We shall soon fix a precise $\varepsilon = \varepsilon(N)$ so that the notation becomes unambiguous.

Cercignani [4] derived a hierarchy of equations for the marginals (in exactly the same spirit of the BBKGY hierarchy for smooth potentials), and the first of such equations ($j = 1$) is

$$(\partial_t + v \cdot \nabla_x) f_1^N = \text{Coll}\,, \tag{11}$$

where Coll denotes the variation of f_1^N due to the collisions, which takes the form

$$\mathrm{Coll} = (N-1)\,\varepsilon^2 \int dv_2 \int_{S^2} dn\ f_2^N(x, v, x + n\varepsilon, v_2)\,(v_2 - v)\cdot n\ . \tag{12}$$

In the next section, we will comment on the justification of this equation. Here, let us accept it and argue on its consequences.

Two given particles should be (almost) *uncorrelated* if the gas is *rarefied* enough. This leads to the propagation of chaos

$$f_2^N(x, v, x_2, v_2) \simeq f(x, v) f(x_2, v_2)\ , \tag{13}$$

which might seem contradictory at first sight. In fact, if two particles collide, correlations are created. Even assuming (13) at some given time, if particle 1 collides with particle 2, such an equation cannot be satisfied at any time *after* the collision.

Before discussing the propagation of chaos further, we notice that, in practical situations, for a rarefied gas, $N\varepsilon^3$ (total volume occupied by the particles) is very small, while $N\varepsilon^2 = O(1)$. This implies that the collision operator given by (12) is $O(1)$. Therefore, since we are dealing with a huge number of particles, we are tempted to perform the limit $N \to \infty$ and $\varepsilon \to 0$ in such a way that $\varepsilon^2 = O(N^{-1})$. As a consequence, the probability that two tagged particles collide (which is of the order of the surface of a ball, that is $O(\varepsilon^2)$), is negligible. Instead, the probability that a given particle collides with *any* of the remaining $N-1$ particles (which is $O(N\varepsilon^2) = O(1)$) is not negligible. On the other hand, condition (13) refers to two preselected particles (say 1 and 2) and it is not unreasonable to conceive that it holds in the limiting situation in which we work.

Nevertheless, we cannot simply insert (13) into (12), as the integral operator refers to times both before and after the collision. Let us assume (13) only when the pair of velocities (v, v_2) are incoming ($(v - v_2)\cdot n > 0$). If the two particles are initially uncorrelated, it is unlikely that they have collided before a given time t, hence we assume their statistical independence.

This is a standard argument in textbooks of kinetic theory, but some extra care is needed. If particles 1 and 2 have not collided directly before a given time t, this does not imply that they are uncorrelated. Indeed there may exist a chain of collisions involving a group $i_1, i_2, \cdots$ of particles

$$1 \to i_1 \to i_2 \to \cdots \to 2\ ,$$

correlating particles 1 and 2. As we shall see later, this is excluded (at least for a short time) by a more rigorous analysis. The two clusters of particles influencing the dynamics of particles 1 and 2 are disjoint with large probability.

Coming back to (12), for the outgoing pair velocities (v, v_2) (satisfying $(v_2 - v)\cdot n > 0$), we shall make use of the continuity property

$$f_2^N(x, v, x + n\varepsilon, v_2) = f_2^N(x, v', x + n\varepsilon, v_2')\ , \tag{14}$$

where the pair (v', v_2') is precollisional. On the two-particle distribution expressed in terms of precollisional variables, we apply now condition (13), obtaining

$$\text{Coll} = (N-1)\varepsilon^2 \int dv_2 \int_{S^2_+} dn\,(v-v_2)\cdot n[f(x,v')f(x-n\varepsilon,v_2')-f(x,v)f(x+n\varepsilon,v_2)] \tag{15}$$

after a change $n \to -n$ in the positive part of Coll (remind the notation S^2_+ for the hemisphere $\{n \in S^2 \mid (v-v_2)\cdot n > 0\}$).

Finally, in the limit $N \to \infty$ and $\varepsilon \to 0$ with $N\varepsilon^2 = \lambda^{-1} > 0$, we find:

$$(\partial_t + v\cdot\nabla_x)f = \lambda^{-1} \int dv_2 \int_{S_+} dn\,(v-v_2)\cdot n\,[f(x,v')f(x,v_2')-f(x,v)f(x,v_2)]. \tag{16}$$

The parameter λ represents, roughly, the typical length a particle can cover without undergoing any collision (*mean free path*). (In (6), we just chose $\lambda = 1$.)

It may be worth remarking that, after having taken the limit $N \to \infty$ and $\varepsilon \to 0$, there is no way to distinguish between incoming and outgoing pair velocities. This is because no trace of the parameter ε is left in (16) and n plays the role of a random variable. However, keeping in mind the way the Boltzmann equation was derived, one shall conventionally maintain the name *incoming* for velocities satisfying the condition $(v - v_2)\cdot n > 0$ (and consequently the pair (v', v_2') would be outgoing in (16)).

Equation (16) (or equivalently (5)-(6)) is the Boltzmann equation for hard spheres. Such an equation has a statistical nature, and it is not equivalent to the hamiltonian dynamics from which it has been derived. Indeed the H theorem shows that it is not reversible in time in contrast with the laws of mechanics.

By the analysis on the order of magnitude of the quantities in the game, we deduced that the Boltzmann equation works in special situations only. The condition $N\varepsilon^2 = O(1)$ means that we consider a rarefied gas, with almost vanishing volume density. After Boltzmann established the equation, Harold Grad [8, 9] postulated its validity in the limit $N \to \infty$ and $\varepsilon \to 0$ with $N\varepsilon^2 \to O(1)$ as discussed above (this is often called, indeed, the Boltzmann-Grad limit).

There is no contradiction in the irreversibility or in the trend to equilibrium obtained after the limit, when they are strictly speaking false for mechanical systems. However, the arguments above are delicate and require a rigorous, deeper analysis. If the Boltzmann equation is not a purely phenomenological model derived by assumptions ad hoc and justified by its practical relevance, but rather a *consequence* of a mechanical model, we must derive it rigorously. In particular, the propagation of chaos should not be a hypothesis but the statement of a theorem.

After the formulation of the mathematical validity problem by Grad, Cercignani [4] obtained the evolution equation (hierarchy) for the marginals of a hard-sphere system, and this was the starting point to rigorously derive the Boltzmann equation, as accomplished by Lanford in his famous paper [14], even though only for a short time interval.

Lanford's theorem is probably the most relevant result regarding the mathematical foundations of kinetic theory. In fact, it dispelled the many previous doubts on the validity of the Boltzmann equation (although some authors refuse a priori the problem of deriving the equation starting from mechanical systems [22]).

Unfortunately, the short-time limitation is a serious one. Only for special systems, as is the case of a very rarefied gas expanding in a vacuum, can we obtain a global validity [11, 12]. The possibility of deriving the Boltzmann equation globally in time, at least in cases when we have a global existence of good solutions, is still an open, challenging problem.

We conclude this section with a few historical remarks. Before Boltzmann, Maxwell proposed a kinetic equation that is just the Boltzmann equation integrated against test functions [16, 17]. He considered also more general potentials, in particular, inverse-power-law potentials, motivated essentially by the special properties of their cross-section. After Lanford's result, the case of smooth short-range potentials has been studied by other authors [13, 7, 19]. It is a nontrivial extension, in particular when the interacting potential is not "close enough" to a hard-sphere potential. The validity (or nonvalidity) of the Boltzmann equation in the case of genuine long-range interactions is open, in absence of techniques suited to deal with collisional and mean-field terms simultaneously.

2.1 Hard-sphere hierarchies

In this and in the following section we give more details on the derivation of (5) from N hard spheres of diameter ε, discussed above heuristically. We remind the reader that we are interested in the behaviour of the system in the limit $N \to \infty, \varepsilon \to 0$ fixing $\varepsilon^2 N = 1$ (1 chosen for simplicity), according to the Boltzmann-Grad limit. Namely we have a single scaling parameter ε (or N), and we study the asymptotics $\varepsilon \to 0$ ($N \to \infty$).

We start with the justification of (1). Let A be a measurable set in $\mathbb{R}^{3N} \times \mathbb{R}^{3N}$. Then the probability of finding the system in A at time $t > 0$ is given by

$$\mathbf{P}_t(A) = \mathbf{P}_0(\Phi^{-t}(A))$$

where

$$\Phi^{-t}(A) = \{Z_N \mid \Phi^t(Z_N) \in A\}$$

(dropping the dependence on $N = \varepsilon^{-2}$). If χ_A is the characteristic function of A, we have that

$$\int W^N(Z_N, t)\chi_A(Z_N) = \int W_0^N(Z_N)\chi_{\Phi^{-t}(A)}(Z_N) = \int W_0^N(Z_N)\chi_A(\Phi^t(Z_N))\,,$$

which implies that

$$\int W^N(Z_N, t)\, u(Z_N) = \int W_0^N(Z_N)\, u(\Phi^t(Z_N)) \tag{17}$$

for any bounded Borel function u. Here the integral is extended over all the phase space Γ_N. By using the Liouville theorem on the transformation $Z_N \to \Phi^t(Z_N)$, it

follows that

$$W^N(\Phi^t(Z_N),t) = W_0^N(Z_N)\,,$$

or (1) by the invertibility of the same transformation.

This probability distribution is not expected to converge. Thus, we focus immediately on the collection of marginal distributions $(f_j^N)_{j\geq 1}$, given by (2), for which the evolution equation has the form

$$(\partial_t + \mathcal{L}_j^\varepsilon) f_j^N = (N-j)\,\varepsilon^2\, C_{j+1}^\varepsilon f_{j+1}^N\,, \quad j = 1,\cdots,N-1\,. \tag{18}$$

Here $\mathcal{L}_j^\varepsilon$ is the generator of the dynamics of j hard spheres of diameter ε (Liouville operator of a $j-$particle system), while

$$C_{j+1}^\varepsilon = \sum_{k=1}^{j} C_{k,j+1}^\varepsilon\,, \tag{19}$$

$$C_{k,j+1}^\varepsilon f_{j+1}^N(Z_j) = \int dv_{j+1} \int_{S^2} dn\,(v_{j+1}-v_k)\cdot n\; f_{j+1}^N(Z_j, x_k+\varepsilon n, v_{j+1}) \tag{20}$$

is the j-particle collision operator (generalizing (12) to higher orders). For $j = N$, we are left with the Liouville equation in a differential form, namely $f_N^N = W^N$ and

$$(\partial_t + \mathcal{L}_N^\varepsilon) W^N = 0\,. \tag{21}$$

To derive Eq. (18) formally, we would like to give some description of $\mathcal{L}_j^\varepsilon$ as differential operator. This poses a difficulty, in fact $\mathcal{L}_j^\varepsilon = \sum_{i=1}^j v_i\cdot\nabla_{x_i}$ on functions vanishing on $\partial\Gamma_j$ and the interacting dynamics is completely coded on the boundary. In [4, 6], boundary conditions are imposed using (14), and its higher order versions, and Eq. (18) is derived integrating by parts over Γ_N. However if one is not afraid of working with delta functions, it is more convenient to use the following compact description:

$$\mathcal{L}_j^\varepsilon = \sum_{i=1}^{j} v_i\cdot\nabla_{x_i} - \mathcal{T}_j^\varepsilon \tag{22}$$

where

$$\mathcal{T}_j^\varepsilon = \sum_{\substack{i<k\\ i,k=1,\cdots,j}} \mathcal{T}_j^{\varepsilon;i,k} \tag{23}$$

and

$$\mathcal{T}_j^{\varepsilon;i,k} f_j^N(Z_j) = \varepsilon^2 \int_{S^2} dn\,\big(U_{i,k}\cdot n\big)_+ \left[\delta(R_{i,k}-\varepsilon n) b_n^{i,k} - \delta(R_{i,k}+\varepsilon n)\right] f_j^N(Z_j)\,, \tag{24}$$

with $U_{i,k} = v_i - v_k$, $R_{i,k} = x_i - x_k$, $(\cdot)_+$ is the positive part, and

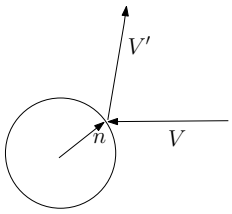
$$b_n^{i,k} f_j^N(Z_j) = f_j^N(x_1, v_1 \cdots x_i, v_i' \cdots x_k, v_k' \cdots x_j, v_j)\,.$$

The last operator transforms the incoming pair (v_i, v_k) into the outgoing (v_i', v_k') after a scattering with impact parameter n. Note finally that the operator (20) can be as well expressed in terms of $\mathcal{T}_j^\varepsilon$:

$$\varepsilon^2 C_{k,j+1}^\varepsilon f_{j+1}^N = \int dz_{j+1} \mathcal{T}_{j+1}^{\varepsilon;k,j+1} f_{j+1}^N \ . \tag{25}$$

We should remind here that the marginals are supported on the space of non-overlapping hard spheres ($|x_i - x_k| \geq \varepsilon$ for $i \neq k$). Therefore, when we think of (18) (and (21)) as equations over the whole space $\mathbb{R}^{3j} \times \mathbb{R}^{3j}$, we should always complement them with the condition $|x_i - x_k| < \varepsilon \Rightarrow f_j^N = 0$.

Let us now check the expression given for the Liouville equation, based on $\mathcal{T}_N^\varepsilon$. Consider a point particle hitting a sphere of diameter ε of infinite mass, centred at the origin. Let $g(X,V,t)$ be the probability distribution of the point particle, with initial datum $g(t=0) = g_0$. Let V and V' denote the incoming and outgoing velocity, respectively. It is $V' = V - 2(V \cdot n)n$, where $n \in S^2$ is the impact vector.



We denote by $Z = (X,V) \to Z(t) = (X(t), V(t))$ the dynamical flow. For any test function $\varphi = \varphi(X,V)$ we have that

$$\frac{d}{dt}\varphi(Z(t)) = V \cdot \nabla_X \varphi(Z(t)) + \delta(t-\tau)[\varphi(X + V\tau, V') - \varphi(X + V\tau, V)]$$

where τ is the hitting time. The term $[\cdots]$ describes the jump in velocity. Proceeding as in (17) we deduce that

$$\frac{d}{dt} \iint g(Z,t)\, \varphi(Z) = - \iint V \cdot \nabla_X g(Z,t)\, \varphi(Z) + \int_{A_{coll}} g_0(Z)\, \delta(t-\tau)\, [\cdots]. \tag{26}$$

Here A_{coll} is the set of configurations Z delivering a collision in the future. Introducing the change of variables

$$(X,V) \in A_{coll} \to (n\varepsilon, \tau, V)\,,$$

which has jacobian determinant of modulus $\varepsilon^2 |V \cdot n|$, the last term in Eq. (26) becomes

$$\begin{aligned}
&\varepsilon^2 \int_0^\infty d\tau \int dV \int_{V\cdot n<0} dn\, |V\cdot n|\, g_0(Z(n\varepsilon,\tau,V))\, \delta(t-\tau)\, [\varphi(n\varepsilon,V') - \varphi(n\varepsilon,V)] \\
&= \varepsilon^2 \int dV \int_{V\cdot n<0} dn \iint d\tilde{Z} |V\cdot n| g_0(\tilde{Z}) \delta(\tilde{Z} - Z(n\varepsilon,t,V))\, [\varphi(n\varepsilon,V') - \varphi(n\varepsilon,V)] \\
&= \varepsilon^2 \int dV \int_{V\cdot n<0} dn \iint d\tilde{Z}\, |V\cdot n|\, g(\tilde{Z},t)\, \delta(\tilde{Z} - (n\varepsilon,V))\, [\varphi(n\varepsilon,V') - \varphi(n\varepsilon,V)] \\
&= \varepsilon^2 \int dV \Big[\int_{V\cdot n>0} dn\, |V\cdot n|\, g(n\varepsilon,V',t) - \int_{V\cdot n<0} dn\, |V\cdot n|\, g(n\varepsilon,V,t) \Big] \varphi(n\varepsilon,V)\ .
\end{aligned}$$

Note that in the last step we changed again variables, $V \to V'$, in the positive term. To identify the time derivative in strong form, we now write $g(n\varepsilon,\cdot)\varphi(n\varepsilon,\cdot) = \int dX \delta(X - n\varepsilon) g(X,\cdot)\varphi(X,\cdot)$, exchange the integrals. and make a last change of variables $n \to -n$ in the negative term. We conclude that

$$\frac{d}{dt} g(Z,t) = -V\cdot\nabla_X g(Z,t) + \varepsilon^2 \int_{V\cdot n>0} dn\, |V\cdot n|\, [\delta(x-\varepsilon n)\, b_n - \delta(x+\varepsilon n)]\, g(Z,t)\,,$$

where b_n flips V into V'. The general form of $\mathcal{T}_N^\varepsilon$ follows easily from this computation.

In order to arrive to (18), it is enough to proceed as in the standard derivation of the BBGKY hierarchy for smooth potentials. We split the sums in $\mathcal{T}_N^\varepsilon$ given by Eq. (23), as

$$\sum_{\substack{i<k \\ i,k=1,\cdots,N}} = \sum_{i<k\le j} + \sum_{i\le j,k>j} + \sum_{j<i<k}\,,$$

and integrate in $dz_{j+1}\cdots dz_N$. The first sum produces $\mathcal{L}_j^\varepsilon$ immediately. The second sum gives the collision operator $\varepsilon^2 C_{j+1}^\varepsilon$, multiplied by a symmetry factor $(N-j)$. The last sum vanishes by exact compensation of gain and loss in (24).

Eq. (18) is the starting point for Lanford's validity theorem, as we shall see in the following section.

2.2 Lanford's Theorem

The iteration of the Duhamel formula for Eq. (18) leads to express $f_j^N(t)$ as a sum:

$$\begin{aligned}
f_j^N(t) = \sum_{n=0}^{N-j} \alpha_n(N,j) \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \qquad (27)\\
S^\varepsilon(t-t_1)\, C_{j+1}^\varepsilon\, S^\varepsilon(t_1-t_2)\cdots S^\varepsilon(t_n)\, f_{j+n}^N(0)\,,
\end{aligned}$$

where $S^\varepsilon(t)F(Z_j) = F(\Phi^{-t}Z_j)$ is the j-particle interacting flow and

$$\alpha_n(N,j) = (N-j)\cdots(N-j-n+1)\,\varepsilon^{2n}.$$

The main ingredient for the theorem of Lanford stated below, is just this explicit representation for the solution of the N-particle hierarchy. Actually, this identity can be rigorously proved directly, without making use of (18) [20, 18].

On the other hand, a similar formula can be established for the tensor product of solutions to the Boltzmann equation $f_j(t) := f(t)^{\otimes j}$. Namely we have that

$$f_j(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \; S(t-t_1)\, C_{j+1}\, S(t_1-t_2) \cdots S(t_n)\, f_{j+n}(0). \tag{28}$$

Here $S(t)F(X_j, V_j) = F(X_j - V_j t, V_j)$ is the free-flow operator, and $C_{j+1} = \sum_{k=1}^{j} C_{k,j+1}$, where $C_{k,j+1}$ is the formal limit for $\varepsilon \to 0$ of (25).

Since $\alpha_n(N, j) \to 1$ and $S^\varepsilon(t) \to S(t)$ almost everywhere in the limit, each term in the right-hand side of (28) is the limit of the corresponding term in (27), provided that we require a good behaviour of the initial datum $f^N_{j+n}(0)$.

We cannot simply require that $f^N_{j+n}(0) = f_j(0)$, because the hard-core condition induces correlations at time zero. Let f_0 be a one-particle probability distribution, and the initial datum for the Boltzmann equation. We make the following assumptions.

Hypothesis 1. $f_0 \in C(\mathbb{R}^6 \to \mathbb{R}^+)$, $\int f_0 = 1$. Moreover

$$f_0(x, v) \le h(x) e^{-\beta v^2}$$

where $h \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $\|h\|_{L^\infty} = z_0$ and $\beta > 0$.

Hypothesis 2. Let $\Gamma_j^{\neq}$ be the subset of $\mathbb{R}^{3j} \times \mathbb{R}^{3j}$ fulfilling the condition

$$x_i \neq x_k \quad \text{for} \quad i \neq k,\ 1 \le i, k \le j.$$

Then, the marginals of the hard-sphere system satisfy

$$\lim_{\varepsilon \to 0} f_j^N(t = 0) = f_0^{\otimes j}\ , \tag{29}$$

uniformly on compact subsets of $\Gamma_j^{\neq}$ and the bound

$$f_j^N(x_1, v_1, \cdots x_j, v_j, t = 0) \le \prod_{i=1}^{j} h(x_i) e^{-\beta v_i^2}.$$

Theorem [14]. *Under the hypotheses 1 and 2, there exists $t_0 > 0$ (depending only on z_0, β) such that, for $t < t_0$ we have, for all $j \ge 1$,*

$$\lim_{\varepsilon \to 0} f_j^N(t) = f(t)^{\otimes j} \tag{30}$$

where $f(t)$ is the unique solution to the Boltzmann equation. The convergence holds almost everywhere.

Following Lanford, the proof can be organized in two steps.

We first give an a priori bound on the series expansions (27) (uniform in ε) and (28), using that the time t is small enough. To give a rough idea of this step, let us cutoff large velocities. In particular, we ignore the factors $|v_{j+1} - v_k|$ in (20). Then, the string of operators can be estimated brutally by

$$|S^\varepsilon(t-t_1)\, C^\varepsilon_{j+1}\, S^\varepsilon(t_1-t_2)\cdots S^\varepsilon(t_n)\, f^N_{j+n}(0)| \leq C^{j+n}\, j(j+1)\cdots(j+n-1)$$

for some $C > 0$, where the factorial growth comes from the sum in (19). On the other hand, the ordered time-integration yields $t^n/n!$, so that the series expansion is bounded by a geometric series $\sum_n C_1^j (C_2 t)^n$, for positive C_1, C_2.

In the second step, one shows the term by term convergence of (27) to (28). Here the short time restriction does not enter anymore.

For more details on Lanford's proof, we refer to [14, 21, 6].

We conclude with some remarks.

1. The time t_0 is explicitly computable. It turns out to be a fraction of the mean free time between collisions. This time limitation is purely technical.
2. Lanford's original proof was qualitative: it does not make explicit the rate of convergence. This can be obtained with some extra care, along the same arguments [7, 19].
3. Initial conditions fulfilling Hypotheses 1 and 2 can be easily constructed. The most natural initial state is maximally factorized, meaning that the only source of correlation is due to the hard-core exclusion. In this case, the N-particle measure is

$$W_0^N(Z_N) := \frac{1}{\mathcal{Z}_N} f_0^{\otimes N}(Z_N) \prod_{1\leq i<k\leq N} \mathbf{1}_{\{|x_i-x_k|>\varepsilon\}}(Z_N)\,,$$

where

$$\mathcal{Z}_N := \int_{\mathbb{R}^{3N}\times\mathbb{R}^{3N}} dZ_N\, f_0^{\otimes N}(Z_N) \prod_{1\leq i<k\leq N} \mathbf{1}_{\{|x_i-x_k|>\varepsilon\}}(Z_N)$$

is a normalization factor, and f_0 satisfies Hypothesis 1. For this state, the verification of (29) is a simple exercise.

3 Weak-coupling limit and Landau equation

The Boltzmann equation is suited to the description of rarefied gases, and one can ask whether a useful kinetic analysis can be applied also to the case of a *dense* gas. To introduce the problem, let us revisit the Boltzmann-Grad limit in an alternative way. Let ε be a small scale parameter denoting the ratio between the microscopic and the macroscopic scale, for instance the inverse number of atomic diameters necessary to cover 1 meter, or the inverse number of atomic characteristic times necessary to cover 1 second. Then, scale space and time by ε in the equations of motion (in

our case, the hard-sphere hierarchy). We need to specify the number of particles N. In a box of side 1, there should be $N \approx \varepsilon^{-3}$ particles if one assumes that the intermolecular distance is of the same order of the molecular diameter. The number of collisions of a given particle per macroscopic unit time would be ε^{-1}. As we have seen, in a low-density regime, N scales differently, namely $N \approx \varepsilon^{-2}$, the number of collisions per unit time is finite and the one-particle distribution function satisfies the Boltzmann equation.

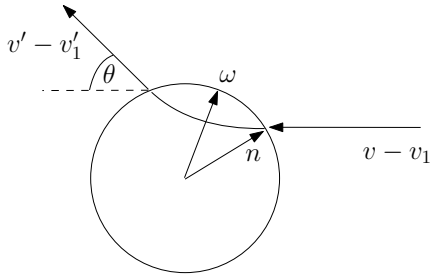
A variety of possible scalings describes different physical situations (see the next section). For instance, the gas may be dense, $N = O(\varepsilon^{-3})$ and the particles are *weakly interacting* via a smooth two-body potential ϕ. To express the weakness of the interaction, we assume that ϕ is rescaled by $\sqrt{\varepsilon}$. Since ϕ varies on a scale ε (in macroscopic unities), the force will be $O(\frac{1}{\sqrt{\varepsilon}})$ and act on a time interval $O(\varepsilon)$. The variation of momentum due to the single scattering is $O(\sqrt{\varepsilon})$, and the number of particles met by a typical particle is $O(\frac{1}{\varepsilon})$. Hence, the total momentum variation for unit time is $O(\frac{1}{\sqrt{\varepsilon}})$. However, in the case of a homogeneous gas and symmetric forces, this variation should be zero in the average. The computation of the variance leads to a result $\frac{1}{\varepsilon}O(\sqrt{\varepsilon})^2 = O(1)$. Therefore, based on a *central-limit* type of argument, we expect that in the kinetic limit a diffusion equation in the velocity variable holds.

At the level of kinetic equations, consider a collision operator of Boltzmann type, for a spherically symmetric, smooth potential $\phi = \phi(x)$. We assume for simplicity the potential to be short-range, namely $\phi(x) = 0$ if $|x| > 1$. The collision operator Q_B is given by (6), with (7) replaced by

$$v' = v - \omega[\omega \cdot (v - v_1)]$$

$$v_1' = v_1 + \omega[\omega \cdot (v - v_1)] \tag{31}$$

where ω is the unit vector in the direction of the transferred momentum, while n is the impact parameter[1]. The potential ϕ enters in the determination of ω.



According to the weak-coupling-limit prescription discussed above, we rescale the potential as $\phi_\varepsilon = \sqrt{\varepsilon}\phi(\frac{x}{\varepsilon})$, and simultaneously increase the density. The new

[1] Note that this is not the conventional form for the Boltzmann equation and usually the factor $(v - v_1) \cdot n$ is rewritten in terms of ω, which amounts to introduce the differential cross-section.

collision operator reads

$$Q_B^\varepsilon(f,f)(x,v) = \frac{1}{\varepsilon}\int dv_1 \int_{S^2_+} dn\,(v-v_1)\cdot n\ \{f(x,v+p)f(x,v_1-p)-f(x,v)f(x,v_1)\} \tag{32}$$

where $p = -\omega\cdot(v-v_1)\,\omega$ is the transferred momentum, which is typically $O(\sqrt{\varepsilon})$.

It follows that, for any smooth test function $u = u(v)$, setting $U = v - v_1$ (omitting the spatial dependence),

$$\begin{aligned}
&\int dv\, u(v)\, Q_B^\varepsilon(f,f)(v)\\
&= \frac{1}{2\varepsilon}\iint dv dv_1 \int_{S^2_+} dn\, U\cdot n\, \Big\{u(v+p)+u(v_1-p)-u(v)-u(v_1)\Big\}\, f(v)f(v_1)\\
&\approx \frac{1}{2\varepsilon}\iint dv dv_1 \int_{S^2_+} dn\, U\cdot n\ \Big\{p\cdot(\nabla_v u(v)-\nabla_{v_1}u(v_1)) + \\
&\quad \sum_{\alpha,\alpha'}\left(\frac{1}{2}\partial^2_{\alpha,\alpha'}u(v)p_\alpha p_{\alpha'} + \frac{1}{2}\partial^2_{\alpha,\alpha'}u(v_1)p_\alpha p_{\alpha'}\right)\Big\} f(v)f(v_1)\,,
\end{aligned} \tag{33}$$

where we Taylor-expanded up to second order in p in order to compensate the divergence $\frac{1}{\varepsilon}$, and the α's run over the three vector components.

We first analyze the second order. Let $x(s)$ be the trajectory of one particle scattering in the central potential ϕ_ε with incoming velocity U and initial time fixed by $x(0) = \varepsilon n$. To evaluate

$$T_{\alpha,\alpha'}(U) := \frac{1}{2\varepsilon}\int_{S^2_+} dn\, U\cdot n\ p_\alpha p_{\alpha'}\ ,$$

we write

$$p_\alpha = -\int_{-\infty}^{+\infty} ds\, \frac{1}{\sqrt{\varepsilon}}\,\nabla_{x_\alpha}\phi\left(\frac{x(s)}{\varepsilon}\right) = -\left(\frac{1}{\sqrt{2\pi}}\right)^3 \frac{1}{\sqrt{\varepsilon}}\int ds\int_{\mathbb{R}^3} dk\, \mathrm{i}\, k_\alpha e^{\mathrm{i}\,k\cdot\frac{x(s)}{\varepsilon}}\hat\phi(k)\ .$$

Then

$$\begin{aligned}
T_{\alpha,\alpha'} = &-\left(\frac{1}{2\pi}\right)^3 \frac{1}{2\varepsilon^2}\int_{S^2_+} dn\, U\cdot n\\
&\int ds_1\int ds_2\int dk_1\int dk_2\ (k_1)_\alpha (k_2)_{\alpha'}\ e^{\mathrm{i}k_1\cdot\frac{x(s_1)}{\varepsilon}}\ e^{\mathrm{i}k_2\cdot\frac{x(s_2)}{\varepsilon}}\ \hat\phi(k_1)\hat\phi(k_2).
\end{aligned}$$

But $\frac{x(s)}{\varepsilon}\approx n + \frac{Us}{\varepsilon}$. Therefore, setting $y(s) = n + Us$ (after rescaling times) we have that

$$T_{\alpha,\alpha'} = -\left(\frac{1}{2\pi}\right)^3 \frac{1}{2} \int_{S^2_+} dn\, U\cdot n \int ds_1 \int ds_2 \int dk_1 \int dk_2\ (k_1)_\alpha (k_2)_{\alpha'}\ e^{\mathrm{i}k_1\cdot y(s_1)}\ e^{\mathrm{i}k_2\cdot y(s_2)}\ \hat\phi(k_1)\hat\phi(k_2).$$

Next we write

$$e^{\mathrm{i}k_1\cdot y(s_1)}\ e^{\mathrm{i}k_2\cdot y(s_2)} = e^{\mathrm{i}(k_1+k_2)\cdot y(s_1)}\ e^{\mathrm{i}k_2\cdot U(s_2-s_1)}$$

and change variables in the following way. Setting $\tau = s_2 - s_1$, parametrize the points ξ of the cylinder with axis $-U/|U|$ and basis the unit circle through the origin, by $(n, s_1) \longrightarrow n + Us_1$. Then $d\xi = dn\, ds_1\, (U\cdot n)_+$ and

$$T_{\alpha,\alpha'} \approx -\left(\frac{1}{2\pi}\right)^3 \frac{1}{2} \int d\xi\, d\tau\, dk_1\, dk_2\, (k_1)_\alpha (k_2)_{\alpha'} e^{\mathrm{i}(k_1+k_2)\cdot\xi} e^{\mathrm{i}k_2\cdot U\tau} \hat\phi(k_1)\hat\phi(k_2)\,, \tag{34}$$

hence we arrive to

$$T_{\alpha,\alpha'} \approx -\frac{(2\pi)}{2} \int dk\, \hat\phi^2(k)\, \delta(k\cdot U)\, k_\alpha k_{\alpha'} =: a_{\alpha,\alpha'}(U)\,. \tag{35}$$

This matrix can be handled conveniently by means of polar coordinates $k = \hat k\rho$, $\hat k = \frac{k}{|k|}$:

$$a_{\alpha,\alpha'}(U) = -\frac{(2\pi)^4}{2} \frac{1}{|U|} \int d\rho\, \hat\phi^2(\rho)\, \rho^3 \int d\hat k\ \ \delta(\hat U\cdot \hat k)\,, \tag{36}$$

where $\hat U$ is the versor of U . Here we are using that, due to the spherical symmetry, $\hat\phi$ depends on k through $|k|$ only. Setting

$$B = \pi \int_0^\infty d\rho\, \hat\phi^2(\rho)\, \rho^3 \tag{37}$$

and computing $\int d\hat k\, \delta(\hat U\cdot \hat k)$, we conclude that

$$a_{\alpha,\alpha'}(U) = \frac{B}{|U|}\left(\delta_{\alpha,\alpha'} - \hat U_\alpha \hat U_{\alpha'}\right)\,. \tag{38}$$

B is the kinetic constant coding all the information on the microscopic potential.

We turn now to the evaluation of the first order terms in (33), i.e.

$$T(U) := \frac{1}{2\varepsilon} \int_{S^2_+} dn\, U\cdot n\ p_U, \qquad T_\perp(U) := \frac{1}{2\varepsilon} \int_{S^2_+} dn\, U\cdot n\ p^\perp_U$$

where $p = (p_U, p^\perp_U)$ and $p_U > 0$ is the projection of p over $-U/|U|$. Note that $T_\perp$ is vanishing by symmetry. On the other hand, $p_U = (\omega\cdot U)^2/|U| = p^2/|U|$ so that

$$T(U) = \frac{1}{|U|}\sum_\alpha T_{\alpha,\alpha}(U) \approx \frac{1}{|U|}\sum_\alpha a_{\alpha,\alpha}(U) = \frac{2B}{|U|^2}\,.$$

In conclusion,

$$\int dv\, u(v)\, Q_B^{\varepsilon}(f,f)(v) \approx \iint dv dv_1\, Lu(v,v_1)\, f(v) f(v_1) \tag{39}$$

where

$$Lu(v,v_1) := -2\frac{B}{|v-v_1|^3}(v-v_1)\cdot(\nabla_v u(v) - \nabla_{v_1} u(v_1)) + Tr(a\otimes D^2 u)(v,v_1)$$

where $Tr(a\otimes D^2 u)(v,v_1) = \sum_{\alpha,\alpha'} a_{\alpha,\alpha'}(v-v_1)\partial^2_{v_\alpha,v_{\alpha'}} u(v)$ and $a = (a_{\alpha,\alpha'})_{\alpha,\alpha'}$ is given by (38).

This leads to introduce the Landau operator, defined by

$$Q_L(f,f)(x,v) := \int dv_1 \nabla_v\, a(v-v_1)\,(\nabla_v - \nabla_{v_1}) f(x,v) f(x,v_1)\ . \tag{40}$$

By a straightforward integration by parts, we get that

$$\int dv\, u(v)\, Q_B^{\varepsilon}(f,f)(v) \approx \int dv\, u(v)\, Q_L(f,f)(v)\ . \tag{41}$$

The collision operator Q_L has been introduced by Landau in 1936 for the study of a weakly interacting dense plasma [15] and

$$(\partial_t + v\cdot\nabla_x) f = Q_L(f,f)$$

is called the *Landau equation* (sometimes, Fokker-Planck-Landau equation)[2].

The qualitative properties of the solutions to the Landau equation are the same as for the Boltzmann equation regarding the basic conservation laws and the H theorem.

The procedure described above is a *grazing collision* limit. To the best of our knowledge, there is no rigorous version of the formal statement (41). The available rigorous results on grazing collision limits concern a suitable rescaling of the differential cross-section (rather than the potential): see [10] and references therein.

Even a rigorous proof of (41) would be not completely satisfactory. Indeed the Landau equation is expected to be a fundamental equation, derivable from particle systems in the weak-coupling limit. A rigorous proof of this fact seems to be hard, even for short times. We will present a formal derivation, outlining the difficulties, in Section 3.2.

[2] The Landau equation was obtained from the Boltzmann equation for cutoffed Coulomb potential (truncated both at short and large distances). Actually the word "Coulomb" is frequently used for the Landau equation with kernel singularity $\frac{1}{|U|}$ (see (38)), which is somehow misleading. In fact as we have seen, this singularity is always present.

3.1 Remarks on the scaling limits

Let us give a unified picture of the different regimes discussed so far, leading to the Boltzmann and the Landau equation.

The starting point is always a classical system of N identical particles of unit mass. Microscopic positions and velocities are denoted by $q_1, \cdots, q_N$ and $v_1, \cdots, v_N$. Let τ be the microscopic time. The Newton's equations read:

$$\frac{d}{d\tau} q_i = v_i \ , \qquad \frac{d}{d\tau} v_i = \sum_{\substack{j=1,\cdots,N \\ j\neq i}} F(q_i - q_j) \tag{42}$$

where $F = -\nabla\phi$ denotes the interparticle (conservative) force, ϕ the two-body, spherically symmetric potential.

There is a unique scaling parameter ε, which can be interpreted as the ratio between typical macroscopic and microscopic units. In practice we introduce macroscopic variables

$$x = \varepsilon\, q, \quad t = \varepsilon\, \tau \ ,$$

and ε has to be sent to zero to extract the essential macroscopic features. Note that the velocity remains unscaled. In these new variables, the system reads:

$$\frac{d}{dt} x_i = v_i \ , \qquad \frac{d}{dt} v_i = \frac{1}{\varepsilon} \sum_{\substack{j=1,\cdots,N \\ j\neq i}} F\left(\frac{x_i - x_j}{\varepsilon}\right) \ . \tag{43}$$

In order to have a finite density we should postulate $N \sim \varepsilon^{-3}$ in three dimensions. Instead, in the low-density limit we chose $N \sim \varepsilon^{-2}$ so that, for a test particle, the change of momentum (or velocity) for each collision is $\delta v \sim \frac{1}{\varepsilon}\delta t = O(1)$, where the typical interaction time δt is $O(\varepsilon)$ (if F has short range); on the other hand the collision frequency scales as the number of particles in the tube of radius ε (which has volume ε^2), therefore it is finite in the Boltzmann-Grad limit. We have been dealing with this scaling in the most favourable situation, the system of hard spheres. In this case, the collision is instantaneous with transferred momentum of $O(1)$.

We are now interested in a situation where the interaction is very weak for which we rescale the potential as $\phi \longrightarrow \varepsilon^\alpha \phi$, $\alpha \in (0, 1)$ and the equations of motion become

$$\frac{d}{dt} x_i = v_i \ , \qquad \frac{d}{dt} v_i = \varepsilon^{\alpha-1} \sum_{\substack{j=1,\cdots,N \\ j\neq i}} F\left(\frac{x_i - x_j}{\varepsilon}\right) \ . \tag{44}$$

We should scale the number density as $N^{-\beta}$ with suitable β, to get a kinetic equation. The heuristic argument for the weak-coupling limit discussed in the previous section implies that, setting $\beta = 2(1 + \alpha)$, one should get diffusion in velocity, preserving mass, momentum and energy. Thus we expect that this regime is ruled out by the Landau equation, with the only exception $\alpha = 0$, for which we recover the low-density

scaling and the Boltzmann equation. Frequently, "weak-coupling limit" refers to the special case $\alpha = 1/2$, which is also the case considered in the next section.

3.2 Weak-coupling limit for classical systems

We start from the weak-coupling dynamics in macroscopic variables

$$\frac{d}{dt}x_i = v_i\,, \qquad \frac{d}{dt}v_i = -\frac{1}{\sqrt{\varepsilon}}\sum_{\substack{j=1,\cdots,N\\ j\neq i}} \nabla\phi\left(\frac{x_i - x_j}{\varepsilon}\right)\,, \tag{45}$$

where we pose $N = \varepsilon^{-3}$.

Once again, $W^N = W^N(Z_N)$ is a symmetric probability density on the phase space $\mathbb{R}^{3N}\times\mathbb{R}^{3N}$, obeying the Liouville equation

$$\left(\partial_t + \sum_{i=1}^N v_i\cdot\nabla_{x_i}\right)W^N = \frac{1}{\sqrt{\varepsilon}}\,T_N^\varepsilon W^N \tag{46}$$

where

$$T_N^\varepsilon W^N = \sum_{1\le k<\ell\le N} T_{k,\ell}^\varepsilon W^N\,, \tag{47}$$

$$T_{k,\ell}^\varepsilon W^N = \nabla\phi\left(\frac{x_k - x_\ell}{\varepsilon}\right)\cdot(\nabla_{v_k} - \nabla_{v_\ell})W^N. \tag{48}$$

The marginals $\left(f_j^N(t)\right)_{j=1}^N$ satisfy the BBGKY hierarchy

$$\left(\partial_t + \sum_{k=1}^j v_k\cdot\nabla_k\right)f_j^N = \frac{1}{\sqrt{\varepsilon}}T_j^\varepsilon f_j^N + \frac{N-j}{\sqrt{\varepsilon}}\varepsilon^3\,C_{j+1}^\varepsilon f_{j+1}^N\,, \tag{49}$$

where the operator C_{j+1}^ε is now defined by

$$C_{j+1}^\varepsilon = \sum_{k=1}^j C_{k,j+1}^\varepsilon\,,$$

$$\begin{aligned} C_{k,j+1}^\varepsilon f_{j+1}^N(Z_j) &= -\varepsilon^{-3}\iint dv_{j+1}dx_{j+1}F\left(\frac{x_k - x_{j+1}}{\varepsilon}\right)\cdot\nabla_{v_k}f_{j+1}^N(Z_j, x_{j+1}, v_{j+1})\\ &= -\iint dv_{j+1}dX\,F(X)\cdot\nabla_{v_k}f_{j+1}^N(Z_j, x_k - \varepsilon X, v_{j+1})\,. \end{aligned} \tag{50}$$

One should note that this hierarchy has the same structure of (18), but now we are considering a smooth and weakly rescaled potential ϕ. In fact $C_{k,j+1}^\varepsilon$ describes the "collision" of particle k, belonging to the j-particle subsystem, with a particle

outside the subsystem, conventionally $j+1$. The dynamics of the j-particle subsystem is governed by three effects: the free-streaming operator, the collisions "inside" the subsystem (the T_j^ε term), and the collisions with particles "outside" the subsystem (the C_{j+1}^ε term).

We can complement the above equations with the initial condition

$$f_j^N(t=0) = f_0^{\otimes j} \,, \tag{51}$$

where f_0 is a given one-particle density. Particles are statistically uncorrelated at time zero, and statistical independence breaks at time $t > 0$ because of the dynamics. Since the interaction between two given particles is vanishing in the limit $\varepsilon \to 0$, we can hope for propagation of chaos. The physical mechanism producing chaos is however quite different from the one discussed in Section 2. Here, two given particles can interact, the force is strong but the net effect of the collision is small (because the interaction time is small), while in the low-density regime collisions are always strong and unlikely.

Let us investigate the convergence of f_1^N to the Landau equation, in the limit $\varepsilon \to 0$, using the hierarchy (49).

Expanding $f_j^N(t)$ as a perturbation of the free flow $S(t)$ (as in (28)) we find that

$$\begin{aligned} f_j^N(t) =& S(t) f_0^{\otimes j} + \frac{N-j}{\sqrt{\varepsilon}} \varepsilon^3 \int_0^t S(t-t_1) C_{j+1}^\varepsilon f_{j+1}^N(t_1) dt_1 + \\ & \frac{1}{\sqrt{\varepsilon}} \int_0^t S(t-t_1) T_j^\varepsilon f_j^N(t_1) dt_1 \,. \end{aligned} \tag{52}$$

It is now reasonable to assume that

$$\int dX \, F(X) = 0 \,,$$

which implies $C_{j+1}^\varepsilon f_{j+1}^N = O(\varepsilon)$, *provided* that the second derivatives $D_v^2 f_{j+1}^N(t)$ are bounded uniformly in ε. Since $\frac{N-j}{\sqrt{\varepsilon}} = O(\varepsilon^{-\frac{7}{2}})$, we see that the second term in the right-hand side of (52) does not give any contribution in the limit. In the same assumptions,

$$\begin{aligned} & \int_0^t S(t-t_1) T_j^\varepsilon f_j^N(t_1) dt_1 = \\ & \sum_{i \neq k} \int_0^t F\left(\frac{(x_i - x_k) - (v_i - v_k)(t-t_1)}{\varepsilon} \right) \cdot g(Z_j, t_1) dt_1 \end{aligned}$$

where g is a smooth j-particle function, which is again $O(\varepsilon)$ so that the last term in the right-hand side of (52) is also vanishing in the limit. We are therefore facing the alternative: either the limit is trivial, or the time evolved marginals are not smooth. This is indeed bad news: a rigorous derivation of the (expected) Landau equation seems problematic.

The above difficulty suggests to split $f_j^N(t)$ into two parts, namely we conjecture that:

$$f_j^N = g_j^N + \gamma_j^N ,$$

where g_j^N is the main part of f_j^N and is smooth, while γ_j^N is small, but strongly oscillating (hence with large derivatives). The two parts satisfy, by definition,

$$\left(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla_{x_k}\right) g_j^N = \frac{N-j}{\sqrt{\varepsilon}} \varepsilon^3 C_{j+1}^{\varepsilon} g_{j+1}^N + \frac{N-j}{\sqrt{\varepsilon}} \varepsilon^3 C_{j+1}^{\varepsilon} \gamma_{j+1}^N ,$$

$$\left(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla_{x_k}\right) \gamma_j^N = \frac{1}{\sqrt{\varepsilon}} T_j^{\varepsilon} \gamma_j^N + \frac{1}{\sqrt{\varepsilon}} T_j^{\varepsilon} g_j^N ,$$

with initial data

$$g_j^N = f_0^{\otimes j}, \quad \gamma_j^N = 0 .$$

The remarkable feature of this decomposition is that the singular part can be eliminated. In fact we have that

$$f_1^N(t) = S(t) f_0 + \frac{N-1}{\sqrt{\varepsilon}} \varepsilon^3 \int_0^t S(t-t_1)\, C_2^{\varepsilon} \left(g_2^N(t_1) + \gamma_2^N(t_1) \right) dt_1 ,$$

where

$$\gamma_2^N(t) = \frac{1}{\sqrt{\varepsilon}} \int_0^t ds\, U_2^{\varepsilon}(s)\, T_2^{\varepsilon} g_2^N(t-s)$$

and U_2^{ε} is just the two-particle interacting flow. Indicating by $\left(Z_2^{\varepsilon}(-s)\right)_{s\in(0,t)}$ this flow with final condition $Z_2^{\varepsilon}(0) = Z_2$, we have that

$$\gamma_2^N(Z_2, t) = \frac{1}{\sqrt{\varepsilon}} \int_0^t ds\, \nabla\phi \left(\frac{x_1^{\varepsilon}(-s) - x_2^{\varepsilon}(-s)}{\varepsilon} \right) \cdot \left(\nabla_{v_1} - \nabla_{v_2} \right)\, g_2^N(Z_2^{\varepsilon}(-s), t-s).$$

Based on the conjecture, we present now a formal derivation of the Landau equation (assuming g_2^N smooth). We have that

$$\left(\partial_t + v_1 \cdot \nabla_{x_1}\right) f_1^N(t) = \frac{N-1}{\sqrt{\varepsilon}} \varepsilon^3 C_2^{\varepsilon} g_2^N(t) + \frac{N-1}{\varepsilon} \varepsilon^3 C_2^{\varepsilon} \int_0^t ds\, U_2^{\varepsilon}(s)\, T_2^{\varepsilon} g_2^N(t-s) .$$

Let $u \in \mathcal{D}$ be a test function. As already mentioned the first term on the right-hand side is negligible:

$$\frac{N-1}{\sqrt{\varepsilon}} \varepsilon^3 \left(u, C_2^{\varepsilon} g_2^N(t) \right) = O\left(\sqrt{\varepsilon}\right) .$$

The last term gives

$$
\begin{aligned}
&-\frac{N-1}{\varepsilon}\int dz_1\int dz_2\int_0^t ds\,\nabla_{v_1}u(z_1)\\
&\quad F\left(\frac{x_1-x_2}{\varepsilon}\right)F\left(\frac{x_1(-s)-x_2(-s)}{\varepsilon}\right)\cdot(\nabla_{v_1}-\nabla_{v_2})g_2^N(Z_2^\varepsilon(-s),t-s)\\
&\approx -\int dz_1\int dr\,dv_2\int_0^\infty ds\,\nabla_{v_1}u(z_1)\\
&\quad F(r)\,F\left(\frac{x_1^\varepsilon(-\varepsilon s)-x_2^\varepsilon(-\varepsilon s)}{\varepsilon}\right)\cdot(\nabla_{v_1}-\nabla_{v_2})g_2^N(x_1,v_1,x_1,v_2,t)\,,
\end{aligned}
$$

after having changed to variables $r=\frac{x_1-x_2}{\varepsilon}$ and $s\to\frac{s}{\varepsilon}$. Here, setting $U=v_1-v_2$,

$$
\frac{x_1^\varepsilon(-\varepsilon s)-x_2^\varepsilon(-\varepsilon s)}{\varepsilon}\approx r+Us\,.
$$

This term is then approximately equal to

$$
\begin{aligned}
&-\int dz_1\int dr dv_2\int_0^\infty ds\nabla_{v_1}u(z_1)F(r)F\left(r+Us\right)\cdot(\nabla_{v_1}-\nabla_{v_2})g_2^N(x_1,v_1,x_1,v_2,t)\\
&\approx\left(u,Q_L(g_1^N,g_1^N)\right)\,,
\end{aligned}
$$

where in the last step we invoked propagation of chaos ($g_2^N\approx(g_1^N)^{\otimes 2}$) and used definition (40). Indeed it is not hard to show that

$$
\int dr\int_0^\infty dsF(r)F(r+Us)=\frac{1}{2}\int dr\int_{-\infty}^\infty dsF(r)F(r+Us)=a(U)
$$

where $a(U)$ is the matrix given by (38). Indeed expressing the above identity in terms of the Fourier transforms we readily arrive to the right-hand side of (34).

Unfortunately, very little is known about the mathematical derivation. We mention here the only result we are aware of.

Consider the first order (in time) approximation $\tilde{g}_j^N$ of g_j^N given by

$$
\begin{aligned}
\tilde{g}_j^N(t)&=S(t)f_0^{\otimes j}+\frac{N-j}{\sqrt{\varepsilon}}\varepsilon^3\int_0^t S(t-\tau)C_{j+1}^\varepsilon S(\tau)g_{j+1}^N\,d\tau\\
&\quad+\frac{N-j}{\varepsilon}\varepsilon^3\int_0^t d\tau\int_0^\tau d\sigma S(t-\tau)C_{j+1}^\varepsilon U_{j+1}^\varepsilon(\tau-\sigma)T_{j+1}^\varepsilon S(\sigma)f_0^{\otimes(j+1)}\,. \quad (53)
\end{aligned}
$$

Then we can prove:

Theorem [2]. *Suppose that $f_0\in C_0^3(\mathbb{R}^3\times\mathbb{R}^3)$ is the initial probability density satisfying:*

$$
|D^r f_0(x,v)|\le Ce^{-b|v|^2}\quad for\qquad r=0,1,2 \quad (54)
$$

where D^r is any derivative of order r and $b>0$. Assume $\phi\in C^2(\mathbb{R}^3)$ and $\phi(x)=0$ if $|x|>1$. Assume that the marginals factorize exactly at time zero. Then

$$\lim_{\varepsilon\to 0} \tilde{g}_1^N(t) = S(t)f_0 + \int_0^t d\tau S(t-\tau)Q_L(S(\tau)f_0, S(\tau)f_0) \tag{55}$$

where $N\varepsilon^3 = 1$ *and the above limit is considered in* $\mathcal{D}'$.

Since the right-hand side of Eq. (55) is the first order approximation of the Landau equation, we can consider the theorem as a consistency result.

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