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▶ To cite this version:

Mahadi Ddamulira. Repdigits as sums of three balancing numbers. Mathematica Slovaca, 2020, 70 (3), pp.557-566. 10.1515/ms-2017-0371 . hal-02405969v2

HAL Id: hal-02405969

https://hal.science/hal-02405969v2

Submitted on 8 Jun 2020

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Mathematica Slovaca

DOI: 10.1515/ms-2017-0371
Math. Slovaca **70** (2020), No. 3, 557–566

REPDIGITS AS SUMS OF THREE BALANCING NUMBERS

Mahadi Ddamulira

(*Communicated by Milan Paštéka*)

ABSTRACT. Let $\{B_n\}_{n\geq 0}$ be the sequence of Balancing numbers defined by $B_0 = 0$, $B_1 = 1$, and $B_{n+2} = 6B_{n+1} - B_n$ for all $n \geq 0$. In this paper, we find all repdigits in base 10 which can be written as a sum of three Balancing numbers.

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1. Introduction

The concept of *balancing numbers* was first introduced by Bahera and Panda [1] in connection with the Diophantine equation that consists of finding a natural number n such that

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),$$

for some natural number r. The number n is called the *balancing number* and the number r is called the *balancer* corresponding to n. For example $6, 35$, and 204 are balacing numbers with balancers $2, 14$, and 84, respectively.

We can also define the balancing numbers as a recurrence sequence. Let $\{B_n\}_{n\geq 0}$ be the sequence of balancing numbers given by

$$B_0 = 0,\ B_1 = 1, \quad \text{and} \quad B_{n+2} = 6B_{n+1} - B_n \qquad \text{for all } n \geq 0.$$

This is sequence *A*001109 on the On-Line Encyclopedia of Integer Sequences (OEIS) [10]. The first few terms of this sequence are

$$\{B_n\}_{n\geq 0} = 0, 1, 6, 35, 204, 1189, 6930, 40391, 235416, 1372105, 7997214, 46611179, \ldots.$$

The Binet formula for the general terms of this sequence is given by

$$B_n = \frac{\gamma^n - \delta^n}{4\sqrt{2}}, \tag{1.1}$$

where $(\gamma, \delta) = (3 + 2\sqrt{2},\ 3 - 2\sqrt{2})$ are the roots of $x^2 - 6x + 1 = 0$, which is the characteristic equation of the balancing sequence.

Furthermore, it can be noted that $5 < \gamma < 6$, $0 < \delta < 1$ and we can prove by induction that

$$\gamma^{n-1} \leq B_n < \gamma^n, \tag{1.2}$$

holds for all $n \geq 1$.

2010 Mathematics Subject Classification: Primary 11B39, 11D45; Secondary 11D61, 11J86.
Keywords: Balancing number, repdigit, linear form in logarithms, reduction method.
This research was supported by the Austrian Science Fund (FWF) projects: F5510-N26 – Part of the special research program (SFB), “Quasi-Monte Carlo Methods: Theory and Applications” and W1230 – “Doctoral Program Discrete Mathematics”.

A repdigit is a positive integer N that has only one distinct digit when written in base 10. That is, N is of the form

$$N = d\Big(\frac{10^\ell - 1}{9}\Big), \tag{1.3}$$

for some positive integers d, ℓ with $1 \leq d \leq 9$. The sequence of repdigits is sequence $A010785$ on the OEIS.

2. Main result

In this paper, we study the problem of writing repdigits as sums of three balancing numbers. More precisely, we completely solve the Diophantine equation

$$N = B_{n_1} + B_{n_2} + B_{n_3} = d\left(\frac{10^\ell - 1}{9}\right), \tag{2.1}$$

in non-negative integers $(N, n_1, n_2, n_3, d, \ell)$ with $n_1 \geq n_2 \geq n_3 \geq 0$, $\ell \geq 1$, and $1 \leq d \leq 9$.

Our main result is the following.

Theorem 2.1. *All non-negative integer solutions $(N, n_1, n_2, n_3, d, \ell)$ of the Diophantine equation (2.1) with $n_1 \geq n_2 \geq n_3 \geq 0$, $\ell \geq 1$, and $1 \leq d \leq 9$, arise from*

$$N \in \{1, 2, 3, 6, 7, 8\}.$$

This paper serves as a continuation of the results in [5–7, 9]. The method of proof involves the application of Baker's theory for linear forms in logarithms of algebraic numbers, and the Baker-Davenport reduction procedure. Computations are done with the help of a simple computer program in *Mathematica*.

3. Preliminary results

3.1. Linear forms in logarithms

Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (x - \eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. Then the *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d}\Big(\log a_0 + \sum_{i=1}^{d} \log\big(\max\{|\eta^{(i)}|, 1\}\big)\Big).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log \max\{|p|, q\}$. The following are some of the properties of the logarithmic height function $h(\cdot)$, which will be used in the next sections of this paper without reference:

$$\begin{aligned} h(\eta_1 \pm \eta_2) &\leq h(\eta_1) + h(\eta_2) + \log 2, \\ h(\eta_1 \eta_2^{\pm 1}) &\leq h(\eta_1) + h(\eta_2), \\ h(\eta^s) &= |s| h(\eta) \qquad\qquad (s \in \mathbb{Z}). \end{aligned} \tag{3.1}$$

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We recall the result of Bugeaud, Mignotte, and Siksek ([2: Theorem 9.4, pp. 989]), which is a modified version of the result of Matveev [8], which is one of our main tools in this paper.

Theorem 3.1. *Let $\eta_1, \dots, \eta_t$ be positive real algebraic numbers in a real algebraic number field $\mathbb{K} \subset \mathbb{R}$ of degree $D_{\mathbb{K}}$, $b_1, \dots, b_t$ be nonzero integers, and assume that*

$$\Lambda := \eta_1^{b_1} \cdots \eta_t^{b_t} - 1, \tag{3.2}$$

is nonzero. Then

$$\log |\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D_{\mathbb{K}}^2 (1 + \log D_{\mathbb{K}})(1 + \log B) A_1 \dots A_t,$$

where

$$B \geq \max\{|b_1|, \dots, |b_t|\},$$

and

$$A_i \geq \max\{D_{\mathbb{K}} h(\eta_i), |\log \eta_i|, 0.16\}, \qquad \text{for all } i = 1, \dots, t.$$

3.2. Reduction procedure

During the calculations, we get upper bounds on our variables which are too large, thus we need to reduce them. To do so, we use some result from the theory of continued fractions.

For a nonhomogeneous linear form in two integer variables, we use a slight variation of a result due to Dujella and Pethő (see [3: Lemma 5a]). For a real number X, we write $\|X\| := \min\{|X - n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 3.1. *Let M be a positive integer, $\frac{p}{q}$ be a convergent of the continued fraction expansion of the irrational number τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Furthermore, let $\varepsilon := \|\mu q\| - M\|\tau q\|$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v, and w with

$$u \leq M \qquad \text{and} \qquad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

Finally, the following Lemma is also useful. It is [4: Lemma 7].

Lemma 3.2. *If $r \geq 1$, $H > (4r^2)^r$, and $H > L/(\log L)^r$, then $L < 2^r H (\log H)^r$.*

4. Bounding the variables

We assume that $n_1 \geq n_2 \geq n_3$. From (1.2) and (2.1), we have

$$\gamma^{n_1 - 1} \leq B_{n_1} \leq B_{n_1} + B_{n_2} + B_{n_3} = d\Big(\frac{10^\ell - 1}{9}\Big) \leq 10^\ell$$

and

$$10^{\ell - 1} \leq d\Big(\frac{10^\ell - 1}{9}\Big) = B_{n_1} + B_{n_2} + B_{n_3} \leq 3B_{n_1} < \gamma^{n_1 + 1},$$

where we use $\gamma > 3$. Thus,

$$(n_1 - 1)\frac{\log \gamma}{\log 10} \leq \ell \quad \text{and} \quad \ell - 1 \leq (n_1 + 1)\frac{\log \gamma}{\log 10}.$$

Since $\log \gamma / \log 10 = 0.76555 \cdots < 4/5$, we can conclude from the above that

$$\frac{3}{5}(n_1 - 1) < \ell < 1 + \frac{4}{5}(n_1 + 1). \tag{4.1}$$

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Running a *Mathematica* program in the range $0 \le n_3 \le n_2 \le n_1 \le 100$, $1 \le d \le 9$, and $1 \le \ell \le 100$ we obtain only the solutions listed in Theorem 2.1. From now onwards, we assume that $n_1 > 100$.

Equation (2.1) can be written as

$$\frac{\gamma^{n_1}-\delta^{n_1}}{4\sqrt{2}}+\frac{\gamma^{n_2}-\delta^{n_2}}{4\sqrt{2}}+\frac{\gamma^{n_3}-\delta^{n_3}}{4\sqrt{2}}=d\Big(\frac{10^\ell-1}{9}\Big). \tag{4.2}$$

Then, we rewrite (4.2) in three ways and apply lower bounds for linear forms in logarithms in three steps as follows:

4.1. Step 1

We have that

$$\frac{\gamma^{n_1}-\delta^{n_1}}{4\sqrt{2}}+\frac{\gamma^{n_2}-\delta^{n_2}}{4\sqrt{2}}+\frac{\gamma^{n_3}-\delta^{n_3}}{4\sqrt{2}}-\frac{d\cdot 10^\ell}{9}=-\frac{d}{9}.$$

This is equivalent to

$$\frac{\gamma^{n_1}}{4\sqrt{2}}\left(1+\gamma^{n_2-n_1}+\gamma^{n_3-n_1}\right)-\frac{d\cdot 10^\ell}{9}=-\frac{d}{9}+\frac{1}{4\sqrt{2}}(\delta^{n_1}+\delta^{n_2}+\delta^{n_3}).$$

Thus, we have

$$\left|\frac{\gamma^{n_1}}{4\sqrt{2}}\left(1+\gamma^{n_2-n_1}+\gamma^{n_3-n_1}\right)-\frac{d\cdot 10^\ell}{9}\right|\le\frac{d}{9}+\frac{1}{4\sqrt{2}}(\delta^{n_1}+\delta^{n_2}+\delta^{n_3})<\frac{1}{4\sqrt{2}}(4\sqrt{2}+3),$$

and so

$$\left|\frac{\gamma^{n_1}}{4\sqrt{2}}\left(1+\gamma^{n_2-n_1}+\gamma^{n_3-n_1}\right)-\frac{d\cdot 10^\ell}{9}\right|<\frac{\gamma}{2\sqrt{2}}. \tag{4.3}$$

We divide through (4.3) by $\alpha^{n_1}(1+\gamma^{n_2-n_1}+\gamma^{n_3-n_1})/(4\sqrt{2})$ to get

$$\left|10^\ell\gamma^{-n_3}\left(\frac{4d\sqrt{2}}{9(1+\gamma^{n_1-n_3}+\gamma^{n_2-n_3})}\right)-1\right|<\frac{2\gamma^{1-n_1}}{1+\gamma^{n_2-n_1}+\gamma^{n_3-n_1}}<\gamma^{2-n_1}.$$

Thus, we have

$$\left|10^\ell\gamma^{-n_3}\left(\frac{4d\sqrt{2}}{9(1+\gamma^{n_1-n_3}+\gamma^{n_2-n_3})}\right)-1\right|<\gamma^{2-n_1}. \tag{4.4}$$

We put

$$\Lambda_1:=10^\ell\gamma^{-n_3}\left(\frac{4d\sqrt{2}}{9(1+\gamma^{n_1-n_3}+\gamma^{n_2-n_3})}\right)-1.$$

In order to apply Theorem 3.1 we need to check that $\Lambda_1\neq 0$. Suppose that $\Lambda_1=0$, then we have

$$\gamma^{n_1}+\gamma^{n_2}+\gamma^{n_3}=\frac{10^\ell\cdot 4d\sqrt{2}}{9}.$$

By conjugating the above relation in $\mathbb{Q}(\sqrt{2})$, we get

$$\delta^{n_1}+\delta^{n_2}+\delta^{n_3}=-\frac{10^\ell\cdot 4d\sqrt{2}}{9}.$$

Taking absolute values on both sides of the above equation, we obtain that

$$\frac{10^\ell\cdot 4d\sqrt{2}}{9}=|\delta^{n_1}+\delta^{n_2}+\delta^{n_3}|\le|\delta|^{n_1}+|\delta|^{n_2}+|\delta|^{n_3}<3,$$

which is false for $\ell\ge 1$ and $d\ge 1$. Therefore, $\Lambda_1\neq 0$.

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So we apply Theorem 3.1 with the data

$$\begin{aligned} &t := 3, \qquad \eta_1 := 10, \qquad \eta_2 := \gamma, \qquad \eta_3 := \frac{4d\sqrt{2}}{9(1+\gamma^{n_1-n_3}+\gamma^{n_2-n_3})}, \\ &b_1 := \ell, \qquad b_2 := -n_3, \qquad b_3 := 1. \end{aligned}$$

Since $\eta_1, \eta_2, \eta_2 \in \mathbb{Q}(\sqrt{2})$, we take the field $\mathbb{K} := \mathbb{Q}(\sqrt{2})$ with degree $D_{\mathbb{K}} := 2$. Since $\max\{1, \ell, n_3\} \leq n_1$, we take $B := n_1$. The minimal polynomial of γ over $\mathbb{Z}$ is $x^2 - 6z + 1$ has roots γ and δ with $5 < \gamma < 6$ and $0 < \delta < 1$. Thus, we can take $h(\gamma) = \frac{1}{2}\log\gamma$, $h(10) = \log 10$, and $h(\sqrt{2}) = \frac{1}{2}\log 2$. Also,

$$\begin{aligned} h(\eta_3) &\leq h(4d\sqrt{2}) + h(9((1+\gamma^{n_1-n_3}+\gamma^{n_2-n_3}))) \\ &\leq h(4) + h(d) + h(\sqrt{2}) + h(9) + h((1+\gamma^{n_1-n_3}+\gamma^{n_2-n_3})) \\ &\leq h(4) + h(d) + h(\sqrt{2}) + h(9) + h(1) + ((n_1-n_2)+(n_2-n_3))h(\gamma) + 2\log 2 \\ &\leq \frac{9}{2}\log 2 + 4\log 3 + \frac{1}{2}(n_1-n_3)\log\gamma \leq 8 + \frac{1}{2}(n_1-n_3)\log\gamma. \end{aligned}$$

Thus, we can take $A_1 := 2\log 10$, $A_2 := \log\gamma$, and $A_3 := 16 + (n_1-n_3)\log\gamma$. So, Theorem 3.1 tells us that the left-hand side of (4.4) is bounded below by

$$\begin{aligned} \log|\Lambda_1| > &-1.4 \times 30^6 \times 3^{4.5} \times 2^2(1+\log 2)(1+\log n_1) \\ &\times (2\log 10)(\log\gamma)(16+(n_1-n_3)\log\gamma) \\ > &-4.37 \times 10^{14}(n_1-n_3)\log n_1. \end{aligned}$$

By comparing the above inequality with the right-hand side of (4.4) we get that

$$n_1 \leq 2.48 \times 10^{14}(n_1-n_3)\log n_1. \tag{4.5}$$

4.2. Step 2

We have that

$$\frac{\gamma^{n_1}-\delta^{n_1}}{4\sqrt{2}} + \frac{\gamma^{n_2}-\delta^{n_2}}{4\sqrt{2}} - \frac{d\cdot 10^\ell}{9} = -\frac{d}{9} - \frac{\gamma^{n_3}-\delta^{n_3}}{4\sqrt{2}}.$$

This is equivalent to

$$\frac{\gamma^{n_1}}{4\sqrt{2}}\left(1+\gamma^{n_2-n_1}\right) - \frac{d\cdot 10^\ell}{9} = -\frac{d}{9} - \frac{\gamma^{n_3}}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}(\delta^{n_1}+\delta^{n_2}+\delta^{n_3}).$$

Thus, we can conclude that

$$\begin{aligned} \left|\frac{\gamma^{n_1}}{4\sqrt{2}}\left(1+\gamma^{n_2-n_1}\right) - \frac{d\cdot 10^\ell}{9}\right| &\leq \frac{d}{9} + \frac{\gamma^{n_3}}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}(\delta^{n_1}+\delta^{n_2}+\delta^{n_3}) \\ &\leq 1 + \frac{\gamma^{n_3}}{4\sqrt{2}} + \frac{3}{4\sqrt{2}} = \frac{3+4\sqrt{2}+\gamma^{n_3}}{4\sqrt{2}} < \frac{\gamma^{n_3+1}}{4\sqrt{2}}. \end{aligned}$$

Thus, we have

$$\left|\frac{\gamma^{n_1}}{4\sqrt{2}}\left(1+\gamma^{n_2-n_1}\right) - \frac{d\cdot 10^\ell}{9}\right| < \frac{\gamma^{n_3+1}}{4\sqrt{2}}. \tag{4.6}$$

Dividing through (4.6) by $\alpha^{n_2}(1+\gamma^{n_1-n_2})/(4\sqrt{2})$, we get

$$\left|10^\ell\gamma^{-n_2}\left(\frac{4d\sqrt{2}}{9(1+\gamma^{n_1-n_2})}\right) - 1\right| < \frac{\gamma^{n_3-n_2+1}}{1+\gamma^{n_1-n_2}} < \gamma^{n_3-n_2+1}. \tag{4.7}$$

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We put

$$\Lambda_2 := 10^\ell \gamma^{-n_2}\Big(\frac{4d\sqrt{2}}{9(1+\gamma^{n_1-n_2})}\Big) - 1.$$

As before, in order to apply Theorem 3.1 we need to check that $\Lambda_2 \neq 0$. Suppose that $\Lambda_2 = 0$, then we have

$$\gamma^{n_1} + \gamma^{n_2} = \frac{10^\ell \cdot 4d\sqrt{2}}{9}.$$

By conjugating the above relation in $\mathbb{Q}(\sqrt{2})$, we get

$$\delta^{n_1} + \delta^{n_2} = -\frac{10^\ell \cdot 4d\sqrt{2}}{9}.$$

Taking absolute values on both sides of the above equation, we obtain that

$$\frac{10^\ell \cdot 4d\sqrt{2}}{9} = |\delta^{n_1} + \delta^{n_2}| \leq |\delta|^{n_1} + |\delta|^{n_2} < 2,$$

which is false for $\ell \geq 1$ and $d \geq 1$. Therefore, $\Lambda_2 \neq 0$. So we apply Theorem 3.1 with the data

$$t := 3, \quad \eta_1 := 10, \quad \eta_2 := \gamma, \quad \eta_3 := \frac{4d\sqrt{2}}{9(1+\gamma^{n_1-n_2})}, \quad b_1 := \ell, \quad b_2 := -n_2, \quad b_3 := 1.$$

We take $D := 2$, $B := n_1$, $A_1 := 2\log 10$, $A_2 := \log\gamma$ and

$$\begin{aligned} A_3 := 2h(\eta_3) &= 2(h(4d\sqrt{2}) + h(9) + h(1+\gamma^{n_1-n_2})) \\ &\leq 2(h(4) + h(d) + h(\sqrt{2}) + h(9) + (n_1-n_2)h(\gamma) + \log 2) \leq 14 + (n_1-n_2)\log\gamma. \end{aligned}$$

So, Theorem 3.1, tells us that

$$\begin{aligned} \log|\Lambda_2| > & -1.4\times 30^6 \times 3^{4.5} \times 2^2(1+\log 2)(1+\log n_1)(2\log 10) \\ & \times (\log\gamma)(14 + (n_1-n_2)\log\gamma) \\ > & -2.48\times 10^{14}(n_1-n_2)\log n_1. \end{aligned}$$

By comparing the above ineqguality with (4.7), we conclude that

$$n_2 - n_3 < 1.42\times 10^{14}(n_1-n_2)\log n_1. \tag{4.8}$$

4.3. Step 3

We have that

$$\frac{\gamma^{n_1}-\delta^{n_1}}{4\sqrt{2}} - \frac{d\cdot 10^\ell}{9} = -\frac{d}{9} - \frac{\gamma^{n_2}-\delta^{n_2}}{4\sqrt{2}} - \frac{\gamma^{n_3}-\delta^{n_3}}{4\sqrt{2}}.$$

This is equivalent to

$$\frac{\gamma^{n_1}}{4\sqrt{2}} - \frac{d\cdot 10^\ell}{9} = -\frac{d}{9} - \frac{\gamma^{n_2}}{4\sqrt{2}} - \frac{\gamma^{n_3}}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}(\delta^{n_1}+\delta^{n_2}+\delta^{n_3}).$$

Thus, we can conclude that

$$\begin{aligned} \left|\frac{\gamma^{n_1}}{4\sqrt{2}} - \frac{d\cdot 10^\ell}{9}\right| &\leq \frac{d}{9} + \frac{\gamma^{n_2}}{4\sqrt{2}} + \frac{\gamma^{n_3}}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}(\delta^{n_1}+\delta^{n_2}+\delta^{n_3}) \\ &\leq 1 + \frac{\gamma^{n_2}+\gamma^{n_3}}{4\sqrt{2}} + \frac{3}{4\sqrt{2}} = \frac{3+4\sqrt{2}+\gamma^{n_2}+\gamma^{n_3}}{4\sqrt{2}} < \frac{\gamma^{n_2+2}}{4\sqrt{2}}. \end{aligned}$$

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Thus, we have

$$\left|\frac{\gamma^{n_1}}{4\sqrt{2}} - \frac{d\cdot 10^{\ell}}{9}\right| < \frac{\gamma^{n_2+2}}{4\sqrt{2}}. \tag{4.9}$$

Dividing through (4.9) by $\alpha^{n_1}/(4\sqrt{2})$, we get

$$\left|10^{\ell}\alpha^{-n_1}\left(\frac{4d\sqrt{2}}{9}\right) - 1\right| < \alpha^{n_2-n_1+2}. \tag{4.10}$$

We put

$$\Lambda_3 = 10^{\ell}\alpha^{-n_1}\left(\frac{4d\sqrt{2}}{9}\right) - 1.$$

As in the previous cases, in order to apply Theorem 3.1 we need to check that $\Lambda_3 \neq 0$. Suppose that $\Lambda_3 = 0$, then we have

$$\gamma^{n_1} = \frac{10^{\ell}\cdot 4d\sqrt{2}}{9}.$$

By conjugating the above relation in $\mathbb{Q}(\sqrt{2})$, we get

$$\delta^{n_1} = -\frac{10^{\ell}\cdot 4d\sqrt{2}}{9}.$$

Taking absolute values on both sides of the above equation, we obtain that

$$\frac{10^{\ell}\cdot 4d\sqrt{2}}{9} = |\delta^{n_1}| \leq |\delta|^{n_1} < 1,$$

which is false for $\ell \geq 1$ and $d \geq 1$. Thus, $\Lambda_3 \neq 0$. So, we apply Theorem 3.1 with the data

$$t := 3, \quad \eta_1 := 10, \quad \eta_2 := \gamma, \quad \eta_3 := \frac{4d\sqrt{2}}{9}, \quad b_1 := \ell, \quad b_2 := -n_2, \quad b_3 := 1.$$

We take $D := 2$, $B := n_1$, $A_1 := 2\log 10$, $A_2 := \log\gamma$ and $A_3 := 2h(\eta_3) \leq 11$. Theorem 3.1, tells us that

$$\begin{aligned}\log|\Lambda_3| &> -1.4\times 30^6\times 3^{4.5}\times 2^2(1+\log 2)(1+\log n_1)(2\log 10)(\log\gamma)\times 11\\ &> -1.73\times 10^{14}\log n_1.\end{aligned}$$

By comparing this with (4.10), we obtain that

$$n_1 - n_2 < 9.8\times 10^{13}\log n_1. \tag{4.11}$$

Now we combine the inequalities (4.5), (4.8) and (4.11) to obtain the bound on n_1 as follows:

$$\begin{aligned}n_1 &< 2.48\times 10^{14}\left(1.42\times 10^{14}\left(9.8\times 10^{13}\log n_1\right)\log n_1\right)\log n_1\\ &< 3.5\times 10^{42}(\log n_1)^3.\end{aligned} \tag{4.12}$$

Now, we apply Lemma 3.2 on the above inequality (4.12) with the data: $r := 3$, $H := 3.5\times 10^{42}$, $L := n_1$. We obtain that $n_1 < 2.6\times 10^{49}$. We record what we have proved.

Lemma 4.1. *Let $(N, n_1, n_2, n_3, d, \ell)$ be the nonnegative integer solutions to the equation* (2.1) *with $n_1 \geq n_2 \geq n_3 \geq 0$, $1 \leq d \leq 9$, and $\ell \geq 1$. Then we have*

$$\ell < n_1 < 10^{50}.$$

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5. Reducing the bounds

The bounds ontained in Lemma 4.1 are too large to carry out meaningful computations on the computer. Thus, we need to reduce these bounds. To do so, we return to (4.4), (4.7), and (4.10) and apply Lemma 3.1 via the following procedure.

First, we put

$$\Gamma_3 := \ell \log 10 - n_1 \log \gamma + \log \left(\frac{4d\sqrt{2}}{9} \right), \qquad 1 \le d \le 9.$$

For technical reasons, we assume that $n_1 - n_2 \ge 20$ for the moment and go to (4.10). We will discuss how to get rid of this condition later. Note that $\mathrm{e}^{\Gamma_3} - 1 = \Lambda_3 \neq 0$. Thus, $\Gamma_3 \neq 0$. If $\Gamma_3 < 0$ then

$$0 < |\Gamma_3| < \mathrm{e}^{|\Gamma_3|} - 1 = |\Lambda_3| < \frac{1}{\gamma^{n_1 - n_2 - 2}}.$$

If $\Gamma_3 > 0$, then we have that $|\mathrm{e}^{\Gamma_3} - 1| < 1/2$. Hence $\mathrm{e}^{\Gamma_3} < 2$. Thus, we get that

$$0 < \Gamma_3 < \mathrm{e}^{\Gamma_3} - 1 = \mathrm{e}^{\Gamma_3} |\Lambda_3| < \frac{2}{\gamma^{n_1 - n_2 - 2}}.$$

Therefore, in both cases, we have that

$$0 < |\Gamma_3| = \left| \ell \log 10 - n_1 \log \gamma + \log \left(\frac{4d\sqrt{2}}{9} \right) \right| < \frac{2\gamma^2}{\gamma^{n_1 - n_2}}.$$

Dividing through the above inequality by $\log \gamma$, we get

$$0 < \left| \ell \frac{\log 10}{\log \gamma} - n_1 + \frac{\log((4d\sqrt{2})/9)}{\log \gamma} \right| < \frac{2\gamma^2}{\gamma^{n_1 - n_2} \log \gamma} \tag{5.1}$$

If we put

$$\tau := \frac{\log 10}{\log \gamma} \quad \text{and} \quad \mu_d := \frac{\log((4d\sqrt{2})/9)}{\log \gamma}, \qquad 1 \le d \le 9,$$

we can rewrite (5.1) as

$$0 < |\ell \tau - n_1 + \mu_d| < 39 \cdot \gamma^{-(n_1 - n_2)}. \tag{5.2}$$

We now apply Lemma 3.1 on (5.2). We put $M := 10^{50}$. A quick computer search in *Mathematica* reveals that the convergent

$$\frac{p_{108}}{q_{108}} = \frac{270384374044310841180242135951625722325900840522010 6}{206993128158920399059536403326757427773138324323195 1}$$

of τ is such that $q_{108} > 6M$ and $\varepsilon_d \ge 0.0623392 > 0$. Therefore, with $A := 46$ and $B := \gamma$ we calculated each value of $\log(39 q_{108}/\varepsilon_d)/\log \gamma$ and found that all of them are at most 70. Thus, we have that $n_1 - n_2 \le 70$. In the case that $n_1 - n_2 < 20$, we would have that $n_1 - n_2 < 20 < 70$. Thus, $n_1 - n_2 < 70$ holds in both cases.

Next, we put

$$\Gamma_2 := \ell \log 10 - n_2 \log \gamma + \log \left(\frac{4d\sqrt{2}}{9(1 + \gamma^{n_1 - n_2})} \right), \quad 1 \le d \le 9.$$

For technical reasons, as before we assume that $n_2 - n_3 \ge 20$ for the moment and go to (4.7). We will discuss how to get rid of this condition later. Note that $\mathrm{e}^{\Gamma_2} - 1 = \Lambda_2 \neq 0$. Thus, $\Gamma_2 \neq 0$. If $\Gamma_2 < 0$, then

$$0 < |\Gamma_2| < \mathrm{e}^{|\Gamma_2|} - 1 = |\Lambda_2| < \frac{1}{\gamma^{n_2 - n_3 - 1}}.$$

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If $\Gamma_2 > 0$, then we have that $|e^{\Gamma_2} - 1| < 1/2$. Hence $e^{\Gamma_2} < 2$. Thus, we get that

$$0 < \Gamma_2 < e^{\Gamma_2} - 1 = e^{\Gamma_2}|\Lambda_2| < \frac{2}{\gamma^{n_2-n_3-1}}.$$

Therefore, in both cases, we have that

$$0 < |\Gamma_2| = \left|\ell \log 10 - n_1 \log \gamma + \log\left(\frac{4d\sqrt{2}}{9(1+\gamma^{n_1-n_2})}\right)\right| < \frac{2\gamma}{\gamma^{n_2-n_3}}.$$

Dividing through the above inequality by $\log \gamma$, we get

$$0 < \left|\ell \frac{\log 10}{\log \gamma} - n_2 + \frac{\log((4d\sqrt{2})/9(1+\gamma^{n_1-n_2}))}{\log \gamma}\right| < \frac{2\gamma}{\gamma^{n_2-n_3}\log \gamma}. \tag{5.3}$$

We put

$$\tau := \frac{\log 10}{\log \gamma} \quad \text{and} \quad \mu_{d,k} := \frac{\log((4d\sqrt{2})/9(1+\gamma^{k}))}{\log \gamma}, \qquad 1 \le d \le 9, \quad 1 \le k \le 70,$$

where $k := n_1 - n_2$. We can rewrite (5.3) as

$$0 < |\ell\tau - n_2 + \mu_{d,k}| < 7 \cdot \gamma^{-(n_2-n_3)}. \tag{5.4}$$

We now apply Lemma 3.1 on (5.4). We put $M := 10^{50}$. A quick computer search in *Mathematica* reveals that the 108-th convergent of τ is such that $q_{108} > 6M$ and $\varepsilon_{d,k} \ge 0.00139047 > 0$. Therefore, with $A := 7$ and $B := \gamma$ we calculated each value of $\log(7q_{108}/\varepsilon_{d,k})/\log\gamma$ and found that all of them are at most 72. Thus, we have that $n_2 - n_3 \le 72$. In the case that $n_2 - n_3 < 20$, we would have that $n_2 - n_3 < 20 < 72$. Thus, $n_2 - n_3 \le 72$ holds in both cases.

Lastly, we put

$$\Gamma_1 := \ell \log 10 - n_3 \log \gamma + \log\left(\frac{4d\sqrt{2}}{9(1+\gamma^{n_1-n_3}+\gamma^{n_2-n_3})}\right), \qquad 1 \le d \le 9.$$

We use the original assumption that $n_1 > 100$ and go to (4.4). Note that $e^{\Gamma_1} - 1 = \Lambda_1 \ne 0$. Thus, $\Gamma_1 \ne 0$. If $\Gamma_1 < 0$, then

$$0 < |\Gamma_1| < e^{|\Gamma_1|} - 1 = |\Lambda_1| < \frac{1}{\gamma^{n_1-2}}.$$

If $\Gamma_1 > 0$, then we have that $|e^{\Gamma_1} - 1| < 1/2$. Hence $e^{\Gamma_1} < 2$. Thus, we get that

$$0 < \Gamma_1 < e^{\Gamma_1} - 1 = e^{\Gamma_1}|\Lambda_1| < \frac{2}{\gamma^{n_1-2}}.$$

Therefore, in both cases, we have that

$$0 < |\Gamma_1| = \left|\ell \log 10 - n_3 \log \gamma + \log\left(\frac{4d\sqrt{2}}{9(1+\gamma^{n_1-n_2}+\gamma^{n_2-n_3})}\right)\right| < \frac{2\gamma^2}{\gamma^{n_1}}.$$

Dividing through the above inequality by $\log \gamma$, we get

$$0 < \left|\ell \frac{\log 10}{\log \gamma} - n_3 + \frac{\log((4d\sqrt{2})/9(1+\gamma^{n_1-n_3}+\gamma^{n_2-n_3}))}{\log \gamma}\right| < \frac{2\gamma^2}{\gamma^{n_1}\log \gamma}. \tag{5.5}$$

We put

$$\tau := \frac{\log 10}{\log \gamma} \quad \text{and} \quad \mu_{d,k,s} := \frac{\log((4d\sqrt{2})/9(1+\gamma^{k}+\gamma^{s}))}{\log \gamma}, \qquad 1 \le d \le 9,$$

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where $1 \leq k := n_1 - n_3 = (n_1 - n_2) + (n_2 - n_3) \leq 142$ and $1 \leq s := n_2 - n_3 \leq 72$. We can rewrite (5.5) as

$$0 < |\ell\tau - n_3 + \mu_{d,k,s}| < 39 \cdot \gamma^{-n_1}. \tag{5.6}$$

We again apply Lemma 3.1 on (5.6). We also put $M := 10^{50}$. A quick computer search in *Mathematica* reveals that the 108-th convergent of τ is such that $q_{108} > 6M$ and $\varepsilon_{d,k,s} \geq 0.00125 > 0$. Therefore, with $A := 39$ and $B := \gamma$ we calculated each value of $\log(39q_{108}/\varepsilon_{d,k,s})/\log\gamma$ and found that all of them are at most 76. Thus, we have that $n_1 \leq 76$. This contradicts our assumption that $n_1 > 100$. Hence, Theorem 2.1 holds. □

Acknowledgement. The author thanks the referee for the useful comments and suggestions that greatly improved the quality of presentation of the paper.

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Received 8. 7. 2019
Accepted 12. 12. 2019

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