



Repdigits as sums of three balancing numbers

Mahadi Ddamulira

► To cite this version:

Mahadi Ddamulira. Repdigits as sums of three balancing numbers. *Mathematica Slovaca*, 2020, 70 (3), pp.557-566. 10.1515/ms-2017-0371 . hal-02405969v1

HAL Id: hal-02405969

<https://hal.science/hal-02405969v1>

Submitted on 12 Dec 2019 (v1), last revised 8 Jun 2020 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Copyright

REPDIGITS AS SUMS OF THREE BALANCING NUMBERS

MAHADI DDAMULIRA

ABSTRACT. Let $\{B_n\}_{n \geq 0}$ be the sequence of Balancing numbers defined by $B_0 = 0$, $B_1 = 1$, and $B_{n+2} = 6B_{n+1} - B_n$ for all $n \geq 0$. In this paper, we find all repdigits in base 10 which can be written as a sum of three Balancing numbers.

1. INTRODUCTION

The concept of *balancing numbers* was first introduced by Bahera and Panda [1] in connection with the Diophantine equation that consists of finding a natural number n such that

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),$$

for some natural number r . The number n is called the *balancing number* and the number r is called the *balancer* corresponding to n . For example 6, 35, and 204 are balancing numbers with balancers 2, 14, and 84, respectively.

We can also define the balancing numbers as a recurrence sequence. Let $\{B_n\}_{n \geq 0}$ be the sequence of balancing numbers given by

$$B_0 = 0, B_1 = 1, \text{ and } B_{n+2} = 6B_{n+1} - B_n \text{ for all } n \geq 0.$$

This is sequence A001109 on the Online Encyclopedia of Integer Sequences (OEIS) [10]. The first few terms of this sequence are

$$\{B_n\}_{n \geq 0} = 0, 1, 6, 35, 204, 1189, 6930, 40391, 235416, 1372105, 7997214, 46611179, \dots$$

The Binet formula for the general terms of this sequence is given by

$$B_n = \frac{\gamma^n - \delta^n}{4\sqrt{2}}, \quad (1.1)$$

where $(\gamma, \delta) = (3 + 2\sqrt{2}, 3 - 2\sqrt{2})$ are the roots of $x^2 - 6x + 1 = 0$, which is the characteristic equation of the Balancing sequence.

Furthermore, it can be noted that $5 < \gamma < 6$, $0 < \delta < 1$ and we can prove by induction that

$$\gamma^{n-1} \leq B_n < \gamma^n, \quad (1.2)$$

holds for all $n \geq 1$.

A repdigit is a positive integer N that has only one distinct digit when written in base 10. That is, N is of the form

$$N = d \left(\frac{10^\ell - 1}{9} \right), \quad (1.3)$$

2010 *Mathematics Subject Classification*. Primary 11B39, 11D45; Secondary 11D61, 11J86.

Key words and phrases. Balancing number, repdigit, linear form in logarithms, reduction method.

This research was supported by the Austrian Science Fund (FWF) projects: F5510-N26 – Part of the special research program (SFB), “Quasi-Monte Carlo Methods: Theory and Applications” and W1230 – “Doctoral Program Discrete Mathematics”.

for some positive integers d, ℓ with $1 \leq d \leq 9$. The sequence of repdigits is sequence A010785 on the OEIS.

2. MAIN RESULT

In this paper, we study the problem of writing repdigits as sums of three balancing numbers. More precisely, we completely solve the Diophantine equation

$$N = B_{n_1} + B_{n_2} + B_{n_3} = d \left(\frac{10^\ell - 1}{9} \right), \quad (2.1)$$

in non-negative integers $(N, n_1, n_2, n_3, d, \ell)$ with $n_1 \geq n_2 \geq n_3 \geq 0$, $\ell \geq 1$, and $1 \leq d \leq 9$.

Our main result is the following.

Theorem 2.1. *All non-negative integer solutions $(N, n_1, n_2, n_3, d, \ell)$ of the Diophantine equation (2.1) with $n_1 \geq n_2 \geq n_3 \geq 0$, $\ell \geq 1$, and $1 \leq d \leq 9$, arise from*

$$N \in \{1, 2, 3, 6, 7, 8\}.$$

This paper serves as a continuation of the results in [5, 6, 7, 9]. The method of proof involves the application of Baker's theory for linear forms in logarithms of algebraic numbers, and the Baker-Davenport reduction procedure. Computations are done with the help of a simple computer program in *Mathematica*.

3. PRELIMINARY RESULTS

3.1. Linear forms in logarithms. Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^d (x - \eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η . Then the *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d} \left(\log a_0 + \sum_{i=1}^d \log \left(\max\{|\eta^{(i)}|, 1\} \right) \right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log \max\{|p|, q\}$. The following are some of the properties of the logarithmic height function $h(\cdot)$, which will be used in the next sections of this paper without reference:

$$\begin{aligned} h(\eta_1 \pm \eta_2) &\leq h(\eta_1) + h(\eta_2) + \log 2, \\ h(\eta_1 \eta_2^{\pm 1}) &\leq h(\eta_1) + h(\eta_2), \\ h(\eta^s) &= |s| h(\eta) \quad (s \in \mathbb{Z}). \end{aligned} \quad (3.1)$$

We recall the result of Bugeaud, Mignotte, and Siksek ([2], Theorem 9.4, pp. 989), which is a modified version of the result of Matveev [8], which is one of our main tools in this paper.

Theorem 3.1. *Let η_1, \dots, η_t be positive real algebraic numbers in a real algebraic number field $\mathbb{K} \subset \mathbb{R}$ of degree $D_{\mathbb{K}}$, b_1, \dots, b_t be nonzero integers, and assume that*

$$\Lambda := \eta_1^{b_1} \cdots \eta_t^{b_t} - 1, \quad (3.2)$$

is nonzero. Then

$$\log |\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D_{\mathbb{K}}^2 (1 + \log D_{\mathbb{K}}) (1 + \log B) A_1 \cdots A_t,$$

where

$$B \geq \max\{|b_1|, \dots, |b_t|\},$$

and

$$A_i \geq \max\{D_{\mathbb{K}}h(\eta_i), |\log \eta_i|, 0.16\}, \quad \text{for all } i = 1, \dots, t.$$

3.2. Reduction procedure. During the calculations, we get upper bounds on our variables which are too large, thus we need to reduce them. To do so, we use some result from the theory of continued fractions. For a nonhomogeneous linear form in two integer variables, we use a slight variation of a result due to Dujella and Pethő (see [3], Lemma 5a). For a real number X , we write $\|X\| := \min\{|X - n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 3.1. *Let M be a positive integer, $\frac{p}{q}$ be a convergent of the continued fraction of the irrational number τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Furthermore, let $\varepsilon := \|\mu q\| - M\|\tau q\|$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v , and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

Finally, the following Lemma is also useful. It is Lemma 7 in [4].

Lemma 3.2. *If $r \geq 1$, $H > (4r^2)^r$, and $H > L/(\log L)^r$, then*

$$L < 2^r H (\log H)^r.$$

4. BOUNDING THE VARIABLES

We assume that $n_1 \geq n_2 \geq n_3$. From (1.2) and (2.1), we have

$$\gamma^{n_1-1} \leq B_{n_1} \leq B_{n_1} + B_{n_2} + B_{n_3} = d \left(\frac{10^\ell - 1}{9} \right) \leq 10^\ell$$

and

$$10^{\ell-1} \leq d \left(\frac{10^\ell - 1}{9} \right) = B_{n_1} + B_{n_2} + B_{n_3} \leq 3B_{n_1} < \gamma^{n_1+1},$$

where we use $\gamma > 3$. Thus,

$$(n_1 - 1) \frac{\log \gamma}{\log 10} \leq \ell \quad \text{and} \quad \ell - 1 \leq (n_1 + 1) \frac{\log \gamma}{\log 10}.$$

Since $\log \gamma / \log 10 = 0.76555\dots < 4/5$, we can conclude from the above that

$$\frac{3}{5}(n_1 - 1) < \ell < 1 + \frac{4}{5}(n_1 + 1). \tag{4.1}$$

Running a *Mathematica* program in the range $0 \leq n_3 \leq n_2 \leq n_1 \leq 100$, $1 \leq d \leq 9$, and $1 \leq \ell \leq 100$ we obtain only the solutions listed in Theorem 2.1. From now onwards, we assume that $n_1 > 100$.

Equation (2.1) can be written as

$$\frac{\gamma^{n_1} - \delta^{n_1}}{4\sqrt{2}} + \frac{\gamma^{n_2} - \delta^{n_2}}{4\sqrt{2}} + \frac{\gamma^{n_3} - \delta^{n_3}}{4\sqrt{2}} = d \left(\frac{10^\ell - 1}{9} \right). \tag{4.2}$$

Then, we rewrite (4.2) in three ways and apply lower bounds for linear forms in logarithms in three steps as follows:

4.1. **Step 1.** We have that

$$\frac{\gamma^{n_1} - \delta^{n_1}}{4\sqrt{2}} + \frac{\gamma^{n_2} - \delta^{n_2}}{4\sqrt{2}} + \frac{\gamma^{n_3} - \delta^{n_3}}{4\sqrt{2}} - \frac{d \cdot 10^\ell}{9} = -\frac{d}{9}.$$

This is equivalent to

$$\frac{\gamma^{n_1}}{4\sqrt{2}} (1 + \gamma^{n_2-n_1} + \gamma^{n_3-n_1}) - \frac{d \cdot 10^\ell}{9} = -\frac{d}{9} + \frac{1}{4\sqrt{2}}(\delta^{n_1} + \delta^{n_2} + \delta^{n_3}).$$

Thus, we have

$$\begin{aligned} \left| \frac{\gamma^{n_1}}{4\sqrt{2}} (1 + \gamma^{n_2-n_1} + \gamma^{n_3-n_1}) - \frac{d \cdot 10^\ell}{9} \right| &\leq \frac{d}{9} + \frac{1}{4\sqrt{2}}(\delta^{n_1} + \delta^{n_2} + \delta^{n_3}) \\ &< \frac{1}{4\sqrt{2}}(4\sqrt{2} + 3), \end{aligned}$$

and so

$$\left| \frac{\gamma^{n_1}}{4\sqrt{2}} (1 + \gamma^{n_2-n_1} + \gamma^{n_3-n_1}) - \frac{d \cdot 10^\ell}{9} \right| < \frac{\gamma}{2\sqrt{2}}. \quad (4.3)$$

We divide through (4.3) by $\alpha^{n_1}(1 + \gamma^{n_2-n_1} + \gamma^{n_3-n_1})/(4\sqrt{2})$ to get

$$\left| 10^\ell \gamma^{-n_3} \left(\frac{4d\sqrt{2}}{9(1 + \gamma^{n_1-n_3} + \gamma^{n_2-n_3})} \right) - 1 \right| < \frac{2\gamma^{1-n_1}}{1 + \gamma^{n_2-n_1} + \gamma^{n_3-n_1}} < \gamma^{2-n_1}.$$

Thus, we have

$$\left| 10^\ell \gamma^{-n_3} \left(\frac{4d\sqrt{2}}{9(1 + \gamma^{n_1-n_3} + \gamma^{n_2-n_3})} \right) - 1 \right| < \gamma^{2-n_1}. \quad (4.4)$$

We put

$$\Lambda_1 := 10^\ell \gamma^{-n_3} \left(\frac{4d\sqrt{2}}{9(1 + \gamma^{n_1-n_3} + \gamma^{n_2-n_3})} \right) - 1.$$

In order to apply Theorem 3.1 we need to check that $\Lambda_1 \neq 0$. Suppose that $\Lambda_1 = 0$, then we have

$$\gamma^{n_1} + \gamma^{n_2} + \gamma^{n_3} = \frac{10^\ell \cdot 4d\sqrt{2}}{9}.$$

By conjugating the above relation in $\mathbb{Q}(\sqrt{2})$, we get

$$\delta^{n_1} + \delta^{n_2} + \delta^{n_3} = -\frac{10^\ell \cdot 4d\sqrt{2}}{9}.$$

Taking absolute values on both sides of the above equation, we obtain that

$$\frac{10^\ell \cdot 4d\sqrt{2}}{9} = |\delta^{n_1} + \delta^{n_2} + \delta^{n_3}| \leq |\delta|^{n_1} + |\delta|^{n_2} + |\delta|^{n_3} < 3,$$

which is false for $\ell \geq 1$ and $d \geq 1$. Therefore, $\Lambda_1 \neq 0$.

So we apply Theorem 3.1 with the data

$$\begin{aligned} t &:= 3, \quad \eta_1 := 10, \quad \eta_2 := \gamma, \quad \eta_3 := \frac{4d\sqrt{2}}{9(1 + \gamma^{n_1-n_3} + \gamma^{n_2-n_3})}, \\ b_1 &:= \ell, \quad b_2 := -n_3, \quad b_3 := 1. \end{aligned}$$

Since $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\sqrt{2})$, we take the field $\mathbb{K} := \mathbb{Q}(\sqrt{2})$ with degree $D_{\mathbb{K}} := 2$. Since $\max\{1, \ell, n_3\} \leq n_1$, we take $B := n_1$. The minimal polynomial of γ over \mathbb{Z} is $x^2 - 6x + 1$ has roots γ and δ with

$5 < \gamma < 6$ and $0 < \delta < 1$. Thus, we can take $h(\gamma) = \frac{1}{2} \log \gamma$, $h(10) = \log 10$, and $h(\sqrt{2}) = \frac{1}{2} \log 2$. Also,

$$\begin{aligned}
 h(\eta_3) &\leq h(4d\sqrt{2}) + h(9((1 + \gamma^{n_1-n_3} + \gamma^{n_2-n_3}))) \\
 &\leq h(4) + h(d) + h(\sqrt{2}) + h(9) + h((1 + \gamma^{n_1-n_3} + \gamma^{n_2-n_3})) \\
 &\leq h(4) + h(d) + h(\sqrt{2}) + h(9) + h(1) + ((n_1 - n_2) + (n_2 - n_3))h(\gamma) + 2 \log 2 \\
 &\leq \frac{9}{2} \log 2 + 4 \log 3 + \frac{1}{2}(n_1 - n_3) \log \gamma \\
 &\leq 8 + \frac{1}{2}(n_1 - n_3) \log \gamma.
 \end{aligned}$$

Thus, we can take $A_1 := 2 \log 10$, $A_2 := \log \gamma$, and $A_3 := 16 + (n_1 - n_3) \log \gamma$. So, Theorem 3.1 tells us that the left-hand side of (4.4) is bounded below by

$$\begin{aligned}
 \log |\Lambda_1| &> -1.4 \times 30^6 \times 3^{4.5} \times 2^2(1 + \log 2)(1 + \log n_1) \\
 &\quad \times (2 \log 10)(\log \gamma)(16 + (n_1 - n_3) \log \gamma) \\
 &> -4.37 \times 10^{14}(n_1 - n_3) \log n_1.
 \end{aligned}$$

By comparing the above inequality with the right-hand side of (4.4) we get that

$$n_1 \leq 2.48 \times 10^{14}(n_1 - n_3) \log n_1. \quad (4.5)$$

4.2. Step 2. We have that

$$\frac{\gamma^{n_1} - \delta^{n_1}}{4\sqrt{2}} + \frac{\gamma^{n_2} - \delta^{n_2}}{4\sqrt{2}} - \frac{d \cdot 10^\ell}{9} = -\frac{d}{9} - \frac{\gamma^{n_3} - \delta^{n_3}}{4\sqrt{2}}.$$

This is equivalent to

$$\frac{\gamma^{n_1}}{4\sqrt{2}} (1 + \gamma^{n_2-n_1}) - \frac{d \cdot 10^\ell}{9} = -\frac{d}{9} - \frac{\gamma^{n_3}}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}(\delta^{n_1} + \delta^{n_2} + \delta^{n_3}).$$

Thus, we can conclude that

$$\begin{aligned}
 \left| \frac{\gamma^{n_1}}{4\sqrt{2}} (1 + \gamma^{n_2-n_1}) - \frac{d \cdot 10^\ell}{9} \right| &\leq \frac{d}{9} + \frac{\gamma^{n_3}}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}(\delta^{n_1} + \delta^{n_2} + \delta^{n_3}) \\
 &\leq 1 + \frac{\gamma^{n_3}}{4\sqrt{2}} + \frac{3}{4\sqrt{2}} = \frac{3 + 4\sqrt{2} + \gamma^{n_3}}{4\sqrt{2}} \\
 &< \frac{\gamma^{n_3+1}}{4\sqrt{2}}.
 \end{aligned}$$

Thus, we have

$$\left| \frac{\gamma^{n_1}}{4\sqrt{2}} (1 + \gamma^{n_2-n_1}) - \frac{d \cdot 10^\ell}{9} \right| < \frac{\gamma^{n_3+1}}{4\sqrt{2}}. \quad (4.6)$$

Dividing through (4.6) by $\alpha^{n_2}(1 + \gamma^{n_1-n_2})/(4\sqrt{2})$, we get

$$\left| 10^\ell \gamma^{-n_2} \left(\frac{4d\sqrt{2}}{9(1 + \gamma^{n_1-n_2})} \right) - 1 \right| < \frac{\gamma^{n_3-n_2+1}}{1 + \gamma^{n_1-n_2}} < \gamma^{n_3-n_2+1}. \quad (4.7)$$

We put

$$\Lambda_2 := 10^\ell \gamma^{-n_2} \left(\frac{4d\sqrt{2}}{9(1 + \gamma^{n_1-n_2})} \right) - 1.$$

As before, in order to apply Theorem 3.1 we need to check that $\Lambda_2 \neq 0$. Suppose that $\Lambda_2 = 0$, then we have

$$\gamma^{n_1} + \gamma^{n_2} = \frac{10^\ell \cdot 4d\sqrt{2}}{9}.$$

By conjugating the above relation in $\mathbb{Q}(\sqrt{2})$, we get

$$\delta^{n_1} + \delta^{n_2} = -\frac{10^\ell \cdot 4d\sqrt{2}}{9}.$$

Taking absolute values on both sides of the above equation, we obtain that

$$\frac{10^\ell \cdot 4d\sqrt{2}}{9} = |\delta^{n_1} + \delta^{n_2}| \leq |\delta|^{n_1} + |\delta|^{n_2} < 2,$$

which is false for $\ell \geq 1$ and $d \geq 1$. Therefore, $\Lambda_2 \neq 0$. So we apply Theorem 3.1 with the data

$$t := 3, \quad \eta_1 := 10, \quad \eta_2 := \gamma, \quad \eta_3 := \frac{4d\sqrt{2}}{9(1 + \gamma^{n_1 - n_2})}, \quad b_1 := \ell, \quad b_2 := -n_2, \quad b_3 := 1.$$

We take $D := 2$, $B := n_1$, $A_1 := 2 \log 10$, $A_2 := \log \gamma$ and

$$\begin{aligned} A_3 := 2h(\eta_3) &= 2(h(4d\sqrt{2}) + h(9) + h(1 + \gamma^{n_1 - n_2})) \\ &\leq 2(h(4) + h(d) + h(\sqrt{2}) + h(9) + (n_1 - n_2)h(\gamma) + \log 2) \\ &\leq 14 + (n_1 - n_2) \log \gamma. \end{aligned}$$

So, Theorem 3.1, tells us that

$$\begin{aligned} \log |\Lambda_2| &> -1.4 \times 30^6 \times 3^{4.5} \times 2^2(1 + \log 2)(1 + \log n_1)(2 \log 10) \\ &\quad \times (\log \gamma)(14 + (n_1 - n_2) \log \gamma) \\ &> -2.48 \times 10^{14}(n_1 - n_2) \log n_1. \end{aligned}$$

By comparing the above inequality with (4.7), we conclude that

$$n_2 - n_3 < 1.42 \times 10^{14}(n_1 - n_2) \log n_1. \quad (4.8)$$

4.3. Step 3. We have that

$$\frac{\gamma^{n_1} - \delta^{n_1}}{4\sqrt{2}} - \frac{d \cdot 10^\ell}{9} = -\frac{d}{9} - \frac{\gamma^{n_2} - \delta^{n_2}}{4\sqrt{2}} - \frac{\gamma^{n_3} - \delta^{n_3}}{4\sqrt{2}}.$$

This is equivalent to

$$\frac{\gamma^{n_1}}{4\sqrt{2}} - \frac{d \cdot 10^\ell}{9} = -\frac{d}{9} - \frac{\gamma^{n_2}}{4\sqrt{2}} - \frac{\gamma^{n_3}}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}(\delta^{n_1} + \delta^{n_2} + \delta^{n_3}).$$

Thus, we can conclude that

$$\begin{aligned} \left| \frac{\gamma^{n_1}}{4\sqrt{2}} - \frac{d \cdot 10^\ell}{9} \right| &\leq \frac{d}{9} + \frac{\gamma^{n_2}}{4\sqrt{2}} + \frac{\gamma^{n_3}}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}(\delta^{n_1} + \delta^{n_2} + \delta^{n_3}) \\ &\leq 1 + \frac{\gamma^{n_2} + \gamma^{n_3}}{4\sqrt{2}} + \frac{3}{4\sqrt{2}} = \frac{3 + 4\sqrt{2} + \gamma^{n_2} + \gamma^{n_3}}{4\sqrt{2}} \\ &< \frac{\gamma^{n_2+2}}{4\sqrt{2}}. \end{aligned}$$

Thus, we have

$$\left| \frac{\gamma^{n_1}}{4\sqrt{2}} - \frac{d \cdot 10^\ell}{9} \right| < \frac{\gamma^{n_2+2}}{4\sqrt{2}}. \quad (4.9)$$

Dividing through (4.9) by $\alpha^{n_1}/(4\sqrt{2})$, we get

$$\left| 10^\ell \alpha^{-n_1} \left(\frac{4d\sqrt{2}}{9} \right) - 1 \right| < \alpha^{n_2 - n_1 + 2}. \quad (4.10)$$

We put

$$\Lambda_3 = 10^\ell \alpha^{-n_1} \left(\frac{4d\sqrt{2}}{9} \right) - 1.$$

As in the previous cases, in order to apply Theorem 3.1 we need to check that $\Lambda_3 \neq 0$. Suppose that $\Lambda_3 = 0$, then we have

$$\gamma^{n_1} = \frac{10^\ell \cdot 4d\sqrt{2}}{9}.$$

By conjugating the above relation in $\mathbb{Q}(\sqrt{2})$, we get

$$\delta^{n_1} = -\frac{10^\ell \cdot 4d\sqrt{2}}{9}.$$

Taking absolute values on both sides of the above equation, we obtain that

$$\frac{10^\ell \cdot 4d\sqrt{2}}{9} = |\delta^{n_1}| \leq |\delta|^{n_1} < 1,$$

which is false for $\ell \geq 1$ and $d \geq 1$. Thus, $\Lambda_3 \neq 0$. So, we apply Theorem 3.1 with the data

$$t := 3, \quad \eta_1 := 10, \quad \eta_2 := \gamma, \quad \eta_3 := \frac{4d\sqrt{2}}{9}, \quad b_1 := \ell, \quad b_2 := -n_2, \quad b_3 := 1.$$

We take $D := 2$, $B := n_1$, $A_1 := 2 \log 10$, $A_2 := \log \gamma$ and $A_3 := 2h(\eta_3) \leq 11$. Theorem 3.1, tells us that

$$\begin{aligned} \log |\Lambda_3| &> -1.4 \times 30^6 \times 3^{4.5} \times 2^2 (1 + \log 2) (1 + \log n_1) (2 \log 10) (\log \gamma) \times 11 \\ &> -1.73 \times 10^{14} \log n_1. \end{aligned}$$

By comparing this with (4.10), we obtain that

$$n_1 - n_2 < 9.8 \times 10^{13} \log n_1. \quad (4.11)$$

Now we combine the inequalities (4.5), (4.8) and (4.11) to obtain the bound on n_1 as follows:

$$\begin{aligned} n_1 &< 2.48 \times 10^{14} (1.42 \times 10^{14} (9.8 \times 10^{13} \log n_1) \log n_1) \log n_1 \\ &< 3.5 \times 10^{42} (\log n_1)^3. \end{aligned} \quad (4.12)$$

Now, we apply Lemma 3.2 on the above inequality (4.12) with the data: $r := 3$, $H := 3.5 \times 10^{42}$, $L := n_1$. We obtain that $n_1 < 2.6 \times 10^{49}$. We record what we have proved

Lemma 4.1. *Let $(N, n_1, n_2, n_3, d, \ell)$ be the nonnegative integer solutions to the equation (2.1) with $n_1 \geq n_2 \geq n_3 \geq 0$, $1 \leq d \leq 9$, and $\ell \geq 1$. Then we have*

$$\ell < n_1 < 10^{50}.$$

5. REDUCING THE BOUNDS

The bounds obtained in Lemma 4.1 are too large to carry out meaningful computations on the computer. Thus, we need to reduce these bounds. To do so, we return to (4.4), (4.7), and (4.10) and apply Lemma 3.1 via the following procedure.

First, we put

$$\Gamma_3 := \ell \log 10 - n_1 \log \gamma + \log \left(\frac{4d\sqrt{2}}{9} \right), \quad 1 \leq d \leq 9.$$

For technical reasons, we assume that $n_1 - n_2 \geq 20$ and go to (4.10). Note that $e^{\Gamma_3} - 1 = \Lambda_3 \neq 0$. Thus, $\Gamma_3 \neq 0$. If $\Gamma_3 < 0$ then

$$0 < |\Gamma_3| < e^{|\Gamma_3|} - 1 = |\Lambda_3| < \frac{1}{\gamma^{n_1 - n_2 - 2}}.$$

If $\Gamma_3 > 0$, then we have that $|e^{\Gamma_3} - 1| < 1/2$. Hence $e^{\Gamma_3} < 2$. Thus, we get that

$$0 < \Gamma_3 < e^{\Gamma_3} - 1 = e^{\Gamma_3} |\Lambda_3| < \frac{2}{\gamma^{n_1 - n_2 - 2}}.$$

Therefore, in both cases, we have that

$$0 < |\Gamma_3| = \left| \ell \log 10 - n_1 \log \gamma + \log \left(\frac{4d\sqrt{2}}{9} \right) \right| < \frac{2\gamma^2}{\gamma^{n_1 - n_2}}.$$

Dividing through the above inequality by $\log \gamma$, we get

$$0 < \left| \ell \frac{\log 10}{\log \gamma} - n_1 + \frac{\log((4d\sqrt{2})/9)}{\log \gamma} \right| < \frac{2\gamma^2}{\gamma^{n_1 - n_2} \log \gamma} \quad (5.1)$$

If we put

$$\tau := \frac{\log 10}{\log \gamma} \quad \text{and} \quad \mu_d := \frac{\log((4d\sqrt{2})/9)}{\log \gamma}, \quad 1 \leq d \leq 9,$$

we can rewrite (5.1) as

$$0 < |\ell\tau - n_1 + \mu_d| < 39 \cdot \gamma^{-(n_1 - n_2)}. \quad (5.2)$$

We now apply Lemma 3.1 on (5.2). We put $M := 10^{50}$. A quick computer search in *Mathematica* reveals that the convergent

$$\frac{p_{108}}{q_{108}} = \frac{2703843740443108411802421359516257223259008405220106}{2069931281589203990595364033267574277731383243231951}$$

of τ is such that $q_{108} > 6M$ and $\varepsilon_d \geq 0.0623392 > 0$. Therefore, with $A := 46$ and $B := \gamma$ we calculated each value of $\log(39q_{108}/\varepsilon_d)/\log \gamma$ and found that all of them are at most 70. Thus, we have that $n_1 - n_2 \leq 70$. In the case that $n_1 - n_2 < 20$, we would have that $n_1 - n_2 < 20 < 70$. Thus, $n_1 - n_2 < 70$ in both cases.

Next, we put

$$\Gamma_2 := \ell \log 10 - n_2 \log \gamma + \log \left(\frac{4d\sqrt{2}}{9(1 + \gamma^{n_1 - n_2})} \right), \quad 1 \leq d \leq 9.$$

For technical reasons, as before we assume that $n_2 - n_3 \geq 20$ and go to (4.7). Note that $e^{\Gamma_2} - 1 = \Lambda_2 \neq 0$. Thus, $\Gamma_2 \neq 0$. If $\Gamma_2 < 0$ then

$$0 < |\Gamma_2| < e^{|\Gamma_2|} - 1 = |\Lambda_2| < \frac{1}{\gamma^{n_2 - n_3 - 1}}.$$

If $\Gamma_2 > 0$, then we have that $|e^{\Gamma_2} - 1| < 1/2$. Hence $e^{\Gamma_2} < 2$. Thus, we get that

$$0 < \Gamma_2 < e^{\Gamma_2} - 1 = e^{\Gamma_2} |\Lambda_2| < \frac{2}{\gamma^{n_2-n_3-1}}.$$

Therefore, in both cases, we have that

$$0 < |\Gamma_2| = \left| \ell \log 10 - n_1 \log \gamma + \log \left(\frac{4d\sqrt{2}}{9(1 + \gamma^{n_1-n_2})} \right) \right| < \frac{2\gamma}{\gamma^{n_2-n_3}}.$$

Dividing through the above inequality by $\log \gamma$, we get

$$0 < \left| \ell \frac{\log 10}{\log \gamma} - n_2 + \frac{\log((4d\sqrt{2})/9(1 + \gamma^{n_1-n_2}))}{\log \gamma} \right| < \frac{2\gamma}{\gamma^{n_2-n_3} \log \gamma}. \quad (5.3)$$

We put

$$\tau := \frac{\log 10}{\log \gamma} \quad \text{and} \quad \mu_{d,k} := \frac{\log((4d\sqrt{2})/9(1 + \gamma^k))}{\log \gamma}, \quad 1 \leq d \leq 9, \quad 1 \leq k \leq 70,$$

where $k := n_1 - n_2$. We can rewrite (5.3) as

$$0 < |\ell\tau - n_2 + \mu_{d,k}| < 7 \cdot \gamma^{-(n_2-n_3)}. \quad (5.4)$$

We now apply Lemma 3.1 on (5.4). We put $M := 10^{50}$. A quick computer search in *Mathematica* reveals that the 108-th convergent of τ is such that $q_{108} > 6M$ and $\varepsilon_{d,k} \geq 0.00139047 > 0$. Therefore, with $A := 7$ and $B := \gamma$ we calculated each value of $\log(7q_{108}/\varepsilon_{d,k})/\log \gamma$ and found that all of them are at most 72. Thus, we have that $n_2 - n_3 \leq 72$. In the case that $n_2 - n_3 < 20$, we would have that $n_2 - n_3 < 20 < 72$. Thus, $n_2 - n_3 \leq 72$ holds in both cases.

Lastly, we put

$$\Gamma_1 := \ell \log 10 - n_3 \log \gamma + \log \left(\frac{4d\sqrt{2}}{9(1 + \gamma^{n_1-n_3} + \gamma^{n_2-n_3})} \right), \quad 1 \leq d \leq 9.$$

We use the original assumption that $n_1 > 100$ and go to (4.4). Note that $e^{\Gamma_1} - 1 = \Lambda_1 \neq 0$. Thus, $\Gamma_1 \neq 0$. If $\Gamma_1 < 0$, then

$$0 < |\Gamma_1| < e^{|\Gamma_1|} - 1 = |\Lambda_1| < \frac{1}{\gamma^{n_1-2}}.$$

If $\Gamma_1 > 0$, then we have that $|e^{\Gamma_1} - 1| < 1/2$. Hence $e^{\Gamma_1} < 2$. Thus, we get that

$$0 < \Gamma_1 < e^{\Gamma_1} - 1 = e^{\Gamma_1} |\Lambda_1| < \frac{2}{\gamma^{n_1-2}}.$$

Therefore, in both cases, we have that

$$0 < |\Gamma_1| = \left| \ell \log 10 - n_3 \log \gamma + \log \left(\frac{4d\sqrt{2}}{9(1 + \gamma^{n_1-n_2} + \gamma^{n_2-n_3})} \right) \right| < \frac{2\gamma^2}{\gamma^{n_1}}.$$

Dividing through the above inequality by $\log \gamma$, we get

$$0 < \left| \ell \frac{\log 10}{\log \gamma} - n_3 + \frac{\log((4d\sqrt{2})/9(1 + \gamma^{n_1-n_3} + \gamma^{n_2-n_3}))}{\log \gamma} \right| < \frac{2\gamma^2}{\gamma^{n_1} \log \gamma}. \quad (5.5)$$

We put

$$\tau := \frac{\log 10}{\log \gamma} \quad \text{and} \quad \mu_{d,k,s} := \frac{\log((4d\sqrt{2})/9(1 + \gamma^k + \gamma^s))}{\log \gamma}, \quad 1 \leq d \leq 9,$$

where $1 \leq k := n_1 - n_3 = (n_1 - n_2) + (n_2 - n_3) \leq 142$ and $1 \leq s := n_2 - n_3 \leq 72$. We can rewrite (5.5) as

$$0 < |\ell\tau - n_3 + \mu_{d,k,s}| < 39 \cdot \gamma^{-n_1}. \quad (5.6)$$

We again apply Lemma 3.1 on (5.6). We also put $M := 10^{50}$. A quick computer search in *Mathematica* reveals that the 108-th convergent of τ is such that $q_{108} > 6M$ and $\varepsilon_{d,k,s} \geq 0.00125 > 0$. Therefore, with $A := 39$ and $B := \gamma$ we calculated each value of $\log(39q_{108}/\varepsilon_{d,k,s})/\log \gamma$ and found that all of them are at most 76. Thus, we have that $n_1 \leq 76$. This contradicts our assumption that $n_1 > 100$. Hence, Theorem 2.1 holds. \square

Acknowledgement. The author thanks the referee for the useful comments and suggestions that greatly improved the quality of presentation of the paper.

REFERENCES

- [1] BAHERA, A. — PANDA, G. K.: On the square roots of triangular numbers. *The Fibonacci Quarterly*, **37**(2):98–105, 1999.
- [2] BUGEAUD, Y. — MIGNOTTE, M. — SIKSEK, S.: Classical Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers. *Annals of Mathematics*, **163**(2):969–1018, 2006.
- [3] DUJELLA, A. — PETHŐ, A.: A generalization of a theorem of Baker and Davenport. *The Quarterly Journal of Mathematics*, **49**(195):291–306, 1998.
- [4] GÚZMAN SÁNCHEZ, S. — LUCA, F.: Linear combinations of factorials and s -units in a binary recurrence sequence. *Annales Mathématiques du Québec*, **38**(2):169–188, 2014.
- [5] GARCÍA LOMELÍ, A. C. — HERNÁNDEZ HERNÁNDEZ, S.: Repdigits as sums of two Padovan numbers. *Journal of Integer Sequences*, **22**(2): Article 19.2.3, 2019.
- [6] LUCA, F.: Repdigits as sums of three Fibonacci numbers. *Math. Commun.*, **17**:1–11, 2012.
- [7] LUCA, F. — NORMENYO, B. V. — TOGBÉ, A.: Repdigits as sums of three Lucas numbers. *Colloquium Mathematicum*, **156**(2):255–265, 2019.
- [8] MATVEEV, E. M.: An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers II. *Izv. Ross. Akad. Nauk Ser. Mat.*, **64**(6):125–180, 2000, in Russian: English translation in *Izvestiya Mathematics*, **64**(6):1217–1269, 2000.
- [9] NORMENYO, B. V. — LUCA, F. — TOGBÉ, A.: Repdigits as sums of three Pell numbers. *Periodica Mathematica Hungarica*, **77**(2):318–328, 2018.
- [10] OEIS FOUNDATION INC.: The On-Line Encyclopedia of Integer Sequences, <https://oeis.org>, 2019.

INSTITUTE OF ANALYSIS AND NUMBER THEORY
 GRAZ UNIVERSITY OF TECHNOLOGY
 KOPERNIKUSGASSE 24/II
 A-8010 GRAZ
 AUSTRIA
 E-mail address: mddamulira@tugraz.at; mahadi@aims.edu.gh