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Canonical extensions - an algebraic approach to Stone duality

Mai Gehrke

In memoriam Bjarni Jónsson

Abstract. This note aims to highlight some of the conceptual contributions to duality theory made by Bjarni Jónsson through the theory of canonical extensions.

1. Introduction

In the fall of 1987 Bjarni Jónsson invited me to apply for a postdoctoral position at Vanderbilt University. It was clear from our conversation at the time of the interview that he was very interested in exploring the content and meaning of Stone and Priestley duality for bounded distributive lattices. As this was the topic of my PhD, I was very enthusiastic about the subject. By the time I came to Vanderbilt the following fall, Bjarni had received a preprint from Robert Goldblatt, which was to become Goldblatt's 1989 paper *Complex Varieties* [17]. As a first task, Bjarni asked me to lecture in the seminar on this preprint. I knew nothing about Boolean Algebras with Operators (the name alone seemed to me so long and clunky), nor about modal logic, and the results in the paper were many and difficult. I am afraid I gave some extremely uninspired talks drudging through the paper supplying very little insight. Bjarni sat in the front row and slept through a couple of months of such talks, waking up at the end of each session to ask the most pertinent and challenging questions. Nevertheless I was not able to see or understand any of it really. By late November, Bjarni changed tack and handed me a short write-up entitled *Twenty Four Functors*, with a cubic diagram of functors, some more defined than others, asking me if I would study these. Now this seemed very exciting to me and I understood the task immediately. I worked

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diligently for the next 18 months handing off quite a substantial packet of work in the spring of 1990 as my postdoc was ending.

The next thing I heard from Bjarni came to my office at New Mexico State University in the spring of 1994 in a brown manila envelope. It was an essentially complete manuscript of our 1994 paper on canonical extensions of distributive lattices with operators [14]. I only knew this topic from my talks about Goldblatt's paper and, at first, I could not imagine what this might have to do with me. Nevertheless, as I started reading, I recognized many of the results from my twenty four functors work, including what was now phrased as the definition of the canonical extension of a bounded distributive lattice and the fact that canonical extension acted compositionally on order preserving maps whose canonical extensions are Scott continuous.

Feeling embarrassed by having a paper with the famous Bjarni Jónsson on a topic I knew nothing about, I started studying BAOs and DLOs in earnest and this is when our close collaboration, which lasted through the early 2000's, really started.

Robert Goldblatt's contribution to this volume [19] is also about canonical extensions and it begins with an excellent historical overview of the subject, which I highly recommend for its comprehensive and succinct analysis of the development of the theory of canonical extensions. He uses a metaphor of Bjarni's of an 'acorn that has grown into a forest'. I have been working in this large and beautiful forest most of the time since my collaboration with Bjarni. Even my more recent work, e.g. with van Gool on sheaf representations [9] or with Grigorieff and Pin on formal language theory [10, 8], owes very much to the algebraic view of topological duality, which Bjarni imparted to me through the work on canonical extensions.

This contribution is non-technical. Neither am I giving a historical overview of the development of the subject as this is done in [19]. What I would like to do with this paper is to highlight some of the conceptual contributions to duality theory made by Bjarni through the theory of canonical extensions. The paper is organized around four sections: Section 2 on the pointfree, algebraic approach to duality theory provided by Jónsson-style canonical extensions, Section 3 on canonical extension of maps as upper and lower semicontinuous envelopes, Section 4 on the functoriality of canonical extension, and finally Section 5 on the decomposition of Cartesian products into Boolean products and ultraproducts (also the fundamental ingredient in the famous Jónsson's Lemma) and its consequence in duality theory.

2. Canonical extension as pointfree Stone duality

A topological space is a pair, (X, \mathcal{O}) , where X is a set and \mathcal{O} is a collection of subsets of X satisfying certain properties. In the pointfree study of topology, the set X is discarded and just the complete lattice \mathcal{O} is kept. If the topological space is sufficiently separated (sober), then the points can be

recovered from the lattice of opens. However, a characteristic feature of topology is precisely the tension between the discrete world of points and the only finite-intersections-closed collection of opens. The canonical extension point of view of topology corresponds to getting rid of the points without getting rid of the complete lattice $\mathcal{P}(X)$. Thus instead of studying the pair (X, \mathcal{O}) , one studies the embedding $\mathcal{O} \hookrightarrow \mathcal{P}(X)$. Further, in the case of Stone duality for Boolean algebras, since \mathcal{O} is generated by the dual Boolean algebra B , the data $\mathcal{O} \hookrightarrow \mathcal{P}(X)$, in turn, amounts to giving an embedding $B \hookrightarrow \mathcal{P}(X)$. This latter formulation brings the entire duality within the setting of lattice theory, making the proper translation of additional structure such as operations on the Boolean algebra more transparent. The key insight needed here is that this embedding may be uniquely characterized among the completions of B .

We may view this problem as a typical duality exercise: We are given the following data

$$\begin{array}{ccc}
 \text{BA} & \overset{?}{\dashrightarrow} & \text{BA}^+ \\
 \text{\scriptsize St} \updownarrow & \text{\scriptsize $Clop$} & \updownarrow \text{\scriptsize At} \\
 \text{BStone} & \xrightarrow{U} & \text{Set}
 \end{array}$$

and we want to transport the forgetful functor U across the dualities. Here $Clop$ and St are the functors of Stone duality for Boolean algebras, assigning to a Boolean algebra its Stone space and to a Boolean Stone space its Boolean algebra of clopen sets, whereas the functors \mathcal{P} and At are the functors of the discrete or Birkhoff duality between sets and complete and atomic Boolean algebras assigning to a set its powerset and to a complete and atomic Boolean algebra its set of atoms. Thus the dual of the forgetful functor $U: \text{BStone} \rightarrow \text{Set}$ from Boolean spaces to sets is transported by duality to the functor $\text{BA} \rightarrow \text{BA}^+$ which sends a Boolean algebra B to $[\mathcal{P} \circ U \circ St](B)$, that is, the powerset of the set of ultrafilters of B . However, we want an intrinsic algebraic characterization rather than the one using the detour through Stone spaces and sets.

This problem was solved by Jónsson and Tarski in their famous paper [24] on Boolean Algebras with Operators (BAOs). They introduced what they called the *perfect extension* of a Boolean algebra. The name was later changed to the *canonical extension*, see [19] for more details. The notation for the canonical extension of A also changed from A^+ to A^σ . More recently, through work with H. A. Priestley, we believe it more apt to denote the canonical extension of a Boolean algebra or bounded distributive lattice by A^δ as the sigma and pi extensions are equal for objects. Incidentally, the translation of the forgetful functor $\text{BA}^+ \rightarrow \text{BA}$, to a functor $\text{Set} \rightarrow \text{BStone}$ is given by the Stone-Čech compactification. In fact, these functor pairs,

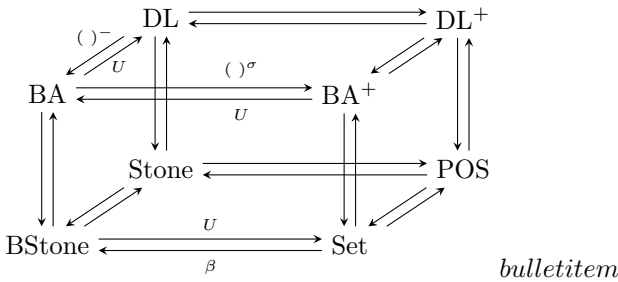
$$U: \text{BA}^+ \rightleftarrows \text{BA}: ()^\delta \quad \text{and} \quad U: \text{BStone} \rightleftarrows \text{Set}: \beta,$$

are very nice categorical adjunctions which are dual to each other.

The key to the power of the Jónsson–Tarski canonical extension is precisely that one no longer needs to travel around the square. They gave an intrinsic characterization of the unit $\eta_B: B \rightarrow B^\delta$ of the adjunction as a completion of B which is ‘compact’ and ‘separating’. This description is a powerful tool for working with duality theory in an algebraic form and may be seen as a form of pointfree topology well before it appeared elsewhere. However, the ‘separating’ property refers to the existence of sufficiently many atoms in the extension, so to show it requires the Axiom of Choice and thus the Jónsson–Tarski characterization does not satisfy the objective of later pointfree topology. That being said, as we observed later in joint work with Harding [11], the separation property may be replaced by a density property making the canonical extension available in an Axiom-of-Choice-free setting. More details of this Jónsson-style pointfree topology (in the setting of distributive lattices and Heyting algebras) may be found in [7, Section 2].

2.1. Twenty four functors

Thus, what Bjarni asked me to do in the fall of 1988, was to study the following diagram.



The back square should be a bounded distributive lattice version of the front one just discussed above. DL^+ is a bit harder to describe than BA^+ . It consists of those complete lattices which are completely distributive and have enough completely join irreducible or, because of the complete distributivity, this is equivalent to having enough completely join prime elements, see also [13, Theorem 2.1] for other equivalent characterizations. Stone is the category of Stone or spectral spaces with proper maps [26]. The functors between the back and the front are inclusion and forgetful functors and Booleanization (which on the topological level corresponds to the patch topology construction). For example, the dual of the forgetful functor from DL^+ to DL is an ordered version of the Stone–Čech compactification first studied by Banaschewski in [1]. While studying this cube, I rediscovered the results of Banaschewski and many more things about these twenty four functors and the categories on which they are defined. One problem that had not been solved before was the problem ‘parallel’ to the one solved by Jónsson–Tarski. Namely giving an intrinsic characterization of the dual of the forgetful functor $Stone \rightarrow POS$, which sends the Stone dual of a distributive lattice to its underlying poset

of points in the specialization order (formulated in terms of the Priestley duality this is even simpler as it just forgets the topology of the space while retaining the order component). For a duality theorist this was really quite an easy exercise and I never realized that I had done much of anything even though, in later years, Bjarni always asked me how I figured out what the canonical extension of a bounded distributive lattice should be. I always just told him it was easy given his cubic diagram and the Boolean case as input.

3. Naturality of the extension of additional operations

A key point, in the work of Jónsson and Tarski as well as in our subsequent work, is that canonical extension is not just applied to bounded lattice homomorphisms but to a much bigger clone of functions. The original construction in Jónsson–Tarski of the sigma extension [24, Definition 2.1 and Theorem 2.3] applies to arbitrary order preserving maps. There are however several problems with this:

- It is not general enough as we cannot extend order-reversing operations – or even worse – compositions, such as $x \mapsto x \vee \neg x$, which may be neither order preserving or reversing.
- It is too general because on the level of arbitrary order-preserving maps sigma extension is not functorial as it does not commute with composition. This point is a serious hindrance as it makes inductive proofs very difficult if not impossible.
- The sigma extension, and its dual (known as the pi extension), are in general just two among (infinitely) many possible extensions.

The first two points are serious practical problems but Bjarni also cared very much about the third point, which he brought up often in our discussions. Even though he did not advocate the perspective of category theory per se, it has been my experience, again and again, in working with Bjarni, that it was centrally important for him that an object of study should be uniquely determined by an abstract property rather than just an ad hoc construction. Interestingly (and probably not surprisingly for mathematicians), the solution to the third bullet point was in the end closely connected to improvement on the two first issues.

As a first step, we realized that sigma extension is functorial on the clone of all (necessarily order preserving) maps whose sigma extensions are Scott continuous:

Theorem 3.1. [14, Theorem 4.3] *Let $g: A \rightarrow B$ be order preserving and $f: B \rightarrow C$ be such that f^σ is Scott continuous. Then $(f \circ g)^\sigma = f^\sigma \circ g^\sigma$.*

This, along with the fact that the sigma extension of any operator is Scott continuous, is what allowed us to give a very simple (inductive) proof of the Jónsson–Tarski canonicity theorem in the setting of bounded distributive lattices with operators [14, Theorem 4.6].

3.1. Topology on canonical extensions

In the summer of 1998, while I was visiting Bjarni and Harriet at their cabin in Minnesota, we finally unblocked the question of generalizing beyond order preserving maps by describing in which sense the sigma and pi extensions of maps are also natural (or ‘canonical’). The answer involves putting topology on canonical extensions. Once we realized this, Bjarni was in quite a state of puzzlement for a while: He told me that it had been suggested to him already in the late 1940s that ‘adding topology’ might be the way to understand extension of maps. However, viewing canonical extensions as topological algebras *does not work* and he had abandoned the idea. One has to consider *distinct topologies* on the domains and codomains of operations, and, as we shall see, this explains the great complexity of questions of canonicity.

The answer comes from a classical concept from real analysis! Given a topological space (X, \mathcal{O}) , a complete lattice L and a function $f: A \rightarrow L$ defined on a dense subset $A \subseteq X$, the *upper and lower semicontinuous envelopes* of f are defined by

$$\begin{aligned}\underline{\text{lim}}(f)(x) &= \bigvee \{ \bigwedge f[U \cap A] \mid x \in U \in \mathcal{O} \} \\ \overline{\text{lim}}(f)(x) &= \bigwedge \{ \bigvee f[U \cap A] \mid x \in U \in \mathcal{O} \}.\end{aligned}$$

Under mild assumptions, which are satisfied whenever L is the canonical extension of a distributive lattice, these are, respectively, the greatest function (in the pointwise order) which is below f on A and which is upper semicontinuous and the least function which is above f on A and which is lower semicontinuous [13, Theorem 2.11]. Upper and lower semicontinuity refer to the two halves of the interval topology on the poset L .

Definition 3.2. Let P be a poset. The *upper*, respectively, *lower topology* on P is generated by the complements of principal downsets, respectively, upsets, and the *interval topology* is the join of these two topologies. That is,

$$\iota^\uparrow = \langle (\downarrow p)^c \mid p \in P \rangle, \quad \iota^\downarrow = \langle (\uparrow p)^c \mid p \in P \rangle, \quad \text{and} \quad \iota = \iota^\uparrow \vee \iota^\downarrow.$$

A function from a topological space to P is *upper*, respectively, *lower semicontinuous* provided it is continuous when viewed as a function into the topological space (P, ι^\uparrow) , respectively, (P, ι^\downarrow) .

The standard topology of the reals is the interval topology for the usual order on the reals. However, for the interval topology, a ‘best continuous approximant’ of a function does not exist in general. The fact that such canonical upper and lower envelopes exist for the two halves of the interval topology often plays an important role in analysis. With the appropriate topology on the domain one may see the sigma and the pi extensions of a map between lattices as the upper and lower semicontinuous envelopes of the map, respectively.

Definition 3.3. Let A be a bounded distributive lattice and $\eta: A \rightarrow A^\delta$ its canonical extension. An element $x \in A^\delta$ is called a *filter element* provided it is in the complete meet closure of A in A^δ , while an element $y \in A^\delta$ is called

an *ideal element* provided it is in the complete join closure of A in A^δ . The *strong topology* on A^δ is generated by the sets of the form

$$[x, y] = \{u \in A^\delta \mid x \leq u \leq y\}$$

with x ranging over filter elements of A^δ and y ranging over ideal elements. This topology is not akin to the interval topology since the basic opens are *closed* intervals.

One can show that A^δ equipped with the interval topology is a Priestley space (in fact, for these lattices, the interval topology is equal to the double Scott topology and to the Lawson topology and this topology is the unique topology for the given order making it into a Priestley space, see the discussion leading up to [13, Theorem 2.8] and the further references given there). The strong topology on A^δ contains the interval topology properly (for A infinite) thus it follows that the strong topology on A^δ makes it into a totally order disconnected space which is not compact. Further, one can show that $\eta: A \rightarrow A^\delta$ embeds A in the space as its set of isolated points and the image is topologically dense in A^δ , again see [13, Theorem 2.8]. This is very convenient as it implies that any function with domain A is continuous with respect to the strong topology on A^δ and has unique upper and lower semicontinuous extensions to A^δ . This is central in showing that $\underline{\text{lim}}(f)$ and $\overline{\text{lim}}(f)$ are always extensions of f . For further details, see [13, Section 2]. See also [16, Section 4] for details on how this works for lattices that are not necessarily distributive.

Canonical extensions of maps, at least if they are operators or dual operators, are of interest because they allow us to define Stone duals of these maps. In particular, consider a bounded distributive lattice A with an additional n -ary operation f which is a normal operator. That is, f preserves finite joins in each coordinate. In this situation, one can show that the sigma extension of f , obtained as $f^\sigma = \underline{\text{lim}}(f)$, is a complete operator. That is, it is completely join preserving in each coordinate. Invoking the connection with Stone duality as discussed in Section 2.1, we know that $A^\delta \cong \mathcal{U}(X, \leq)$ where X is the dual space of A with its specialization (or Priestley) order. Now identifying A^δ with $\mathcal{U}(X, \leq)$ and using the fact that f^σ is a complete operator, it follows that f^σ is completely determined by its restriction to the tuples whose coordinates are principal upsets $\uparrow x$ for $x \in X$; these being the completely join irreducible elements of A^δ . Thus, defining an $(n+1)$ -ary relation on X by $(y, x_1, \dots, x_n) \in R$ if and only if $\uparrow y \leq f^\sigma(\uparrow x_1, \dots, \uparrow x_n)$, one may show that f^σ is given by inverse image under R , i.e. $f^\sigma: (U_1, \dots, U_n) \mapsto R^{-1}[U_1 \times \dots \times U_n]$. It follows that we can recapture f from the relation R . In the case of Boolean algebras with operators, this is the Jónsson–Tarski duality, exploited in their work on canonical extensions [24] in order to obtain set representations of various classes of BAOs, and foreshadowing Kripke semantics of modal logic about a decade before it was introduced. For further details, see [19] in this volume and [18, Section 3.3].

3.2. Syntactic conditions for canonicity

An important issue concerning canonical extension is which properties are preserved when moving from a lattice or Boolean algebra with additional operations to its canonical extension. When a property is preserved, it is said to be canonical. It follows that the dual of a canonical property is purely order theoretic and thus independent of topology. Accordingly, if a class is given by canonical properties then canonical extension provides concrete set-theoretic representation *within the class*.

An important algebraic condition closely related to canonicity is the question of when canonical extension for maps commutes with composition of maps. In fact, as we will explain below, in order to be able to give inductive proofs of canonicity for large classes of varieties, it is particularly important to identify whole clones (classes containing all projections and closed under composition) with this property.

Consider the diagrams

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 h \downarrow & & \downarrow g \\
 C & \xrightarrow{k} & D
 \end{array}
 \qquad
 \begin{array}{ccc}
 A^\delta & \xrightarrow{f^\sigma} & B^\delta \\
 h^\sigma \downarrow & & \downarrow g^\sigma \\
 C^\delta & \xrightarrow{k^\sigma} & D^\delta
 \end{array}$$

Given that the first diagram commutes, we want to be able to conclude that the second diagram also commutes. An obvious way to prove this is to show that

$$g^\sigma f^\sigma = (gf)^\sigma = (kh)^\sigma = k^\sigma h^\sigma.$$

Here the second equality holds by hypothesis. The question is therefore, when canonical extension distributes over composition, i.e., under what conditions it is the case that $(gf)^\sigma = g^\sigma f^\sigma$?

The topological description of sigma (and pi) extensions clearly exhibits the problem with composition since the topology on the domain (the strong topology) and on the codomain (the Scott (or dual Scott) topology) are different thus causing problems when we want to compose. In fact the Scott and dual Scott topologies are contained in the strong topology. Nevertheless, the topological description of sigma (and pi) extensions, points us towards two special cases where things go better:

- (1) when the extension is continuous with respect to the Scott (or dual Scott) topology on both the domain and codomain;
- (2) when it is continuous with respect to the strong topology on both the domain and codomain.

The situation (1) covers the case of operators as discussed above, while (2) is essential for dealing with join or meet preserving or reversing maps. Combining these two special cases in various ways leads to Sahlqvist-type theorems. This algebraic approach to Sahlqvist theorems, using algebraic induction and

based on the properties of basic operations and the syntactic shape of the terms involved, has been studied extensively, starting with Bjarni's paper [23]. See among others [15, 5, 4, 3].

4. Functoriality and Bjarni's PH property

As we have seen above, the view of canonical extensions of maps as upper and lower topological envelopes leads to a natural dichotomy, depending on whether one considers maps that are continuous with respect to a weaker topology on the domain or with respect to a stronger one on the codomain. Also, as discussed above, the first option, which leads one to focus on sigma extensions that are Scott continuous, may be seen to be responsible for the original Jónsson–Tarski canonicity result as well as the positive version from [14] and the subsequent work on Sahlqvist theorems.

The second choice, of imposing continuity with respect to a stronger topology on the codomain, leads to Bjarni's PH (as in 'Preserving Homomorphisms') property, discussed in this section, and through that to the model theoretic results on canonicity, which we will discuss in Section 5.

The PH property comes about when we ask whether canonical extension is functorial. First of all, there is quite a bit of good news. Canonical extension *is* functorial even on the category of bounded (not-necessarily-distributive) lattices with homomorphisms. In fact, the upper and lower envelopes of homomorphisms agree thus providing the unique doubly continuous extensions, which are in fact complete homomorphisms. Furthermore, these extensions are continuous both with respect to the weaker topology on both domain and codomain and with respect to the stronger topology on both. In fact, in the setting of the category of bounded distributive lattices with their homomorphisms (and also for Boolean algebras), canonical extension is not only a functor but a reflection into the category DL^+ (respectively BA^+). However, if we are interested in distributive lattices or Boolean algebras with additional operations, so called Distributive Lattice Expansions (DLEs), then canonical extension is not functorial in general. This may be seen with a simple example due to Bjarni: Let L be a bounded chain with a bounded subchain A so that both A and its complement are dense in L (e.g. the unit interval with A the set of rational points). Further, consider as a unary operation on L the characteristic function of A . Then the embedding of A equipped with the constant function 1 into this algebra does not extend to a DLE homomorphism as it does not commute with the sigma extensions of the additional operations.

The fact that the above counterexample is provided by an embedding is no coincidence: by the First Isomorphism Theorem, every homomorphism between abstract algebras factors into a quotient map followed by an embedding and one may show that the extensions of surjective homomorphisms *are always* homomorphisms. Thus, if functoriality of canonical extension is

violated, it is violated by the associated embedding. The characteristic insight of Bjarni is that this allows us to see functoriality *as a local property* at an algebra: a DLE has the PH property provided the canonical extension of the inclusion of each of its subalgebras is a subalgebra inclusion. Then it follows from the factorization of morphisms and the fact that quotients are well behaved that canonical extension on a category of DLEs with their homomorphisms is functorial if and only if each individual algebra in the class has the PH property.

In our paper [13], we showed that large classes of DLEs have the property PH. In particular, any DLE whose basic operations are monotone, that is, preserve or reverse order in each coordinate (it may preserve in some and reverse in others), has the PH property. Also, any DLE whatsoever which lies in a finitely generated variety of DLEs has the PH property. Much of this analysis goes through for lattices but things are more complicated – mainly because the various topologies (in particular, the one generated by the principal upsets of completely join irreducible elements, the Scott topology, and the upper topology) no longer are equal. However, some results are provided in [11, 16] and it was studied in the PhD thesis of Jacob Vosmaer [27].

5. Boolean products and model theoretic conditions for canonicity

The most fundamental model theoretic result about algebras is Birkhoff's variety theorem. In a formula this may be stated as follows

$$\mathcal{V}(\mathcal{K}) = HSP(\mathcal{K}).$$

Here \mathcal{K} is a class of similar algebras and $\mathcal{V}(\mathcal{K})$ is the class of all algebras satisfying all the equations simultaneously satisfied by all the algebras in \mathcal{K} . H , S , and P are the operators on classes of similar algebras taking all homomorphic images, isomorphic copies of subalgebras, and isomorphic copies of Cartesian products, respectively. An important fact that Bjarni often has used to great advantage in his work, is that one can further decompose the operator P into $P = P_B P_\mu$, where P_B takes all possible *Boolean products* of algebras from a class \mathcal{K} and P_μ takes all possible *ultraproducts* of algebras from \mathcal{K} . This decomposition comes about as follows. Let

$$h: A \cong \prod_{i \in I} B_i$$

be a direct decomposition of an algebra A , let X be the Boolean space of all ultrafilters on I , that is, $X = \beta(I)$, the Stone-Čech compactification of the discrete space on I . For each $x \in X$ let $C_x = \prod_{i \in I} B_i / x$. That is, C_x is the ultraproduct obtained from the product $\prod_{i \in I} B_i$ by considering two tuples as equivalent if they are equal on a subset of I which belongs to the ultrafilter x . For $a \in A$ and $x \in X$, let $h'(a)(x) = h(a)/x$. Then the map

$$h': A \rightarrow \prod_{x \in X} C_x$$

is a Boolean product decomposition of A . A weak Boolean product decomposition is simply a sheaf representation over a Boolean space, and thus a Boolean product decomposition is a special sheaf representation over a Boolean space. For more details see [2] or more specifically [13, Section 3.3].

In [13], we obtained a canonical extension formulation of a result on Boolean products from my PhD dissertation: The canonical extension of a weak Boolean product decomposition is the *full Cartesian product* of the canonical extensions of the factors [6]. That is,

Theorem 5.1. [13, Theorem 3.15] *Suppose*

$$g: B \rightarrow \prod_{x \in X} C_x$$

is a weak Boolean product decomposition of the DLE B . If all the induced homomorphisms $g_x: B \rightarrow C_x$ are preserved by canonical extension, then

$$g': B^\sigma \simeq \prod_{x \in X} C_x^\sigma$$

where, for all $b \in B$ and $x \in X$, $g'(b)(x) = g_x^\sigma(b)$.

Of course a similar result holds for canonical extensions using pi extensions.

This result, combined with the decomposition $P = P_B P_\mu$ yields a number of model theoretic results on the PH property, on canonicity, and on sheaf representations. The first of these are in [13]. In particular, we showed that every distributive lattice with additional operations (DLE) that is either monotone (i.e. each additional operation is either order preserving or reversing in each coordinate) or smooth (i.e. the sigma and pi extensions of each additional operation agree) has the property PH, see [13, Theorem 3.24 and Theorem 3.26]. We showed that the property PH is preserved by a number of constructions on monotone DLEs such as homomorphic images [13, Theorem 3.12], subalgebras [13, Theorem 3.11], binary joins [13, Theorem 3.27], and weak Boolean products [13, Theorem 3.16]. Finally, we showed that a direct product has the PH property if and only if all the ultraproducts on it have it, and thus ultraproducts is the one badly behaved construction in the lot. This leads to a number of canonicity results [13, Section 4.1], including what one may see as an algebraic version of one of the main results of [17].

The line of inquiry into model theoretic conditions for canonicity started, at least for Bjarni and me, with Robert Goldblatt's paper [17], which, in 1989, as Hilary Priestley has put it 'made the wall come down' between the model theory of modal logic and the study of canonicity via lattices and duality. Since then a great number of contributions in this direction have been made, such as [20, 12, 21], the most recent of which is Goldblatt's contribution to this volume [19]. For further discussion and a historical overview we refer the reader to [19].

A somewhat different direction of generalization of this work is the duality theoretic investigation of sheaf representations of universal algebras and DLEs in particular. The fact that the canonical extension of a weak Boolean product is the Cartesian product of the canonical extensions of the factors

[13, Theorem 3.15] may be seen as a first step in this direction, while [9] is the most recent contribution in this still active direction.

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