Comment on ”Thermodynamic uncertainty relation for time-delayed Langevin systems”
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An extension of the thermodynamic uncertainty relation (TUR) to time-delayed Langevin systems has been recently proposed in [1]. Here we show that the derivation is erroneous.

An important recent development in the field of stochastic thermodynamics has been the discovery of the so-called thermodynamic uncertainty relations (TURs) that provide general lower bounds on the fluctuations of time-integrated currents in nonequilibrium systems (see e.g. [2] and references therein). Such relations have been so far established for Markov processes only but in a recent work [1] Vu and Hasegawa have presented an extension to time-delayed Langevin systems in a steady state. If correct, this would be an interesting result since delays are ubiquitous in real-world processes, for instance in biology. Unfortunately, the arguments in [1] are incorrect, as we show in the present comment.

The steady-state TUR for a general Markovian dynamics is expressed as

$$\epsilon^2(T) \equiv \frac{(\langle \Theta^2 \rangle - \langle \Theta \rangle^2)}{\langle \Theta \rangle^2} \geq \frac{2}{\langle \Sigma \rangle},$$

(1)

where $\Theta$ is an arbitrary current integrated over some observation time $T$ and $\langle \Sigma \rangle$ is the total entropy production accumulated by $T$ (in units where Boltzmann’s constant is set to $k_B = 1$). According to [1], this relation remains valid for a time-delayed Langevin dynamics provided $\langle \Sigma \rangle$ is replaced by a “generalized” dissipation $\langle \Sigma_g \rangle$ (defined by Eq. (13) in [1] or Eq. (6) below). This is an intriguing result, but we here show that it follows from an incorrect treatment of the non-Markovian character of the dynamics. Specifically, the original Langevin equation for the $N$-dimensional random variable $x(t)$ (cf. Eq. (2) in [1]),

$$\dot{x} = F(x, x_r) + \sqrt{2D} \xi,$$

(2)

where $\tau$ is the delay, $x_r \equiv x(t-\tau)$, $F(x, x_r)$ is a drift force, and $\xi$ is a Gaussian white noise, has been mistakenly replaced by

$$\dot{x} = \overline{F}(x) + \sqrt{2D} \xi,$$

(3)

where $\overline{F}(x)$ is the (instantaneous) effective force defined by $\overline{F}(x) P_{ss}(x) = \int F(x, x_r) P_{ss}(x, t; x_r, t-\tau) \, dx_r$ (here, $P_{ss}(x)$ and $P_{ss}(x, t; x_r, t-\tau)$ are the steady-state one time and two-time probability distributions, respectively). This replacement allows Vu and Hasegawa to express the probability density of a stochastic trajectory as $P(\Gamma) \propto \exp \left[ -\frac{(\epsilon^2(T))}{2} \right]$ and to obtain a lower bound on $\epsilon^2(T)$ by repeating the derivation performed in [3] for a Markovian Langevin dynamics. The point we want to stress is that $P(\Gamma)$ is not the probability of observing a trajectory generated by the non-Markovian dynamics described by Eq. (2) in a steady state, despite the fact that Eq. (9) in [1] leads to the same probability distribution $P_{ss}(x)$ as Eq. (2). The same mistake was made in Ref. [1] and signaled in [5] where $P(\Gamma)$ as given above was shown to differ from the exact path probability computed for a linear time-delayed Langevin equation (in the case $T \leq \tau$). In other words, we argue that the inequality derived in [1] applies to an effective stochastic dynamics that is not the true one. For the same reason, and contrary to the claim in Ref. [4], which is repeated in [1], the quantity $\Delta S_{tot}^\Gamma$ (cf. Eq. (9) in [1]) does not satisfy an integral fluctuation theorem (IFT) with the actual dynamics described by [2]. In fact, as shown in [3], there is another candidate for the entropy production in time-delayed systems, which is obtained from time inversion and satisfies a proper IFT.

To illustrate our point, we explicitly show that $2/\langle \Sigma_g \rangle$ is not a lower bound on the squared relative uncertainty $\epsilon^2(T)$. To this aim, we consider a two-dimensional version of Eq. (2) with

$$F(x, x_r) = \begin{pmatrix} -a_{11}x_1 - a_{12}x_{2, \tau} \\ -a_{21}x_1,_{\tau} - a_{22}x_2 \end{pmatrix},$$

(4)

and we choose $\Theta = -\int_0^T \{ [a_{11}x_1(t) + a_{12}x_2(t)] \, \hat{x}_1(t) + [a_{12}(t)x_1(t) + a_{22}x_2(t)] \, \hat{x}_2(t) \} \, dt$ as the current, where $\odot$ denotes the Stratonovich product. The model studied in section IV.C of [1] corresponds to the symmetric case $a_{11} = a_{22}$ and $a_{12} = -a_{21}$. We here focus on the model recently studied in [6] in which there is no feedback from 1 to 2. Specifically, we take $a_{11} = a$, $a_{22} = b$, $a_{12} = -c$, and $a_{21} = 0$. Note that these models are exactly solvable in a steady state due to the linearity of the force $F(x, x_r)$ and the Gaussian character of the white noise, which makes all probability distributions Gaussian. Therefore, there is no need to restrict the study to the small-$\tau$ limit, as done in [1].

In particular, using the same method as [7], one can easily compute the steady-state correlation functions $\phi_{ij}(t) \equiv \langle x_i(0)x_j(t) \rangle$ for $0 \leq t \leq \tau$. For instance, we

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find $\phi_{21}(t) = Dc/[b(a+b)]e^{b(-\tau)}$, from which we get

$$\frac{1}{T}(\Theta) = c^2\phi_{21}(0) = D\frac{c^2}{a+b}e^{-b\tau}. \quad (5)$$

The calculation of $\langle \Sigma_g \rangle$, defined in Ref. [1] as

$$\frac{1}{T}(\Sigma_g) = \frac{1}{D}(\mathbf{F}(x_t) \circ \dot{x_t}), \quad (6)$$

is also quite easy because the effective force is linear, i.e., $\mathbf{F}_1 = -K_1x_1 - K_2x_2$, $\mathbf{F}_2 = F_2 = -a_2x_2$, and the unknown coefficients $K_{11}$ and $K_{12}$ can be readily obtained by solving the steady-state Fokker-Planck equation $\sum_i=1,2 \partial_{x_i}[-(\mathbf{F}_i(x))P^{ss}(x) + D\partial_{x_i}P^{ss}(x)] = 0$ where $P^{ss}(x) \propto \exp\left[-(1/2)x^T\Sigma^{-1}x\right]$ and $\Sigma$ is the covariance matrix with elements $\sigma_{ij} = \phi_{ij}(0)$. $K_{11}$ and $K_{12}$ are then expressed in terms of the $\sigma_{ij}$'s. This eventually yields $K_{11} = ab[(a+b)^2e^{b\tau} + c^2e^{-b\tau}]/[(a+b)^2 + c^2](a+b)e^{b\tau} - ac^2e^{-b\tau}]$, $K_{12} = -bc[(a+b)^2 + c^2]/[(a+b)^2 + c^2](a+b)e^{b\tau} - ac^2e^{-b\tau}]$, and in turn

$$\langle \Sigma_g \rangle / T = \frac{bc^2[(a+b)^2 + c^2]e^{-b\tau}}{(a+b)[(a+b)^2 + c^2](a+b)e^{b\tau} - ac^2e^{-b\tau}]}. \quad (7)$$

In the more general case of the force defined by Eq. [4], solving the Fokker-Planck equation does not fully determine $\mathbf{F}(x)$, but one can then use the expression of the transition probability of Gaussian stationary processes in terms of the correlation functions (see Eq. (A1) in [6]). (In passing, we also note that $\mathbf{F}(x)$ at the order $\tau$ is not obtained by simply taking the $\tau = 0$ limit of the transition probability, as defined in Eq. (30) in [1]. For instance, in the model considered in section IV.C of [1], the exact calculation shows that the coefficient of $x_1$ in $\mathbf{F}_1$, and of $x_2$ in $\mathbf{F}_2$, is $-a + b^2\tau + \mathcal{O}(\tau^2)$. Accordingly, one should have $A = -b^2\tau$ in the expression (43) of $P^{ss}(x)$, implying that the variance of $x_1$ and $x_2$ increases with $\tau$ instead of decreasing. This error suggests that the small-$\tau$ limit is also incorrect in the two other examples considered in [1]. However, this may be undetectable at the scale of the figures displayed in [1].)

Finally, we compute the variance of $\Theta$, and for simplicity we focus on the long-time limit. Then

$$\lim_{T \to \infty} T^{-1} \langle (\Theta^2 - \langle \Theta \rangle)^2 \rangle = \chi_2(0)$$

where $\chi_2(k)$ is the scaled cumulant generating function defined by

$$\chi_2(k) = \lim_{T \to \infty} T^{-1} \ln(e^{k\Theta}).$$

A standard calculation using discrete Fourier series (see e.g. [8]) yields

$$\chi_2(k) = -1/(2\pi) \int_0^\infty d\omega \ln[1 - F_k(\omega)]$$

with $F_k(\omega) = 4kDc^2\omega[a \sin(\omega\tau) + \omega \cos(\omega\tau) + \omega^2kD\omega]/[(a^2 + \omega^2)(b^2 + \omega^2)]$. This leads to

$$\chi_2''(0) = D^2\frac{c^2}{b(a+b)^3} \left[ (a+b)[2ab + 2b^2 + c^2] + c^2(b + 2b\tau) - (a - 2b\tau)c e^{-2b\tau} \right]. \quad (8)$$

An example of the behavior of the quantity $R_\Theta = \lim_{T \to \infty} T[e^2(T) - 2/\langle \Sigma_g \rangle]$ as a function of $\tau$ is shown in Fig. 1. We observe that $R_\Theta$ becomes negative for large values of $\tau$, thus invalidating the TUR derived in [1] (more generally, the parabolic lower bound (25) on $\chi_2(k)$ is invalid). On the other hand, as expected, $R_\Theta$ is always positive if $e^2(T)$ is calculated with the effective stochastic dynamics defined by Eq. (3). We have confirmed these results by performing numerical simulations of the two dynamics.

In conclusion, the extension of the TUR to time-delayed Langevin systems is still an open problem. Whether or not the connection between TUR and Fisher information recently discussed in [9–11] offers a possible solution remains to be seen.