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A generalization of Ramos-Louzada distribution: Properties and estimation

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Abstract: In this paper, a new two-parameter model called generalized Ramos-Louzada (GRL) distribution is proposed. The new model provides more flexibility in modeling data with increasing, decreasing, j shaped and reversed-J shaped hazard rate function. Several statistical and reliability properties of the GRL model are also presented in this paper. The unknown parameters of the GRL distribution are discussed using eight frequentist estimation approaches. These approaches are important to develop a guideline to choose the best method of estimation for the GRL parameters, that would be of great interest to practitioners and applied statisticians. A detailed numerical simulation study is carried out to examine the bias and the mean square error of the proposed estimators. We illustrate the performance of the GRL distribution using two real data sets from the fields of medicine and geology and both data sets show that the new model is more appropriate as compared to the gamma, Marshall-Olkin exponential, exponentiated exponential, beta exponential, generalized Lindley, Poisson-Lomax, Lindley geometric and Lindley distributions, among others.

Keywords: Cramér–von Mises Estimation; Maximum Likelihood Estimation; Maximum Product of Spacing Estimation; Right–Tail Anderson-Darling Estimation.

1. Introduction

The probability distributions have great importance for modeling data in several areas such as medicine, engineering, and life testing, among others. Ramos and Louzada (2019) recently introduced the one-parameter distribution called Ramos-Louzada (RL) distribution with survival function (SF) given by

$$S(t|\lambda) = \left(\frac{1}{\lambda - 1} \right) \left(\lambda - 1 + \frac{t}{\lambda} \right) e^{-\frac{t}{\lambda}}, \quad t > 0, \quad (1)$$

where $\lambda \geq 2$.

The two most common one-parameter distributions are the exponential and Lindley distributions. The important generalizations of the exponential distribution are the Weibull (Weibull, 1951) and exponentiated exponential (Gupta and Kundu, 2001) models. In the case of the Lindley distribution, the power Lindley (Ghitany et al., 2013) and generalized Lindley (Nadarajah et al., 2011) models have play an important role in survival analysis. These two generalizations are obtained by considering a power parameter in the exponential and Lindley distributions. Ramos and Louzada (2019) showed that (1) outperforms the common exponential and Lindley distributions in many situations. Therefore, we will propose a new two-parameter extension of the RL distribution by including a power parameter in the baseline model (1). The new proposed model is called a generalized Ramos-Louzada (GRL) distribution.

Let T be a non-negative random variable follows the GRL model, the SF of random variable T is given by

$$S(t|\lambda, \alpha) = \left(\frac{1}{\lambda - 1} \right) \left(\lambda - 1 + \frac{t^\alpha}{\lambda} \right) e^{-\frac{t^\alpha}{\lambda}}, \quad (2)$$

where $\lambda (\geq 2)$ and $\alpha (> 0)$ are shape parameters.

Some mathematical properties, parameters estimation by eight different methods, simulations and applications are studied and proposed in this paper.

We can summarize the motivations of this proposed model as: (i) Its cumulative distribution function (CDF) and hazard rate function (HRF) have simple closed forms, hence, it can be utilized to analyze censored data; (ii) It can be represented as a mixture of Weibull distribution and a particular case of the generalized gamma distribution (Stacy, 1962) (See Section 2); (iii) The GRL distribution exhibits increasing, decreasing, reversed-J shaped and J shaped hazard rates, whereas the RL model exhibits only increasing hazard rate; and (iv) The GRL distribution outperforms many of the well-known distributions namely: gamma, Marshall-Olkin exponential, exponentiated exponential, Beta exponential, generalized Lindley, Poisson-Lomax, Lindley geometric and Lindley distributions, using two real data sets from the fields of medicine and geology. Furthermore, another important goal of this paper is to show how several frequentist estimators of the GRL parameters perform to choose the best parameter estimation method for the proposed model, which would be a great interest to practitioners and applied statisticians.

The skewness of the GRL distribution varies within the interval $(-0.68158, 5.17333)$, whereas the skewness of the RL distribution can only range in the interval $(1.41421, 1.85648)$ when the parameter λ takes values $(2, 3.1, 4, 5.5)$. Furthermore, the spread of the kurtosis of the GRL distribution is much larger ranging, which is from 2.69447 to 52.6597, whereas the spread of the kurtosis of the RL distribution can only varies from 6.00 to 8.04 for the same values shown above for the parameter λ .

This paper is organized as follows: Section 2 introduces the GRL distribution and its properties such as: quantile function, moments, order statistics and HRF. Section 3 presents the estimators of the GRL unknown parameters based on eight classical estimation methods. Simulation study, to evaluate and compare the behavior of the eight classical estimation methods, is discussed in Section 4. Section 5 illustrates the relevance of GRL model for two real lifetime data sets. Section 6 summarizes the present study.

2. Properties

Let T be a random variable follows the GRL model with SF given in (2), the probability density function (PDF) of the random variable T is given by

$$f(t; \phi) = \frac{\alpha}{\lambda(\lambda - 1)} t^{\alpha-1} \left(\lambda + \frac{t^\alpha}{\lambda} - 2 \right) e^{-\frac{t^\alpha}{\lambda}}, \quad t > 0, \quad \lambda \geq 2, \quad \alpha > 0, \quad (3)$$

where $\phi = (\lambda, \alpha)^\top$. Note that, the RL model can be obtained from (3) when $\alpha = 1$.

2.1. Shapes

The behavior of the PDF in (3) when $t \rightarrow 0$ and $t \rightarrow \infty$ are, respectively, given by

$$\lim_{t \rightarrow 0} f(t; \phi) = \begin{cases} \infty, & \text{if } \alpha < 1 \\ \frac{(\lambda - 2)}{\lambda(\lambda - 1)}, & \text{if } \alpha = 1, \\ 0, & \text{if } \alpha > 1 \end{cases}$$

$$\lim_{t \rightarrow \infty} f(t; \phi) = 0.$$

In Figure 1, we present the shapes of the PDF for different values of the parameters λ and α . The shape of PDF of the GRL model can be right-skewed and reversed-J shaped.

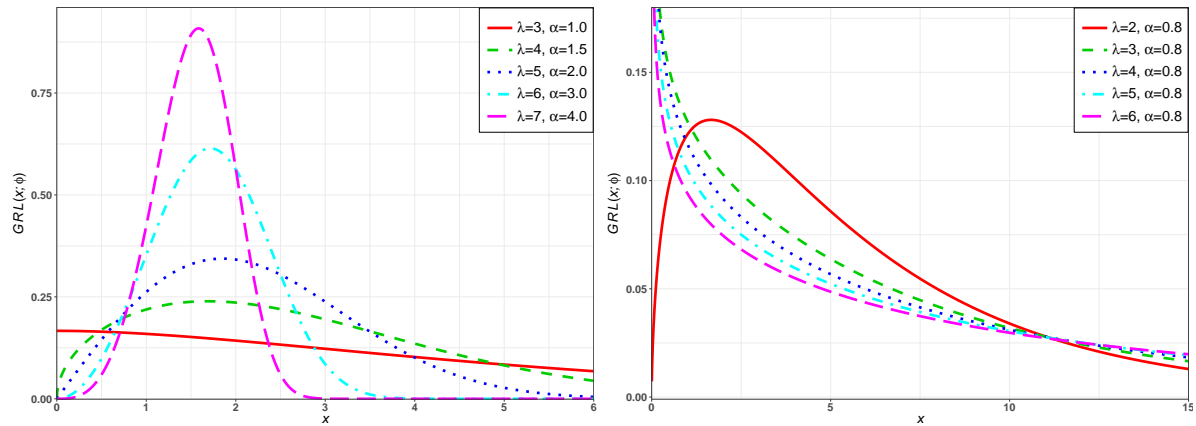


Figure 1. PDF shapes for the GRL distribution considering different values of λ and α .

The CDF of the GRL distribution is given by

$$F(t; \boldsymbol{\phi}) = 1 - \left(\frac{1}{\lambda - 1} \right) \left(\lambda - 1 + \frac{t^\alpha}{\lambda} \right) e^{-\frac{t^\alpha}{\lambda}}, \quad t > 0, \quad \lambda \geq 2, \quad \alpha > 0. \quad (4)$$

The HRF of T is given by

$$h(t; \boldsymbol{\phi}) = \frac{f(t; \boldsymbol{\phi})}{S(t; \boldsymbol{\phi})} = \frac{\alpha t^{\alpha-1}}{\lambda} \frac{(\lambda^2 + t^\alpha - 2\lambda)}{(\lambda^2 + t^\alpha - \lambda)}. \quad (5)$$

Figure 2 displays some possible shapes of HRF of the GRL for some selected values of λ and α . The shape of HRF can be increasing, decreasing, reversed-J shaped and J shaped hazard rates.

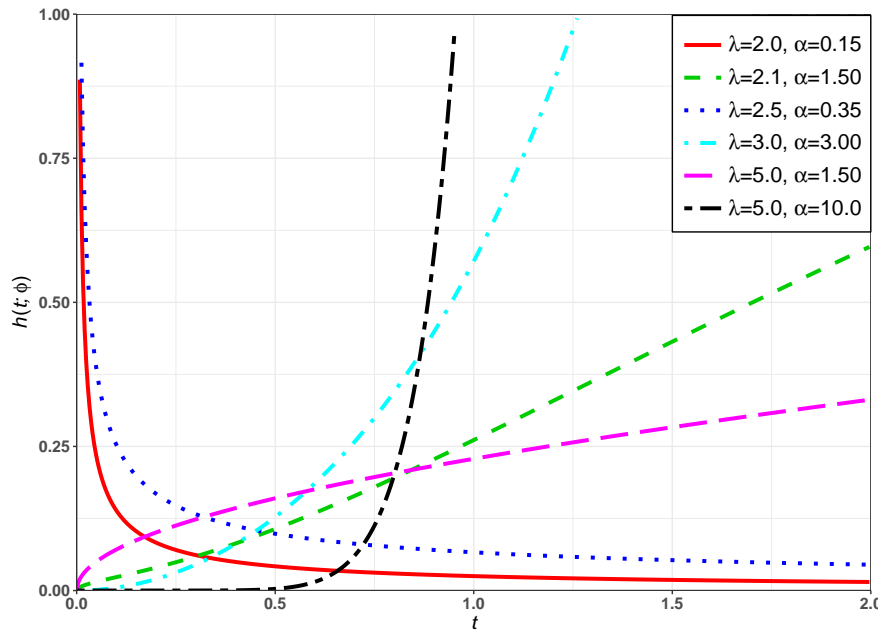


Figure 2. Hazard rate function shapes for the GRL considering different values of λ and α .

The GRL distribution can be expressed as a two-component mixture

$$f(t; \boldsymbol{\phi}) = p f_1(t; \boldsymbol{\phi}) + (1 - p) f_2(t; \boldsymbol{\phi}), \quad (6)$$

where $1 - p = 1/(\lambda - 1)$ (or $p = (\lambda - 2)/(\lambda - 1)$) and

$$f_j(t; \boldsymbol{\phi}) = \frac{\alpha}{\lambda^j} t^{j\alpha-1} e^{-\frac{t^\alpha}{\lambda}} \quad \text{for } j = 1, 2. \quad (7)$$

Note that, $f_1(\cdot)$ is a Weibull distribution and $f_2(\cdot)$ is a particular case of the generalized gamma distribution (Stacy, 1962). Then, after some algebra, Equation (6) reduces to the PDF in (3).

2.2. Quantile function

The quantile function (QF) of the GRL distribution defined in (3), say $Q(p)$ where $0 < p < 1$, it can be obtained by solving the equation $F(Q(p)) = p$ in (4) for $Q(p)$ in terms of p , and this implies

$$Q(p) = \left(-\lambda \left[W_{-1} \left[(\lambda - 1)(p - 1)e^{1-\lambda} \right] + \lambda - 1 \right] \right)^{1/\alpha}, \quad (8)$$

where $W_{-1}(x)$ is the negative branch of the Lambert function.

2.3. Moments

Moments play an important role in statistical theory, in this section we provide the r -th moment, the mean and variance for the GRL distribution.

Proposition 1. *For the random variable T follows the GRL distribution, the r -th moment is given by*

$$\mu_r = E[T^r] = \frac{r\lambda^{\frac{r}{\alpha}}}{\alpha(\lambda - 1)} \left(\lambda + \frac{r}{\alpha} - 1 \right) \Gamma\left(\frac{r}{\alpha}\right), \quad \text{for } r \in \mathbb{N}. \quad (9)$$

Proof. Note that, the r -th moment for the random variable in (7) is given by

$$E[T^r; \alpha, \lambda] = \lambda^{\frac{r}{\alpha}} \Gamma\left(\frac{r}{\alpha} + j\right), \quad \text{for } j = 1, 2.$$

Since the GRL model can be expressed as a two-component mixture, as in (6), so we have

$$\begin{aligned} \mu_r = E[T^r] &= \int_0^\infty t^r f(t|\alpha, \lambda) dt = pE[X^r; \alpha, \lambda] + (1 - p)E[X^r; \alpha + 1, \lambda] \\ &= \left(\frac{\lambda - 2}{\lambda - 1} \right) \lambda^{\frac{r}{\alpha}} \Gamma\left(\frac{r}{\alpha} + 1\right) + \frac{1}{(\lambda - 1)} \lambda^{\frac{r}{\alpha}} \Gamma\left(\frac{r}{\alpha} + 2\right) \\ &= \frac{r\lambda^{\frac{r}{\alpha}}}{\alpha(\lambda - 1)} \left(\lambda + \frac{r}{\alpha} - 1 \right) \Gamma\left(\frac{r}{\alpha}\right). \end{aligned}$$

□

Proposition 2. *The random variable T follows the PDF in (3), its mean and variance, respectively, are given by*

$$\mu = \frac{\lambda^{\frac{1}{\alpha}}}{\alpha(\lambda - 1)} \Gamma\left(\frac{1}{\alpha}\right) \left(\lambda + \frac{1}{\alpha} - 1 \right) \quad \text{and} \quad (10)$$

$$\sigma^2 = \frac{\lambda^{\frac{2}{\alpha}}}{\alpha^2(\lambda - 1)^2} \left[2\alpha(\lambda - 1) \Gamma\left(\frac{2}{\alpha}\right) \left(\lambda + \frac{2}{\alpha} - 1 \right) - \Gamma^2\left(\frac{1}{\alpha}\right) \left(\lambda + \frac{1}{\alpha} - 1 \right)^2 \right]. \quad (11)$$

Proof. From (9) and considering $r = 1$, it follows that $\mu_1 = \mu$. The second result can be obtained by using $\sigma^2 = E[T^2] - \mu^2$ and with some algebra the proof is completed. □

Proposition 3. The r -th central moment for the GRL distribution is given by

$$M_r = E[T - \mu]^r = \sum_{i=0}^r \binom{r}{i} (-\mu)^{r-i} E[T^i] \quad (12)$$

$$= \sum_{i=0}^r \binom{r}{i} \left[-\frac{\lambda^{\frac{1}{\alpha}} \Gamma(\alpha^{-1})}{\alpha(\lambda - 1)} \left(\lambda + \frac{1}{\alpha} - 1 \right) \right]^{r-i} \left[\frac{i \lambda^{\frac{i}{\alpha}}}{\alpha(\lambda - 1)} \left(\lambda + \frac{i}{\alpha} - 1 \right) \Gamma\left(\frac{i}{\alpha}\right) \right].$$

Proof. The result follows directly from the Proposition 1. \square

The mean, variance, skewness and kurtosis of the GRL distribution are computed numerically for different values of the parameters λ and α , using the R software. Table 1 displays these numerical values. From Table 1 we can indicate that the skewness of the GRL distribution can range in the interval $(-0.68158, 5.17333)$. The spread for the GRL kurtosis is much larger ranging from 2.69447 to 52.6597. Further, the GRL model can be left skewed or right skewed. Hence, the GRL distribution is a flexible distribution which can be used in modelling skewed data.

Table 1. Mean, variance, skewness and kurtosis of the GRL distribution for different values of the parameters λ and α

ϕ^T	Mean	Variance	Skewness	Kurtosis
$(\lambda = 2.0, \alpha = 0.5)$	24.00	1344.00	4.30	37.41
$(\lambda = 2.0, \alpha = 0.7)$	8.27	72.44	2.39	12.65
$(\lambda = 2.0, \alpha = 2.5)$	1.64	0.23	0.20	2.89
$(\lambda = 2.0, \alpha = 3.5)$	1.41	0.09	-0.04	2.88
$(\lambda = 3.1, \alpha = 0.5)$	37.52	5030.15	5.17	52.66
$(\lambda = 3.1, \alpha = 0.7)$	10.71	186.23	2.85	16.24
$(\lambda = 3.1, \alpha = 2.5)$	1.66	0.42	0.18	2.73
$(\lambda = 3.1, \alpha = 3.1)$	1.49	0.23	-0.03	2.71
$(\lambda = 4.0, \alpha = 1.5)$	2.78	3.19	0.92	3.89
$(\lambda = 4.0, \alpha = 3.5)$	1.46	0.20	-0.07	2.71
$(\lambda = 4.0, \alpha = 5.0)$	1.29	0.08	-0.35	2.95
$(\lambda = 4.0, \alpha = 10)$	1.13	0.02	-0.73	3.74
$(\lambda = 5.5, \alpha = 3.5)$	1.56	0.23	-0.03	2.69
$(\lambda = 5.5, \alpha = 5.0)$	1.35	0.09	-0.30	2.90
$(\lambda = 5.5, \alpha = 5.5)$	1.31	0.07	-0.36	2.99
$(\lambda = 5.5, \alpha = 10)$	1.15	0.02	-0.68	3.64

2.4. Order statistics

Let X_1, X_2, \dots, X_n be a random sample from (3) and $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ denote the corresponding order statistics. It is well known that the PDF and the CDF of the r -th order statistic say $X_{r:n}$; $1 \leq r \leq n$, respectively, are given by

$$\begin{aligned} f_{r:n}(x) &= \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1} [1-F(x)]^{n-r} f(x) \\ &= \frac{n!}{(r-1)!(n-r)!} \sum_{u=0}^{n-r} (-1)^u \binom{n-r}{u} [F(x)]^{r-1+u} f(x) \end{aligned} \quad (13)$$

and

$$F_{r:n}(x) = \sum_{l=k}^n \binom{n}{l} [F(x)]^l [1-F(x)]^{n-l} = \sum_{l=k}^n \sum_{u=0}^{n-r} (-1)^u \binom{n}{l} \binom{n-r}{u} [F(x)]^{l+u}, \quad (14)$$

for $k = 1, 2, \dots, n$. It follows, from (13) and (14), that the PDF and CDF of the r -th order statistic of the GRL reduce to

$$\begin{aligned} f_{r:n}(x) &= \frac{n!}{(\lambda-1)(r-1)!(n-r)!} \left(\lambda + \frac{t}{\lambda} - 2 \right) e^{-\frac{t}{\lambda}} \sum_{u=0}^{n-r} (-1)^u \binom{n-r}{u} \times \\ &\quad \left[1 - \left(\frac{1}{\lambda-1} \right) \left(\lambda - 1 + \frac{t^\alpha}{\lambda} \right) e^{-\frac{t^\alpha}{\lambda}} \right]^{r-1+u} \end{aligned}$$

and

$$F_{r:n}(x) = \sum_{l=k}^n \sum_{u=0}^{n-r} (-1)^u \binom{n}{l} \binom{n-r}{u} \left[1 - \left(\frac{1}{\lambda-1} \right) \left(\lambda - 1 + \frac{t^\alpha}{\lambda} \right) e^{-\frac{t^\alpha}{\lambda}} \right]^{l+u}.$$

3. Inference

In this section, we estimate of the GRL parameters λ and α using eight frequentist approaches. These methods are: the weighted least-squares (WLSE), ordinary least squares (OLSE), maximum likelihood (MLE), maximum product of spacing (MPSE), Cramér-von Mises (CVME), Anderson-Darling (ADE), Right-tail Anderson-Darling (RADE) and percentile based (PCE) estimators.

3.1. Maximum likelihood estimation

In this sub-section we present the maximum likelihood estimator (MLE) of the parameters λ and α of the GRL distribution.

Let T_1, \dots, T_n be a random sample such that T has PDF given in (3). In this case, for $\boldsymbol{\phi} = (\lambda, \alpha)^\top$, the likelihood function from (3) is given by

$$L(\boldsymbol{\phi}; \mathbf{t}) = \frac{\alpha^n}{\lambda^{n+1}(\lambda-1)^n} \prod_{i=1}^n t_i^{\alpha-1} \prod_{i=1}^n \left(\lambda^2 + t_i^\alpha - 2\lambda \right) \exp \left(-\frac{1}{\lambda} \sum_{i=1}^n t_i^\alpha \right). \quad (15)$$

The log-likelihood function $l(\boldsymbol{\phi}; \mathbf{t}) = \log L(\boldsymbol{\phi}; \mathbf{t})$ is given by

$$\begin{aligned} l(\boldsymbol{\phi}; \mathbf{t}) &= n \log(\alpha) - (n+1) \log(\lambda) - n \log(\lambda-1) - \frac{1}{\lambda} \sum_{i=1}^n t_i^\alpha + (\alpha-1) \sum_{i=1}^n \log(t_i) \\ &\quad + \sum_{i=1}^n \log \left(\lambda^2 + t_i^\alpha - 2\lambda \right). \end{aligned} \quad (16)$$

From the expressions $\frac{\partial}{\partial \lambda} l(\boldsymbol{\phi}; \mathbf{t}) = 0$, $\frac{\partial}{\partial \alpha} l(\boldsymbol{\phi}; \mathbf{t}) = 0$, we get the likelihood equations

$$-\frac{n+1}{\lambda} - \frac{n}{\lambda-1} + \frac{1}{\lambda^2} \sum_{i=1}^n t_i^\alpha + \sum_{i=1}^n \frac{2(\lambda-1)}{\lambda^2 + t_i^\alpha - 2\lambda} = 0$$

and

$$\frac{n}{\alpha} - \frac{1}{\lambda} \sum_{i=1}^n t_i^\alpha \log(t_i) + \sum_{i=1}^n \log(t_i) + \sum_{i=1}^n \frac{t_i^\alpha \log(t_i)}{\lambda^2 + t_i^\alpha - 2\lambda} = 0.$$

Under mild conditions (Migon, 2014) the ML estimates are asymptotically normal distributed with a bivariate normal distribution given by

$$(\hat{\lambda}, \hat{\alpha}) \sim N_2[(\lambda, \alpha), H^{-1}(\lambda, \alpha)] \text{ for } n \rightarrow \infty,$$

where the elements of the observed Fisher information matrix $H(\lambda, \alpha)$ are given by

$$\begin{aligned} h_{11}(\lambda, \alpha) &= -\frac{n+1}{\lambda^2} - \frac{n}{(\lambda-1)^2} + \frac{2}{\lambda^3} \sum_{i=1}^n t_i^\alpha - \sum_{i=1}^n \frac{2(t_i^\alpha - \lambda^2 + 2\lambda - 2)}{(\lambda^2 + t_i^\alpha - 2\lambda)^2}, \\ h_{12}(\lambda, \alpha) &= h_{21}(\alpha, \lambda) = -\frac{1}{\lambda^2} \sum_{i=1}^n t_i^\alpha \log(t_i) + \sum_{i=1}^n \frac{2(\lambda-1)t_i^\alpha \log(t_i)}{(\lambda^2 + t_i^\alpha - 2\lambda)^2}, \\ h_{22}(\lambda, \alpha) &= +\frac{n}{\alpha^2} + \frac{1}{\lambda} \sum_{i=1}^n t_i^\alpha \log(t_i)^2 - \sum_{i=1}^n \frac{\lambda(\lambda-2)t_i^\alpha \log(t_i)^2}{(\lambda^2 + t_i^\alpha - 2\lambda)^2}. \end{aligned}$$

This can also be done by using different programs namely R (optim function), SAS (PROC NLMIXED) or by solving the nonlinear likelihood equations obtained by differentiating ℓ .

3.2. Ordinary and weighted least-square estimators

Let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be the order statistics of the random sample of size n from $F(\mathbf{x}; \lambda, \alpha)$ in (4). The ordinary least square estimators (OLSEs) (Swain et al., 1988). $\hat{\lambda}_{LSE}$ and $\hat{\alpha}_{LSE}$ can be obtained by minimizing

$$V(\lambda, \alpha) = \sum_{i=1}^n \left[F(x_{(i)} | \lambda, \alpha) - \frac{i}{n+1} \right]^2,$$

with respect to λ and α . Or equivalently, the OLSEs follow by solving the non-linear equations

$$\sum_{i=1}^n \left[F(x_{(i)} | \lambda, \alpha) - \frac{i}{n+1} \right] \Delta_s(x_{(i)} | \lambda, \alpha) = 0, \quad s = 1, 2,$$

where

$$\Delta_1(x_{(i)} | \lambda, \alpha) = \frac{\partial}{\partial \lambda} F(x_{(i)} | \lambda, \alpha) \text{ and } \Delta_2(x_{(i)} | \lambda, \alpha) = \frac{\partial}{\partial \alpha} F(x_{(i)} | \lambda, \alpha). \quad (17)$$

Note that the solution of Δ_s for $s = 1, 2$ can be obtained numerically.

The weighted least-squares estimators (WLSEs) (Swain et al., 1988), $\hat{\lambda}_{WLSE}$ and $\hat{\alpha}_{WLSE}$, can be obtained by minimizing the following equation

$$W(\lambda, \alpha) = \sum_{i=1}^n \frac{(n+1)^2 (n+2)}{i(n-i+1)} \left[F(x_{(i)} | \lambda, \alpha) - \frac{i}{n+1} \right]^2.$$

Further, the WLSEs can also be derived by solving the non-linear equations

$$\sum_{i=1}^n \frac{(n+1)^2 (n+2)}{i(n-i+1)} \left[F(x_{(i)}|\lambda, \alpha) - \frac{i}{n+1} \right] \Delta_s(x_{(i)}|\lambda, \alpha) = 0, \quad s = 1, 2,$$

99 where $\Delta_1(\cdot|\lambda, \alpha)$ and $\Delta_2(\cdot|\lambda, \alpha)$ are provided in (17).

100 3.3. Method of maximum product of spacing

101 The maximum product of spacings (MPS) method (Cheng and Amin, 1979 and Cheng and Amin,
102 1983 and Ranneby, 1984), as an approximation to the Kullback-Leibler information measure, is a good
103 alternative to the MLE method.

Let $D_i(\lambda, \alpha) = F(x_{(i)}|\lambda, \alpha) - F(x_{(i-1)}|\lambda, \alpha)$, for $i = 1, 2, \dots, n+1$, be the uniform spacing of a random sample from the GRL distribution, where $F(x_{(0)}|\lambda, \alpha) = 0$, $F(x_{(n+1)}|\lambda, \alpha) = 1$ and $\sum_{i=1}^{n+1} D_i(\lambda, \alpha) = 1$. The maximum product of spacing estimators (MPSEs) for $\hat{\lambda}_{MPSE}$ and $\hat{\alpha}_{MPSE}$ can be obtained by maximizing the geometric mean of the spacing

$$G(\lambda, \alpha) = \left[\prod_{i=1}^{n+1} D_i(\lambda, \alpha) \right]^{\frac{1}{n+1}}$$

with respect to λ and α , or, equivalently, by maximizing the logarithm of the geometric mean of sample spacings

$$H(\lambda, \alpha) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log D_i(\lambda, \alpha).$$

The MPSEs of the GRL parameters can be obtained by solving the nonlinear equations defined by

$$\frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(\lambda, \alpha)} \left[\Delta_s(x_{(i)}|\lambda, \alpha) - \Delta_s(x_{(i-1)}|\lambda, \alpha) \right] = 0, \quad s = 1, 2,$$

104 where $\Delta_1(\cdot|\lambda, \alpha)$ and $\Delta_2(\cdot|\lambda, \alpha)$ are defined in (17).

105 3.4. The Cramér-von Mises minimum distance estimators

The Cramér-von Mises estimators (CVMEs) as a type of minimum distance estimators have less bias than the other minimum distance estimators (Macdonald, 1971). The CVMEs are obtained based on the difference between the estimates of the CDF and the empirical distribution function (Luceño, 2006). The CVMEs of the GRL parameters are obtained by minimizing

$$C(\lambda, \alpha) = \frac{1}{12n} + \sum_{i=1}^n \left[F(x_{(i)}|\lambda, \alpha) - \frac{2i-1}{2n} \right]^2,$$

with respect to λ and α . Further, the CVMEs follow by solving the non-linear equations

$$\sum_{i=1}^n \left[F(x_{(i)}|\lambda, \alpha) - \frac{2i-1}{2n} \right] \Delta_s(x_{(i)}|\lambda, \alpha) = 0, \quad s = 1, 2,$$

106 where $\Delta_1(\cdot|\lambda, \alpha)$ and $\Delta_2(\cdot|\lambda, \alpha)$ are provided in (17).

3.5. The Anderson-Darling and right-tail Anderson-Darling estimators

The Anderson-Darling statistic or Anderson-Darling estimator is another type of minimum distance estimators. The Anderson-Darling estimators (ADEs) of the GRL parameters are obtained by minimizing

$$A(\lambda, \alpha) = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) \left[\log F(x_{(i)}|\lambda, \alpha) + \log S(x_{(i)}|\lambda, \alpha) \right],$$

with respect to λ and α . These ADEs can also be obtained by solving the non-linear equations

$$\sum_{i=1}^n (2i-1) \left[\frac{\Delta_s(x_{(i)}|\lambda, \alpha)}{F(x_{(i)}|\lambda, \alpha)} - \frac{\Delta_j(x_{(n+1-i)}|\lambda, \alpha)}{S(x_{(n+1-i)}|\lambda, \alpha)} \right] = 0, \quad s = 1, 2.$$

The right-tail Anderson-Darling estimators (RADEs) of the GRL parameters are obtained by minimizing

$$R(\lambda, \alpha) = \frac{n}{2} - 2 \sum_{i=1}^n F(x_{i:n}|\lambda, \alpha) - \frac{1}{n} \sum_{i=1}^n (2i-1) \log S(x_{n+1-i:n}|\lambda, \alpha),$$

with respect to λ and α . The RADEs can also be obtained by solving the non-linear equations

$$-2 \sum_{i=1}^n \Delta_s(x_{i:n}|\lambda, \alpha) + \frac{1}{n} \sum_{i=1}^n (2i-1) \frac{\Delta_s(x_{n+1-i:n}|\lambda, \alpha)}{S(x_{n+1-i:n}|\lambda, \alpha)} = 0, \quad s = 1, 2.$$

where $\Delta_1(\cdot|\lambda, \alpha)$ and $\Delta_2(\cdot|\lambda, \alpha)$ are defined in Equation (17).

3.6. Method of percentile estimation

This method was originally suggested by Kao (1958, 1959). Let $u_i = i/(n+1)$ be an unbiased estimator of $F(x_{(i)}|\lambda, \alpha)$. Then, the PCE of the parameters of GRL distribution are obtained by minimizing the following function

$$P(\lambda, \alpha) = \sum_{i=1}^n \left(x_{(i)} - \left[-\lambda \left(W_{-1}((\lambda-1)(u_i-1)e^{1-\lambda}) + \lambda - 1 \right) \right]^{1/\alpha} \right)^2,$$

with respect to λ and α , where $W_{-1}(x)$ is the negative branch of the Lambert function.

4. Simulation analysis

A simulation study is conducted to evaluate and compare the behavior of the estimates with respect to their: average of absolute value of biases ($|Bias(\hat{\phi})|$), $|Bias(\hat{\phi})| = \frac{1}{N} \sum_{i=1}^N |\hat{\phi} - \phi|$, the average of mean square errors (MSEs), $MSEs = \frac{1}{N} \sum_{i=1}^N (\hat{\phi} - \phi)^2$, and average of mean relative errors (MREs), $MREs = \frac{1}{N} \sum_{i=1}^N |\hat{\phi} - \phi|/\phi$.

We generate $N = 5000$ random samples: X_1, X_2, \dots, X_N of sizes $n = 30, 50, 80, 100$ and 200 from GRL model by using equation (8) by choosing $\lambda = \{2.0, 4.5\}$ and $\alpha = \{0.5, 2.5, 0.7, 3.5\}$, we used R software (version 3.6.1) (R Core Team, 2019) For each parameters combination and each sample, we estimate of the GRL parameters λ and α using eight frequentist approaches: WLSE (weighted least-squares), OLSE (ordinary least squares), MLE (maximum likelihood), MPSE (maximum product of spacing), CVME (Cramer-von Mises), ADE (Anderson-Darling), RADE (right-tail Anderson-Darling) and PCE (percentile based). Then, the MSEs and MREs of the parameter estimates are computed. Simulated outcomes are listed in Tables 2-9 (see Appendix A). Furthermore, these tables show the rank of each of the estimators among all the estimators in each row, which is the superscript indicators, and the $\sum Ranks$, which is the partial sum of the ranks for each column in a certain sample size. Table 11 shows the partial and overall rank of the estimators.

From tables 2-9, we observe that:

- All Estimation methods show the property of consistency i.e., the MSEs and MREs decrease as sample size increase, for all parameter combinations, except the estimator method WLSE.
- WLSE Estimation method shows the property of consistency for all parameter combinations, except the combinations $\phi = (\lambda = 3.1, \alpha = 0.7)^T$ and $\phi = (\lambda = 3.1, \alpha = 3.5)^T$, for the parameter λ .

Form Table 10, and for the parameter combinations, we can conclude that the MPSE estimator method outperforms all the other estimator methods (overall score of 62). Therefore, depends on our study, we can confirm the superiority of MPSE and ADE estimator methods for the GRL distribution.

5. Data analysis

In this section, we illustrate the importance of the GRL distribution in modelling skewed data using two real data sets from the medicine and geology fields. The first data represent the survival times, in weeks, of 33 patients suffering from acute myelogeneous leukaemia (Feigl and Zelen, 1965). These data have been analyzed by Abouelmagd et al. (2018), Nassar et al. (2018) and Sen et al. (2019). The second data is used to evaluate the risks associated with earthquakes occurring close to the central site of a nuclear power plant. This data set refers to the distances, in miles, to the nuclear power plant of the most recent 8 earthquakes of intensity larger than a given value (Castillo, 2012) and it consists of 60 observations.

We consider some measures of goodness-of-fit namely, minus maximized log-likelihood ($-\hat{\ell}$), Cramér-Von Mises (W^*), Anderson-Darling (A^*) and Kolmogorov Smirnov (KS) statistics with its bootstrapped p -value (PV), to compare the fits of the GRL distribution with other competitive models given in Tables 12 and 13. We draw 999999 bootstrap samples to obtain the KS bootstrapped PV.

The fitted competitive models are namely given in 11, and their densities (for $x > 0$) are given by:

$$\text{MOEx: } f(x) = \alpha \lambda \exp(-\lambda x) [1 - (1 - \alpha) \exp(-\lambda x)]^{-2}.$$

$$\text{BEx: } f(x) = \frac{\lambda}{B(a,b)} \exp(-b\lambda x) [1 - \exp(-\lambda x)]^{a-1}.$$

$$\text{EEx: } f(x) = \alpha \lambda \exp(-\lambda x) [1 - \exp(-\lambda x)]^{\alpha-1}.$$

$$\text{Ga: } f(x) = \frac{b^{-a}}{\Gamma(a)} x^{a-1} \exp(-x/b).$$

$$\text{GLi: } f(x) = \frac{\alpha \lambda^2}{\lambda+1} (1+x) \exp(-\lambda x) \left[1 - \frac{1+\lambda+\lambda x}{\lambda+1} \exp(-\lambda x)\right]^{\alpha-1}.$$

$$\text{TTLi: } f(x) = \left(\frac{a^2}{a+a} (1+ax) \exp(-ax)\right) (1+\lambda - 2\lambda (1 - \frac{\alpha+a+\alpha ax}{a+a} \exp(-ax))).$$

$$\text{PLx: } f(x) = \alpha \beta \lambda (1+\beta x)^{-\alpha-1} \exp[-\lambda (1+\beta x)^{-\alpha}] [1 - \exp(-\lambda)]^{-1}.$$

$$\text{LiGc: } f(x) = [1 - (1 + \frac{ax}{a+1}) \exp(-ax)] / [1 - \alpha (1 + \frac{ax}{a+1}) \exp(-ax)].$$

$$\text{Li: } f(x) = \frac{\lambda^2}{\lambda+1} (1+x) \exp(-\lambda x).$$

The parameters of the above densities are all positive real numbers except $|\lambda| \leq 1$ for the TTLi distribution and $\alpha \in (0, 1)$ for the LiGc distribution.

The numerical values of $-\hat{\ell}$, W^* , A^* , KS, Bootstrapped PV, the MLEs and their corresponding standard errors (SEs) (given in parentheses) of the fitted models are listed in Tables 12 and 13, for both data sets, respectively. The figures in Tables 12 and 13 show that the GRL distribution has the lowest values for all goodness-of-fit statistics among all fitted models.

Tables 14 and 15 display the parameter estimates under various estimation methods and the goodness-of-fit statistics for both data sets, respectively. From Table 14 and based on the $K-S$ bootstrapped PV, we recommend to use the MPSE method to estimate the parameters of the GRL distribution for leukaemia data, while the OLS method is recommended to estimate the GRL parameters for epicenter data, based on the $K-S$ bootstrapped PV in Table 15.

Table 10. Partial and overall ranks of all the methods of estimation for various combination of ϕ

ϕ^T	n	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
$(\lambda = 2.0, \alpha = 0.5)$	30	5.5	7.5	2	3	5.5	1	4	7.5
	50	4	8	2	3	5	1	6	7
	80	4	7	1	2	5	3	6	8
	120	3	6.5	2	1	6.5	4	5	8
	200	4	7	1.5	1.5	5	3	6	8
$(\lambda = 2.0, \alpha = 2.5)$	30	5.5	8	3	4	7	1	5.5	2
	50	5	8	3.5	2	6	1	7	3.5
	80	5	8	2.5	1	6	4	7	2.5
	120	3	8	2	1	6	4.5	7	4.5
	200	4	7.5	2	1	7.5	5	6	3
$(\lambda = 2.0, \alpha = 0.7)$	30	3	8	5	2	6	1	4	7
	50	1	8	4	2	5.5	3	5.5	7
	80	2	8	3	1	6	4	5	7
	120	3	7.5	1	2	7.5	4	5	6
	200	3	8	2	1	6	4	5	7
$(\lambda = 2.0, \alpha = 3.5)$	30	2	8	4	5	6	1	7	3
	50	1	8	5	4	6.5	2.5	6.5	2.5
	80	1	8	3.5	2	6.5	5	6.5	3.5
	120	2.5	8	1	2.5	6	4	7	5
	200	3.5	7.5	2	1	7.5	5	6	3.5
$(\lambda = 3.1, \alpha = 0.5)$	30	6	3.5	7	1	5	2	3.5	8
	50	5	3	7	1	6	2	4	8
	80	5.5	3.5	7	1	5.5	2	3.5	8
	120	5	3	7	1	6	2	4	8
	200	5	3	7	1	5	2	5	8
$(\lambda = 3.1, \alpha = 2.5)$	30	2	5	8	1	6	3	7	4
	50	3	4	8	1	6	5	7	2
	80	4	3	8	1	6	5	7	2
	120	6	3	8	1	4.5	2	7	4.5
	200	5.5	2.5	8	1	5.5	2.5	7	4
$(\lambda = 3.1, \alpha = 0.7)$	30	3.5	2	8	1	5	6	3.5	7
	50	4	3	8	1	5	6	2	7
	80	4	3	7	1	5	6	2	8
	120	4	3	7	1	5	6	2	8
	200	5	2	7	1	6	4	3	8
$(\lambda = 3.1, \alpha = 3.5)$	30	2	6	7	1	8	4	5	3
	50	2	5	8	1	7	4	6	3
	80	2	4.5	8	1	6	4.5	7	3
	120	2	3	8	1	6	4	7	5
	200	2	3	8	1	5	4	7	6
\sum Ranks		142.5	222.5	203	62	236.5	137	216.5	220
Overall Rank		3	7	4	1	8	2	5	6

Table 11. The fitted competitive models

Distribution	Author(s)
Ramos-Louzada (RL) (Special case)	Ramos and Louzada (2019)
Marshall-Olkin exponential (MOEx)	Marshall and Olkin (1997)
Beta exponential (BEx)	Jones (2004)
Exponentiated exponential (EEx)	Gupta and Kundu (2001)
Gamma (Ga)	–
Generalized Lindley (GLi)	Nadarajah et al. (2011)
Transmuted two-parameter Lindley (TTLi)	Kemaloglu and Yilmaz (2017)
Poisson-Lomax (PLx)	Al-Zahrani and Sagor (2014)
Lindley geometric (LiGc)	Zakerzadeh and Mahmoudi (2012)
Lindley (Li)	Lindley (1958)

Table 12. Goodness-of-fit statistics, MLEs and SEs for leukaemia data

Model	$-\hat{\ell}$	W^*	A^*	K-S	Bootstrapped PV	Estimates (SE)	
GRL	153.58031	0.09469	0.65053	0.13637	0.15573	$\hat{\lambda}$	14.6996 (7.67698)
						$\hat{\alpha}$	0.77410 (0.10927)
MOEx	153.59511	0.09725	0.65062	0.14919	0.13112	$\hat{\alpha}$	0.30374 (0.21450)
						$\hat{\lambda}$	0.01344 (0.00677)
BEx	153.65124	0.09676	0.67015	0.13830	0.23219	\hat{a}	0.67358 (0.15375)
						\hat{b}	0.79022 (1.59942)
						$\hat{\lambda}$	0.02421 (0.05263)
EEx	153.65164	0.09664	0.66905	0.13834	0.19718	$\hat{\alpha}$	0.67805 (0.14476)
						$\hat{\lambda}$	0.01880 (0.00476)
Ga	153.67366	0.09662	0.66842	0.13901	0.19166	\hat{a}	0.68776 (0.14403)
						\hat{b}	59.4374 (17.6235)
GLi	154.70797	0.11038	0.76662	0.14254	0.30774	$\hat{\alpha}$	0.36486 (0.07752)
						$\hat{\lambda}$	0.02603 (0.00606)
TTLi	154.85217	0.09695	0.66683	0.20278	0.01875	$\hat{\alpha}$	0.00007 (0.00891)
						\hat{a}	0.02071 (0.01043)
						$\hat{\lambda}$	0.36761 (0.36088)
PLx	155.92974	0.15942	0.95843	0.14182	0.10734	$\hat{\alpha}$	0.74788 (0.19910)
						$\hat{\beta}$	2.38873 (8.77439)
						$\hat{\lambda}$	9.70373 (19.3366)
RL	155.45330	0.09726	0.67297	0.21831	0.00762	$\hat{\lambda}$	39.8689 (7.11473)
LiGc	161.98422	0.13034	0.85743	0.24283	0.00001	$\hat{\alpha}$	0.91431 (0.07170)
						\hat{a}	0.02303 (0.00888)
Li	168.83368	0.11041	0.76655	0.32512	0.00001	$\hat{\lambda}$	0.04781 (0.00589)

Table 13. Goodness-of-fit statistics, MLEs and SEs for epicenter data

Model	$-\hat{\ell}$	W^*	A^*	K-S	Bootstrapped PV	Estimates (SE)	
GRL	323.74634	0.17905	1.04395	0.13180	0.12386	$\hat{\lambda}$	3728110.76443 (8389.87450)
Ga	325.52256	0.24286	1.40568	0.15528	0.00124	\hat{a}	2.97246 (0.02463)
						\hat{a}	6.18451 (1.10003)
BEx	326.12308	0.25892	1.50278	0.16002	0.00445	\hat{b}	23.37182 (4.33053)
						\hat{a}	7.13835 (1.56344)
GLi	326.54668	0.26913	1.56480	0.15616	0.00117	\hat{b}	2.19531 (1.21066)
						$\hat{\lambda}$	0.01133 (0.00433)
EEx	326.98566	0.28122	1.63873	0.16018	0.00110	\hat{a}	3.82547 (0.93381)
						$\hat{\lambda}$	0.02387 (0.00247)
MOEx	327.99000	0.22196	1.29842	0.13454	0.00156	\hat{a}	8.64049 (2.25238)
						$\hat{\lambda}$	0.01913 (0.00216)
TTLi	329.75618	0.25369	1.47104	0.14685	0.01108	\hat{a}	30.37537 (12.01109)
						$\hat{\lambda}$	0.02479 (0.00253)
PLx	331.17805	0.38592	2.27644	0.18476	0.00001	\hat{a}	177.96005 (3753.42055)
						\hat{a}	0.01876 (0.00518)
Li	340.54657	0.24249	1.40349	0.19602	0.00056	$\hat{\lambda}$	-0.99999 (1.37433)
						\hat{a}	3.63997 (0.85273)
LiGc	340.54657	0.24249	1.40349	0.19603	0.00009	$\hat{\beta}$	0.01796 (0.01428)
						$\hat{\lambda}$	53.41620 (59.51819)
RL	358.41349	0.24322	1.40786	0.33146	0.00000	$\hat{\lambda}$	0.01374 (0.00125)
						\hat{a}	0.00001 (0.41139)
						\hat{a}	0.01374 (0.00289)
						$\hat{\lambda}$	143.52395 (18.66117)

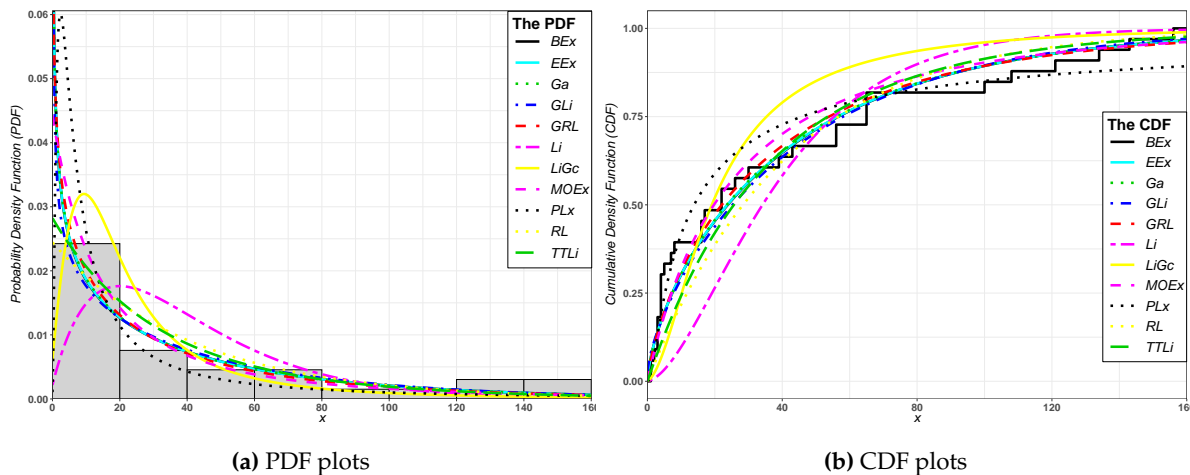
Table 14. The parameter estimates under various estimation methods and the goodness-of-fit statistics for leukaemia data

Method	$\hat{\lambda}$	\hat{a}	$-\hat{\ell}$	W	A	$K - S$	Bootstrapped PV
WLSE	10.92982	0.69340	153.92720	0.09412	0.64348	0.11093	0.36749
OLSE	8.26873	0.62355	154.77563	0.09401	0.63966	0.09917	0.36766
MLE	14.03083	0.76522	153.58430	0.09465	0.65005	0.13133	0.15634
MPSE	11.97607	0.71768	153.75038	0.09424	0.64529	0.11903	0.55489
CVME	9.09894	0.64955	154.37521	0.09403	0.64104	0.09309	0.32234
ADE	10.34346	0.68310	153.99337	0.09411	0.64310	0.10464	0.38431
RADE	10.39537	0.68317	154.00034	0.09411	0.64301	0.10549	0.43200
PCE	24.31768	0.86231	154.07402	0.09492	0.65389	0.18744	0.08984

Table 15. The parameter estimates under various estimation methods and the goodness-of-fit statistics for epicenter data

Method	$\hat{\lambda}$	$\hat{\alpha}$	$-\hat{\ell}$	W	A	$K - S$	Bootstrapped PV
WLSE	1125196.57183	2.73252	324.154	0.18569	1.07845	0.11796	0.04276
OLSE	267870.74460	2.44310	325.819	0.19434	1.12489	0.09761	0.32650
MLE	3728110.76443	2.97253	323.746	0.17904	1.04392	0.13192	0.12335
MPSE	1947372.47807	2.84345	323.858	0.18251	1.06180	0.12630	0.15794
CVME	357125.58188	2.50034	325.407	0.19263	1.11557	0.10028	0.12011
ADE	650623.70779	2.62361	324.591	0.18879	1.09491	0.11313	0.13961
RADE	2842081.10850	2.90677	323.923	0.18149	1.05625	0.10840	0.19406
PCE	2842081.10830	2.91651	323.778	0.18065	1.05215	0.12569	0.08414

The histogram of the fitted GRL distribution and the other distributions are displayed in Figures 3 and 4 for the two data sets, respectively. Figures 3 and 4 show the plots of PDFs and CDFs of the fitted models for leukaemia and epicenter data. The HRF plot of the GRL distribution and the TTT plot of leukaemia data are displayed in Figure 5, whereas the HRF plot of the GRL distribution and the TTT plot of epicenter data are displayed in Figure 6. It is shown that, the HRF is decreasing for leukaemia data, whereas the HRF is increasing for epicenter data. Furthermore, the scaled TTT plot for the leukemia data is convex which indicates a decreasing HRF and it is concave for epicenter data which indicates an increasing HRF. Then, the GRL distribution is a suitable for modeling leukaemia and epicenter data.

**Figure 3.** PDFs and CDFs of the fitted models for leukaemia data

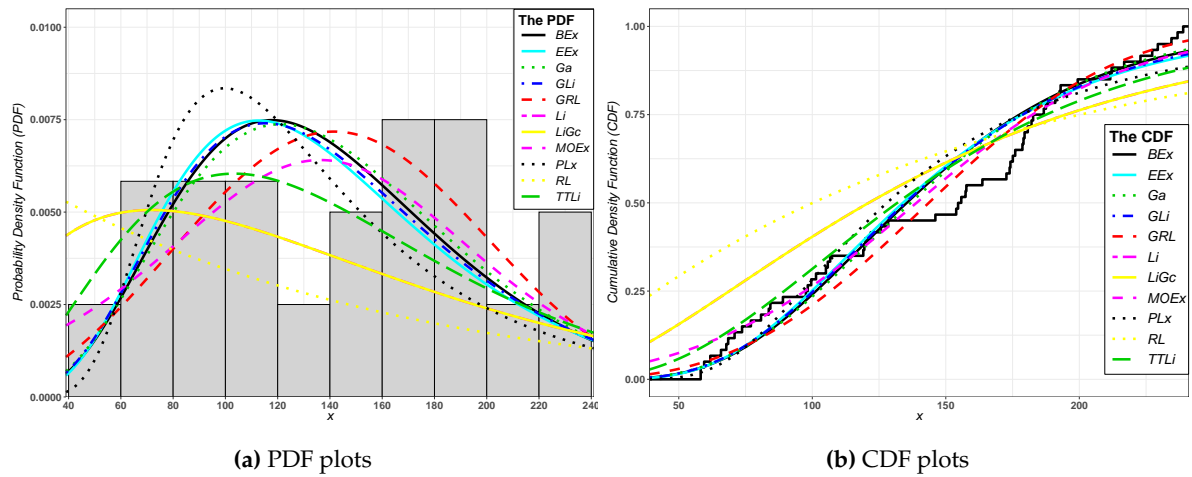


Figure 4. PDFs and CDFs of the fitted models for epicenter data

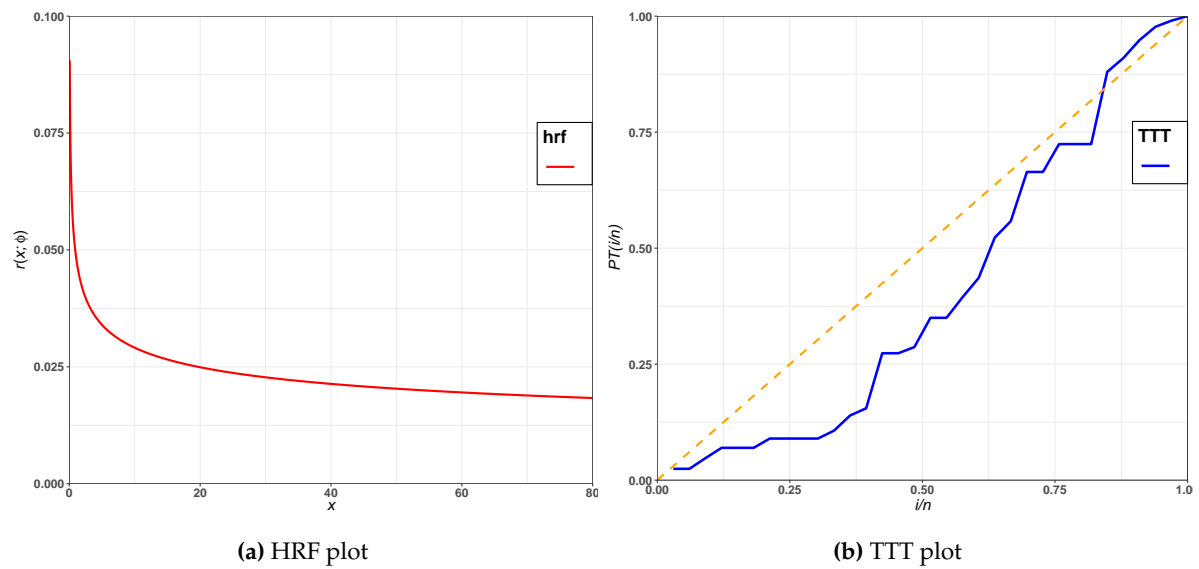


Figure 5. The HRF plot of the GRL distribution and TTT plot for leukaemia data

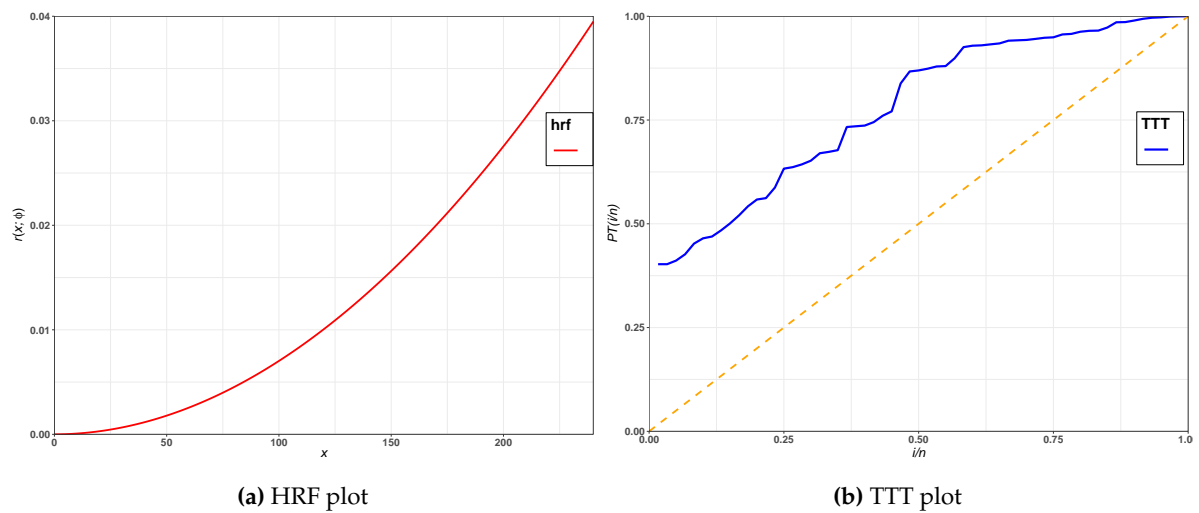


Figure 6. The HRF plot of the GRL distribution and TTT plot for epicenter data

6. Concluding remarks

In this paper, we introduce a new two-parameter distribution called generalized Ramos-Louzada (GRL) distribution. Further, the mathematical properties of the GRL model are studied in detail. The GRL parameters are estimated by eight estimation methods namely: the weighted least-squares, ordinary least squares, maximum likelihood, maximum product of spacing, Cramér-von Mises, Anderson-Darling, Right-tail Anderson-Darling and percentile based estimators. The simulation study illustrates that the maximum product of spacing estimation method outperforms all other estimation methods. Therefore, depends on our study, we can confirm the superiority of the maximum product of spacing method for the GRL distribution. Finally, the practical importance of GRL model was reported in two real applications. The goodness of fit for the proposed data sets showed that our model returned better fitting in comparison with other well-known distributions. Further, the two real data applications show that the maximum product of spacing estimator for leukemia data and least-square estimator for epicenter data return the best estimates for the parameters of the GRL distribution.

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Conflicts of Interest: The authors declare no conflict of interest.

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Sample Availability: Samples of the compounds are available from the authors.

Table 2. Simulation results for $\phi = (\lambda = 2.0, \alpha = 0.5)^\top$

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE	
30	BIAS	$\hat{\lambda}$	0.21560 ^{4}	0.25992 ^{7}	0.19097 ^{2}	0.24975 ^{6}	0.19471 ^{3}	0.18910 ^{1}	0.23949 ^{5}	0.60664 ^{8}	
		$\hat{\alpha}$	0.04161 ^{6}	0.04378 ^{8}	0.03703 ^{2}	0.03455 ^{1}	0.04313 ^{7}	0.03896 ^{4}	0.03856 ^{3}	0.03931 ^{5}	
	MSE	$\hat{\lambda}$	0.18356 ^{3}	0.23965 ^{5}	0.36520 ^{7}	0.27292 ^{6}	0.17386 ^{2}	0.16253 ^{1}	0.22387 ^{4}	2.80485 ^{8}	
		$\hat{\alpha}$	0.00315 ^{5}	0.00347 ^{7}	0.00283 ^{4}	0.00210 ^{1}	0.00337 ^{6}	0.00262 ^{3}	0.00259 ^{2}	0.00516 ^{8}	
	MRE	$\hat{\lambda}$	0.10780 ^{4}	0.12996 ^{7}	0.09549 ^{2}	0.12488 ^{6}	0.09736 ^{3}	0.09455 ^{1}	0.11975 ^{5}	0.30332 ^{8}	
		$\hat{\alpha}$	0.08322 ^{6}	0.08755 ^{8}	0.07406 ^{2}	0.06909 ^{1}	0.08626 ^{7}	0.07791 ^{4}	0.07713 ^{3}	0.07861 ^{5}	
	$\Sigma Ranks$		28 ^{5.5}	42 ^{7.5}	19 ^{2}	21 ^{3}	28 ^{5.5}	14 ^{1}	22 ^{4}	42 ^{7.5}	
	50	BIAS	$\hat{\lambda}$	0.13639 ^{3}	0.18362 ^{6}	0.12166 ^{1}	0.15871 ^{5}	0.14529 ^{4}	0.12558 ^{2}	0.1852 ^{7}	0.45425 ^{8}
			$\hat{\alpha}$	0.02936 ^{5}	0.03369 ^{8}	0.02734 ^{2}	0.02600 ^{1}	0.03270 ^{7}	0.02913 ^{4}	0.02955 ^{6}	0.02903 ^{3}
		MSE	$\hat{\lambda}$	0.09622 ^{2}	0.14769 ^{4}	0.19069 ^{7}	0.14789 ^{5}	0.11450 ^{3}	0.08491 ^{1}	0.15437 ^{6}	0.67882 ^{8}
$\hat{\alpha}$			0.00150 ^{3}	0.00197 ^{8}	0.00156 ^{4.5}	0.00123 ^{1}	0.00186 ^{6}	0.00144 ^{2}	0.00156 ^{4.5}	0.00187 ^{7}	
MRE		$\hat{\lambda}$	0.06819 ^{3}	0.09181 ^{6}	0.06083 ^{1}	0.07936 ^{5}	0.07264 ^{4}	0.06279 ^{2}	0.09260 ^{7}	0.22712 ^{8}	
		$\hat{\alpha}$	0.05873 ^{5}	0.06739 ^{8}	0.05469 ^{2}	0.05200 ^{1}	0.06540 ^{7}	0.05826 ^{4}	0.05910 ^{6}	0.05806 ^{3}	
$\Sigma Ranks$		21 ^{4}	40 ^{8}	17.5 ^{2}	18 ^{3}	31 ^{5}	15 ^{1}	36.5 ^{6}	37 ^{7}		
80		BIAS	$\hat{\lambda}$	0.08834 ^{2}	0.12514 ^{6}	0.05967 ^{1}	0.09455 ^{4}	0.10202 ^{5}	0.08968 ^{3}	0.12941 ^{7}	0.42545 ^{8}
			$\hat{\alpha}$	0.02248 ^{4}	0.02583 ^{8}	0.01925 ^{2}	0.01906 ^{1}	0.02481 ^{7}	0.02247 ^{3}	0.02265 ^{5}	0.02466 ^{6}
		MSE	$\hat{\lambda}$	0.04546 ^{2}	0.07616 ^{6}	0.06131 ^{5}	0.05756 ^{3}	0.05795 ^{4}	0.04391 ^{1}	0.08631 ^{7}	0.66542 ^{8}
	$\hat{\alpha}$		0.00087 ^{3.5}	0.00114 ^{7}	0.00072 ^{2}	0.00064 ^{1}	0.00108 ^{6}	0.00087 ^{3.5}	0.00091 ^{5}	0.00140 ^{8}	
	MRE	$\hat{\lambda}$	0.04417 ^{2}	0.06257 ^{6}	0.02983 ^{1}	0.04728 ^{4}	0.05101 ^{5}	0.04484 ^{3}	0.06471 ^{7}	0.21272 ^{8}	
		$\hat{\alpha}$	0.04496 ^{4}	0.05165 ^{8}	0.03849 ^{2}	0.03812 ^{1}	0.04962 ^{7}	0.04494 ^{3}	0.04529 ^{5}	0.04932 ^{6}	
	$\Sigma Ranks$		17.5 ^{4}	41 ^{7}	13 ^{1}	14 ^{2}	34 ^{5}	16.5 ^{3}	36 ^{6}	44 ^{8}	
	120	BIAS	$\hat{\lambda}$	0.06011 ^{3}	0.08382 ^{6}	0.03817 ^{1}	0.05747 ^{2}	0.07447 ^{5}	0.06168 ^{4}	0.09059 ^{7}	0.36554 ^{8}
			$\hat{\alpha}$	0.01795 ^{5}	0.02018 ^{6}	0.01511 ^{2}	0.01504 ^{1}	0.02037 ^{7}	0.01791 ^{4}	0.01785 ^{3}	0.02127 ^{8}
		MSE	$\hat{\lambda}$	0.01744 ^{2}	0.03412 ^{6}	0.02904 ^{4}	0.01501 ^{1}	0.03090 ^{5}	0.01893 ^{3}	0.04248 ^{7}	0.38789 ^{8}
$\hat{\alpha}$			0.00053 ^{3.5}	0.00068 ^{6}	0.00043 ^{2}	0.00037 ^{1}	0.00070 ^{7}	0.00053 ^{3.5}	0.00055 ^{5}	0.00095 ^{8}	
MRE		$\hat{\lambda}$	0.03005 ^{3}	0.04191 ^{6}	0.01909 ^{1}	0.02874 ^{2}	0.03724 ^{5}	0.03084 ^{4}	0.04530 ^{7}	0.18277 ^{8}	
		$\hat{\alpha}$	0.03590 ^{5}	0.04036 ^{6}	0.03022 ^{2}	0.03008 ^{1}	0.04074 ^{7}	0.03582 ^{4}	0.03570 ^{3}	0.04253 ^{8}	
$\Sigma Ranks$		21.5 ^{3}	36 ^{6.5}	12 ^{2}	8 ^{1}	36 ^{6.5}	22.5 ^{4}	32 ^{5}	48 ^{8}		
200		BIAS	$\hat{\lambda}$	0.04146 ^{4}	0.05483 ^{6}	0.02171 ^{1}	0.03768 ^{2}	0.04827 ^{5}	0.04074 ^{3}	0.06524 ^{7}	0.29846 ^{8}
			$\hat{\alpha}$	0.01373 ^{4}	0.01528 ^{7}	0.01158 ^{2}	0.01126 ^{1}	0.01502 ^{6}	0.01344 ^{3}	0.01374 ^{5}	0.01693 ^{8}
		MSE	$\hat{\lambda}$	0.00755 ^{4}	0.01424 ^{6}	0.00364 ^{1}	0.00503 ^{2}	0.00929 ^{5}	0.00656 ^{3}	0.01827 ^{7}	0.25762 ^{8}
	$\hat{\alpha}$		0.00030 ^{4}	0.00039 ^{7}	0.00022 ^{2}	0.00021 ^{1}	0.00036 ^{6}	0.00029 ^{3}	0.00031 ^{5}	0.00062 ^{8}	
	MRE	$\hat{\lambda}$	0.02073 ^{4}	0.02742 ^{6}	0.01086 ^{1}	0.01884 ^{2}	0.02413 ^{5}	0.02037 ^{3}	0.03262 ^{7}	0.14923 ^{8}	
		$\hat{\alpha}$	0.02745 ^{4}	0.03056 ^{7}	0.02316 ^{2}	0.02252 ^{1}	0.03004 ^{6}	0.02689 ^{3}	0.02748 ^{5}	0.03385 ^{8}	
	$\Sigma Ranks$		24 ^{4}	39 ^{7}	9 ^{1.5}	9 ^{1.5}	33 ^{5}	18 ^{3}	36 ^{6}	48 ^{8}	

Appendix A: Tables

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Table 3. Simulation results for $\phi = (\lambda = 2.0, \alpha = 2.5)^T$

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
30	BIAS	$\hat{\lambda}$	0.21244 ^{4}	0.26189 ^{8}	0.19438 ^{1}	0.25088 ^{7}	0.19742 ^{3}	0.19522 ^{2}	0.24708 ^{6}	0.24484 ^{5}
		$\hat{\alpha}$	0.19894 ^{6}	0.22582 ^{8}	0.18894 ^{3}	0.16969 ^{1}	0.21786 ^{7}	0.19693 ^{5}	0.19001 ^{4}	0.18423 ^{2}
	MSE	$\hat{\lambda}$	0.19082 ^{3}	0.24426 ^{6}	0.37099 ^{8}	0.27879 ^{7}	0.17947 ^{2}	0.17346 ^{1}	0.23687 ^{5}	0.22556 ^{4}
		$\hat{\alpha}$	0.07158 ^{5}	0.09018 ^{8}	0.07436 ^{6}	0.05047 ^{1}	0.08317 ^{7}	0.06861 ^{4}	0.06401 ^{3}	0.05827 ^{2}
	MRE	$\hat{\lambda}$	0.10622 ^{4}	0.13095 ^{8}	0.09719 ^{1}	0.12544 ^{7}	0.09871 ^{3}	0.09761 ^{2}	0.12354 ^{6}	0.12242 ^{5}
		$\hat{\alpha}$	0.07958 ^{6}	0.09033 ^{8}	0.07558 ^{3}	0.06788 ^{1}	0.08715 ^{7}	0.07877 ^{5}	0.07600 ^{4}	0.07369 ^{2}
	$\Sigma Ranks$		28 ^{5.5}	46 ^{8}	22 ^{3}	24 ^{4}	29 ^{7}	19 ^{1}	28 ^{5.5}	20 ^{2}
50	BIAS	$\hat{\lambda}$	0.15218 ^{4}	0.19041 ^{8}	0.12832 ^{1}	0.16578 ^{5}	0.14117 ^{3}	0.13719 ^{2}	0.17754 ^{7}	0.16591 ^{6}
		$\hat{\alpha}$	0.15254 ^{6}	0.16622 ^{8}	0.13759 ^{3}	0.13128 ^{1}	0.16422 ^{7}	0.14949 ^{5}	0.14755 ^{4}	0.13745 ^{2}
	MSE	$\hat{\lambda}$	0.12789 ^{3}	0.15222 ^{6}	0.22581 ^{8}	0.15456 ^{7}	0.10804 ^{2}	0.10193 ^{1}	0.14774 ^{5}	0.13513 ^{4}
		$\hat{\alpha}$	0.04130 ^{5}	0.04890 ^{8}	0.04144 ^{6}	0.03102 ^{1}	0.04880 ^{7}	0.03916 ^{4}	0.03824 ^{3}	0.03382 ^{2}
	MRE	$\hat{\lambda}$	0.07609 ^{4}	0.09520 ^{8}	0.06416 ^{1}	0.08289 ^{5}	0.07058 ^{3}	0.06859 ^{2}	0.08877 ^{7}	0.08296 ^{6}
		$\hat{\alpha}$	0.06102 ^{6}	0.06649 ^{8}	0.05504 ^{3}	0.05251 ^{1}	0.06569 ^{7}	0.05980 ^{5}	0.05902 ^{4}	0.05498 ^{2}
	$\Sigma Ranks$		28 ^{5}	46 ^{8}	22 ^{3.5}	20 ^{2}	29 ^{6}	19 ^{1}	30 ^{7}	22 ^{3.5}
80	BIAS	$\hat{\lambda}$	0.09792 ^{5}	0.11640 ^{7}	0.06223 ^{1}	0.09687 ^{4}	0.09837 ^{6}	0.08741 ^{2}	0.12992 ^{8}	0.09557 ^{3}
		$\hat{\alpha}$	0.11506 ^{6}	0.12703 ^{8}	0.09758 ^{2}	0.09640 ^{1}	0.12599 ^{7}	0.11229 ^{4}	0.11275 ^{5}	0.10036 ^{3}
	MSE	$\hat{\lambda}$	0.06450 ^{5}	0.06699 ^{6}	0.07769 ^{7}	0.06419 ^{4}	0.05569 ^{3}	0.04415 ^{1}	0.08704 ^{8}	0.04739 ^{2}
		$\hat{\alpha}$	0.02399 ^{6}	0.02743 ^{8}	0.01923 ^{3}	0.01682 ^{1}	0.02688 ^{7}	0.02139 ^{4}	0.02313 ^{5}	0.01729 ^{2}
	MRE	$\hat{\lambda}$	0.04896 ^{5}	0.05820 ^{7}	0.03111 ^{1}	0.04843 ^{4}	0.04919 ^{6}	0.04370 ^{2}	0.06496 ^{8}	0.04778 ^{3}
		$\hat{\alpha}$	0.04602 ^{6}	0.05081 ^{8}	0.03903 ^{2}	0.03856 ^{1}	0.05039 ^{7}	0.04492 ^{4}	0.04510 ^{5}	0.04014 ^{3}
	$\Sigma Ranks$		33 ^{5}	44 ^{8}	16 ^{2.5}	15 ^{1}	36 ^{6}	17 ^{4}	39 ^{7}	16 ^{2.5}
120	BIAS	$\hat{\lambda}$	0.06067 ^{2}	0.08408 ^{7}	0.04208 ^{1}	0.06183 ^{3}	0.06947 ^{6}	0.06212 ^{4}	0.09096 ^{8}	0.06412 ^{5}
		$\hat{\alpha}$	0.08937 ^{5}	0.10170 ^{8}	0.07782 ^{2}	0.07594 ^{1}	0.09938 ^{7}	0.08778 ^{4}	0.08961 ^{6}	0.08093 ^{3}
	MSE	$\hat{\lambda}$	0.02061 ^{1}	0.03392 ^{6}	0.03763 ^{7}	0.02264 ^{3}	0.02543 ^{5}	0.02114 ^{2}	0.04016 ^{8}	0.02293 ^{4}
		$\hat{\alpha}$	0.01348 ^{5}	0.01715 ^{8}	0.01167 ^{3}	0.01002 ^{1}	0.01616 ^{7}	0.01292 ^{4}	0.01362 ^{6}	0.01103 ^{2}
	MRE	$\hat{\lambda}$	0.03033 ^{2}	0.04204 ^{7}	0.02104 ^{1}	0.03091 ^{3}	0.03474 ^{6}	0.03106 ^{4}	0.04548 ^{8}	0.03206 ^{5}
		$\hat{\alpha}$	0.03575 ^{5}	0.04068 ^{8}	0.03113 ^{2}	0.03037 ^{1}	0.03975 ^{7}	0.03511 ^{4}	0.03584 ^{6}	0.03237 ^{3}
	$\Sigma Ranks$		20 ^{3}	44 ^{8}	16 ^{2}	12 ^{1}	38 ^{6}	22 ^{4.5}	42 ^{7}	22 ^{4.5}
200	BIAS	$\hat{\lambda}$	0.03782 ^{2}	0.05703 ^{7}	0.02118 ^{1}	0.03827 ^{3}	0.04951 ^{6}	0.03908 ^{4}	0.06286 ^{8}	0.04085 ^{5}
		$\hat{\alpha}$	0.06707 ^{5}	0.07506 ^{7}	0.05763 ^{2}	0.05660 ^{1}	0.07580 ^{8}	0.06811 ^{6}	0.0667 ^{4}	0.05915 ^{3}
	MSE	$\hat{\lambda}$	0.00618 ^{3}	0.01328 ^{7}	0.00639 ^{4}	0.00498 ^{1}	0.01080 ^{6}	0.00681 ^{5}	0.01851 ^{8}	0.00576 ^{2}
		$\hat{\alpha}$	0.00738 ^{4}	0.00911 ^{7}	0.00559 ^{3}	0.00520 ^{1}	0.00931 ^{8}	0.00739 ^{5}	0.00745 ^{6}	0.00554 ^{2}
	MRE	$\hat{\lambda}$	0.01891 ^{2}	0.02852 ^{7}	0.01059 ^{1}	0.01913 ^{3}	0.02476 ^{6}	0.01954 ^{4}	0.03143 ^{8}	0.02043 ^{5}
		$\hat{\alpha}$	0.02683 ^{5}	0.03002 ^{7}	0.02305 ^{2}	0.02264 ^{1}	0.03032 ^{8}	0.02724 ^{6}	0.02668 ^{4}	0.02366 ^{3}
	$\Sigma Ranks$		21 ^{4}	42 ^{7.5}	13 ^{2}	10 ^{1}	42 ^{7.5}	30 ^{5}	38 ^{6}	20 ^{3}

Table 4. Simulation results for $\phi = (\lambda = 2.0, \alpha = 0.7)^T$

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
30	BIAS	$\hat{\lambda}$	0.21445 ^{4}	0.25938 ^{7}	0.19354 ^{2}	0.24781 ^{6}	0.19844 ^{3}	0.18907 ^{1}	0.24632 ^{5}	0.34260 ^{8}
		$\hat{\alpha}$	0.05581 ^{3}	0.08771 ^{8}	0.07506 ^{5}	0.04727 ^{1}	0.08616 ^{7}	0.07770 ^{6}	0.05562 ^{2}	0.07092 ^{4}
	MSE	$\hat{\lambda}$	0.18452 ^{3}	0.23540 ^{4}	0.36081 ^{7}	0.27295 ^{6}	0.18169 ^{2}	0.16400 ^{1}	0.23850 ^{5}	0.37356 ^{8}
		$\hat{\alpha}$	0.00563 ^{3}	0.01365 ^{8}	0.01143 ^{6}	0.00388 ^{1}	0.01301 ^{7}	0.01055 ^{5}	0.00535 ^{2}	0.00939 ^{4}
	MRE	$\hat{\lambda}$	0.10723 ^{4}	0.12969 ^{7}	0.09677 ^{2}	0.12391 ^{6}	0.09922 ^{3}	0.09454 ^{1}	0.12316 ^{5}	0.17130 ^{8}
		$\hat{\alpha}$	0.07972 ^{6}	0.08771 ^{8}	0.07506 ^{3}	0.06753 ^{1}	0.08616 ^{7}	0.07770 ^{4}	0.07946 ^{5}	0.07092 ^{2}
	$\Sigma Ranks$		23 ^{3}	42 ^{8}	25 ^{5}	21 ^{2}	29 ^{6}	18 ^{1}	24 ^{4}	34 ^{7}
	BIAS	$\hat{\lambda}$	0.13755 ^{3}	0.18644 ^{6}	0.12791 ^{1}	0.15661 ^{5}	0.14790 ^{4}	0.13583 ^{2}	0.18659 ^{7}	0.26307 ^{8}
		$\hat{\alpha}$	0.04117 ^{2}	0.06716 ^{8}	0.05516 ^{5}	0.03626 ^{1}	0.06619 ^{7}	0.05860 ^{6}	0.04165 ^{3}	0.05464 ^{4}
	$\Sigma Ranks$		16 ^{1}	40 ^{8}	23 ^{4}	18 ^{2}	32 ^{5.5}	21 ^{3}	32 ^{5.5}	34 ^{7}
80	BIAS	$\hat{\lambda}$	0.08945 ^{3}	0.12477 ^{6}	0.06659 ^{1}	0.09352 ^{4}	0.09986 ^{5}	0.08396 ^{2}	0.12731 ^{7}	0.19421 ^{8}
		$\hat{\alpha}$	0.03115 ^{3}	0.05100 ^{8}	0.03940 ^{4}	0.02693 ^{1}	0.05047 ^{7}	0.04567 ^{6}	0.03092 ^{2}	0.04264 ^{5}
	MSE	$\hat{\lambda}$	0.04583 ^{2}	0.07469 ^{5}	0.07839 ^{7}	0.05851 ^{4}	0.05740 ^{3}	0.03994 ^{1}	0.07827 ^{6}	0.14794 ^{8}
		$\hat{\alpha}$	0.00168 ^{2.5}	0.00447 ^{8}	0.00311 ^{4}	0.00129 ^{1}	0.00428 ^{7}	0.00354 ^{6}	0.00168 ^{2.5}	0.00314 ^{5}
	MRE	$\hat{\lambda}$	0.04473 ^{3}	0.06239 ^{6}	0.03329 ^{1}	0.04676 ^{4}	0.04993 ^{5}	0.04198 ^{2}	0.06366 ^{7}	0.09710 ^{8}
		$\hat{\alpha}$	0.04450 ^{5}	0.05100 ^{8}	0.03940 ^{2}	0.03848 ^{1}	0.05047 ^{7}	0.04567 ^{6}	0.04417 ^{4}	0.04264 ^{3}
	$\Sigma Ranks$		18.5 ^{2}	41 ^{8}	19 ^{3}	15 ^{1}	34 ^{6}	23 ^{4}	28.5 ^{5}	37 ^{7}
	BIAS	$\hat{\lambda}$	0.06209 ^{3}	0.08450 ^{6}	0.03534 ^{1}	0.06315 ^{4}	0.07257 ^{5}	0.06097 ^{2}	0.09661 ^{7}	0.15318 ^{8}
		$\hat{\alpha}$	0.02490 ^{2}	0.04039 ^{7}	0.03054 ^{4}	0.02139 ^{1}	0.04080 ^{8}	0.03519 ^{6}	0.02575 ^{3}	0.03373 ^{5}
	$\Sigma Ranks$		18 ^{3}	39 ^{7.5}	13 ^{1}	16 ^{2}	39 ^{7.5}	21 ^{4}	33 ^{5}	37 ^{6}
120	BIAS	$\hat{\lambda}$	0.02010 ^{3}	0.03307 ^{6}	0.01861 ^{2}	0.02640 ^{4}	0.03033 ^{5}	0.01851 ^{1}	0.04877 ^{7}	0.09039 ^{8}
		$\hat{\alpha}$	0.00103 ^{2}	0.00273 ^{7}	0.00166 ^{4}	0.00080 ^{1}	0.00287 ^{8}	0.00205 ^{6}	0.00118 ^{3}	0.00201 ^{5}
	MSE	$\hat{\lambda}$	0.03105 ^{3}	0.04225 ^{6}	0.01767 ^{1}	0.03157 ^{4}	0.03628 ^{5}	0.03048 ^{2}	0.04830 ^{7}	0.07659 ^{8}
		$\hat{\alpha}$	0.03556 ^{5}	0.04039 ^{7}	0.03054 ^{1}	0.03056 ^{2}	0.04080 ^{8}	0.03519 ^{4}	0.03679 ^{6}	0.03373 ^{3}
	$\Sigma Ranks$		18 ^{3}	39 ^{7.5}	13 ^{1}	16 ^{2}	39 ^{7.5}	21 ^{4}	33 ^{5}	37 ^{6}
	BIAS	$\hat{\lambda}$	0.04049 ^{3}	0.05716 ^{6}	0.02130 ^{1}	0.03670 ^{2}	0.04973 ^{5}	0.04186 ^{4}	0.06346 ^{7}	0.09598 ^{8}
		$\hat{\alpha}$	0.01894 ^{2}	0.03005 ^{8}	0.02270 ^{4}	0.01579 ^{1}	0.02988 ^{7}	0.02714 ^{6}	0.01898 ^{3}	0.02502 ^{5}
	MSE	$\hat{\lambda}$	0.00618 ^{3}	0.01315 ^{6}	0.00335 ^{1}	0.00383 ^{2}	0.01143 ^{5}	0.00719 ^{4}	0.01828 ^{7}	0.03563 ^{8}
		$\hat{\alpha}$	0.00057 ^{2}	0.00146 ^{8}	0.00083 ^{4}	0.00040 ^{1}	0.00145 ^{7}	0.00117 ^{6}	0.00060 ^{3}	0.00107 ^{5}
	$\Sigma Ranks$		17 ^{3}	42 ^{8}	13 ^{2}	9 ^{1}	36 ^{6}	30 ^{4}	32 ^{5}	37 ^{7}
200	BIAS	$\hat{\lambda}$	0.02025 ^{3}	0.02858 ^{6}	0.01065 ^{1}	0.01835 ^{2}	0.02487 ^{5}	0.02093 ^{4}	0.03173 ^{7}	0.04799 ^{8}
		$\hat{\alpha}$	0.02706 ^{4}	0.03005 ^{8}	0.02270 ^{2}	0.02255 ^{1}	0.02988 ^{7}	0.02714 ^{6}	0.02711 ^{5}	0.02502 ^{3}
	MSE	$\hat{\lambda}$	0.00618 ^{3}	0.01315 ^{6}	0.00335 ^{1}	0.00383 ^{2}	0.01143 ^{5}	0.00719 ^{4}	0.01828 ^{7}	0.03563 ^{8}
		$\hat{\alpha}$	0.00057 ^{2}	0.00146 ^{8}	0.00083 ^{4}	0.00040 ^{1}	0.00145 ^{7}	0.00117 ^{6}	0.00060 ^{3}	0.00107 ^{5}
	MRE	$\hat{\lambda}$	0.02025 ^{3}	0.02858 ^{6}	0.01065 ^{1}	0.01835 ^{2}	0.02487 ^{5}	0.02093 ^{4}	0.03173 ^{7}	0.04799 ^{8}
		$\hat{\alpha}$	0.02706 ^{4}	0.03005 ^{8}	0.02270 ^{2}	0.02255 ^{1}	0.02988 ^{7}	0.02714 ^{6}	0.02711 ^{5}	0.02502 ^{3}
	$\Sigma Ranks$		17 ^{3}	42 ^{8}	13 ^{2}	9 ^{1}	36 ^{6}	30 ^{4}	32 ^{5}	37 ^{7}
	BIAS	$\hat{\lambda}$	0.02025 ^{3}	0.02858 ^{6}	0.01065 ^{1}	0.01835 ^{2}	0.02487 ^{5}	0.02093 ^{4}	0.03173 ^{7}	0.04799 ^{8}
		$\hat{\alpha}$	0.02706 ^{4}	0.03005 ^{8}	0.02270 ^{2}	0.02255 ^{1}	0.02988 ^{7}	0.02714 ^{6}	0.02711 ^{5}	0.02502 ^{3}
	$\Sigma Ranks$		17 ^{3}	42 ^{8}	13 ^{2}	9 ^{1}	36 ^{6}	30 ^{4}	32 ^{5}	37 ^{7}

Table 5. Simulation results for $\phi = (\lambda = 2.0, \alpha = 3.5)^T$

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
30	BIAS	$\hat{\lambda}$	0.22261 ^{4}	0.24906 ^{7}	0.20385 ^{3}	0.25031 ^{8}	0.19876 ^{2}	0.19204 ^{1}	0.23014 ^{6}	0.22406 ^{5}
		$\hat{\alpha}$	0.08131 ^{1}	0.30396 ^{7}	0.26007 ^{4}	0.24244 ^{2}	0.30685 ^{8}	0.27002 ^{5}	0.27073 ^{6}	0.25486 ^{3}
	MSE	$\hat{\lambda}$	0.20562 ^{4}	0.22146 ^{6}	0.40083 ^{8}	0.27667 ^{7}	0.18213 ^{2}	0.17251 ^{1}	0.21507 ^{5}	0.19057 ^{3}
		$\hat{\alpha}$	0.01216 ^{1}	0.16235 ^{7}	0.14066 ^{6}	0.10401 ^{2}	0.16608 ^{8}	0.12814 ^{4}	0.12825 ^{5}	0.11077 ^{3}
	MRE	$\hat{\lambda}$	0.11131 ^{4}	0.12453 ^{7}	0.10193 ^{3}	0.12516 ^{8}	0.09938 ^{2}	0.09602 ^{1}	0.11507 ^{6}	0.11203 ^{5}
		$\hat{\alpha}$	0.08131 ^{6}	0.08685 ^{7}	0.07431 ^{3}	0.06927 ^{1}	0.08767 ^{8}	0.07715 ^{4}	0.07735 ^{5}	0.07282 ^{2}
	$\Sigma Ranks$		20 ^{2}	41 ^{8}	27 ^{4}	28 ^{5}	30 ^{6}	16 ^{1}	33 ^{7}	21 ^{3}
50	BIAS	$\hat{\lambda}$	0.14076 ^{3}	0.17315 ^{7}	0.13497 ^{2}	0.16017 ^{6}	0.15183 ^{4}	0.13429 ^{1}	0.17667 ^{8}	0.15499 ^{5}
		$\hat{\alpha}$	0.05954 ^{1}	0.23178 ^{7}	0.19221 ^{4}	0.17824 ^{2}	0.23547 ^{8}	0.21257 ^{6}	0.20455 ^{5}	0.19197 ^{3}
	MSE	$\hat{\lambda}$	0.10117 ^{1}	0.13015 ^{5}	0.25058 ^{8}	0.13783 ^{6}	0.12098 ^{4}	0.10176 ^{2}	0.14206 ^{7}	0.11956 ^{3}
		$\hat{\alpha}$	0.00625 ^{1}	0.09500 ^{7}	0.08184 ^{6}	0.05664 ^{2}	0.09878 ^{8}	0.07837 ^{5}	0.07283 ^{4}	0.06472 ^{3}
	MRE	$\hat{\lambda}$	0.07038 ^{3}	0.08657 ^{7}	0.06749 ^{2}	0.08008 ^{6}	0.07591 ^{4}	0.06715 ^{1}	0.08834 ^{8}	0.07750 ^{5}
		$\hat{\alpha}$	0.05954 ^{5}	0.06622 ^{7}	0.05492 ^{3}	0.05093 ^{1}	0.06728 ^{8}	0.06074 ^{6}	0.05844 ^{4}	0.05485 ^{2}
	$\Sigma Ranks$		14 ^{1}	40 ^{8}	25 ^{5}	23 ^{4}	36 ^{6,5}	21 ^{2,5}	36 ^{6,5}	21 ^{2,5}
80	BIAS	$\hat{\lambda}$	0.08361 ^{2}	0.11900 ^{7}	0.07242 ^{1}	0.09968 ^{5}	0.10484 ^{6}	0.08950 ^{3}	0.12894 ^{8}	0.09510 ^{4}
		$\hat{\alpha}$	0.04465 ^{1}	0.17299 ^{7}	0.14306 ^{3}	0.13813 ^{2}	0.17890 ^{8}	0.15684 ^{5}	0.15773 ^{6}	0.14474 ^{4}
	MSE	$\hat{\lambda}$	0.03898 ^{1}	0.06942 ^{6}	0.09479 ^{8}	0.06796 ^{5}	0.06324 ^{4}	0.04614 ^{2}	0.08308 ^{7}	0.05042 ^{3}
		$\hat{\alpha}$	0.00342 ^{1}	0.05242 ^{7}	0.04352 ^{6}	0.03421 ^{2}	0.05496 ^{8}	0.04257 ^{4}	0.04335 ^{5}	0.03620 ^{3}
	MRE	$\hat{\lambda}$	0.04180 ^{2}	0.05950 ^{7}	0.03621 ^{1}	0.04984 ^{5}	0.05242 ^{6}	0.04475 ^{3}	0.06447 ^{8}	0.04755 ^{4}
		$\hat{\alpha}$	0.04465 ^{4}	0.04943 ^{7}	0.04087 ^{2}	0.03947 ^{1}	0.05111 ^{8}	0.04481 ^{5}	0.04507 ^{6}	0.04135 ^{3}
	$\Sigma Ranks$		11 ^{1}	41 ^{8}	21 ^{3,5}	20 ^{2}	40 ^{6,5}	22 ^{5}	40 ^{6,5}	21 ^{3,5}
120	BIAS	$\hat{\lambda}$	0.06074 ^{4}	0.08677 ^{7}	0.03610 ^{1}	0.06042 ^{3}	0.07107 ^{6}	0.05939 ^{2}	0.09382 ^{8}	0.06159 ^{5}
		$\hat{\alpha}$	0.03569 ^{1}	0.14059 ^{8}	0.10621 ^{2}	0.10639 ^{3}	0.13868 ^{7}	0.12474 ^{5}	0.12599 ^{6}	0.11254 ^{4}
	MSE	$\hat{\lambda}$	0.01968 ^{2}	0.03739 ^{7}	0.02450 ^{5}	0.02157 ^{4}	0.02708 ^{6}	0.01586 ^{1}	0.04333 ^{8}	0.02021 ^{3}
		$\hat{\alpha}$	0.00211 ^{1}	0.03329 ^{8}	0.02044 ^{3}	0.01922 ^{2}	0.03175 ^{7}	0.02530 ^{5}	0.02727 ^{6}	0.02135 ^{4}
	MRE	$\hat{\lambda}$	0.03037 ^{4}	0.04338 ^{7}	0.01805 ^{1}	0.03021 ^{3}	0.03554 ^{6}	0.02970 ^{2}	0.04691 ^{8}	0.03079 ^{5}
		$\hat{\alpha}$	0.03569 ^{5}	0.04017 ^{8}	0.03034 ^{1}	0.03040 ^{2}	0.03962 ^{7}	0.03564 ^{4}	0.03600 ^{6}	0.03216 ^{3}
	$\Sigma Ranks$		17 ^{2,5}	45 ^{8}	13 ^{1}	17 ^{2,5}	39 ^{6}	19 ^{4}	42 ^{7}	24 ^{5}
200	BIAS	$\hat{\lambda}$	0.04023 ^{4}	0.05513 ^{7}	0.02195 ^{1}	0.03743 ^{2}	0.04930 ^{6}	0.04285 ^{5}	0.06439 ^{8}	0.03858 ^{3}
		$\hat{\alpha}$	0.02731 ^{1}	0.10492 ^{7}	0.07878 ^{3}	0.07874 ^{2}	0.10681 ^{8}	0.09556 ^{6}	0.09326 ^{5}	0.08512 ^{4}
	MSE	$\hat{\lambda}$	0.0057 ^{3}	0.01226 ^{7}	0.00576 ^{4}	0.00391 ^{1}	0.01159 ^{6}	0.00702 ^{5}	0.01860 ^{8}	0.00498 ^{2}
		$\hat{\alpha}$	0.00119 ^{1}	0.01769 ^{7}	0.01038 ^{3}	0.00970 ^{2}	0.01836 ^{8}	0.01455 ^{5}	0.01471 ^{6}	0.01170 ^{4}
	MRE	$\hat{\lambda}$	0.02012 ^{4}	0.02756 ^{7}	0.01097 ^{1}	0.01871 ^{2}	0.02465 ^{6}	0.02142 ^{5}	0.03220 ^{8}	0.01929 ^{3}
		$\hat{\alpha}$	0.02731 ^{6}	0.02998 ^{7}	0.02251 ^{2}	0.02250 ^{1}	0.03052 ^{8}	0.02730 ^{5}	0.02664 ^{4}	0.02432 ^{3}
	$\Sigma Ranks$		19 ^{3,5}	42 ^{7,5}	14 ^{2}	10 ^{1}	42 ^{7,5}	31 ^{5}	39 ^{6}	19 ^{3,5}

Table 6. Simulation results for $\phi = (\lambda = 3.1, \alpha = 0.5)^T$

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
30	BIAS	$\hat{\lambda}$	0.71536 ^{6}	0.42777 ^{2}	1.54000 ^{7}	0.34691 ^{1}	0.49932 ^{4}	0.47997 ^{3}	0.64058 ^{5}	3.17604 ^{8}
		$\hat{\alpha}$	0.05196 ^{4}	0.05374 ^{5}	0.06587 ^{7}	0.03983 ^{1}	0.05536 ^{6}	0.04617 ^{3}	0.04508 ^{2}	0.10036 ^{8}
	MSE	$\hat{\lambda}$	5.59864 ^{6}	0.53566 ^{4}	7.76717 ^{7}	0.45223 ^{3}	0.44662 ^{2}	0.42723 ^{1}	2.97747 ^{5}	40.41914 ^{8}
		$\hat{\alpha}$	0.00488 ^{5}	0.00471 ^{4}	0.00846 ^{7}	0.00246 ^{1}	0.00507 ^{6}	0.00325 ^{2}	0.00328 ^{3}	0.02050 ^{8}
	MRE	$\hat{\lambda}$	0.23076 ^{6}	0.13799 ^{2}	0.49677 ^{7}	0.11191 ^{1}	0.16107 ^{4}	0.15483 ^{3}	0.20664 ^{5}	1.02453 ^{8}
		$\hat{\alpha}$	0.10391 ^{4}	0.10749 ^{5}	0.13173 ^{7}	0.07965 ^{1}	0.11072 ^{6}	0.09235 ^{3}	0.09017 ^{2}	0.20072 ^{8}
	$\Sigma Ranks$		31 ^{6}	22 ^{3,5}	42 ^{7}	8 ^{1}	28 ^{5}	15 ^{2}	22 ^{3,5}	48 ^{8}
50	BIAS	$\hat{\lambda}$	0.56014 ^{5}	0.41998 ^{2}	1.13188 ^{7}	0.29988 ^{1}	0.47322 ^{4}	0.46099 ^{3}	0.57534 ^{6}	2.37341 ^{8}
		$\hat{\alpha}$	0.04086 ^{4}	0.04371 ^{5}	0.05129 ^{7}	0.03127 ^{1}	0.04460 ^{6}	0.03870 ^{3}	0.03857 ^{2}	0.08268 ^{8}
	MSE	$\hat{\lambda}$	0.98435 ^{5}	0.43159 ^{3}	3.60984 ^{7}	0.35181 ^{1}	0.45373 ^{4}	0.39471 ^{2}	1.15095 ^{6}	20.21575 ^{8}
		$\hat{\alpha}$	0.00268 ^{4}	0.00291 ^{5}	0.00500 ^{7}	0.00160 ^{1}	0.00300 ^{6}	0.00229 ^{2}	0.00232 ^{3}	0.01332 ^{8}
	MRE	$\hat{\lambda}$	0.18069 ^{5}	0.13548 ^{2}	0.36512 ^{7}	0.09673 ^{1}	0.15265 ^{4}	0.14871 ^{3}	0.18559 ^{6}	0.76562 ^{8}
		$\hat{\alpha}$	0.08172 ^{4}	0.08743 ^{5}	0.10258 ^{7}	0.06254 ^{1}	0.08920 ^{6}	0.07741 ^{3}	0.07713 ^{2}	0.16535 ^{8}
	$\Sigma Ranks$		27 ^{5}	22 ^{3}	42 ^{7}	6 ^{1}	30 ^{6}	16 ^{2}	25 ^{4}	48 ^{8}
80	BIAS	$\hat{\lambda}$	0.49817 ^{6}	0.38717 ^{2}	0.92557 ^{7}	0.22727 ^{1}	0.44267 ^{4}	0.41883 ^{3}	0.48775 ^{5}	1.86629 ^{8}
		$\hat{\alpha}$	0.03429 ^{4}	0.03576 ^{5}	0.04238 ^{7}	0.02437 ^{1}	0.03657 ^{6}	0.03264 ^{3}	0.03232 ^{2}	0.07193 ^{8}
	MSE	$\hat{\lambda}$	0.71931 ^{6}	0.32886 ^{2}	2.09433 ^{7}	0.23205 ^{1}	0.35316 ^{4}	0.33507 ^{3}	0.52189 ^{5}	10.99028 ^{8}
		$\hat{\alpha}$	0.00184 ^{4}	0.00193 ^{5}	0.00327 ^{7}	0.00102 ^{1}	0.00200 ^{6}	0.00164 ^{3}	0.00159 ^{2}	0.00959 ^{8}
	MRE	$\hat{\lambda}$	0.16070 ^{6}	0.12489 ^{2}	0.29857 ^{7}	0.07331 ^{1}	0.14280 ^{4}	0.13511 ^{3}	0.15734 ^{5}	0.60203 ^{8}
		$\hat{\alpha}$	0.06858 ^{4}	0.07153 ^{5}	0.08476 ^{7}	0.04873 ^{1}	0.07313 ^{6}	0.06527 ^{3}	0.06464 ^{2}	0.14387 ^{8}
	$\Sigma Ranks$		30 ^{5,5}	21 ^{3,5}	42 ^{7}	6 ^{1}	30 ^{5,5}	18 ^{2}	21 ^{3,5}	48 ^{8}
120	BIAS	$\hat{\lambda}$	0.44545 ^{5}	0.37929 ^{2}	0.78722 ^{7}	0.14305 ^{1}	0.41506 ^{4}	0.40455 ^{3}	0.44877 ^{6}	1.61673 ^{8}
		$\hat{\alpha}$	0.02999 ^{4}	0.03100 ^{5}	0.03649 ^{7}	0.01830 ^{1}	0.03209 ^{6}	0.02922 ^{3}	0.02843 ^{2}	0.06428 ^{8}
	MSE	$\hat{\lambda}$	0.49028 ^{6}	0.31036 ^{3}	1.40623 ^{7}	0.13170 ^{1}	0.32146 ^{4}	0.30324 ^{2}	0.38210 ^{5}	7.38471 ^{8}
		$\hat{\alpha}$	0.00139 ^{4}	0.00146 ^{5}	0.00238 ^{7}	0.00063 ^{1}	0.00151 ^{6}	0.00128 ^{3}	0.00124 ^{2}	0.00741 ^{8}
	MRE	$\hat{\lambda}$	0.14369 ^{5}	0.12235 ^{2}	0.25394 ^{7}	0.04615 ^{1}	0.13389 ^{4}	0.13050 ^{3}	0.14476 ^{6}	0.52153 ^{8}
		$\hat{\alpha}$	0.05998 ^{4}	0.06199 ^{5}	0.07297 ^{7}	0.03660 ^{1}	0.06418 ^{6}	0.05844 ^{3}	0.05687 ^{2}	0.12856 ^{8}
	$\Sigma Ranks$		28 ^{5}	22 ^{3}	42 ^{7}	6 ^{1}	30 ^{6}	17 ^{2}	23 ^{4}	48 ^{8}
200	BIAS	$\hat{\lambda}$	0.38991 ^{5}	0.34392 ^{2}	0.67812 ^{7}	0.06967 ^{1}	0.36830 ^{3}	0.36835 ^{4}	0.41231 ^{6}	1.34104 ^{8}
		$\hat{\alpha}$	0.02586 ^{4}	0.02621 ^{5}	0.03106 ^{7}	0.01231 ^{1}	0.02664 ^{6}	0.02427 ^{2}	0.02514 ^{3}	0.05693 ^{8}
	MSE	$\hat{\lambda}$	0.30051 ^{5}	0.22845 ^{2}	0.95732 ^{7}	0.05801 ^{1}	0.24094 ^{3}	0.25248 ^{4}	0.31978 ^{6}	4.48475 ^{8}
		$\hat{\alpha}$	0.00102 ^{4}	0.00103 ^{5}	0.00167 ^{7}	0.00031 ^{1}	0.00105 ^{6}	0.00089 ^{2}	0.00097 ^{3}	0.00538 ^{8}
	MRE	$\hat{\lambda}$	0.12578 ^{5}	0.11094 ^{2}	0.21875 ^{7}	0.02247 ^{1}	0.11881 ^{3}	0.11882 ^{4}	0.13300 ^{6}	0.43259 ^{8}
		$\hat{\alpha}$	0.05173 ^{4}	0.05241 ^{5}	0.06211 ^{7}	0.02461 ^{1}	0.05328 ^{6}	0.04854 ^{2}	0.05028 ^{3}	0.11387 ^{8}
	$\Sigma Ranks$		27 ^{5}	21 ^{3}	42 ^{7}	6 ^{1}	27 ^{5}	18 ^{2}	27 ^{5}	48 ^{8}

Table 7. Simulation results for $\phi = (\lambda = 3.1, \alpha = 2.5)^T$

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
30	BIAS	$\hat{\lambda}$	0.51721 ^{2}	0.52263 ^{3}	1.49692 ^{8}	0.38546 ^{1}	0.69528 ^{4}	0.79067 ^{5}	1.09366 ^{7}	0.79091 ^{6}
		$\hat{\alpha}$	0.24577 ^{5}	0.25919 ^{6}	0.30569 ^{8}	0.19785 ^{1}	0.27143 ^{7}	0.22402 ^{3}	0.23879 ^{4}	0.22054 ^{2}
	MSE	$\hat{\lambda}$	0.75318 ^{2}	1.88313 ^{3}	7.25731 ^{7}	0.57621 ^{1}	4.65183 ^{5}	3.60637 ^{4}	10.39735 ^{8}	4.93878 ^{6}
		$\hat{\alpha}$	0.10220 ^{4}	0.11081 ^{5}	0.17793 ^{8}	0.05913 ^{1}	0.12940 ^{7}	0.08607 ^{3}	0.11464 ^{6}	0.08595 ^{2}
	MRE	$\hat{\lambda}$	0.16684 ^{2}	0.16859 ^{3}	0.48288 ^{8}	0.12434 ^{1}	0.22428 ^{4}	0.25506 ^{5}	0.35279 ^{7}	0.25513 ^{6}
		$\hat{\alpha}$	0.09831 ^{5}	0.10368 ^{6}	0.12228 ^{8}	0.07914 ^{1}	0.10857 ^{7}	0.08961 ^{3}	0.09552 ^{4}	0.08821 ^{2}
	$\Sigma Ranks$		20 ^{2}	26 ^{5}	47 ^{8}	6 ^{1}	34 ^{6}	23 ^{3}	36 ^{7}	24 ^{4}
50	BIAS	$\hat{\lambda}$	0.49964 ^{4}	0.43208 ^{2}	1.13544 ^{8}	0.29748 ^{1}	0.49464 ^{3}	0.64148 ^{6}	0.74256 ^{7}	0.62661 ^{5}
		$\hat{\alpha}$	0.20103 ^{5}	0.20767 ^{6}	0.24650 ^{8}	0.15655 ^{1}	0.21426 ^{7}	0.19105 ^{4}	0.18936 ^{3}	0.18306 ^{2}
	MSE	$\hat{\lambda}$	0.61440 ^{2}	0.74280 ^{4}	3.52138 ^{8}	0.33120 ^{1}	0.66637 ^{3}	1.87302 ^{6}	2.52074 ^{7}	1.60356 ^{5}
		$\hat{\alpha}$	0.06302 ^{5}	0.06803 ^{6}	0.10993 ^{8}	0.03846 ^{1}	0.07194 ^{7}	0.05884 ^{3}	0.06276 ^{4}	0.05668 ^{2}
	MRE	$\hat{\lambda}$	0.16118 ^{4}	0.13938 ^{2}	0.36627 ^{8}	0.09596 ^{1}	0.15956 ^{3}	0.20693 ^{6}	0.23953 ^{7}	0.20213 ^{5}
		$\hat{\alpha}$	0.08041 ^{5}	0.08307 ^{6}	0.09860 ^{8}	0.06262 ^{1}	0.08570 ^{7}	0.07642 ^{4}	0.07574 ^{3}	0.07323 ^{2}
	$\Sigma Ranks$		25 ^{3}	26 ^{4}	48 ^{8}	6 ^{1}	30 ^{6}	29 ^{5}	31 ^{7}	21 ^{2}
80	BIAS	$\hat{\lambda}$	0.47763 ^{4}	0.40157 ^{2}	0.96450 ^{8}	0.23448 ^{1}	0.43315 ^{3}	0.54244 ^{6}	0.63030 ^{7}	0.52713 ^{5}
		$\hat{\alpha}$	0.17005 ^{5}	0.17508 ^{6}	0.20869 ^{8}	0.12369 ^{1}	0.17998 ^{7}	0.16035 ^{3}	0.16493 ^{4}	0.15601 ^{2}
	MSE	$\hat{\lambda}$	0.53403 ^{4}	0.31557 ^{2}	2.27547 ^{8}	0.21461 ^{1}	0.35941 ^{3}	0.91225 ^{6}	1.42029 ^{7}	0.90393 ^{5}
		$\hat{\alpha}$	0.04402 ^{4}	0.04643 ^{6}	0.07861 ^{8}	0.02486 ^{1}	0.04881 ^{7}	0.04039 ^{3}	0.04535 ^{5}	0.04032 ^{2}
	MRE	$\hat{\lambda}$	0.15408 ^{4}	0.12954 ^{2}	0.31113 ^{8}	0.07564 ^{1}	0.13973 ^{3}	0.17498 ^{6}	0.20332 ^{7}	0.17004 ^{5}
		$\hat{\alpha}$	0.06802 ^{5}	0.07003 ^{6}	0.08348 ^{8}	0.04948 ^{1}	0.07199 ^{7}	0.06414 ^{3}	0.06597 ^{4}	0.06240 ^{2}
	$\Sigma Ranks$		26 ^{4}	24 ^{3}	48 ^{8}	6 ^{1}	30 ^{6}	27 ^{5}	34 ^{7}	21 ^{2}
120	BIAS	$\hat{\lambda}$	0.45850 ^{4}	0.37313 ^{2}	0.81338 ^{8}	0.17010 ^{1}	0.40144 ^{3}	0.47439 ^{6}	0.54149 ^{7}	0.46981 ^{5}
		$\hat{\alpha}$	0.15390 ^{5}	0.15422 ^{6}	0.18040 ^{8}	0.09789 ^{1}	0.15624 ^{7}	0.14126 ^{2}	0.14128 ^{3}	0.14157 ^{4}
	MSE	$\hat{\lambda}$	0.47784 ^{4}	0.26374 ^{2}	1.44563 ^{8}	0.12242 ^{1}	0.29037 ^{3}	0.59249 ^{5}	0.81415 ^{7}	0.64080 ^{6}
		$\hat{\alpha}$	0.03625 ^{7}	0.03609 ^{6}	0.05588 ^{8}	0.01636 ^{1}	0.03603 ^{5}	0.03066 ^{2}	0.03234 ^{3}	0.03374 ^{4}
	MRE	$\hat{\lambda}$	0.14790 ^{4}	0.12037 ^{2}	0.26238 ^{8}	0.05487 ^{1}	0.12950 ^{3}	0.15303 ^{6}	0.17467 ^{7}	0.15155 ^{5}
		$\hat{\alpha}$	0.06156 ^{5}	0.06169 ^{6}	0.07216 ^{8}	0.03916 ^{1}	0.06250 ^{7}	0.05650 ^{2}	0.05651 ^{3}	0.05663 ^{4}
	$\Sigma Ranks$		29 ^{6}	24 ^{3}	48 ^{8}	6 ^{1}	28 ^{4.5}	23 ^{2}	30 ^{7}	28 ^{4.5}
200	BIAS	$\hat{\lambda}$	0.40531 ^{4}	0.35698 ^{2}	0.67928 ^{8}	0.10091 ^{1}	0.37161 ^{3}	0.41205 ^{6}	0.47736 ^{7}	0.41018 ^{5}
		$\hat{\alpha}$	0.13048 ^{5}	0.13128 ^{6}	0.15507 ^{8}	0.07165 ^{1}	0.13240 ^{7}	0.11847 ^{2}	0.12448 ^{4}	0.11966 ^{3}
	MSE	$\hat{\lambda}$	0.33842 ^{4}	0.23149 ^{2}	0.93473 ^{8}	0.06746 ^{1}	0.24402 ^{3}	0.38206 ^{5}	0.56987 ^{7}	0.43944 ^{6}
		$\hat{\alpha}$	0.02643 ^{7}	0.02567 ^{5}	0.04022 ^{8}	0.00916 ^{1}	0.02586 ^{6}	0.02145 ^{2}	0.02479 ^{4}	0.02431 ^{3}
	MRE	$\hat{\lambda}$	0.13075 ^{4}	0.11515 ^{2}	0.21912 ^{8}	0.03255 ^{1}	0.11987 ^{3}	0.13292 ^{6}	0.15399 ^{7}	0.13232 ^{5}
		$\hat{\alpha}$	0.05219 ^{5}	0.05251 ^{6}	0.06203 ^{8}	0.02866 ^{1}	0.05296 ^{7}	0.04739 ^{2}	0.04979 ^{4}	0.04786 ^{3}
	$\Sigma Ranks$		29 ^{5.5}	23 ^{2.5}	48 ^{8}	6 ^{1}	29 ^{5.5}	23 ^{2.5}	33 ^{7}	25 ^{4}

Table 8. Simulation results for $\phi = (\lambda = 3.1, \alpha = 0.7)^\top$

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
30	BIAS	$\hat{\lambda}$	0.51044 ^{4}	0.41892 ^{2}	1.49321 ^{8}	0.35271 ^{1}	0.49185 ^{3}	0.73176 ^{6}	0.56808 ^{5}	1.25874 ^{7}
		$\hat{\alpha}$	0.07146 ^{3}	0.10634 ^{5}	0.12671 ^{8}	0.05599 ^{1}	0.10847 ^{6}	0.08794 ^{4}	0.06330 ^{2}	0.12407 ^{7}
	MSE	$\hat{\lambda}$	0.64958 ^{4}	0.35564 ^{1}	6.96203 ^{7}	0.44103 ^{3}	0.43581 ^{2}	3.25229 ^{6}	1.32980 ^{5}	7.09123 ^{8}
		$\hat{\alpha}$	0.00833 ^{3}	0.01841 ^{5}	0.03025 ^{8}	0.00473 ^{1}	0.01896 ^{6}	0.01283 ^{4}	0.00615 ^{2}	0.03006 ^{7}
	MRE	$\hat{\lambda}$	0.16466 ^{4}	0.13514 ^{2}	0.48168 ^{8}	0.11378 ^{1}	0.15866 ^{3}	0.23605 ^{6}	0.18325 ^{5}	0.40604 ^{7}
		$\hat{\alpha}$	0.10208 ^{4}	0.10634 ^{5}	0.12671 ^{8}	0.07998 ^{1}	0.10847 ^{6}	0.08794 ^{2}	0.09044 ^{3}	0.12407 ^{7}
	$\Sigma Ranks$		22 ^{3,5}	20 ^{2}	47 ^{8}	8 ^{1}	26 ^{5}	28 ^{6}	22 ^{3,5}	43 ^{7}
50	BIAS	$\hat{\lambda}$	0.52873 ^{5}	0.41577 ^{2}	1.18465 ^{8}	0.29751 ^{1}	0.46496 ^{3}	0.60832 ^{6}	0.50598 ^{4}	1.08938 ^{7}
		$\hat{\alpha}$	0.05677 ^{3}	0.08631 ^{5}	0.10430 ^{8}	0.04526 ^{1}	0.08710 ^{6}	0.07469 ^{4}	0.05241 ^{2}	0.10276 ^{7}
	MSE	$\hat{\lambda}$	0.77634 ^{5}	0.33606 ^{2}	3.91632 ^{7}	0.32608 ^{1}	0.38912 ^{3}	1.32804 ^{6}	0.52610 ^{4}	4.29311 ^{8}
		$\hat{\alpha}$	0.00527 ^{3}	0.01163 ^{6}	0.02036 ^{8}	0.00316 ^{1}	0.01161 ^{5}	0.00870 ^{4}	0.00419 ^{2}	0.02002 ^{7}
	MRE	$\hat{\lambda}$	0.17056 ^{5}	0.13412 ^{2}	0.38214 ^{8}	0.09597 ^{1}	0.14999 ^{3}	0.19623 ^{6}	0.16322 ^{4}	0.35141 ^{7}
		$\hat{\alpha}$	0.08110 ^{4}	0.08631 ^{5}	0.10430 ^{8}	0.06466 ^{1}	0.08710 ^{6}	0.07469 ^{2}	0.07487 ^{3}	0.10276 ^{7}
	$\Sigma Ranks$		25 ^{4}	22 ^{3}	47 ^{8}	6 ^{1}	26 ^{5}	28 ^{6}	19 ^{2}	43 ^{7}
80	BIAS	$\hat{\lambda}$	0.52408 ^{5}	0.39229 ^{2}	0.93465 ^{7}	0.22702 ^{1}	0.43290 ^{3}	0.53716 ^{6}	0.47085 ^{4}	0.94348 ^{8}
		$\hat{\alpha}$	0.04917 ^{3}	0.07192 ^{5}	0.08450 ^{7}	0.03557 ^{1}	0.07491 ^{6}	0.06463 ^{4}	0.04587 ^{2}	0.08939 ^{8}
	MSE	$\hat{\lambda}$	0.76622 ^{5}	0.29465 ^{2}	2.06808 ^{7}	0.21289 ^{1}	0.33855 ^{3}	0.91016 ^{6}	0.42668 ^{4}	2.71063 ^{8}
		$\hat{\alpha}$	0.00384 ^{3}	0.00792 ^{5}	0.01263 ^{7}	0.00206 ^{1}	0.00832 ^{6}	0.00642 ^{4}	0.00321 ^{2}	0.01472 ^{8}
	MRE	$\hat{\lambda}$	0.16906 ^{5}	0.12655 ^{2}	0.30150 ^{7}	0.07323 ^{1}	0.13964 ^{3}	0.17328 ^{6}	0.15189 ^{4}	0.30435 ^{8}
		$\hat{\alpha}$	0.07025 ^{4}	0.07192 ^{5}	0.08450 ^{7}	0.05082 ^{1}	0.07491 ^{6}	0.06463 ^{2}	0.06552 ^{3}	0.08939 ^{8}
	$\Sigma Ranks$		25 ^{4}	21 ^{3}	42 ^{7}	6 ^{1}	27 ^{5}	28 ^{6}	19 ^{2}	48 ^{8}
120	BIAS	$\hat{\lambda}$	0.47169 ^{5}	0.36477 ^{2}	0.80694 ^{7}	0.17537 ^{1}	0.40635 ^{3}	0.48685 ^{6}	0.45033 ^{4}	0.86447 ^{8}
		$\hat{\alpha}$	0.04261 ^{3}	0.06155 ^{5}	0.07417 ^{7}	0.02865 ^{1}	0.06420 ^{6}	0.05727 ^{4}	0.04126 ^{2}	0.08132 ^{8}
	MSE	$\hat{\lambda}$	0.59285 ^{5}	0.25405 ^{2}	1.43510 ^{7}	0.15462 ^{1}	0.29381 ^{3}	0.62129 ^{6}	0.37322 ^{4}	2.16579 ^{8}
		$\hat{\alpha}$	0.00284 ^{3}	0.00577 ^{5}	0.00959 ^{7}	0.00141 ^{1}	0.00605 ^{6}	0.00500 ^{4}	0.00261 ^{2}	0.01237 ^{8}
	MRE	$\hat{\lambda}$	0.15216 ^{5}	0.11767 ^{2}	0.26030 ^{7}	0.05657 ^{1}	0.13108 ^{3}	0.15705 ^{6}	0.14527 ^{4}	0.27886 ^{8}
		$\hat{\alpha}$	0.06087 ^{4}	0.06155 ^{5}	0.07417 ^{7}	0.04093 ^{1}	0.06420 ^{6}	0.05727 ^{2}	0.05894 ^{3}	0.08132 ^{8}
	$\Sigma Ranks$		25 ^{4}	21 ^{3}	42 ^{7}	6 ^{1}	27 ^{5}	28 ^{6}	19 ^{2}	48 ^{8}
200	BIAS	$\hat{\lambda}$	0.40517 ^{5}	0.34440 ^{2}	0.68797 ^{7}	0.09976 ^{1}	0.37346 ^{3}	0.39717 ^{4}	0.40852 ^{6}	0.73687 ^{8}
		$\hat{\alpha}$	0.03563 ^{3}	0.05148 ^{5}	0.06267 ^{7}	0.02047 ^{1}	0.05340 ^{6}	0.04820 ^{4}	0.03498 ^{2}	0.07080 ^{8}
	MSE	$\hat{\lambda}$	0.36345 ^{5}	0.21845 ^{2}	0.95254 ^{7}	0.07933 ^{1}	0.24571 ^{3}	0.36392 ^{6}	0.30293 ^{4}	1.38079 ^{8}
		$\hat{\alpha}$	0.00196 ^{3}	0.00398 ^{5}	0.00660 ^{7}	0.00077 ^{1}	0.00424 ^{6}	0.00355 ^{4}	0.00189 ^{2}	0.00895 ^{8}
	MRE	$\hat{\lambda}$	0.13070 ^{5}	0.11110 ^{2}	0.22192 ^{7}	0.03218 ^{1}	0.12047 ^{3}	0.12812 ^{4}	0.13178 ^{6}	0.23770 ^{8}
		$\hat{\alpha}$	0.05090 ^{4}	0.05148 ^{5}	0.06267 ^{7}	0.02925 ^{1}	0.05340 ^{6}	0.04820 ^{2}	0.04997 ^{3}	0.07080 ^{8}
	$\Sigma Ranks$		25 ^{5}	21 ^{2}	42 ^{7}	6 ^{1}	27 ^{6}	24 ^{4}	23 ^{3}	48 ^{8}

Table 9. Simulation results for $\phi = (\lambda = 3.1, \alpha = 3.5)^T$

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
30	BIAS	$\hat{\lambda}$	0.52557 ^{2}	1.09506 ^{4}	1.37011 ^{8}	0.38222 ^{1}	1.32080 ^{7}	1.12191 ^{5}	1.26170 ^{6}	0.90329 ^{3}
		$\hat{\alpha}$	0.09999 ^{1}	0.39092 ^{6}	0.40091 ^{7}	0.26417 ^{2}	0.41516 ^{8}	0.34797 ^{5}	0.34765 ^{4}	0.31293 ^{3}
	MSE	$\hat{\lambda}$	0.73033 ^{2}	9.10142 ^{6}	5.90223 ^{4}	0.64502 ^{1}	13.03684 ^{8}	6.92476 ^{5}	10.48503 ^{7}	4.13403 ^{3}
		$\hat{\alpha}$	0.01745 ^{1}	0.32306 ^{7}	0.29950 ^{6}	0.10485 ^{2}	0.37929 ^{8}	0.23413 ^{4}	0.25531 ^{5}	0.18394 ^{3}
	MRE	$\hat{\lambda}$	0.16954 ^{2}	0.35324 ^{4}	0.44197 ^{8}	0.12330 ^{1}	0.42606 ^{7}	0.36191 ^{5}	0.40700 ^{6}	0.29139 ^{3}
		$\hat{\alpha}$	0.09999 ^{5}	0.11169 ^{6}	0.11455 ^{7}	0.07548 ^{1}	0.11862 ^{8}	0.09942 ^{4}	0.09933 ^{3}	0.08941 ^{2}
	$\Sigma Ranks$		13 ^{2}	33 ^{6}	40 ^{7}	8 ^{1}	46 ^{8}	28 ^{4}	31 ^{5}	17 ^{3}
50	BIAS	$\hat{\lambda}$	0.53431 ^{2}	0.77955 ^{4}	1.13495 ^{8}	0.30627 ^{1}	0.84188 ^{6}	0.82885 ^{5}	0.91870 ^{7}	0.76134 ^{3}
		$\hat{\alpha}$	0.08205 ^{1}	0.30568 ^{6}	0.33386 ^{8}	0.21076 ^{2}	0.32089 ^{7}	0.27459 ^{4}	0.28322 ^{5}	0.26937 ^{3}
	MSE	$\hat{\lambda}$	0.75632 ^{2}	3.40283 ^{5}	3.72827 ^{7}	0.35359 ^{1}	3.65598 ^{6}	2.69228 ^{4}	3.74656 ^{8}	2.45117 ^{3}
		$\hat{\alpha}$	0.01083 ^{1}	0.17521 ^{6}	0.20579 ^{8}	0.06882 ^{2}	0.19261 ^{7}	0.13341 ^{3}	0.15065 ^{5}	0.13393 ^{4}
	MRE	$\hat{\lambda}$	0.17236 ^{2}	0.25147 ^{4}	0.36611 ^{8}	0.09880 ^{1}	0.27157 ^{6}	0.26737 ^{5}	0.29635 ^{7}	0.24559 ^{3}
		$\hat{\alpha}$	0.08205 ^{5}	0.08734 ^{6}	0.09539 ^{8}	0.06022 ^{1}	0.09168 ^{7}	0.07845 ^{3}	0.08092 ^{4}	0.07696 ^{2}
	$\Sigma Ranks$		13 ^{2}	31 ^{5}	47 ^{8}	8 ^{1}	39 ^{7}	24 ^{4}	36 ^{6}	18 ^{3}
80	BIAS	$\hat{\lambda}$	0.50939 ^{2}	0.54629 ^{3}	0.91031 ^{8}	0.24136 ^{1}	0.62435 ^{4}	0.68136 ^{6}	0.72675 ^{7}	0.66055 ^{5}
		$\hat{\alpha}$	0.06783 ^{1}	0.24670 ^{6}	0.27980 ^{8}	0.16993 ^{2}	0.25689 ^{7}	0.23484 ^{4}	0.23668 ^{5}	0.23008 ^{3}
	MSE	$\hat{\lambda}$	0.68459 ^{2}	1.28342 ^{3}	1.92419 ^{8}	0.22485 ^{1}	1.52018 ^{4}	1.62341 ^{5}	1.91126 ^{7}	1.62519 ^{6}
		$\hat{\alpha}$	0.00726 ^{1}	0.10084 ^{6}	0.13649 ^{8}	0.04669 ^{2}	0.11013 ^{7}	0.09518 ^{3}	0.10055 ^{5}	0.09793 ^{4}
	MRE	$\hat{\lambda}$	0.16432 ^{2}	0.17622 ^{3}	0.29365 ^{8}	0.07786 ^{1}	0.20140 ^{4}	0.21980 ^{6}	0.23444 ^{7}	0.21308 ^{5}
		$\hat{\alpha}$	0.06783 ^{5}	0.07049 ^{6}	0.07994 ^{8}	0.04855 ^{1}	0.07340 ^{7}	0.06710 ^{3}	0.06762 ^{4}	0.06574 ^{2}
	$\Sigma Ranks$		13 ^{2}	27 ^{4,5}	48 ^{8}	8 ^{1}	33 ^{6}	27 ^{4,5}	35 ^{7}	25 ^{3}
120	BIAS	$\hat{\lambda}$	0.48993 ^{3}	0.45147 ^{2}	0.81321 ^{8}	0.17944 ^{1}	0.51255 ^{4}	0.53393 ^{5}	0.60605 ^{7}	0.55585 ^{6}
		$\hat{\alpha}$	0.06042 ^{1}	0.20977 ^{6}	0.24965 ^{8}	0.13483 ^{2}	0.21551 ^{7}	0.19592 ^{3}	0.20459 ^{5}	0.19930 ^{4}
	MSE	$\hat{\lambda}$	0.60942 ^{2}	0.66596 ^{3}	1.46692 ^{8}	0.12943 ^{1}	0.77617 ^{4}	0.87632 ^{5}	1.10606 ^{7}	1.01317 ^{6}
		$\hat{\alpha}$	0.00570 ^{1}	0.06907 ^{4}	0.10753 ^{8}	0.03039 ^{2}	0.07314 ^{7}	0.06299 ^{3}	0.07099 ^{6}	0.07057 ^{5}
	MRE	$\hat{\lambda}$	0.15804 ^{3}	0.14564 ^{2}	0.26233 ^{8}	0.05788 ^{1}	0.16534 ^{4}	0.17224 ^{5}	0.19550 ^{7}	0.17931 ^{6}
		$\hat{\alpha}$	0.06042 ^{6}	0.05993 ^{5}	0.07133 ^{8}	0.03852 ^{1}	0.06157 ^{7}	0.05598 ^{2}	0.05846 ^{4}	0.05694 ^{3}
	$\Sigma Ranks$		16 ^{2}	22 ^{3}	48 ^{8}	8 ^{1}	33 ^{6}	23 ^{4}	36 ^{7}	30 ^{5}
200	BIAS	$\hat{\lambda}$	0.42021 ^{4}	0.37280 ^{2}	0.69009 ^{8}	0.10805 ^{1}	0.39538 ^{3}	0.45283 ^{5}	0.51238 ^{7}	0.48913 ^{6}
		$\hat{\alpha}$	0.05008 ^{1}	0.17620 ^{5}	0.21339 ^{8}	0.09758 ^{2}	0.18090 ^{7}	0.17105 ^{3}	0.17725 ^{6}	0.17519 ^{4}
	MSE	$\hat{\lambda}$	0.42916 ^{4}	0.30798 ^{2}	0.96479 ^{8}	0.06409 ^{1}	0.35395 ^{3}	0.53591 ^{5}	0.65186 ^{6}	0.69627 ^{7}
		$\hat{\alpha}$	0.00394 ^{1}	0.04657 ^{4}	0.07711 ^{8}	0.01697 ^{2}	0.04883 ^{5}	0.04619 ^{3}	0.05097 ^{6}	0.05417 ^{7}
	MRE	$\hat{\lambda}$	0.13555 ^{4}	0.12026 ^{2}	0.22261 ^{8}	0.03486 ^{1}	0.12754 ^{3}	0.14608 ^{5}	0.16528 ^{7}	0.15778 ^{6}
		$\hat{\alpha}$	0.05008 ^{4}	0.05034 ^{5}	0.06097 ^{8}	0.02788 ^{1}	0.05169 ^{7}	0.04887 ^{2}	0.05064 ^{6}	0.05005 ^{3}
	$\Sigma Ranks$		18 ^{2}	20 ^{3}	48 ^{8}	8 ^{1}	28 ^{5}	23 ^{4}	38 ^{7}	33 ^{6}