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The Ritt-Raudenbush Theorem and Tropical Differential Geometry

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The Ritt-Raudenbush Basis Theorem is proved by Ritt for differential polynomial rings of the form $\mathcal{R} = \mathcal{F}\{u_1, \dots, u_n\}$, where \mathcal{F} is a differential field of characteristic zero. Ritt's proof is provided in [6, chap. I] in the ordinary differential case. In [6, chap. IX] Ritt claims that the theorem holds also in the partial ($m > 1$ derivation operators) case. Separately, a stronger version of this theorem was proved by Kolchin in [4] and generalized in [5, chap. III, 4] under the name of *Basis Theorem*.

In tropical differential geometry, the differential polynomial ring under study often is the differential polynomial ring $K[[x]]\{u_1, \dots, u_n\}$. The Ritt-Raudenbush Basis Theorem is also useful in this context. It is used, for instance, in [1, Proposition 2.1] in order to prove the Fundamental Theorem of tropical differential geometry. The fact that the ring of coefficients is not a field forces the authors to complicate statements.

It turns out that this — let us call it “tropical” — version is actually a consequence of Kolchin's results but is also a consequence of the classical Ritt-Raudenbush version, up to a slight generalization of a minor Lemma.

In this paper, we prove that the tropical version is implied by Kolchin's result. For this, we rely on Kolchin's 1942 paper [4]. The version of the Basis Theorem which is stated in Kolchin's book actually is difficult to cite accurately because it relies on many concepts whose definitions are not equivalent to the ones commonly used (quoted from [5, chap. 0, 6, page 9]). We also show how to modify the classical Ritt-Raudenbush version in order to obtain the tropical version. Last, we will discuss the use made by this theorem in [1].

1 Statement of the Problem

The differential polynomial ring under consideration is $\mathcal{R} = \mathcal{S}\{u_1, \dots, u_n\}$, endowed with $m \geq 1$ derivation operators, where \mathcal{S} is any differential ring of characteristic zero such that, for any nonzero $a \in \mathcal{S}$, there exists some derivative θa of a is invertible in \mathcal{S} .

Examples of such rings \mathcal{S} are differential fields of characteristic zero and rings of formal power series $\mathcal{F}[[x_1, \dots, x_m]]$ where \mathcal{F} is a differential field of constants, with characteristic zero (see Lemma 3 in Section 3).

In the following definition, $\{\Phi\}$ denotes the perfect differential ideal generated by Φ (see Section 2.2.1).

Definition 1 *Let Σ be a subset of \mathcal{R} . A subset Φ of Σ is said to be a basis of Σ if Φ is finite and $\Sigma \subset \{\Phi\}$.*

Theorem 1 (*Ritt-Raudenbush Basis Theorem*) *Every subset of \mathcal{R} has a basis.*

The next Lemma is all we need.

Lemma 1 *Let Σ be any subset of \mathcal{R} . If Σ contains a nonzero element of \mathcal{S} then Σ has a basis.*

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Proof Let a be a nonzero element of $\Sigma \cap \mathcal{S}$ and θa be a derivative of a which is invertible in \mathcal{S} . Then $\theta a \in \{a\}$, the perfect differential ideal $\{a\}$ is equal to \mathcal{R} and the singleton a is a basis of Σ . \square

2 Two Proofs of the Theorem

2.1 Kolchin Proof

In [4, sect. 6], Kolchin defines *regular differential rings* (a concept that does not seem to be kept in [5]) and writes that every differential ring which contains the rational number system (i.e. the field of the rational numbers) is a regular differential ring.

Thus our ring \mathcal{S} is a regular differential ring.

Let us quote [4, Sect. 1]: *if every subset of [some differential ring] \mathcal{R} has a basis we say that the Basis Theorem holds in \mathcal{R}* . Here is, almost as is, the Theorem stated in [4, Sect. 7]:

Theorem 2 (*Kolchin's 1942 Theorem*)

Let \mathcal{S} be a regular commutative differential ring with unit element. Let \mathcal{R} be a commutative differential ring obtained from \mathcal{S} by the differential ring adjunction of a finite number of elements: $\mathcal{R} = \mathcal{S}\{\eta_1, \dots, \eta_m\}$ (the η_i may be hypertranscendental over \mathcal{S} (for example, they may be unknowns) or may satisfy some algebraic relation with coefficients in \mathcal{S}). If the Basis Theorem holds in \mathcal{S} then the Basis Theorem holds in \mathcal{R} .

By Lemma 1, the Basis Theorem holds in \mathcal{S} . Thus, by Theorem 2, it holds in \mathcal{R} .

2.2 Ritt-Raudenbush Proof

2.2.1 Basic Definitions

Reference books are [6] and [5]. The following basic notions are introduced in [5, chap. I, 1].

An operator δ on a ring is called a *derivation operator* if $\delta(a + b) = \delta a + \delta b$ and $\delta(ab) = (\delta a)b + a\delta b$ for all elements a, b of the ring. A *differential ring* \mathcal{R} is defined as a ring with finitely many derivation operators which commute pairwise i.e. such that $\delta_1\delta_2 a = \delta_2\delta_1 a$ for all derivation operators δ_1, δ_2 and all $a \in \mathcal{R}$. In the case $\mathcal{S} = \mathcal{F}[[x_1, \dots, x_m]]$, the m derivation operators $\delta_1, \dots, \delta_m$ are supposed to act on the symbols x_1, \dots, x_m as $\delta_i = \partial/\partial x_i$ for $1 \leq i \leq m$. A *differential field* is a differential ring which is a field. If the number m of derivation operators is equal to 1 then the differential ring is said to be *ordinary*. If it is greater than 1, the differential ring is said to be *partial*. The m derivation operators generate a commutative semigroup Θ , denoted multiplicatively. Each *derivative operator* $\theta \in \Theta$ has the form

$$\theta = \delta_1^{e_1} \dots \delta_m^{e_m}$$

where $e_1, \dots, e_m \in \mathbb{N}$ (the set of the nonnegative integers). Let $U = \{u_1, \dots, u_n\}$ denote a set of n *differential indeterminates*. The semigroup Θ acts on U , giving the infinite set of *derivatives* ΘU . The differential polynomial ring $\mathcal{R} = \mathcal{S}\{u_1, \dots, u_n\}$ is the ring $\mathcal{S}[\Theta U]$ of the polynomials whose indeterminates are the derivatives of the differential indeterminates, with coefficients in \mathcal{S} .

A nonempty subset \mathfrak{A} of \mathcal{R} is said to be a *differential ideal* of \mathcal{R} if it is an ideal of \mathcal{R} and it is stable under the action of the derivations. A differential ideal \mathfrak{A} is said to be *perfect* if it is equal to its radical i.e. if $(\exists d \in \mathbb{N}, p^d \in \mathfrak{A}) \Rightarrow p \in \mathfrak{A}$. A differential ideal \mathfrak{A} is said to be *prime* if it is prime in the usual sense. Let Σ be any subset of \mathcal{R} . One denotes $[\Sigma]$ the *differential ideal of \mathcal{R} generated by Σ* . It is defined as the intersection of all differential ideals of \mathcal{R} containing Σ . It is the set of all finite linear combinations, with arbitrary elements of \mathcal{R} for coefficients, of elements of Σ and their derivatives of any order. One denotes $\{\Sigma\}$ the *perfect differential ideal of \mathcal{R} generated by Σ* . It is defined as the intersection of all perfect differential ideals of \mathcal{R} containing Σ .

The next Proposition is essentially proved in [6, chap. I, 9, Lemma] in the case of a differential field \mathcal{S} of characteristic zero but the proof holds as is the context of this paper.

Proposition 1 *Let Σ be any subset of \mathcal{R} . Then $\{\Sigma\} = \sqrt{[\Sigma]}$.*

2.2.2 Characteristic Sets

This section is a mix between Ritt and Kolchin definitions of characteristic sets. We recall all the definitions and key results without any proof since none of the arguments involves any consideration on the base ring \mathcal{S} .

Let $U = \{u_1, \dots, u_n\}$ be a set of differential indeterminates. A *ranking* [5, chap. I, 8] is a total order on the infinite set ΘU which satisfies the two following axioms, for all derivatives $v, w \in \Theta U$ and every derivative operator $\theta \in \Theta$:

1. $v \leq \theta v$ and
2. $v < w \Rightarrow \theta v < \theta w$.

Proposition 2 *Every ranking is a well-ordering.*

Fix any ranking and consider some differential polynomial $p \in \mathcal{R} \setminus \mathcal{S}$. The *leading derivative* of p is the highest derivative v such that $\deg(p, v) > 0$. Let v be the leading derivative of p and $d = \deg(p, v)$. The *rank* of p is the monomial v^d . The ranking induces a *total ordering on ranks* as follows. A rank v^d is said to be less than a rank w^e if $v < w$ with respect to the ranking or $v = w$ and $d < e$. It is convenient to extend the above definitions by introducing some artificial rank, common to all nonzero elements of \mathcal{S} and considering that it is strictly less than the rank of any element of $\mathcal{R} \setminus \mathcal{S}$. If p, q are two nonzero differential polynomials, we will write $p < q$ to express the fact that the rank of p is strictly less than the one of q . Proposition 2 implies that any such ordering on ranks is a well-ordering. The *initial* of p is the leading coefficient of p , viewed as a univariate polynomial in v . In general, the *initial* of p is a differential polynomial of \mathcal{R} . The *separant* of p is the differential polynomial $\partial p / \partial v$. The second axiom of rankings implies that any proper derivative θp of p has rank θv and that its initial is the separant of p . Let $q \in \mathcal{R}$ and $p \in \mathcal{R} \setminus \mathcal{S}$ be two differential polynomials. Let p have rank v^d . The differential polynomial q is said to be *partially reduced* with respect to p if it does not depend on any proper derivative of v i.e. if, for every proper derivative operator θ , we have $\deg(q, \theta v) = 0$. The differential polynomial q is said to be (fully) *reduced* with respect to p if it is partially reduced with respect to p and $\deg(q, v) < d$. A set of differential polynomials $A \subset \mathcal{R} \setminus \mathcal{S}$ is said to be *autoreduced* if its elements are pairwise reduced with respect to each other. Observe that this definition, which comes from Kolchin, is somewhat different from Ritt's concept of chain, which allows chains to involve elements of the base field.

Proposition 3 *Every autoreduced set is finite.*

Let $A = \{p_1, \dots, p_r\}$ and $A' = \{p'_1, \dots, p'_{r'}\}$ be two autoreduced sets such that $p_1 < \dots < p_r$ and $p'_1 < \dots < p'_{r'}$. The set A' is said to be *lower than* the set A if

1. there exists some index $j \in [1, \min(r, r')]$ such that $p'_j < p_j$ and the two subsets $\{p_1, \dots, p_{j-1}\}$ and $\{p'_1, \dots, p'_{j-1}\}$ have the same set of ranks ; or
2. no such j exists and $r < r'$ (longer sets are lower).

Observe that the above relation is transitive [6, chap. I, 4] and defines a total ordering on autoreduced sets of ranks.

Proposition 4 *Every nonempty set of autoreduced sets contains a minimal element.*

The next Proposition actually is nothing but a reformulation of Proposition 4.

Proposition 5 *Every strictly decreasing sequence of autoreduced sets is finite.*

If Σ is any subset of \mathcal{R} then Σ contains autoreduced subsets, since the empty set is an autoreduced set.

Definition 2 Let Σ be any subset of \mathcal{R} . A characteristic set of Σ is any minimal autoreduced subset of Σ .

The next proposition is emphasized in [6, chap. I, 5].

Proposition 6 Let Σ be any subset of \mathcal{R} , A be a characteristic set of Σ and $p \in \mathcal{R} \setminus \mathcal{S}$ be a differential polynomial reduced with respect to A . Denote $\Sigma + p$ the set obtained by adjoining p to Σ .

The characteristic sets of $\Sigma + p$ are lower than A .

Corollary 1 Let Σ be any subset of \mathcal{R} and A be a characteristic set of Σ . Then Σ does not contain any differential polynomial of $\mathcal{R} \setminus \mathcal{S}$, reduced with respect to A .

2.2.3 Ritt's Reduction Algorithms

Ritt's reduction algorithm are based on the pseudodivision algorithm, which applies for polynomials with coefficients in rings. These algorithms are thus still applicable in the context of this paper.

Let f, g be two polynomials in one indeterminate x and coefficients in a ring. Assume $\deg(g, x) > 0$. One denotes $\text{prem}(f, g, x)$ the pseudoremainder of f by g . It is the polynomial $r(x)$ mentioned in [7, chap. I, 17, Theorem 9, page 30]. Let now $A = \{p_1, \dots, p_r\}$ be an autoreduced set of \mathcal{R} and $f \in \mathcal{R}$ be a differential polynomial. The *partial remainder* of f by A , denoted $\text{partialrem}(f, A)$ is defined inductively as follows:

1. if f is partially reduced with respect to all elements of A then $\text{partialrem}(f, A) = f$ else
2. there must exist some $p \in A$ with leading derivative v and some proper derivative operator θ such that $\deg(f, \theta v) > 0$. Among all such triples (p, v, θ) , choose one such that θv is maximal with respect to the ranking. Then $\text{partialrem}(f, A) = \text{partialrem}(\text{prem}(f, \theta p, \theta v), A)$.

Proposition 7 Let $A \subset \mathcal{R} \setminus \mathcal{S}$ be a finite set of differential polynomials, $f \in \mathcal{R}$ be a differential polynomial and $g = \text{partialrem}(f, A)$. Then g is partially reduced with respect to A and there exists a power product h of the separants of A such that $h f = g \pmod{[A]}$.

The *full remainder* of f by A , denoted $\text{fullrem}(f, A)$, is defined as follows. Assume $p_1 < \dots < p_r$.

1. if f is reduced with respect to all elements of A then $\text{fullrem}(f, A) = f$ else
2. if f is not partially reduced with respect to A then $\text{fullrem}(f, A) = \text{fullrem}(\text{partialrem}(f, A), A)$ else
3. there must exist some index $i \in [1, r]$ such that $\deg(f, v_i) \geq \deg(p_i, v_i)$ where v_i denotes the leading derivative of p_i . Among all such indices i , fix the maximal one. Then define $\text{fullrem}(f, A)$ as $\text{fullrem}(\text{prem}(f, p_i, v_i), A)$.

Proposition 8 Let $A \subset \mathcal{R} \setminus \mathcal{S}$ be a finite set of differential polynomials, $f \in \mathcal{R}$ be a differential polynomial and $g = \text{fullrem}(f, A)$. Then g is reduced with respect to A and there exists a power product h of the initials and the separants of A such that $h f = g \pmod{[A]}$.

2.2.4 The Ritt-Raudenbush Basis Theorem

The two next propositions are slight adaptations of [6, chap. I, 10].

Proposition 9 Let f, g be two differential polynomials and \mathfrak{A} be a perfect differential ideal of \mathcal{R} such that $f g \in \mathfrak{A}$. Then, for all derivative operators θ, φ , the product $(\theta f)(\varphi g) \in \mathfrak{A}$.

Following [6, chap. I, 9], in order to reduce possible confusion on the meaning of curly braces, if p is a differential polynomial and Σ is a set, we denote $\Sigma + p$ the set obtained by adjoining p to Σ .

Proposition 10 Let f, g be two differential polynomials and Σ be a set of differential polynomials of \mathcal{R} . Then $\{\Sigma + f g\} = \{\Sigma + f\} \cap \{\Sigma + g\}$.

The remaining part of this section comes from [6, chap. I, 12-16]. We may now give the proof of Theorem 1. The presentation is borrowed from [2].

Proof We assume that there exists infinite subsets of \mathcal{R} with no basis and seek a contradiction. Let Σ be such a subset and assume moreover that Σ is such that the characteristic sets of Σ are lower than the characteristic sets of any other infinite set which lacks a basis.

By Lemma 1 we have $\Sigma \cap \mathcal{S}$ empty or equal to $\{0\}$.

Let A be a characteristic set of Σ .

We perform Ritt's full reduction algorithm, with respect to A , over all $q \in \Sigma \setminus A$. For each $q \in \Sigma \setminus A$, there exists a power product h_q of initials and separants of A and a differential polynomial g_q , reduced with respect to A such that

$$h_q q = g_q \pmod{[A]}. \quad (1)$$

Introduce the two following sets (the plus sign standing for "union"):

$$\begin{aligned} \Lambda &= \{h_q q \mid q \in \Sigma \setminus A\} + A, \\ \Omega &= \{g_q \mid q \in \Sigma \setminus A\} + A. \end{aligned}$$

The set Ω must have a basis. Indeed, if it contains any nonzero element of \mathcal{S} it has a basis by Lemma 1. Otherwise, since the differential polynomials g_q are reduced with respect to A , its characteristic sets are lower than A by Proposition 6 thus it cannot lack a basis by the minimality assumption on the characteristic sets of Σ .

Thus there exists finitely many differential polynomials $q_1, \dots, q_t \in \Sigma \setminus A$ such that the set $\Phi = \{g_{q_1}, \dots, g_{q_t}\} + A$ is a basis of Ω (observe that it is always possible to enlarge a basis with finitely many further differential polynomials).

Claim: the set $\Psi = \{h_{q_1} q_1, \dots, h_{q_t} q_t\} + A$ is a basis of Λ .

Each $h_{q_i} q_i - g_{q_i}$ ($1 \leq i \leq t$), belongs to the perfect differential ideals $\{\Phi\}$ and $\{\Psi\}$ by Proposition 8 and the fact that A is a subset of both Φ and Ψ .

Thus, since each $g_{q_i} \in \Phi$ ($1 \leq i \leq t$), we see that each $h_{q_i} q_i \in \{\Phi\}$ ($1 \leq i \leq t$) and $\Psi \subset \{\Phi\}$. Conversely, since each $h_{q_i} q_i \in \Psi$, we see that each $g_{q_i} \in \{\Psi\}$ and $\Phi \subset \{\Psi\}$. Thus both perfect differential ideals $\{\Phi\}$ and $\{\Psi\}$ are equal.

Since Φ is a basis of Ω we have $\Omega \subset \{\Phi\}$. Since the full remainder g_q of each $q \in \Sigma$ belongs to Ω , we see that the corresponding product $h_q q$ of each $q \in \Sigma$ belongs to $\{\Omega\}$, which is included in $\{\Phi\} = \{\Psi\}$. Thus $\Lambda \subset \{\Psi\}$ and the claim is proved.

Let f_1, \dots, f_s denote the initials and separants of A . By Lemma 2, there exists an index $1 \leq i \leq s$ such that the set $\Sigma + f_i$ has no basis. The differential polynomial $f_i \notin \mathcal{S}$ by Lemma 1. Thus the set $\Sigma + f_i$ has a characteristic set lower than A by Proposition 6. This contradiction with the minimality assumption on the characteristic sets of Σ completes the proof of the Theorem. \square

The next Lemma is involved in the proof of the Ritt-Raudenbush Basis Theorem. The differential polynomials f_i actually are the initials and separants of some characteristic set of Σ .

Lemma 2 *Let Σ be an infinite subset of \mathcal{R} and f_1, \dots, f_s be differential polynomials of \mathcal{R} . Let*

$$\Lambda = \{h_q q \mid q \in \Sigma \text{ and } h_q \text{ is some power product of } f_1, \dots, f_s\}.$$

If Σ has no basis and Λ has a basis then at least one of the sets $\Sigma + f_i$, for $1 \leq i \leq s$, has no basis.

Proof We assume that all sets $\Sigma + f_i$ ($1 \leq i \leq s$) have a basis and seek a contradiction.

Let $\Psi = \{h_{q_1} q_1, \dots, h_{q_t} q_t\}$ be a basis of Λ . Since a basis can always be enlarged as long as it remains finite, there exists some finite set $\Phi \subset \Sigma$ such that: 1) $\Phi + f_i$ is a basis of $\Sigma + f_i$ ($1 \leq i \leq s$) and; 2) $q_1, \dots, q_t \in \Phi$. Let g denote the product $f_1 \cdots f_s$.

By Proposition 10, the perfect differential ideal $\{\Sigma + g\}$ is the intersection of the perfect differential ideals $\{\Sigma + f_i\}$ ($1 \leq i \leq s$); similarly, the perfect differential ideal $\{\Phi + g\}$ is the intersection of the perfect differential ideals $\{\Phi + f_i\}$. Since each $\Phi + f_i$ is a basis of $\Sigma + f_i$ we have

$$\{\Sigma + g\} = \bigcap_{i=1}^s \{\Sigma + f_i\} \subset \bigcap_{i=1}^s \{\Phi + f_i\} = \{\Phi + g\}.$$

Thus $\Phi + g$ is a basis of $\Sigma + g$. Therefore, for each differential polynomial $p \in \Sigma$, there exists a relation

$$p^d = r + m_1 \theta_1 g + \cdots + m_e \theta_e g$$

where $d \geq 1$, $e \geq 0$, the m_i are differential polynomials of \mathcal{R} and $r \in [\Phi]$. Multiplying by p we get

$$p^{d+1} = rp + m_1 p \theta_1 g + \cdots + m_e p \theta_e g \quad (2)$$

Since $q_1, \dots, q_t \in \Phi$ we have $\Psi \subset \{\Phi\}$. Since, moreover, $p \in \Sigma$ and g is the product of the f_i , we have $pg \in \{\Lambda\} \subset \{\Psi\} \subset \{\Phi\}$. Thus, by Proposition 10, we have $p \theta_i g \in \{\Phi\}$ for $1 \leq i \leq e$. Since $r \in [\Phi]$ we have $rp \in \{\Phi\}$. Thus by (2), we have $p \in \{\Phi\}$, which means that Φ is a basis of Σ : the sought contradiction. \square

2.3 Corollaries to the Ritt-Raudenbush Theorem

The following immediate consequences of the Ritt-Raudenbush Theorem are recalled for the convenience of the reader.

Corollary 2 *Let \mathfrak{A} be a perfect differential ideal of \mathcal{R} . Then there exists a finite subset $\Phi \subset \mathfrak{A}$ such that $\mathfrak{A} = \{\Phi\}$.*

Theorem 3 *Every perfect differential ideal \mathfrak{A} is a finite intersection of prime differential ideals.*

Let \mathfrak{A} be a perfect differential ideal of \mathcal{R} . A representation (3) of \mathfrak{A} as an intersection of prime differential ideals \mathfrak{P}_i is said to be *minimal* if, for all indices $1 \leq i, j \leq \varrho$ such that $i \neq j$ we have $\mathfrak{P}_i \not\subset \mathfrak{P}_j$.

$$\mathfrak{A} = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_\varrho. \quad (3)$$

Theorem 4 *There exists a unique minimal representation of a perfect differential ideal \mathfrak{A} as a finite intersection of prime differential ideals.*

3 A Well Known Property of Formal Power Series

Let us first introduce a few notations. If $\theta = \delta_1^{e_1} \cdots \delta_m^{e_m}$ is a derivative operator then $x^\theta = x_1^{e_1} \cdots x_m^{e_m}$ and $\theta! = e_1! \cdots e_m!$. Every multivariate formal power series with coefficients in some differential field \mathcal{F} of characteristic zero can be denoted

$$S = \sum_{\theta \in \Theta} \frac{1}{\theta!} a_\theta x^\theta. \quad (4)$$

Remark: since derivative operators act multiplicatively, the identity derivative operator is 1 so that the first coefficient of a formal power series is a_1 . In the following Lemma, the hypothesis that \mathcal{F} is a field of constants is not necessary. It simplifies Formula (5).

Lemma 3 *Assume \mathcal{F} is a field of constants. Let S be a nonzero formal power series and θ be a derivative operator such that $a_\theta \neq 0$. Then there exists a formal power series T such that $(\theta S)T = 1$.*

Proof Since \mathcal{F} is a field of constants we have

$$\theta S = \sum_{\varphi \in \Theta} \frac{1}{\varphi!} a_{\varphi\theta} x^\varphi. \quad (5)$$

The first coefficient of θS is thus nonzero. To simplify notations, it is sufficient to assume that the first coefficient a_1 of S is nonzero and prove that there exists a formal power series T such that $ST = 1$. Denote

$$T = \sum_{\theta \in \Theta} \frac{1}{\theta!} b_\theta x^\theta.$$

Order the derivative operators θ with respect to a ranking (more precisely, fix any ranking on Θu and let $\theta < \varphi$ if $\theta u < \varphi u$). Fix the first coefficient of T as $b_1 = 1/a_1$. Expanding the product we get

$$ST = \sum_{\theta \in \Theta} c_\theta x^\theta \quad \text{where} \quad c_\theta = \sum_{\varphi, \psi \text{ s.t. } \varphi\psi = \theta} \frac{1}{\varphi!} \frac{1}{\psi!} a_\varphi b_\psi.$$

Every coefficient c_θ ($\theta \neq 1$) is a sum of terms involving the product $a_1 b_\theta$. For every $\varphi < \theta$, the coefficient c_φ does not depend on b_θ . Thus, assuming that every such coefficient c_φ is annihilated by a suitable choice of some coefficients of T , it is still possible to annihilate c_φ by fixing a suitable value to b_θ . \square

4 On an Application to Tropical Differential Geometry

Quoting [6, page 21], the Proposition 2.1 of [1] claims: *the solution of any infinite system of differential polynomials $\Sigma \subset \mathcal{F}\{u_1, \dots, u_n\}$, where \mathcal{F} is a differential field of characteristic zero, is the solution of some finite subset of the system.*

It should be noticed that, in [6, page 21], the term “solution” probably refers to an abstract solution in some differential field extension of the base field. Such solutions can be interpreted as formal power series expanded over some *non fixed* expansion point. With this interpretation in mind, we may interpret Ritt’s statement as some informal version of a differential Theorem of Zeros [6, chap. II, 7], which can be expressed as follows: *the set of the differential polynomials which annihilate over any solution of a given infinite system Σ is the perfect differential ideal $\{\Sigma\}$ which actually is generated by some finite basis Φ of Σ by Corollary 2.*

In tropical differential geometry, solutions are sought in a ring of formal power series $K[[x]]$, where K is a field of characteristic zero. The sought formal power series are thus expanded at a fixed expansion point: the origin. In such a case, the differential Theorem of Zeros does not apply. See [2] which actually relies on [3] for an example of a differential polynomial p which has no such solution and is such that $\mathcal{R} \neq \{p\}$. However, the Ritt-Raudenbush Theorem still holds. A precise statement would be: *the set of the differential polynomials which annihilate over any formal power series solution expanded at the origin of a given infinite system Σ is a perfect differential ideal \mathfrak{A} which contains the perfect differential ideal $\{\Sigma\}$; the solution set of \mathfrak{A} is equal to the solution set of $\{\Sigma\}$; it is also equal to the solution set of any basis of Σ .*

Thus [1, Proposition 2.1] relies on the Ritt-Raudenbush Basis Theorem. In [1, Sect. 6], this proposition is applied to the differential polynomial ring $\mathcal{F}\{u_1, \dots, u_n\}$ where \mathcal{F} is the field of fractions of $K[[x]]$. The consequences are then applied to the differential polynomial ring $K[[x]]\{u_1, \dots, u_n\}$. The tropical version given in this paper provides a more direct argument.

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