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► **To cite this version:**

Baptiste Nicolas Huguet. Intertwining relations for diffusions in manifolds and applications to functional inequalities. 2019. hal-02402712v1

**HAL Id: hal-02402712**

**<https://hal.science/hal-02402712v1>**

Preprint submitted on 10 Dec 2019 (v1), last revised 8 Jan 2021 (v2)

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# INTERTWINING RELATIONS FOR DIFFUSIONS IN MANIFOLDS AND APPLICATIONS TO FUNCTIONAL INEQUALITIES

BAPTISTE HUGUET

ABSTRACT. We prove intertwining relations by twisted gradients for Markov semi-groups. These relations are applied to Brascamp-Lieb type inequalities and spectral gap results. It generalizes the results of [1] from the Euclidean space to Riemannian manifolds and to non symmetric twisted operators.

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## 1. INTRODUCTION

The aim of this paper is to extend our understanding of intertwining relations between Markov semi-groups in the setting of Riemannian manifolds and its applications in functional inequalities but also the underlying role of stochastic processes as the deformed parallel translation. These relations have been first investigated in the discrete case for birth-death processes in [9] and in the one dimensional case in [6]. The case of the Euclidean space  $\mathbb{R}^n$  is treated in [1] for reversible and ergodic diffusions. In this paper, we also investigate the case of reversible and ergodic diffusions, on a complete connected Riemannian manifold  $M$ , with generator

$$Lf = \Delta f - \langle \nabla V, \nabla f \rangle$$

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*Date:* December 10, 2019 *File:* intertwining'Huguet.tex.

*Key words and phrases.* Intertwining; Diffusion operator on manifold; Brascamp-Lieb types inequalities; spectral gap.

where  $V$  is a smooth potential on  $M$ . Such diffusions admit an unique invariant measure,  $\mu$ , absolutely continuous with respect to the Riemannian measure, with density proportional to  $e^{-V}$ .

We are looking for intertwining relations by differentiation: the goal is to rewrite the derivative of a smooth Markov semi-group acting on functions as a Markov semi-group acting on differential forms. Unlike in the one-dimensional case, where functions and their derivatives have the same nature, in a manifold setting, the two intertwined semi-groups act on different spaces. Actually, we look at semi-groups on 1-forms which restriction on differential forms satisfy an intertwining relation.

At the level of operators, the intertwining relation occurs without further assumptions. The generator  $L$  is intertwined with a weighted Laplacian acting on 1-forms,  $L^W$ , unitary equivalent to the Witten Laplacian. A large study of this operator can be found in the work of Helffer, with application to correlation decay in spin systems (see [11]). At the level of stochastic processes,  $L^W$  is the generator on 1-forms of a diffusion on the tangent bundle: the deformed parallel translation (or geodesic transport in [18]). In [2], this process appears naturally as a spacial derivative of a flow of the diffusion with generator  $L$ . This suggests a stochastic representation of the intertwined semi-group. At the level of semi-groups, intertwining relations are not so obvious: more assumptions are required. In the Euclidean space, the classical assumption is the strong convexity of the potential  $V$  or, in other way to say it, the positiveness of its Hessian. A classical generalization of this condition on Riemannian manifolds is the positiveness of an operator depending on the Hessian and the Ricci curvature, known as the Bakry-Émery criterion (see [3]). In the present paper, we consider twisted gradients, or equivalently, twisted metrics on the tangent space by a section of  $GL(TM)$ . This operation does not change the stochastic diffusion on  $M$  but creates new ones on the tangent space, with associated generators and semi-groups. Under assumptions on these twists, which replace the Bakry-Émery condition, we can obtain intertwining relations at the level of semi-groups. A consequence of these intertwinings is a family of Brascamp-Lieb type inequalities, extending the classical case satisfied under the strict convexity assumption of the potential.

Let us summarize the content of this paper. In Section 2, we recall basic facts about semi-group, deformed parallel translation and the classical commutation at the level of the generators. The semi-group considered is stochastically defined on bounded continuous functions. In Section 3, we see what an intertwining relation can imply in terms of covariance representation. For that, we work under the Bakry-Émery criterion. We obtain some applications to measure concentration. In Section 4, we introduce twistings, associated semi-groups and their generators. The goal of Section 5 is to find conditions for these generators to be decomposable as a sum of a symmetric positive second order generator and a zero order potential. In Sections 6 and 7, we obtain conditions to have intertwining relations for the  $L^2$  semi-groups on 1-forms. Theorem 6.2 is a generalization of theorem 2.2 in [1], in a

manifold setting, with the same kind of assumptions: conditions of symmetry and positiveness of the second order operator and bound on the potential. Theorem 7.1 extends this result when the second order operator is not symmetric non-negative. We achieve to release all assumptions over the second order operator by a stronger bound on the potential. These intertwining are applied in theorems 6.3 and 7.3 to obtain generalized Brascamp-Lieb and Poincaré inequalities.

## 2. DEFORMED PARALLEL TRANSPORT AND COMMUTATION

On a connected complete Riemannian manifold  $(M, g)$ , endowed with its Levi-Civita connexion  $\nabla$ , let  $\mathcal{C}^\infty(M)$  be the space of smooth real-valued functions and  $\mathcal{C}_c^\infty(M)$  its subspace of compactly supported functions. In this paper, we consider the second order diffusion operator defined on  $\mathcal{C}^\infty(M)$  by

$$Lf = \Delta f - \langle \nabla V, \nabla f \rangle, \quad (2.1)$$

where  $V$  is a smooth potential. We denote by  $\mu$  the measure on  $M$  with density  $e^{-V}$ . On  $\mathcal{C}_c^\infty(M)$ , the operator  $L$  is symmetric with respect to  $\mu$ , that is for all  $f, g \in \mathcal{C}_c^\infty(M)$ ,

$$\int_M Lfg \, d\mu = - \int_M \langle df, dg \rangle \, d\mu = \int_M fLg \, d\mu. \quad (2.2)$$

Let  $X_t(x)$  be a diffusion process with generator  $L$ , started at  $x \in M$ . Such a process exists and is unique in law, up to an explosion time  $\tau_x$ . We define a family  $(\mathbf{P}_t)_{t \geq 0}$  of operators on the space of bounded continuous functions by:

$$\mathbf{P}_t f(x) = \mathbb{E}[f(X_t(x)) \mathbf{1}_{t < \tau_x}]. \quad (2.3)$$

The Markov property for diffusion processes implies that  $(\mathbf{P}_t)_{t \geq 0}$  is a semi-group and for all  $f \in \mathcal{C}_c^\infty(M)$ , we have

$$\partial_t \mathbf{P}_t f(x) = L \mathbf{P}_t f(x) = \mathbf{P}_t Lf(x).$$

Above  $X_t(x)$ , one can construct the parallel translation  $\parallel_t(x)$ . It is an isometric isomorphism from  $T_x M$  to  $T_{X_t(x)} M$ . It also can be seen as a diffusion on the tangent bundle with the following generator on 1-forms:

$$\langle L \parallel \alpha, w \rangle = \langle \Delta^h \alpha - \nabla_U \alpha, w \rangle, \quad (2.4)$$

where  $\Delta^h$  is the horizontal Laplacian on 1-forms. According to the Weitzenböck formula, we have:

$$\langle \Delta^h \alpha, w \rangle = \langle \square \alpha, w \rangle + \langle \alpha, \text{Ric}^\sharp(w) \rangle, \quad (2.5)$$

with  $\square$  being the Hodge-de Rham Laplacian. One can look at [12] for instance.

Above  $X_t(x)$  one can also construct the deformed parallel translation  $W_t(x)$ . It is the linear map  $T_x M \rightarrow T_{X_t(x)} M$  determined by the differential equation:

$$\begin{cases} D_t W_t(x)v = -\mathcal{M}^* W_t(x)v dt \\ W_0(x) = id_{T_x M} \end{cases}, \quad (2.6)$$

where

$$D_t W_t(x)v = \parallel_t(x)d(\parallel_t^{-1}(x)W_t(x)v) \quad (2.7)$$

stands for the covariant derivative of  $W_t(x)v$  and the operator  $\mathcal{M}^*$  is a section of  $TM \otimes T^*M$  defined by

$$\mathcal{M}^*w = \nabla_w U + \text{Ric}^\sharp(w). \quad (2.8)$$

Theorem 2.1 in [2] shows that  $W_t(x)v$  can be seen as the spatial derivative of a flow of the diffusion with generator  $L$ , obtained from  $X_t(x)$  by parallel coupling. In some way, the processes  $X_t(x)$  and  $W_t(x)$  are intertwined.

Alternatively, the deformed parallel translation can be seen as a diffusion.

**Proposition 2.1.** *The deformed parallel translation over  $X_t(x)$  is a diffusion on the tangent bundle with generator on 1-forms*

$$\langle L^W \alpha, w \rangle = \langle L^\parallel \alpha(X_t), w \rangle - \langle \mathcal{M} \alpha, w \rangle. \quad (2.9)$$

*Proof.* Let  $\alpha$  a 1-form and  $v \in T_x M$ .

$$d\langle \alpha(X_t(x)), W_t(x)v \rangle = \langle D_t \alpha(X_t(x)), W_t(x)v \rangle + \langle \alpha(X_t(x)), D_t W_t(x)v \rangle, \quad (2.10)$$

where  $D_t \alpha(X_t(x)) = d(\alpha(X_t(x)) \parallel_t(x)) \parallel_t^{-1}$  stands for the covariant differential of  $\alpha(X_t(x))$ . There is no quadratic term because the deformed parallel translation has finite variations. As the parallel translation is a diffusion and as  $W_t(x)$  satisfies equation (2.6), it ends the proof.  $\square$

Using (2.5), one obtains

$$\langle L^W \alpha, w \rangle = \langle \square \alpha - \nabla_U \alpha, w \rangle - \langle \alpha, \nabla_w U \rangle. \quad (2.11)$$

Let  $L^2(\mu)$  be the space of measurable 1-forms  $\alpha$  such that

$$\int_M |\alpha_x|^2 d\mu(x) < +\infty.$$

**Theorem 2.2.** *The operator  $L^W$  is essentially self-adjoint on  $L^2(\mu)$ .*

*Proof.* We denote by  $\delta_V$  the adjoint of the exterior derivative on forms for the scalar product on  $L^2(\mu)$ . Some calculation shows that, for all smooth compactly supported 1-forms, we have:

$$L^W = -(d\delta_V + \delta_V d) \quad (2.12)$$

Then  $-L^W$  is non-negative. Adapting the proof of [11], we obtain the result.  $\square$

Then, without any assumptions, the deformed parallel translation defines a semi-group  $(\mathbf{Q}_t)_{t \geq 0}$  on  $L^2(\mu)$ . Under suitable assumptions over the potential  $\mathcal{M}$ , it generates a semi-group, also denoted by  $\mathbf{Q}$ , on the space of bounded continuous 1-forms, with the following stochastic representation:

$$\langle \mathbf{Q}_t \alpha, v \rangle = \mathbb{E}[\langle \alpha, W_t(x)v \rangle \mathbf{1}_{t < \tau_x}]. \quad (2.13)$$

The generator of the deformed parallel translation satisfies a commutation formula. For all  $f \in C^\infty(M)$ , one has:

$$dLf = L^W df. \quad (2.14)$$

This commutation formula on generators and the intertwining relation at the level of stochastic processes suggest an intertwining relation between the semi-groups  $\mathbf{P}$  and  $\mathbf{Q}$ . This is this kind of relation we want to prove.

### 3. A COVARIANCE INEQUALITY

In this section, we consider  $\mathbf{Q}$  as the semi-group on bounded continuous 1-forms defined by 2.13. We prove an intertwining relation and we use it to obtain an asymmetric Brascamp-Lieb inequality in the spirit of Ledoux (see [16] or [8]). This inequality is called asymmetric because it gives an  $L^1$ - $L^\infty$  bound of the covariance. We firstly have to find a condition so as to properly define the semi-group. As an endomorphism of  $T_x^*M$ ,  $\mathcal{M}(x)$ , defined in (2.8), is symmetric with respect to the metric. We denote by  $\rho(x)$  the smallest eigenvalue of  $\mathcal{M}(x)$  and by  $\rho$ , its infimum over  $M$ :

$$\rho = \inf_{x \in M} \{\text{smallest eigenvalue of } \mathcal{M}(x)\} \quad (3.1)$$

The assumption of this section is the so-called Bakry-Émery criterion (known as  $CD(\rho, \infty)$  condition in [5]).

**Assumption 3.1** (Bakry-Émery criterion). *The operator  $\mathcal{M}$  is uniformly bounded from below, i.e  $\rho > -\infty$ .*

It is a sufficient condition for hypercontractivity of the diffusion and allows to prove Poincaré or Log-Sobolev inequalities (see [4]). Bakry proves in [3] that, under this criterion, the diffusion  $X$  does not explode (i.e for all  $x \in M$ ,  $\tau_x = +\infty$  almost surely). The following result is well known.

**Proposition 3.2.** *Under the Bakry-Émery criterion, the semi-group  $\mathbf{Q}$  has the stochastic representation (2.13) and we have: for all 1-form  $\alpha$ ,*

$$|\mathbf{Q}_t \alpha_x| \leq e^{-\rho t} \|\alpha\|_\infty. \quad (3.2)$$

*Proof.* The heart of the proof is to show that under this criterion, the deformed parallel translation is bounded. For all  $x \in M$  and all  $v \in T_x M$ , one has

$$\begin{aligned} \frac{d}{dt} |W_t(x)v|^2 &= 2\langle W_t(x)v, D_t W_t(x)v \rangle \\ &= -2\langle W_t(x)v, \mathcal{M}^* W_t(x)v \rangle \\ &\leq -2\rho |W_t(x)v|^2. \end{aligned}$$

By Grönwall lemma, this yields

$$|W_t(x)v|_2 \leq e^{-\rho t} |v|, \text{ a.s.} \quad (3.3)$$

This shows that the stochastic representation (2.13) is well-defined and concludes the proof.  $\square$

**Proposition 3.3.** *Under the Bakry-Émery criterion, the semi-groups  $\mathbf{P}$  and  $\mathbf{Q}$  are intertwined by the derivative of functions: for all  $f \in \mathcal{C}_c^\infty(M)$ ,*

$$d\mathbf{P}_t f = \mathbf{Q}_t df. \quad (3.4)$$

*Proof.* The bound for  $|W_t|_2$  obtained in equation 3.3 and the non explosion of the process guarantee the differentiation under the expectation.  $\square$

This intertwining relation results in an asymmetric Brascamp-Lieb inequality.

**Theorem 3.4.** *Assume that  $\rho > 0$ , then for all functions  $f, g \in \mathcal{C}_c^\infty(M)$ , one has*

$$|\text{Cov}_\mu(f, g)| \leq \frac{1}{\rho} \|dg\|_\infty \int_M |df|_2 d\mu. \quad (3.5)$$

*Proof.* Using the ergodicity and proposition 3.3, we have for all  $f, g \in \mathcal{C}_c^\infty(M)$  the following covariance representation:

$$\begin{aligned} \text{Cov}_\mu(f, g) &= \int_M f(g - \mu(g)) d\mu \\ &= - \int_M \int_0^{+\infty} f LP_t g dt d\mu \\ &= \int_0^{+\infty} \left( \int_M \langle df, d\mathbf{P}_t g \rangle d\mu \right) dt \\ &= \int_0^{+\infty} \left( \int_M \langle df, \mathbf{Q}_t dg \rangle d\mu \right) dt. \end{aligned}$$

We conclude with the estimate of 3.2.  $\square$

The heart of the proof, is the integral representation of the covariance. This is the reason why we want to obtain intertwining relations.

In [3], Bakry shows that the condition  $\rho > 0$  implies that  $\mu$  is a finite measure. Up to renormalization, we can assume it is a probability measure. A consequence of theorem 3.4, is the Gaussian concentration of the probability  $\mu$ . This concentration has been shown by Ledoux in [13] for the volume measure of a compact Riemannian manifold under the condition of positive Ricci curvature and in [14] in the Euclidean space under the condition of strictly convex potential. This inequality is deeply exposed in [15]. Our proof gives a new outlook of the result, with only stochastic tools.

**Proposition 3.5.** *If  $\rho > 0$ , then for all 1-Lipschitz  $f \in \mathcal{C}_c^\infty(M)$  and for all  $r > 0$ ,*

$$\mu(|f - \mu(f)| > r) \leq 2e^{-\rho \frac{r^2}{2}}. \quad (3.6)$$

*Proof.* The idea of the proof is to bound the Laplace transform. For any  $\lambda > 0$ , we have:

$$\begin{aligned} \frac{d}{d\lambda} \mathbb{E}_\mu[e^{\lambda f}] &= \text{Cov}_\mu(f, e^{\lambda f}) \\ &\leq \frac{1}{\rho} \|df\|_\infty \mathbb{E}[\lambda |df| e^{\lambda f}] \\ &\leq \frac{\lambda}{\rho} \mathbb{E}[e^{\lambda f}]. \end{aligned}$$

By Grönwall lemma, it yields

$$\mathbb{E}_\mu[e^{\lambda f}] \leq e^{-\lambda^2/2\rho}. \quad (3.7)$$

The proof ends by using Markov's inequality and optimizing in  $\lambda$ .  $\square$

#### 4. TWISTED PROCESSES AND SEMI-GROUPS

Let  $B$  be a smooth invertible section of  $TM \otimes T^*M$ , i.e for all  $x \in M$ ,  $B(x)$  is an isomorphism of  $T_x M$ . The section  $B$  is used to twist the semi-group so as to obtain an intertwining relation even when the Bakry-Émery criterion is not satisfied. Firstly, we define the  $B$ -parallel translation by conjugation as:

$$\parallel_t^B(x) = B(X_t(x)) \parallel_t(x) B(x)^{-1} : T_x M \rightarrow T_{X_t(x)} M. \quad (4.1)$$

**Proposition 4.1.** *The  $B$ -parallel translation is a diffusion on  $TM$  with generator on 1-forms*

$$L^{\parallel, B} \alpha = L^{\parallel} \alpha + 2(B^{-1})^* \nabla B^* \cdot \nabla \alpha + (B^{-1})^* (L^{\parallel} B^*) \alpha.$$

*Proof.*

$$d\langle \alpha(X_t), B(X_t(x)) \parallel_t(x) w \rangle = d\langle B(X_t(x))^* \alpha(X_t), \parallel_t(x) w \rangle \quad (4.2)$$

so

$$\langle L^{\parallel, B} \alpha, w \rangle = \langle L^{\parallel} (B^* \alpha), B^{-1} w \rangle. \quad (4.3)$$

On the other hand writing  $v = B^{-1} w$

$$\langle L^{\parallel} (B^* \alpha), v \rangle = \langle B^* L^{\parallel} \alpha, v \rangle + \langle L^{\parallel} (B^*) \alpha, v \rangle + 2\langle \nabla B^* \cdot \nabla \alpha, v \rangle. \quad (4.4)$$

$\square$

Unlike the parallel translation, the  $B$ -parallel translation is not an isometry for the Riemmanian metric. Yet, it is an isometry for the  $B$ -twisted metric: for all  $v, w \in T_x M$

$$\langle v, w \rangle_B = \langle B^{-1}(x)v, B^{-1}(x)w \rangle. \quad (4.5)$$

Similarly, we define the  $B$ -deformed parallel translation as:

$$W_t^B(x) = B(X_t(x)) W_t(x) B(x)^{-1} : T_x M \rightarrow T_{X_t(x)} M. \quad (4.6)$$



**Proposition 4.2.** *The  $B$ -deformed parallel translation is a diffusion with generator on 1-forms*

$$L^{W,B} = L^{\parallel,B} - \mathcal{M}_B \quad (4.7)$$

where

$$\mathcal{M}_B = (B^{-1})^* \mathcal{M} B^*. \quad (4.8)$$

The  $B$ -deformed parallel translation could also be defined as in 2.6 by a stochastic covariant differential equation:

$$D_t W_t^B v = - (\mathcal{M}_B - (B^{-1})^* L^{\parallel} (B^*)) W_t^B v dt + (\nabla_{d_m X_t} B) (B^{-1})^* W_t^B v. \quad (4.9)$$

Unlike, in 2.6, there is a martingale part in the stochastic covariant derivative. If we want to strip it away, we must the  $B$ -stochastic covariant derivative:

$$D_t^B = \parallel_t^B d \left( \parallel_t^{B^{-1}} \right). \quad (4.10)$$

The generators  $L^{W,B}$  and  $L^W$  are conjugated:  $L^{W,B} = (B^{-1})^* L^W B^*$ . So,  $L^{W,B}$  and  $L$  are intertwined:

$$(B^{-1})^* dL = L^{W,B} (B^{-1})^* L. \quad (4.11)$$

We denote by  $\langle \cdot, \cdot \rangle_B$  the intertwined-metric on 1-forms: for two 1-forms  $\alpha, \beta$ ,

$$\langle \alpha, \beta \rangle_B = \langle B^* \alpha, B^* \beta \rangle, \quad (4.12)$$

and by  $L^2(B, \mu)$  the space of measurable 1-forms  $\alpha$  such that

$$\int_M |\alpha|_B^2 d\mu < +\infty. \quad (4.13)$$

As  $L^W$ ,  $L^{W,B}$  is also essentially self-adjoint, on  $L^2(B, \mu)$  and is associated to a  $L^2$  semi-group of diffusion on 1-forms,  $\mathbf{Q}_t^B$ . Under suitable conditions, it generates a semi-group on smooth compactly supported 1-forms, also denoted by  $\mathbf{Q}_t^B$ , with the stochastic representation

$$\langle \mathbf{Q}_t^B \alpha, v \rangle = \mathbb{E} [\langle \alpha, W_t^B(x) v \rangle \mathbf{1}_{t < \tau_x}]. \quad (4.14)$$

**Proposition 4.3.** *Under the Bakry-Émery criterion, the semi-groups  $\mathbf{P}$  and  $\mathbf{Q}^B$  are intertwined by  $(B^{-1})^* d$ , i.e. for all  $f \in \mathcal{C}_c^\infty(M)$ ,*

$$(B^*)^{-1} d\mathbf{P}_t f = \mathbf{Q}_t^B ((B^*)^{-1} df).$$

*Proof.* As in proposition 3.2, the Bakry-Émery criterion prove the existence of the stochastic representation 4.14. For all  $f \in \mathcal{C}_c^\infty(M)$ , we have:

$$\begin{aligned} (B^{-1})^* d\mathbf{P}_t f &= (B^{-1})^* \mathbf{Q}_t df \\ &= (B^{-1})^* \mathbb{E} [\langle df, W_t(x) \cdot \rangle] \\ &= \mathbb{E} [\langle df, W_t(x) B^{-1}(x) \cdot \rangle] \\ &= \mathbb{E} [\langle (B^{-1})^* df, B(X_t(x)) W_t(x) B^{-1}(x) \cdot \rangle] \\ &= \mathbb{E} [\langle (B^{-1})^* df, W_t^B(x) \cdot \rangle]. \end{aligned}$$

□

The operators  $\mathcal{M}$  and  $\mathcal{M}^B$  are conjugated so they have the same eigenvalues. Then  $\mathcal{M}^B$  does not seem useful to improve inequalities such as Section 3 even if we could obtain the intertwining relation without using the Bakry-Émery criterion. We define the generators on 1-forms

$$L_B^{\parallel} = L^{\parallel} + 2(B^{-1})^* \nabla B^* \cdot \nabla. \quad (4.15)$$

and

$$M_B = \mathcal{M}^B - (B^{-1})^* L^{\parallel} (B^*). \quad (4.16)$$

Then we have

$$L^{W,B} = L_B^{\parallel} - M_B. \quad (4.17)$$

From the operators point of view, this new split seems more satisfying because the potential  $M_B$  contains all the zero-order terms and only them. From the stochastic point of view, it seems also more relevant because  $M_B$  is the drift of the covariant derivative of  $W_t^B$ . So, this potential is the natural candidate for a generalization of Bakry-Émery criterion.

## 5. SYMMETRY AND POSITIVENESS OF $-L_B^{\parallel}$

First, as we noticed,  $L^{W,B}$  is conjugated to  $L^W$ , and so, is self-adjoint in  $L^2(B, \mu)$ . For the same reason, in the subspace of twisted gradients  $\{(B^{-1})^* df : f \in \mathcal{C}_c^\infty(M)\}$ , we additionally have the non-positiveness of  $L^{W,B}$ :

$$\begin{aligned} \int_M \langle (B^{-1})^* df, L^{W,B} (B^{-1})^* df \rangle_B d\mu &= \int_M \langle df, L^W df \rangle d\mu \\ &= \int_M \langle df, d(Lf) \rangle d\mu \\ &= - \int_M (Lf)^2 d\mu. \end{aligned}$$

Now we are looking for conditions such that  $L_B^{\parallel}$  be symmetric with respect to the  $B$ -twisted metric. This is not trivial, even in the subspace of twisted gradients. First, by integration by parts for the horizontal Laplacian, we have

$$\int_M \langle L^{\parallel} \alpha, \beta \rangle d\mu = - \int_M \langle \nabla \alpha, \nabla \beta \rangle d\mu \quad (5.1)$$

with  $\langle \nabla \alpha, \nabla \beta \rangle = \sum_i \langle \nabla_{e_i} \alpha, \nabla_{e_i} \beta \rangle$ , with  $(e_i)$  any orthonormal basis. On one hand,

$$\int_M \langle (-L^{\parallel}) \alpha, \beta \rangle_B d\mu = \int_M \langle (-L^{\parallel}) \alpha, (B^*)^t B^* \beta \rangle d\mu$$

$$\begin{aligned}
&= \int_M \langle \nabla \alpha, \nabla((B^*)^t B^* \beta) \rangle d\mu \\
&= \int_M \langle \nabla \alpha, (B^*)^t B^* \nabla \beta \rangle d\mu + \int_M \langle \nabla \alpha, \nabla((B^*)^t B^*) \beta \rangle d\mu \\
&= \int_M \langle \nabla \alpha, \nabla \beta \rangle_B d\mu + \int_M \langle \nabla \alpha, \nabla((B^*)^t B^*) \beta \rangle d\mu
\end{aligned}$$

where  $(B^*)^t$  denotes the dual map of  $B^*$  with respect to scalar product on  $T^*M$ . On the other hand,

$$\begin{aligned}
- \int_M \langle 2(B^{-1})^* \nabla B^* \cdot \nabla \alpha, \beta \rangle_B d\mu &= - \int_M \langle 2\nabla B^* \cdot \nabla \alpha, B^* \beta \rangle d\mu \\
&= - \int_M \langle \nabla \alpha, 2(\nabla B^*)^t B^* \beta \rangle d\mu.
\end{aligned}$$

This yields

$$\int_M \langle (-L_B^\parallel) \alpha, \beta \rangle_B d\mu = \int_M \langle \nabla \alpha, \nabla \beta \rangle_B d\mu - \int_M \langle B^* \nabla \alpha, \mathcal{B}(B^* \beta) \rangle d\mu. \quad (5.2)$$

where

$$\mathcal{B} = ((\nabla B^*)(B^*)^{-1})^t - (\nabla B^*)(B^*)^{-1} \quad (5.3)$$

We immediately get this first criterion of symmetry and non-negativeness.

**Proposition 5.1.** *If  $(\nabla B^*)(B^{-1})^*$  is symmetric with respect to  $\langle \cdot, \cdot \rangle$  then the generator  $-L_B^\parallel$  is symmetric with respect to  $\langle \cdot, \cdot \rangle_B$ , non-negative and we have:*

$$- \int_M \langle L_B^\parallel \alpha, \beta \rangle_B d\mu = \int_M \langle \nabla \alpha, \nabla \beta \rangle_B d\mu \quad (5.4)$$

The operators  $L^{W,B}$  and  $\mathcal{M}_B$  are symmetric with respect to  $\langle \cdot, \cdot \rangle_B$ . We have

$$L_B^\parallel = L^{W,B} + \mathcal{M}_B - (B^{-1})^* L^\parallel (B^*). \quad (5.5)$$

So a necessary and sufficient condition for the  $B$ -symmetry of  $L_B^\parallel$  is the  $B$ -symmetry of  $(B^{-1})^* L^\parallel (B^*)$ . But unlike the condition of proposition 5.1, this is not a sufficient condition for positiveness. For example, one can look at  $(\mathbb{R}_+^*)^2$  with the potential  $V(x, y) = x + y$  and the twist

$$B^* = \begin{pmatrix} \varphi & \varphi \\ 1 & e^V \end{pmatrix},$$

where  $\varphi$  is positive such that  $L\varphi \neq 0$ . The associated  $L_B^\parallel$  is symmetric but is not non-negative.

The following result is immediate and gives examples satisfying the condition of proposition 5.1:

**Proposition 5.2.** *If  $B(x) = \lambda(x) \text{id}_{T_x M}$  for some smooth positive function  $\lambda$  then  $(\nabla B^*)(B^{-1})^*$  is symmetric with respect to  $\langle \cdot, \cdot \rangle$*

*Proof.* If  $B(x) = \lambda(x) \text{id}_{T_x M}$  then  $B^*(x) = \lambda(x) \text{id}_{T_x^* M}$ ,  $\nabla B^* = d\lambda \otimes \text{id}_{T_x^* M}$  and  $(\nabla B^*)(B^{-1})^* = \lambda^{-1} d\lambda \otimes \text{id}_{T_x^* M}$ . It is clearly symmetric.  $\square$

## 6. INTERTWINING: A SYMMETRIC POSITIVE CASE

In this section, we assume that  $\mathcal{B} = 0$ . According to Proposition 5.1,  $-L_B^{\parallel}$  is symmetric, non-negative, with respect to  $\langle \cdot, \cdot \rangle_B$ . As  $L^{W,B}$  is symmetric with respect to this metric, then  $M_B$ , defined in (4.16), is symmetric too. We denote by  $\rho_B$  the infimum over  $M$  of the smallest eigenvalue of  $B^* M_B (B^*)^{-1}$ :

$$\rho_B = \inf_{x \in M} \left\{ \text{smallest eigenvalue of } B^* M_B (B^*)^{-1} \right\}. \quad (6.1)$$

We also assume that  $\rho_B$  is bounded from below. As we already said, the generator  $L^{W,B}$  is essentially self-adjoint. With this new assumption,  $L^{W,B}$  is the sum of a symmetric non-negative operator  $L_B^{\parallel}$  and a bounded potential  $M_B$ . So we could obtain a new proof of the essential self-adjointness as a generalization of proof of Strichartz in [20]. In order to obtain the intertwining relation, we need to show that  $(B^*)^{-1} d\mathbf{P}_t f$  is the unique  $L^2$  strong solution to the Cauchy problem

$$\begin{cases} \partial_t F = L^{W,B} F \\ F(\cdot, 0) = G \in L^2(B, \mu) \end{cases}$$

where the mapping  $t \mapsto F(\cdot, t)$  is continuous from  $\mathbb{R}_+$  to  $L^2(B, \mu)$ . Remark that we are looking for a strong solution. Actually, as we do not know the domain of  $L^{W,B}$ , we cannot use the uniqueness in the sense of self-adjoint operator.

**Proposition 6.1.** *Assume that  $\mathcal{B} = 0$  and that  $M_B$  is uniformly bounded from below. Let  $F$  be a solution of the  $L^2$  Cauchy problem above. Then, we have*

$$F(\cdot, t) = Q_t^B(G), t \geq 0.$$

*Proof.* We generalize the argument of [17] and [1] which deal respectively with the case of a Laplacian in a Riemannian manifold and the case of our operator  $L^{W,B}$  in  $\mathbb{R}^n$ . By linearity, it is sufficient to show the uniqueness of the solution for the zero initial condition. Replacing the solution  $F$  by  $e^{-\rho_B t} F$ , let assume that  $M_B$  is positive semi-definite. For every  $\phi \in \mathcal{C}_c^\infty(M)$  and  $\tau > 0$ , we have:

$$\begin{aligned} \int_0^\tau \int_M \phi^2 \langle F, L_B^{\parallel} F \rangle_B d\mu dt &= \int_0^\tau \int_M \phi^2 \langle F, (L^{W,B} + M_B) F \rangle_B d\mu dt \\ &= \int_0^\tau \int_M \phi^2 \frac{1}{2} \partial_t |F|_B^2 d\mu dt + \int_0^\tau \int_M \phi^2 \langle F, M_B F \rangle_B d\mu dt \\ &\geq \int_M \phi^2 \frac{1}{2} |F(\cdot, \tau)|_B^2 d\mu. \end{aligned}$$

On the other hand, by the integration by parts formula of proposition 5.1, we have

$$\begin{aligned} \int_0^\tau \int_M \phi^2 \langle F, L_B^\parallel F \rangle_B d\mu dt &= - \int_0^\tau \int_M \langle \nabla(\phi^2 F), \nabla F \rangle_B d\mu dt \\ &= - \int_0^\tau \int_M \phi^2 |\nabla F|_B^2 d\mu dt \\ &\quad - 2 \int_0^\tau \int_M \phi \langle \nabla \phi \otimes F, \nabla F \rangle_B d\mu dt. \end{aligned}$$

By Cauchy-Schwarz inequality, we have for every  $\lambda > 0$ ,

$$2 |\langle \nabla \phi \otimes F, \nabla F \rangle_B| \leq \lambda |\nabla \phi|_2^2 |F|_B^2 + \frac{1}{\lambda} \phi^2 |\nabla F|_B^2. \quad (6.2)$$

Combining the above inequalities, in the particular case of  $\lambda = 2$ , we obtain

$$\begin{aligned} \frac{1}{2} \int_M \phi^2 |F(\cdot, \tau)|_B^2 d\mu &\leq - \frac{1}{2} \int_0^\tau \int_M \phi^2 |\nabla F|_B^2 d\mu dt \\ &\quad + 2 \int_0^\tau \int_M |\nabla \phi|_2^2 |F|_B^2 d\mu dt. \end{aligned}$$

By completeness of  $M$ , there exists a sequence of cut-off functions  $(\phi_n)_{n \in \mathbb{N}}$  in  $\mathcal{C}_c^\infty(M)$  such that  $(\phi_n)_n$  converge to 1 pointwise and  $\|\nabla \phi_n\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ . Plugging this sequence in the previous inequality, gives

$$\int_M |F(\cdot, \tau)|_B^2 d\mu = 0, \tau > 0. \quad (6.3)$$

Hence  $F = 0$  in  $\mathcal{C}^0(\mathbb{R}_+, L^2(B, \mu))$ .  $\square$

**Theorem 6.2.** *Assume that  $\mathcal{B} = 0$  and that  $M_B$  is uniformly bounded from below. Then the semi-groups  $\mathbf{P}$  and  $\mathbf{Q}^B$  are intertwined by  $(B^{-1})^*d$ , i.e for every  $f \in \mathcal{C}_c^\infty(M)$*

$$(B^{-1})^*d\mathbf{P}_t f = \mathbf{Q}_t^B((B^{-1})^*df), t \geq 0.$$

*Proof.* The main argument is to prove that  $F(\cdot, t) = (B^{-1})^*d\mathbf{P}_t f$  is a solution of the previous  $L^2$  Cauchy problem with initial condition  $G = (B^{-1})^*df$ . First,  $G$  is in  $L^2(B, \mu)$  since  $f$  is compactly supported. For every  $t > 0$ , we have:

$$\begin{aligned} \int_M |F(\cdot, t)|_B^2 d\mu &= \int_M |(B^{-1})^*d\mathbf{P}_t f|_B^2 d\mu \\ &= \int_M |d\mathbf{P}_t f|^2 d\mu \\ &= - \int_M \mathbf{P}_t f L \mathbf{P}_t f d\mu, \end{aligned}$$

which is finite since  $\mathbf{P}_t f \in \mathcal{D}(L) \subset L^2(\mu)$ . So  $F(\cdot, t)$  is in  $L^2(B, \mu)$  for every  $t > 0$ . Besides, the  $L^2$  continuity is proven by the same calculus, since for every  $t, s \geq 0$ ,

$$\int_M |F(\cdot, t) - F(\cdot, s)|_B^2 d\mu = - \int_M (\mathbf{P}_t f - \mathbf{P}_s f) L (\mathbf{P}_t f - \mathbf{P}_s f) d\mu. \quad (6.4)$$

By spectral theorem, this is upper bounded by  $(\sup_{x \in \mathbb{R}_+} |\sqrt{x}(e^{-tx} - e^{-sx})|)^2 \|f\|_2^2$  which tends to zero as  $s$  tends to  $t > 0$  (see [19] for more details on spectral theorem). For the right-continuity in  $t = 0$ , we use that

$$\int_M (\mathbf{P}_s f - f) L (\mathbf{P}_s f - f) d\mu = \int_0^s \int_0^s \int_M \mathbf{P}_t L f \mathbf{P}_u L^2 f d\mu du dt. \quad (6.5)$$

Finally, the commutation property 4.11, yields

$$\partial_t F = (B^{-1})^* dL \mathbf{P}_t f = L^{W,B} ((B^{-1})^* d\mathbf{P}_t f) = L^{W,B} F.$$

The result follow by the uniqueness of the solution of the Cauchy problem.  $\square$

Now, we are able to give a covariance representation, as in theorem 3.4, using the semi-group  $\mathbf{Q}_t^B$ : for all  $f, g \in \mathcal{C}_c^\infty(M)$ ,

$$\text{Cov}_\mu(f, g) = \int_0^{+\infty} \left( \int_M \langle (B^{-1})^* df, \mathbf{Q}_t^B((B^{-1})^* dg) \rangle_B d\mu \right) dt. \quad (6.6)$$

The main result is a generalization of an inequality due to Brascamp and Lieb, in [7], known as Brascamp-Lieb inequality.

**Theorem 6.3.** *Assume that  $\mathcal{B} = 0$  and that  $M_B$  is positive definite, then for every  $f \in \mathcal{C}_0^\infty(M)$  we have the generalized Brascamp-Lieb inequality:*

$$\text{Var}_\mu(f) \leq \int_M \langle df, (B^* M_B (B^*)^{-1} df) \rangle d\mu.$$

Firstly, we need a little lemma.

**Lemma 6.4.** *Let  $C$  and  $D$  be symmetric non-negative operators such that  $D$  and  $C + D$  are invertible. Then we have*

$$0 \leq D^{-1} - (C + D)^{-1}.$$

*Proof.* We have:

$$D^{-1} - (C + D)^{-1} = (C + D)^{-1} C D^{-1},$$

and we have

$$\langle (C + D)^{-1} C D^{-1} \alpha, \alpha \rangle = \langle C D^{-1} \alpha, (C + D)^{-1} \alpha \rangle.$$

Letting  $(C + D)^{-1} \alpha = \beta$  this equals

$$\begin{aligned} \langle C D^{-1} (C + D) \beta, \beta \rangle &= \langle C D^{-1} C \beta, \beta \rangle + \langle C \beta, \beta \rangle \\ &= \langle D^{-1} C \beta, C \beta \rangle + \langle C \beta, \beta \rangle \geq 0, \end{aligned}$$

since  $D^{-1}$  and  $C$  are non-negative.  $\square$

*Proof.* First, let assume that  $\rho_B$  is positive. This implies that for all 1-form  $\alpha$ , we have

$$\int_M \langle (-L^{W,B})\alpha, \alpha \rangle_B d\mu \geq \rho_B \int_M |\alpha|_B^2 d\mu. \quad (6.7)$$

So  $-L^{W,B}$  is essentially self-adjoint and bounded from below by  $\rho_B \text{id}$ . Then it is invertible in  $L^2(B, \mu)$  i.e given any smooth compactly supported 1-form  $\alpha$ , the Poisson equation  $-L^{W,B}\beta = \alpha$  admits a unique solution  $\beta$  in the domain of  $L^{W,B}$  which has the following integral representation:

$$\beta = \int_0^{+\infty} \mathbf{Q}_t^B(\alpha) dt = (-L^{W,B})^{-1}\alpha. \quad (6.8)$$

Using the variance representation formula (6.6), we have

$$\begin{aligned} \text{Var}_\mu(f) &= \int_0^\infty \left( \int_M \langle (B^*)^{-1}df, \mathbf{Q}_t^B((B^*)^{-1}df) \rangle_B d\mu \right) dt \\ &= \int_M \left\langle (B^*)^{-1}df, \int_0^\infty \mathbf{Q}_t^B((B^*)^{-1}df) dt \right\rangle_B d\mu \\ &= \int_M \langle (B^*)^{-1}df, (-L^{W,B})^{-1}((B^*)^{-1}df) \rangle_B d\mu \\ &= \int_M \left\langle (B^*)^{-1}df, (-L_B^\parallel + M_B)^{-1}((B^*)^{-1}df) \right\rangle_B d\mu \end{aligned}$$

Using the lemma 6.1 to  $C = -L_B^\parallel$  and  $D = M_B$ , we have:

$$\begin{aligned} \text{Var}_\mu(f) &\leq \int_M \langle (B^*)^{-1}df, M_B^{-1}((B^*)^{-1}df) \rangle_B d\mu \\ &\leq \int_M \left\langle df, (B^*M_B(B^*)^{-1})^{-1}df \right\rangle d\mu \end{aligned}$$

Now, when the operator  $M_B$  is not uniformly bounded from below by a positive constant, we need to regularize. For all  $\varepsilon > 0$ , the operator  $\varepsilon \text{id} - L^{W,B}$  is invertible and we have the following integral representation for all 1-form  $\alpha$ :

$$(\varepsilon \text{id} - L^{W,B})^{-1}\alpha = \int_0^{+\infty} e^{-\varepsilon t} \mathbf{Q}_t^B \alpha dt. \quad (6.9)$$

Similarly,  $(\varepsilon \text{id} - L)$  is invertible on the sub-space of centred functions and we have the integral representation for all centred  $f \in \mathcal{C}_c^\infty(M)$ :

$$(\varepsilon \text{id} - L)^{-1}f = \int_0^{+\infty} e^{-\varepsilon t} \mathbf{P}_t f dt := g_\varepsilon. \quad (6.10)$$

We have:

$$\text{Var}_\mu(f) = \int_M f^2 d\mu$$

$$\begin{aligned}
&= \int_M f(\varepsilon id - L)g_\varepsilon d\mu \\
&= \varepsilon \int_M f g_\varepsilon d\mu + \int_M f(-L) \left( \int_0^{+\infty} e^{-\varepsilon t} \mathbf{P}_t f dt \right) d\mu \\
&= \varepsilon \int_M f g_\varepsilon d\mu + \int_0^{+\infty} e^{-\varepsilon t} \int_M f(-L) \mathbf{P}_t f d\mu dt \\
&= \varepsilon \int_M f g_\varepsilon d\mu + \int_0^{+\infty} e^{-\varepsilon t} \int_M \langle (B^*)^{-1} df, \mathbf{Q}_t^B((B^*)^{-1} df) \rangle_B d\mu dt \\
&= \varepsilon \int_M f g_\varepsilon d\mu + \int_M \left\langle (B^*)^{-1} df, \int_0^{+\infty} e^{-\varepsilon t} \mathbf{Q}_t^B((B^*)^{-1} df) dt \right\rangle_B d\mu \\
&= \varepsilon \int_M f g_\varepsilon d\mu + \int_M \langle (B^*)^{-1} df, (\varepsilon id - L^{W,B})^{-1}((B^*)^{-1} df) \rangle_B d\mu.
\end{aligned}$$

We can apply the lemma to  $\varepsilon id - L^{W,B} = \varepsilon id - L_B^{\parallel} + M_B$ . We have:

$$\text{Var}_\mu(f) \leq \varepsilon \|f\|_{\mathbb{L}^2(\mu)} \|g_\varepsilon\|_{\mathbb{L}^2(\mu)} + \int_M \langle (B^*)^{-1} df, (M_B)^{-1}((B^*)^{-1} df) \rangle_B d\mu.$$

Finally, we have

$$\varepsilon \|g_\varepsilon\|_{\mathbb{L}^2(\mu)} = \left\| \int_0^{+\infty} e^{-t} \mathbf{P}_{t/\varepsilon} f dt \right\|_{\mathbb{L}^2(\mu)} \leq \int_0^{+\infty} \int_M e^{-t} (\mathbf{P}_{t/\varepsilon} f)^2 d\mu dt.$$

By ergodicity of  $\mathbf{P}$  and dominated convergence, this term converges to 0 as  $\varepsilon \rightarrow 0$ . This ends the proof.  $\square$

An immediate corollary of this theorem is the Poincaré inequality.

**Theorem 6.5.** *Assuming that  $\mathcal{B} = 0$  and that  $\rho_B$  is positive, for all  $f \in \mathcal{C}_c^\infty(M)$ , we have*

$$\text{Var}_\mu(f) \leq \frac{1}{\rho_B} \int_M |df|^2 d\mu,$$

In the case where  $M_B$  is only positive and not uniformly bounded from below (i.e  $\rho_B = 0$ ), this inequality is trivially true. Let us give an alternative proof which does not use the generalized Brascamp-Lieb inequality, and thus, avoids the inversion of  $L^{W,B}$  and its integral representation.

*Proof.* Using a time change and the symmetry of the semi-group  $\mathbf{Q}^B$ , we have a new representation of the variance:

$$\begin{aligned}
\text{Var}_\mu(f) &= 2 \int_0^\infty \left( \int_M \langle (B^*)^{-1} df, \mathbf{Q}_{2t}^B((B^*)^{-1} df) \rangle_B d\mu \right) dt \\
&= 2 \int_0^\infty \left( \int_M |\mathbf{Q}_t^B((B^*)^{-1} df)|_B^2 d\mu \right) dt.
\end{aligned}$$



Let

$$\phi(t) = \int_M |\mathbf{Q}_t^B((B^*)^{-1}df)|_B^2 d\mu. \quad (6.11)$$

We have

$$\begin{aligned} \phi'(t) &= 2 \int_M \langle \mathbf{Q}_t^B((B^*)^{-1}df), L^{W,B} \mathbf{Q}_t^B((B^*)^{-1}df) \rangle_B d\mu \\ &= 2 \int_M \langle \mathbf{Q}_t^B((B^*)^{-1}df), (L_B^\parallel) \mathbf{Q}_t^B((B^*)^{-1}df) \rangle_B d\mu \\ &\quad - 2 \int_M \langle \mathbf{Q}_t^B((B^*)^{-1}df), M_B \mathbf{Q}_t^B((B^*)^{-1}df) \rangle_B d\mu \\ &= -2 \int_M |\nabla \mathbf{Q}_t^B((B^*)^{-1}df)|_B^2 d\mu \\ &\quad - 2 \int_M \langle \mathbf{Q}_t^B((B^*)^{-1}df), M_B \mathbf{Q}_t^B((B^*)^{-1}df) \rangle_B d\mu \\ &\leq -2 \int_M \langle \mathbf{Q}_t^B((B^*)^{-1}df), M_B \mathbf{Q}_t^B((B^*)^{-1}df) \rangle_B d\mu \\ &\leq -2\rho_B \phi(t) \end{aligned}$$

By Grönwall lemma, this implies

$$\phi(t) \leq e^{-2\rho_B t} \phi(0) = e^{-2\rho_B t} \int_M |df|^2 d\mu. \quad (6.12)$$

Integrating on  $\mathbb{R}_+$  ends the proof.  $\square$

We finish with an interpretation of the Poincaré inequality in terms of spectral gap.

**Proposition 6.6.** *Assume that  $\mathcal{B} = 0$  and that  $\rho_B$  is positive then the spectral gap  $\lambda_1(-L, \mu)$  satisfies*

$$\lambda_1(-L, \mu) \geq \rho_B \quad (6.13)$$

This is a generalization to Riemannian manifolds of the Chen and Wang formula established in the one dimensional case in [10]. This spectral gap gives an exponential rate of convergence to equilibrium to the ergodic semi-group  $\mathbf{P}$ .

## 7. INTERTWINING: GENERAL CASE

Except the strong condition of proposition 5.1, we do not have relevant conditions for  $L_B^\parallel$  to be symmetric non-negative. We are looking for a proof of the intertwining which does not use this symmetry. Actually, we can release all assumptions on the second order operator if we are ready to strengthen the conditions

on the potential  $M_B$ . In this case, the eigenvalue  $\rho_B$  is not a good criterion anymore. We need to find a quantity which offsets the lack of symmetry. For all 1-form  $\alpha$ , according to 5.3, we have:

$$\begin{aligned} \int_M \langle \alpha, L_B^\parallel \alpha \rangle_B d\mu &= - \int_M |B^* \nabla \alpha|^2 d\mu + \int_M \langle B^* \nabla \alpha, \mathcal{B}(B^* \alpha) \rangle d\mu \\ &= - \int_M \left| B^* \nabla \alpha - \frac{1}{2} \mathcal{B} B^* \alpha \right|^2 d\mu + \int_M \frac{1}{4} |\mathcal{B} B^* \alpha|^2 d\mu \\ &\leq \int_M \langle B^* \alpha, N_B B^* \alpha \rangle d\mu \end{aligned}$$

where

$$N_B(x) = \frac{1}{4} \mathcal{B}_x^t \cdot \mathcal{B}_x \in \text{End}(T_x^* M), \quad (7.1)$$

and

$$\mathcal{B} = ((\nabla B^*)(B^*)^{-1})^t - (\nabla B^*)(B^*)^{-1}. \quad (7.2)$$

Hence, we have:

$$\int_M \langle \alpha, (-L^{W,B}) \alpha \rangle_B d\mu \geq \int_M \langle B^* \alpha, [(B^* M_B (B^*)^{-1})^s - N_B] B^* \alpha \rangle d\mu. \quad (7.3)$$

where  $(B^* M_B (B^*)^{-1})^s$  is the symmetric part of  $B^* M_B (B^*)^{-1}$  with respect to the Riemannian metric. So the quantity we need to control seems to be the following:

$$\tilde{\rho}_B = \inf_{x \in M} \{ \text{smallest eigenvalue of } (B^* M_B (B^*)^{-1})^s - N_B \}. \quad (7.4)$$

First, as in the symmetric case, we show the intertwining relation.

**Theorem 7.1.** *Assume that  $(B^* M_B (B^*)^{-1})^s - (1 + \varepsilon) N_B$  is bounded from below for some  $\varepsilon > 0$ . Then the semi-groups  $\mathbf{P}$  and  $\mathbf{Q}^B$  are intertwined by  $(B^{-1})^* d$ , i.e for every  $f \in \mathcal{C}_c^\infty(M)$*

$$(B^{-1})^* d\mathbf{P}_t f = \mathbf{Q}_t^B ((B^{-1})^* df), \quad t \geq 0.$$

*Proof.* The core of the proof is still the uniqueness of the solution of the same  $L^2$  Cauchy problem. We assume that  $(B^* M_B (B^*)^{-1})^s - (1 + \varepsilon) N_B$  is non-negative without any loss of generality. Let  $F$  be a solution with the zero initial condition. For  $\phi \in \mathcal{C}_c^\infty$  and  $\tau > 0$ , as in the proof of proposition 6.1, we have

$$\int_0^\tau \int_M \phi^2 \langle F, (L_B^\parallel - (1 + \varepsilon)(B^{-1})^* N_B B^*) F \rangle_B d\mu dt \geq \int_M \phi^2 \frac{1}{2} |F(\cdot, \tau)|_B^2 d\mu. \quad (7.5)$$

On the other hand, according to the formula 5.2, we have

$$\begin{aligned} \int_M \phi^2 \langle F, L_B^\parallel F \rangle_B d\mu &= - \int_M \langle \nabla(\phi^2 F), \nabla F \rangle_B d\mu + \int_M \langle B^* \nabla F, \mathcal{B}(B^* \phi^2 F) \rangle d\mu \\ &= - \int_M \phi^2 |\nabla F|_B^2 d\mu + \int_M \phi^2 \langle B^* \nabla F, \mathcal{B}(B^* \phi^2 F) \rangle d\mu \end{aligned}$$

$$\begin{aligned}
& -2 \int_M \langle \nabla \phi \otimes F, \phi \nabla F \rangle_B d\mu \\
&= - \int_M \phi^2 |B^* \nabla F - \frac{1}{2} \mathcal{B} B^* F|^2 d\mu + \int_M \phi^2 \langle F, N_B F \rangle d\mu \\
& \quad -2 \int_M \langle \nabla \phi \otimes F, \phi (\nabla F - \frac{1}{2} (B^{-1})^* \mathcal{B} B^* F) \rangle_B d\mu \\
& \quad -2 \int_M \langle \nabla \phi \otimes F, \phi \frac{1}{2} (B^{-1})^* \mathcal{B} B^* F \rangle_B d\mu.
\end{aligned}$$

According to Cauchy-Schwarz inequality, for every  $\lambda, k > 0$ , we have:

$$\begin{aligned}
2|\langle \nabla \phi \otimes F, \phi (\nabla F - \frac{1}{2} (B^{-1})^* \mathcal{B} B^* F) \rangle_B| &\leq \lambda |\nabla \phi \otimes F|_B^2 + \frac{1}{\lambda} \phi^2 |B^* \nabla F - \frac{1}{2} \mathcal{B} B^* F|_2^2 \\
2|\langle \nabla \phi \otimes F, \phi \frac{1}{2} (B^{-1})^* \mathcal{B} B^* F \rangle_B| &\leq k |\nabla \phi \otimes F|_B^2 + \frac{1}{k} \phi^2 |\frac{1}{2} \mathcal{B} B^* F|_2^2
\end{aligned}$$

So, we have:

$$\begin{aligned}
\int_M \phi^2 \langle F, L_B^\# F \rangle_B d\mu &\leq \left( \frac{1}{\lambda} - 1 \right) \int_M \phi^2 |B^* \nabla F - \frac{1}{2} \mathcal{B} B^* F|^2 d\mu \\
&\quad + \left( 1 + \frac{1}{k} \right) \int_M \phi^2 \langle F, N_B F \rangle d\mu + (\lambda + k) \int_M |\nabla \phi|_2^2 |F|_B^2 d\mu.
\end{aligned}$$

Combining the above inequalities, we obtain that there exists a  $c > 0$  such that for every  $\phi \in \mathcal{C}_c^\infty(M)$ , and every  $\tau > 0$

$$\frac{1}{2} \int_M \phi^2 |F(\cdot, \tau)|_B^2 d\mu \leq c \int_0^\tau \int_M |\nabla \phi|_2^2 |F|_B^2 d\mu dt. \quad (7.6)$$

Using the same sequence of cut-off functions, we prove that  $F = 0$ . The end of the proof follows the proof of theorem 6.2 without any differences.  $\square$

Remark that under the condition of proposition 7.1,  $\tilde{\rho}_B$  is bounded from below, since  $N_B$  is non-negative. But unlike theorem 6.1, this proof requires a stronger condition. The intertwining relation of proposition 7.1 give an new covariance representation as in 6.6. This brings Brascamp-Lieb and Poincaré type inequalities.

**Theorem 7.2.** *Assume that  $(B^* M_B (B^*)^{-1})^s - (1 + \varepsilon) N_B$  is bounded from below for some  $\varepsilon > 0$  and that  $\tilde{\rho}_B$  is positive. Then for all  $f \in \mathcal{C}_c^\infty(M)$ , we have*

$$\text{Var}_\mu(f) \leq \frac{1}{\tilde{\rho}_B} \int_M |df|^2 d\mu,$$

*Proof.* Let  $f \in \mathcal{C}_c^\infty(M)$  and  $F_t = \mathbf{Q}_t^B ((B^*)^{-1}df)$ . As in 6.8, we set

$$\phi(t) = \int_M |F_t|_B^2 d\mu \quad (7.7)$$

and we have the following representation of the variance

$$\text{Var}_\mu(f) = \int_0^{+\infty} \phi(t) dt. \quad (7.8)$$

We have:

$$\begin{aligned} \phi'(t) &= 2 \int_M \langle F_t, L^{W,B} F_t \rangle_B dt \\ &\leq -2 \int_M \langle B^* F_t, [(B^* M_B (B^*)^{-1})^s - N_B] B^* F_t \rangle d\mu \\ &\leq -2\tilde{\rho}_B \phi(t) \end{aligned}$$

So we have

$$\phi(t) \leq e^{-2\tilde{\rho}_B t} \int_M |df|^2 d\mu. \quad (7.9)$$

Integrating on  $\mathbb{R}_+$  gives the results.  $\square$

As for the theorem 6.3, the result still make sense when  $\tilde{\rho}_B = 0$ . With the same kind of hypothesis, we can also prove a generalized Brascamp-Lieb inequality.

**Theorem 7.3.** *Assume that  $(B^* M_B (B^*)^{-1})^s - (1 + \varepsilon)N_B$  is bounded from below for some  $\varepsilon > 0$  and that  $(B^* M_B (B^*)^{-1})^s - N_B$  is positive definite, then for every  $f \in \mathcal{C}_c^\infty(M)$  we have the generalized Brascamp-Lieb inequality:*

$$\text{Var}_\mu(f) \leq \int_M \left\langle df, [(B^* M_B (B^*)^{-1})^s - N_B]^{-1} df \right\rangle d\mu. \quad (7.10)$$

*Proof.* First, let assume that  $\tilde{\rho}_B$  is positive. Equation 7.3 implies that for all 1-form  $\alpha$  we have:

$$\int_M \langle (-L^{W,B})\alpha, \alpha \rangle_B d\mu \geq \tilde{\rho}_B \int_M |\alpha|_B^2 d\mu. \quad (7.11)$$

As in the proof of theorem 6.3,  $-L^{W,B}$  is invertible with the same integral representation. So

$$\text{Var}_\mu(f) = \int_M \langle (B^*)^{-1}df, (-L^{W,B})^{-1}((B^*)^{-1}df) \rangle_B d\mu. \quad (7.12)$$

Furthermore, we have:

$$\begin{aligned} \int_M \langle \alpha, (-L^{W,B})\alpha \rangle_B d\mu &\geq \int_M \langle B^* \alpha, [(B^* M_B (B^*)^{-1})^s - N_B] B^* \alpha \rangle d\mu \\ &\geq \int_M \langle \alpha, (B^*)^{-1} [(B^* M_B (B^*)^{-1})^s - N_B] B^* \alpha \rangle_B d\mu. \end{aligned}$$

As  $(B^*)^{-1} [(B^* M_B (B^*)^{-1})^s - N_B] B^*$  is symmetric with respect to  $\langle \cdot, \cdot \rangle_B$  and positive by assumption, we can use the lemma 6.1 to obtain

$$\text{Var}_\mu(f) \leq \int_M \left\langle (B^*)^{-1} df, (B^*)^{-1} [(B^* M_B (B^*)^{-1})^s - N_B]^{-1} B^* ((B^*)^{-1} df) \right\rangle_B d\mu.$$

Now, if  $\tilde{\rho}_B = 0$ , we regularize as in the proof of theorem 6.3. It ends the proof.  $\square$

We also obtain a bound for the spectral gap.

**Proposition 7.4.** *Assume that  $(B^* M_B (B^*)^{-1})^s - (1+\varepsilon)N_B$  is bounded from below for some  $\varepsilon > 0$  and that  $\tilde{\rho}_B$  is positive. Then the spectral gap  $\lambda_1(-L, \mu)$  satisfies:*

$$\lambda_1(-L, \mu) \geq \tilde{\rho}_B. \quad (7.13)$$

Remark that if the hypothesis of proposition 5.1 are satisfied, then  $N_B = 0$  and  $B^* M_B (B^*)^{-1}$  is symmetric and so  $\rho_B = \tilde{\rho}_B$ . In particular, theorem 7.2 (respectively 7.3 and 7.4) can be applied to small perturbations of generators satisfying the conditions of theorem 6.5 (or 6.3 and 6.6) and the bounds obtained are stable with respect to perturbations.

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