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The Interaction Picture Method for solving the Generalized Non-Linear Schrödinger Equation in Optics

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The Generalized Non-Linear Schrödinger Equation

✗ In a model of light-wave propagation in an optical fibre, the evolution of the slowly varying pulse envelope A obeys the **Generalized Non-Linear Schrödinger Equation** (GNLSE)

$$\frac{\partial}{\partial z} A(z, t) = -\frac{\alpha}{2} A(z, t) + \left(\sum_{n=2}^{n_{\max}} i^{n+1} \frac{\beta_n}{n!} \frac{\partial^n}{\partial t^n} A(z, t) \right) + i\gamma \left(1 + \frac{i}{\omega_0} \frac{\partial}{\partial t} \right) \left[A(z, t) \left((1 - f_R) |A(z, t)|^2 + f_R \int_{-\infty}^{+\infty} h_R(s) |A(z, t-s)|^2 ds \right) \right] \quad (1)$$

taking into account phenomena such as

- ▶ **linear attenuation**
- ▶ **linear dispersion**
- ▶ **non linear effects**: non linear dispersion, instantaneous Kerr effect, delayed Raman effect

✗ The GNLSE is solved for the **initial condition** at $z = 0$

$$\forall t \in \mathbb{R} \quad A(0, t) = a_0(t) \quad (2)$$

where a_0 is a given function and for all $t \in \mathbb{R}$ and all $z \in [0, L]$ where L denotes the length of the fiber.

Theory behind the IP method : a change of unknown

✗ We introduce the **linear operator**

$$\mathcal{D} : A(z) \mapsto \frac{\alpha}{2} A(z) - \sum_{n=2}^{n_{\max}} \beta_n \frac{i^{n+1}}{n!} \partial_t^n A(z),$$

the **non-linear operator** (\star stands for the convolution product)

$$\mathcal{N} : A(z) \mapsto i\gamma \left(1 + \frac{i}{\omega_0} \frac{\partial}{\partial t} \right) \left[A(z) \left((1 - f_R) |A(z)|^2 + f_R (h_R \star |A(z)|^2) \right) \right]$$

and a **subdivision** $z_k, k \in \{0, \dots, K\}$ of $[0, L]$. We set $h_k = z_{k+1} - z_k$ and $z_{k+\frac{1}{2}} = z_k + \frac{h_k}{2}$.

✗ Solving (1)–(2) is equivalent to solving the **sequence of connected problems** $(\mathcal{P}_k)_{k=0, \dots, K-1}$ where (we set $A_{-1} = a_0$)

$$(\mathcal{P}_k) \quad \begin{cases} \frac{\partial}{\partial z} A_k(z) = \mathcal{D} A_k(z) + \mathcal{N}(A_k(z)) & \forall z \in [z_k, z_{k+1}] \\ A_k(z_k) = A_{k-1}(z_k) \end{cases}$$

✗ We introduce as **new unknown** the mapping

$$A_k^{\text{ip}} : (z, t) \in [z_k, z_{k+1}] \times \mathbb{R} \mapsto \exp(-(z - z_{k+\frac{1}{2}}) \mathcal{D}) A_k(z, t) \quad (3)$$

where $\exp((z - z_{k+\frac{1}{2}}) \mathcal{D})$ refers to the continuous group of bounded operators on $\mathbb{L}^2(\mathbb{R}, \mathbb{C})$ defined by \mathcal{D} [1].

✗ The unknown A_k^{ip} is solution to the following **ODE problem** over each subinterval $[z_k, z_{k+1}]$ where t acts as a parameter [1]

$$(\mathcal{Q}_k) \quad \begin{cases} \frac{\partial}{\partial z} A_k^{\text{ip}}(z) = \mathcal{G}_k(z, A_k^{\text{ip}}(z)) & \forall z \in [z_k, z_{k+1}] \\ A_k^{\text{ip}}(z_k) = \exp(-(z_k - z_{k+\frac{1}{2}}) \mathcal{D}) A_k(z_k) \end{cases}$$

where $\mathcal{G}_k(z, \cdot) = \exp(-(z - z_{k+\frac{1}{2}}) \mathcal{D}) \circ \mathcal{N} \circ \exp((z - z_{k+\frac{1}{2}}) \mathcal{D})$.

Implementation of the IP method

✗ Solving pb (\mathcal{P}_k) through pb (\mathcal{Q}_k) is done in **3 steps**:

1. Compute the initial data $A_k^{\text{ip}}(z_k) = \exp(-(z_k - z_{k+\frac{1}{2}}) \mathcal{D}) A_k(z_k)$ corresponding to the **change of unknown** (3)
2. Solve problem (\mathcal{Q}_k) for $A_k^{\text{ip}}(z)$
3. Compute $A_k(z_{k+1}) = \exp((z_k - z_{k+\frac{1}{2}}) \mathcal{D}) A_k^{\text{ip}}(z_{k+1})$ by the **inverse of mapping** (3)

✗ This is equivalent to solving the following **three nested problems**

$$\forall z \in [z_k, z_{k+\frac{1}{2}}] \quad \frac{\partial}{\partial z} A_k^+(z) = \mathcal{D} A_k^+(z), \quad A_k^+(z_k) = A_{k-1}(z_k), \quad (4)$$

where $A_{k-1}(z_k)$ is the solution to (\mathcal{P}_k) at node z_k computed at step $k - 1$,

$$\forall z \in [z_k, z_{k+1}] \quad \frac{\partial}{\partial z} A_k^{\text{ip}}(z) = \mathcal{G}_k(z, A_k^{\text{ip}}(z)), \quad A_k^{\text{ip}}(z_k, t) = A_k^+(z_{k+\frac{1}{2}}), \quad (5)$$

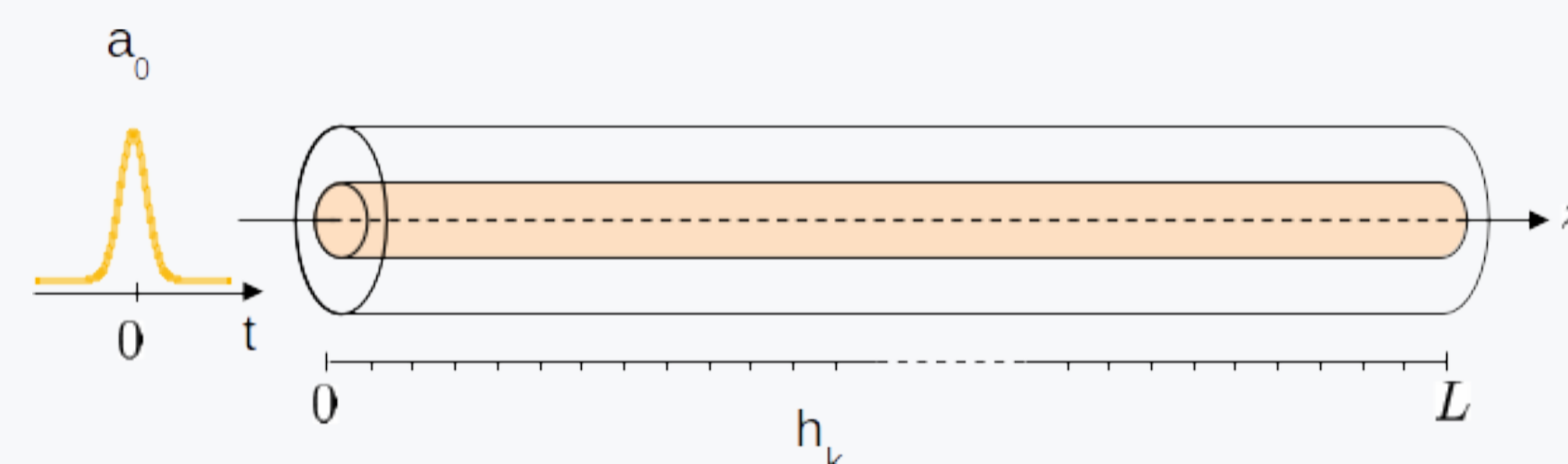
where $A_k^+(z_{k+\frac{1}{2}})$ is the solution to (4) at node $z_{k+\frac{1}{2}}$;

$$\forall z \in [z_{k+\frac{1}{2}}, z_{k+1}] \quad \frac{\partial}{\partial z} A_k^-(z) = \mathcal{D} A_k^-(z), \quad A_k^-(z_{k+\frac{1}{2}}) = A_k^{\text{ip}}(z_{k+\frac{1}{2}}), \quad (6)$$

where $A_k^{\text{ip}}(z_{k+1})$ represents the solution to (5) at node z_{k+1} .

→ The solution of (1) at grid point z_{k+1} is given by $A_k(z_{k+1}) = A_k^-(z_{k+1})$.

✗ Problems (4) and (6) are solved by **Fourier Transforms** whereas problem (5) is solved by the **4th order Runge-Kutta (RK4) method**.



Theoretical comparison to the Symmetric Split-Step Fourier method

✗ The **Symmetric Split-Step method** consists in solving over each subinterval $[z_k, z_{k+1}]$ for $k \in \{0, \dots, K - 1\}$, the following 3 nested problems:

$$\forall z \in [z_k, z_{k+\frac{1}{2}}] \quad \frac{\partial}{\partial z} A_k^+(z) = \mathcal{D} A_k^+(z), \quad A_k^+(z_k) = A_{k-1}(z_k), \quad (7)$$

where $A_{k-1}(z_k)$ is the solution at node z_k computed at step $k - 1$

$$\forall z \in [z_k, z_{k+1}] \quad \frac{\partial}{\partial z} B_k(z) = \mathcal{N}(B_k(z)), \quad B_k(z_k) = A_k^+(z_{k+\frac{1}{2}}), \quad (8)$$

where $A_k^+(z_{k+\frac{1}{2}}, t)$ is the solution to problem (7) at node $z_{k+\frac{1}{2}}$;

$$\forall z \in [z_{k+\frac{1}{2}}, z_{k+1}] \quad \frac{\partial}{\partial z} A_k^-(z) = \mathcal{D} A_k^-(z), \quad A_k^-(z_{k+\frac{1}{2}}) = B_k(z_{k+1}), \quad (9)$$

where $B_k(z_{k+1}, t)$ is the solution to problem (8) at node z_{k+1} .

→ The solution of (1) at grid node z_{k+1} is approx. by $A_k(z_{k+1}) = A_k^-(z_{k+1})$.

✗ In the IP method are solved the nested problems (4)–(5)–(6) instead of (7)–(8)–(9). **The only difference** is (5) replaced by (8).

→ It's very easy to **modify a program implementing the S3F method** to solve the GNLSE into a program implementing the IP method. **It suffices to change \mathcal{N} into \mathcal{G} in the RK4 solver for problem (8).**

Experimental Results and Comparison

✗ **Comparison of convergence order** : we have shown in [1] that the IP-RK4 method is 4th order accurate (RK4 error) whereas the S3F-RK4 method is 2nd order accurate (due to the use of Strang splitting formula).

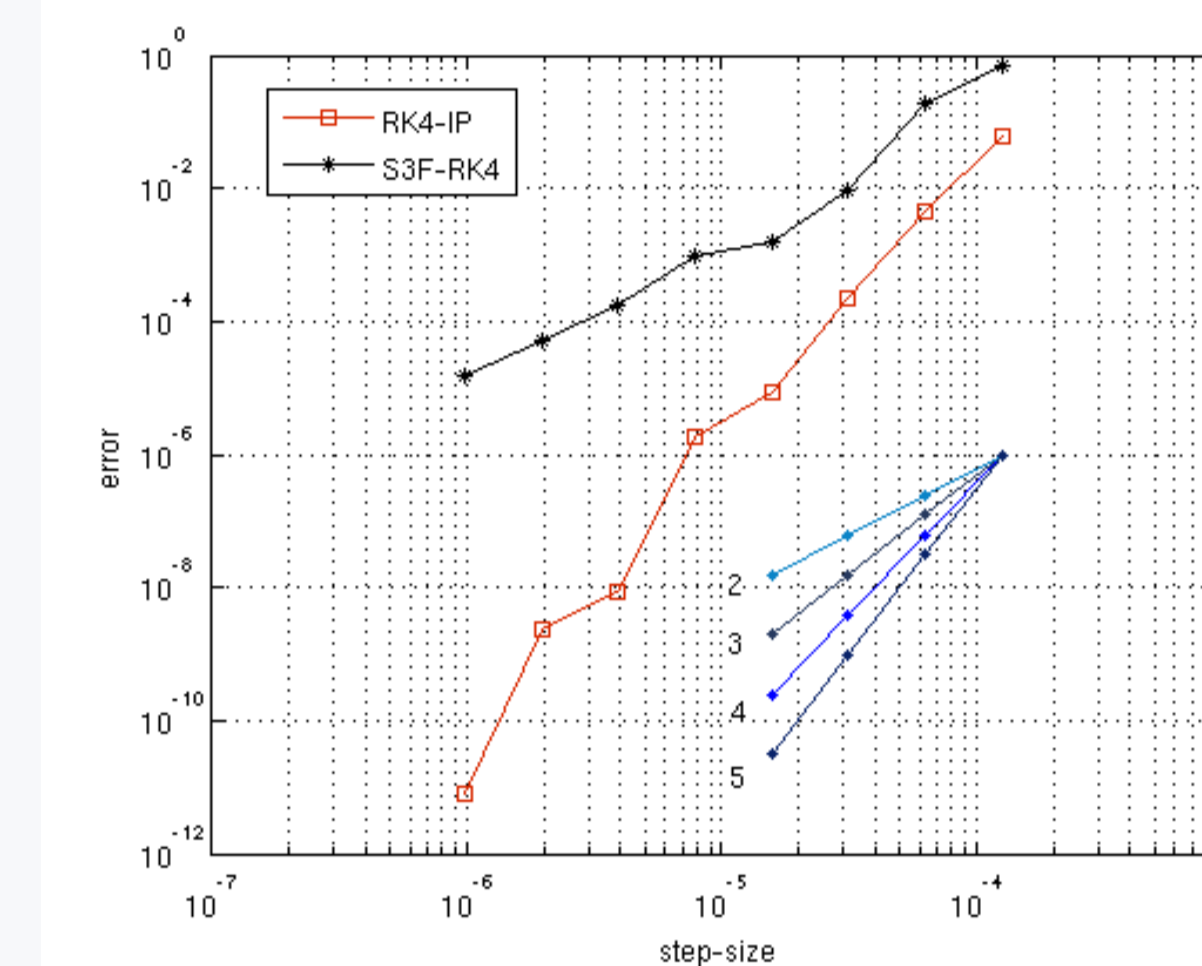


Figure : Experimental convergence curves for the IP-RK4 and S3F-RK4 methods. Quadratic relative error versus step size in logarithmic scale (see [1] for simulation details).

✗ **Comparison of CPU time and relative quadratic error**

$\|A(L) - A_{k-1}(L)\|_{\mathbb{L}^2} / \|A(L)\|_{\mathbb{L}^2}$ on a test example chosen to match with a typical case of high speed data propagation through a $L = 20$ km single mode fibre in optical telecommunication (see [1] for simulation details). Tests were achieved on a Intel Core i5-4200M with 8Go RAM.

Method	Step-size (m)	CPU time (s)	Relative quadratic error
S3F-RK4	100	1.48	$2.5582 \cdot 10^{-6}$
IP-RK4	100	1.42	$1.4957 \cdot 10^{-9}$
S3F-RK4	2.5	70.17	$1.5968 \cdot 10^{-9}$
S3F-RK4	10	14.49	$2.555 \cdot 10^{-8}$
IP-RK4	10	13.85	$4.6192 \cdot 10^{-13}$

Conclusion : Main features of the method

✗ **Accuracy** : the IP-RK4 method is 4th order accurate whereas the S3F-RK4 method is second order accurate (due to the use of Strang splitting formula).

✗ With the IP method, the use of an **adaptive step-size control** is straightforward [2] which is not the case for the S3F-RK4 method [3].

✗ The IP-RK4 method can be **easily implemented** by minor changes on a S3F-RK4 program.

References

- Papers can be downloaded from hal.archives-ouvertes.fr
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