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► **To cite this version:**

Paula Vieira da Silva, Leonor Santos. Construction of triangles: some misconceptions and difficulties. Eleventh Congress of the European Society for Research in Mathematics Education, Utrecht University, Feb 2019, Utrecht, Netherlands. <hal-02402268>

**HAL Id: hal-02402268**

**<https://hal.science/hal-02402268v1>**

Submitted on 10 Dec 2019

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# Construction of triangles: some misconceptions and difficulties

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*This study aims to analyze tasks that involve the construction of triangles and which are included in 6<sup>th</sup> grade national assessment tests in Portugal. More specifically, we analyze the mental processes used by the students, the difficulties experienced and the mistakes made in solving these tasks. The study follows a methodology of an interpretative nature, with data collection from the students' tests, the productions of the students and of interviews recorded in audio. In summary, the results show that, in linear measurement, many students demonstrated a quite acceptable level of sophistication. However, in the construction of angle the causes of their difficulties seem to be the absence of adequate structural components.*

*Keywords: Geometric thinking, construction of triangles, didactic material.*

## Introduction

In constructing a school curriculum it is necessary to make choices, and this need to choose is precisely what makes it so difficult to decide on a coherent and balanced curriculum throughout the period of learning in the school (Mammana & Villani, 1998). In Portugal, these choices are also visible in the emphasis given to certain contents in the 6th grade National Assessment Tests (NAT). In these tests, from 2010 to 2015, a task that involves the construction of triangles was always presented. In the Basic Education Mathematics Program (PMEB) of 2007, in the Program and Curricular Goals of Basic Mathematics Education of 2013 - PMCMEB, and in NAT, the construction of triangles has an obvious centrality. The characteristics of the tasks proposed in the NAT, overestimating certain curricular contents and certain mental processes, can induce the teaching methodologies used by the teachers and, consequently, influence the students' learning, and therefore, their analysis is of great importance. So, the main objective of this study is to analyze the 6<sup>th</sup> grade NAT tasks that involve the construction of triangles. In particular, we consider the following research questions: What mental processes were used by students? What difficulties have been experienced and what mistakes have been made in solving these tasks?

## Theoretical framework

A very particular aspect of the study of geometry is the construction of triangles. The triangle is the simplest flat figure. It comprises, with only three sides, three angles and three vertices, a defined part of the plane. Its simplicity makes it a basic tool for decomposing more complex figures and making measurements, and is therefore considered the basic mosaic of geometry (León & Timón, 2017).

In the process of constructing triangles, the abstraction of the concept of angle and of measurement are essential, since some of the errors produced by the students in the resolution of tasks that involve the construction of triangles are related to the construction of angles given a certain amplitude and with the measurement of line segments.

When constructing triangles with ruler, compass and protractor, various concepts and properties should be used. The previously studied concepts and properties must be related and applied at the time of construction. In the various stages involved in the construction of the

triangle, different difficulties can arise if the properties of the triangles are not taken into account in their construction (Zamagni & Crespo, 2016). The students' difficulties in solving tasks involving the construction of two-dimensional representations of triangles are sometimes also due to the lack of experience and skills in the use of the didactic materials needed for this construction (Van de Walle, 2009). These difficulties are also due to the appeal to their properties and the fact that visualization plays an important role in the decoding and transformation of mathematical information.

### **Understand the concept of measurement**

The concept of measurement is built along the construction of the geometric conceptualization, reasoning and application (Battista, 2007). Based on other authors, Battista (2007) considers that despite the importance of geometric measurement, the performance of some students in measurement tasks is low. That is, they do not adequately establish the connection between numerical measurements and the iteration process of unitary measures. For example, students who incorrectly measure an object's length when one of its edges is not zero-aligned in a ruler do not conceptualize clearly how the numerical ruler marks indicate the iteration of the lengths of the unit. According to this same author, many traditional curricula prematurely teach numerical procedures for geometric measurement, students have little opportunity to think about the adequacy of the numerical procedures they apply and do not have sufficient opportunities to develop skills with spatially structuring units of measurement.

Joram, Subrahmanyam, and Gelman (1998) argue that although younger students can learn some simple measurement skills, real physical measurement, such as determining lengths, for example with physical objects serving as a unit of measure, is a challenge for many, which may mean that students' knowledge of linear measures may be more superficial than it seems.

Battista (2007) integrating investigations that describe levels of sophistication in the development of concepts of measurement of length, developed a characterization of the construction of meaning of the students. In this characterization, there are two fundamentally different types of reasoning about length, *reasoning without measurement* that does not use numbers (involves visual-spatial inferences based on direct or indirect comparisons, imagined transformations of geometric properties) and *measurement reasoning* (involves iteration of units of length, that is, the determination of the number of units of fixed length that fit end to end along the object, without gaps or overlaps). Measurement reasoning includes not only the measurement process, but the reasoning about numerical measures (e.g., adding lengths to find the perimeter of a polygon, making inferences about length measurements based on the properties of the figures). Although students usually develop strategies without measurement before measurement strategies, reasoning without measurement continues to develop in sophistication even after the reasoning of the measurement appears. In addition, a more sophisticated reasoning about length involves the integration of reasoning without measurement and measurement.

The research of several authors suggests that students construct a meaningful understanding of the measure of length as they abstract and reflect on the process of iteration of units of length (Battista, 2007).

## **Understanding the concept of angle**

Students have great difficulty in learning the concept of angle (Clements & Battista, 1992). Angle is considered to be a multifaceted concept (Douek, 1998), since there is a great variety of definitions presented in school textbooks and textbooks for teacher training (Mitchelmore & White, 2000). Some authors consider that the angle can be defined based on three different aspects: as a quantity of rotation between two lines meet at a point (turn), as a union of two rays with a common extreme point (ray) and as the intersection of two half-planes (region) (Mitchelmore & White, 2000; Devichi & Munier, 2013). This concept is constructed slowly and progressively (Lehrer, Jenkins, & Osana, 1998; Mitchelmore, 1998). White and Mitchelmore (2003) consider that it is clear from the research literature that students have great difficulty in coordinating the various facets of the angle concept. In addition, several authors have observed that students also have different angle conceptions (Mitchelmore & White, 2000).

The process of constructing the concept of angle crosses several obstacles (Mitchelmore, 1998), and the difficulties that students encounter in learning angles and the mistakes made were observed in several experimental studies (Berthelot & Salin, 1996; Mitchelmore, 1998; Mitchelmore & White, 2000; Devichi & Munier, 2013). Berthelot & Salin (1996) report, based on a Close study (1982), that when students have to compare pairs of angles with the same amplitude but with different lengths of sides, many students respond that the larger angle is the one that has one or both sides longer, without taking into consideration the space between them. These studies reveal other misconceptions of some students because they consider that one side of the angle must always be horizontal and the direction of the angular aperture must always be counterclockwise, or that the angle is a sector of a circle.

Students face some difficulties in using a protractor, which have been emphasized in the literature. Sometimes students align the base of the protractor's body along one side of the angle, rather than the protractor's own reference line; or they do not place the origin of the protractor on the vertex of the angle to be measured. They can also read the angle measure clockwise when they should read counterclockwise or vice versa (Tanguay, 2012). Mitchelmore and White (2000) have suggested that the difficulty students have in learning to use a protractor may result from the fact that in a protractor several lines may be chosen for the initial side of an angle but the terminal side must be imagined. On the other hand, the cause of the students' difficulties seems to be the absence of structural angular components, which leads to failure to establish adequate structural mappings.

## **Methodology**

The present study follows a qualitative interpretive methodology (Bogdan & Biklen, 1997), since it privileges the interpretation of reality according to the perspective of the actors. In total, 171 students who completed the 6<sup>th</sup> grade participated in the study. The selection of these students was intentional and took into account their performance levels. The fact that students expressed different levels of performance permitted us to access a greater diversity of resolution strategies and, consequently, different mental processes and errors. The tasks were solved after the final internal assessment and in school context, which greatly reduces the probability of interfering factors in the study.

The data collection was carried out in a school context between 2012 and 2015 and used: (i) the collection of documents (tasks of NAT and resolutions of these tasks by the students); and (ii) semi-structured interviews to students recorded on video. The interviews were conducted the day after the task was resolved. Each of the students interviewed had the sheet with the resolution produced by them and, after reading each question, explained how they arrived at the answer.

The selection of the 35 students interviewed was made according to the assumptions of the students' selection for the study.

The analysis of the data for this study initially involved the organization of the information obtained by the written productions of the students and the interviews carried out. Secondly, content analysis was used with pre-defined categories: the mental processes used by the students, the difficulties experienced and the mistakes made.

## Results

Of the five tasks of the NAT of the 6th grade between 2010 and 2015, which involve the construction of triangles, two were chosen (one of the test of 2010 and another of 2012). These two tasks have in common the fact that the required final product is the construction of a triangle, but the method of construction of the triangles is different.

### Task 20 of the 2010 NAT

*The perimeter of an equilateral triangle is 18 cm.  
Construct this triangle in the box below. Shows the calculations you make. Make your construction using pencil, ruler and compass. Do not erase the lines that you do with the compass.*

Figure 1. Task 20

In this task (figure 1), students should know that: in an equilateral triangle the length of the three sides is equal; the perimeter must be divided by the number of sides, obtaining the length of each side. After this procedure, the student can construct the equilateral triangle, proceeding as follows: draw a line segment of 6 cm; draw the circumference arcs at a distance of 6 cm from each end of the line segment until they intersect, marking the point of intersection of the two arcs; draw the two line segments that join the ends of the line segment and the point obtained by intercepting the two lines. The student is not required to mark the vertex designation of the triangle.

Of the 171 students who were asked to solve this task, 106 (62%) correctly constructed the triangle. Of these 106 students, 80 (47%) recorded the calculation of the length of the sides of the triangle. The others 26 (15%) did not register the calculations made, as it was requested in the task. However, some of these students wrote 6 cm on the sides of the triangle. Of the 35 students interviewed, all of them performed the division and correctly constructed the requested triangle. Only the student S1 did not construct the triangle with ruler and compass, instead, he used the protractor, as he explains: "First, I divided 18 by 3 and gave me 6. So, since an equilateral triangle has all angles equal to  $60^\circ$ , I drew a 6 cm line segment and placed the protractor here [the student points to the left end of the line segment he drew] and traced an angle of  $60^\circ$ . So I drew a 6 cm line segment. Here [the student points to the other end of

the initial line segment] I did the same thing and joined the line segments.” When asked if instead of constructing the triangle with ruler and protractor using the amplitude of the inner angles could be constructed with ruler and compass, the student replied "Yes." He was also asked why he did not and the student answered: “Because I just remembered this method”.

Of the 65 students who did not answer correctly, 19 (11%) did not register any work. With the exception of S21, these students did not belong to the group of students interviewed and, therefore, it was not possible to know the reason why they did not solve the task. Student S21 justified himself saying that he had no ruler or compass. However, after the interviewer asked him to reread the question, the student explained step-by-step what he should do to solve the task correctly. The interviewer asked why he did not record the calculations to get the length of each side of the triangle and the student replied that he thought he should draw only the triangle. Although this student knew how to solve the first part of the task, for which the drawing material was not necessary, and to be written in the question "Show the calculations you make", did not make any record.

Of the 46 (27%) students who attempted to solve the task, but were unable to construct the requested triangle, 7 recorded the calculations required to obtain the length of each side of the triangle. It seems that these students understand that in an equilateral triangle the length of its three sides is equal and that the perimeter of the triangle is equal to the sum of the length of its sides. However, when they went to construct the triangle they used other lengths. For example, one student used the lengths 6.5 cm, 5.5 cm and 5.5 cm and records 6 cm on each side of the triangle. In this case there seems to have been some problem with measuring the lengths. The line segment measured with the ruler is 6.5 cm and the other two, which were obtained with the opening of the compass, are 5.5 cm. 10 constructed equilateral triangles whose sum of the lengths of the sides is not 18 cm, for example, equilateral triangles with 4.5 cm on each side. Finally, 6 constructed non-equilateral triangles with perimeter of 18 cm. Other constructions were found in which there is no evidence of the use of the compass. In these constructions, it seems to us that students try to construct the triangle with only the ruler. This strategy is used when students cannot remember the method of construction with the ruler and compass.

### Task 12 of the NAT 2012

*Constructs the triangle [ABC] that obeys the following conditions:*

- $\overline{AB} = 6 \text{ cm}$
- $B\hat{A}C = 30^\circ$
- $C\hat{B}A = 120^\circ$

*Use the appropriate drawing material.*

Figure 2. Task 12

To solve the task 12 (Figure 2), first, students should draw [AB] and mark points A and B at their ends. Then, with the protractor, they should draw at A an angle of  $30^\circ$  and at B an angle of  $120^\circ$ . When they extend the sides of the angles they find the point of intersection, that is C.

This task was solved by 126 students, having correctly constructed the requested triangle only 40 (32%). Of the other 86 students, 30 (24%) students followed all the steps necessary to

construct the triangle, but when they tried to draw the angle of  $120^\circ$  they made incorrectly. For example, student S3 manifested this difficulty, but was able to recognize his own error:

I: Do you think that the measure of this angle is  $120^\circ$ ? [indicating the  $60^\circ$  angle the student had drawn]

S3: No!!

I: Why?

S3: Because it does not reach  $90^\circ$ . [Marking the sides of an angle with  $90^\circ$  at point B] Because I measured from here to here. [indicating the counterclockwise]

I: But if angle that is requested is  $120^\circ$ , and is it greater than...?

S3:  $90^\circ$ !

I: Did not you see that this [the angle of  $60^\circ$  drawn by the student] was less than  $90^\circ$ ? So, how did you draw the  $120^\circ$  angle?

S3: Should start from  $0^\circ$  and count from left to right" [clockwise]

Other type of difficulty in using the protractor was to place the point of reference of the protractor at the point that will be the vertex of the angle and  $0^\circ$  in the extension of the side of the angle. The student S35, after having traced [AB] correctly, cannot explain how to proceed.

I: The angles of your triangle measure all  $45^\circ$  and the sides measure 6 cm. Here are the arcs made by the compass? Didn't you see the two angles of the triangle in the data?

S35: Yes!

I: This point marked in the middle of [AB] for what did it serve?

S35: To place the protractor.

I: And how do you measure the angles from that point?

S35: I didn't know!

Of the 86 students who didn't build the requested triangle, 11 (about 9%) didn't do any work, nor even traced [AB] and 17 (about 13.5%) couldn't geometrically represent the straight segment with the requested measure.

## Discussion

The difficulty in understanding the construction of triangles (on a sheet of paper) begins first by understanding the concept of angle and the concept of measurement, then goes through the use of ruler, compass and transferor, and by understanding its properties. With respect to the first task, a large proportion of the students successfully constructed the triangle. However, the resolution method was not always as expected. A significant number of students find it difficult to use the ruler to draw line segment of a given length. As mentioned by Battista (2007), students who measure the length of an object incorrectly, when one of their ends is not aligned with the zero of the ruler, do not conceptualize clearly how the numerical marks of the ruler indicate the iteration of the length of the unit. Others seem to understand what an equilateral triangle is, but they have been unable to correctly calculate the length of the three sides. We can attribute this difficulty to the fact that some 6<sup>th</sup> graders students still cannot operate the division algorithm or have difficulty calculating mental quotients between integers. Nevertheless, regarding measurement reasoning, and in relation to linear measurement, the majority of the students demonstrated a level of sophistication quite acceptable. However, in relation to the perimeter of a triangle, the level of sophistication of these students is still quite insipient, since a large number of students could not obtain the

measure on the side of an equilateral triangle given its perimeter. This would have been the aspect of the task that requires a higher cognitive challenge.

Comparing the results obtained in the first task with the second, we verified that the percentage of students who correctly solved the second one is close to half of the percentage of students who correctly solved the first one. It seems to us that in the case of the second task the construction of an obtuse angle constituted a greater cognitive obstacle to the correct resolution of the task than others detected in the first task. A considerable number of students were not able to draw the angle of  $120^\circ$  correctly. These students have shown difficulties in coordinating the various aspects of the angle concept (White & Mitchelmore, 2003), namely they read the angle measurement counterclockwise when it should have been clockwise (Tanguay, 2012). The cause of the students' difficulties seems to be the "absence of adequate structural components" (Mitchelmore & White, 2000).

There were students who demonstrated difficulty in the conceptualization of angle and in turn in the use of the protractor (Mitchelmore & White, 2000). These difficulties are manifested in the second step, which is to place the point of reference of the protractor at the point that will be the vertex of the angle and  $0^\circ$  in the extension of the side of the angle. Some students demonstrated this difficulty because they did not know how to place the protractor at the apex of the angle to be measured (Tanguay, 2012). This task required a high cognitive challenge regarding the interpretation of the question, in part because of the symbolic mathematical load that is presented to students.

## Conclusions

The results evidence that, in linear measurement, many students demonstrated a level of sophistication quite acceptable. They confirm that young students can determine lengths and use correctly material (Joram, Subrahmanyam, & Gelman, 1998), namely developing measurement reasoning (Battista, 2007). However, different difficulties may arise in the construction of triangles (Zamagni & Crespo, 2016), namely due to the absence of adequate structural components. Students' difficulties in the conceptualization of measurement and angle were obstacle to the understanding and resolution of the tasks. Additionally, students have difficulty in decoding and transforming mathematical information. The combination and coordination of motor and visual skills in the use of didactic materials is very relevant in solving tasks of triangles on paper, and for this reason the knowledge of the properties of the triangles and the consolidated knowledge of the concept of measurement and of angle are essential.

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