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# **Personal geometrical work of pre-service teachers: a case study based on the theory of Mathematical Working Spaces**

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*Abstract. In this contribution, we intend to revisit the interplay between geometrical paradigms GI and GII in the context of the current French curriculum, which gives an important place to modelling activities in mathematics learning. We asked master students to perform a geometric modelling task on the estimation of the area of a field. The resolution of the task requires an articulation between GI and GII by considering measurement and approximation. From this first study, it results that the vast majority of students have developed a geometric work in the GI paradigm by mobilizing a low use of the classical property-based proof discourse. These results question teacher training and, more broadly, the today's mathematics teaching in France.*

*Keywords: Geometrical paradigms, Teachers training, Mathematical Working Spaces, Geometry teaching and learning*

## **Presentation and motivations of the study**

In this contribution, we revisit the issue of teaching geometry in teacher education, which was the subject of several of our presentations at the CERME conferences in the early 2000s. These presentations enabled the development of the theoretical and methodological framework of geometrical paradigms and Geometric Working Spaces, which is used to describe the forms and conditions of geometric work that is performed by students with the help of their teachers.

Kuzniak and Rauscher (2011) provide a classification of the solutions given by pre-service teachers to a geometry task. The main authors' objective was to raise students' awareness of the interaction between geometrical paradigms GI and GII (see next section). However, their study shows that primary school teachers were not very receptive to this interplay because they were confined to the first paradigm GI. Only secondary school teachers, more experienced in mathematics, understood the relevance of this interaction between paradigms. In the study, it was also questioned the difficulties that future primary school teachers had in teaching geometry because of their often deficient or old knowledge in the field. The goal was both to consolidate their knowledge and make them aware of the existence of several geometric paradigms leading to different forms of validation. It seemed interesting to us to review this work in consideration of the evolution of French curriculum, which has undergone notable fluctuations in recent years. The part given to geometry in education decreases significantly throughout the compulsory curriculum and its teaching is now mainly based on calculation and application, most often mechanical, of Pythagoras and Thales theorems, which are increasingly used to practice algebraic computation techniques. At the same time, more emphasis is put on modelling activities. In addition, from an institutional point of view, primary school teachers are now trained at university and must obtain a specific master's degree in education whereas previously they followed a two-year non-degree vocational training course after passing a competitive examination.

More specifically, this research has two types of objectives that concern students who want to obtain a master's degree in primary education:

- identify the geometric work actually produced by these teacher students in order to understand it and eventually be able to influence it.
- make these students aware of the measurement and area issues in relation with the geometric paradigms GI and GII with the possible interplay between these two paradigms.

In this new study, which extends our recent studies on the teaching of geometry (Kuzniak and Nechache, 2015), teacher students had to perform a geometric modelling task on the estimation of the area of a field. This research is part of a larger project that aims to investigate mathematical work in various institutional contexts. The analysis of students' geometric work is supported on the previously used research framework which, and this is also one of the motivations of this new study, had largely evolved towards what is now referred to as the theory of Mathematical Working Spaces (MWS).

## **Theoretical and methodological elements**

As this research is based on the theoretical framework of Mathematical Working Spaces articulated with the notion of geometrical paradigms, we will briefly outline the main elements of this theory by referring the reader who wishes to know more about it to the literature on the subject, especially that of previous CERMEs.

### **Geometrical paradigms**

According to Houdement and Kuzniak, (2006), in the context of teaching, it is possible to identify three geometrical paradigms named respectively GI, GII and GIII. By taking up the presentation made by Kuzniak (2018) we can introduce them in this way:

“The paradigm called Geometry I is concerned by the world of practice with technology. In this geometry, valid assertions are generated using arguments based upon perception, experiment, and deduction. There is high resemblance between model and reality and any argument is allowed to justify an assertion and to convince the audience. Indeed, mechanical and experimental proofs are acceptable in Geometry I.” (Kuzniak 2018, p. 10)

“The paradigm called Geometry II, whose archetype is classic Euclidean geometry, is built on a model that approaches reality without being fused with it. Once the axioms are set up, proofs have to be developed within the system of axioms to be valid. The system of axioms may be left incomplete as the axiomatic process is dynamic and has modelling at its core.” (Ibidem, p. 10)

Both geometries, I and II, have close links to the real world, albeit in varying ways. In particular, they differ with regard to the type of validation, the nature of figure (unique and specific in Geometry I, general and definition-based in Geometry II) and by their work guidelines. To these two Geometries, it is necessary to add Geometry III, which is usually not present in compulsory schooling, but which is the implicit reference of mathematics teachers who are trained in advanced mathematics. In Geometry III, the system of axioms itself is disconnected from reality, but central. The system is complete and unconcerned with any possible applications to the real world. (Ibidem, p. 11)

The identification of geometrical paradigms contributes to the understanding of the epistemological nature of the work actually produced in a school institution. The study of the circulation of

geometric work within the Mathematical Working Space will provide a more fine-grained characterization of this work.

### Mathematical Working Spaces

The theory of Mathematical Working Spaces (Kuzniak, Tanguay and Elia, 2016; Kuzniak and Nechache, 2015), known as MWS, aims to analyse the mathematical work produced by students or teachers in a specific educational institution. Mathematical Working Spaces are designed and thought out in such a way that their users can carry out their work under conditions that promote the emergence of a coherent and complete work. In this theory, mathematical work is structured through the interactions of epistemological and cognitive planes:

- the epistemological plane, composed of three poles: representamen, artifact, theoretical reference frame. It is used to structure the mathematical content of tasks.
- the cognitive level, composed of three cognitive processes: visualization, construction and proving. It reflects user's cognitive activity during task resolution.

The move from one plane to another is ensured by a set of geneses linked to the poles: a semiotic genesis based on the registers of semiotic representation that gives the tangible objects of the MWS their status of operative mathematical objects; an instrumental genesis, which has for function to make the artifacts operative in the constructive process; a discursive genesis of proof, which allows to give meaning to the properties to be implemented in the mathematical reasoning.

These three geneses facilitate the circulation between the two planes by stimulating an articulation between the respective components of the two planes. This set of relationships can be visualized by means of a diagram in the form of a triangular-based prism.

The analysis of geometric work can therefore be done through one of the three dimensions associated with each of the geneses (semiotic, instrumental and discursive), or through the articulation of two of them: semiotic and instrumental ([Sem-Ins]), semiotic and discursive ([Sem-Dis]), or discursive and instrumental ([Dis-Ins]). These different articulations thus define three vertical planes (Figure 1).

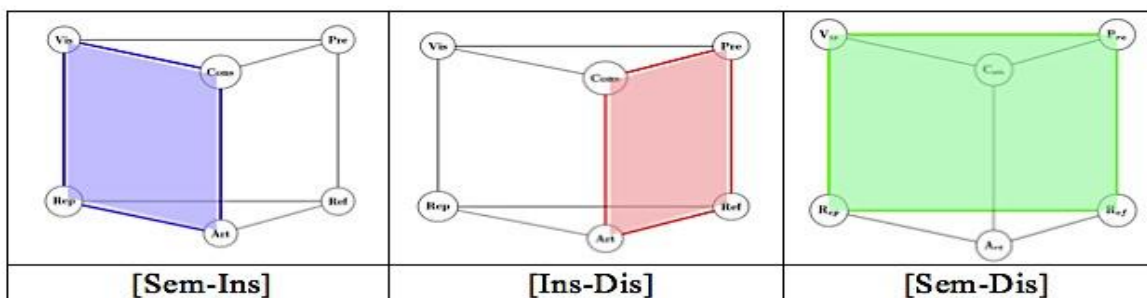


Figure 1. The three vertical planes of the MWS (Kuzniak & Nechache 2014)

The use of these different vertical planes helps to specify the circulation of geometric work in the MWS and the manner in which geometric work is carried out.

## **The task “Le terrain d’Alphonse”**

### **Experimentation**

To carry out the research, the task "Le terrain d'Alphonse" was assigned to two groups of undergraduate and master's students (45 students), who intend to teach in elementary school. The implementation of this task was performed in three phases. In the first phase, the task statement is given to the students in the form of a text to be read.

Alphonse has just returned from a trip in Périgord where he saw a field in the shape of a quadrilateral that had interested his family. He would like to estimate its area. To do this, during his trip, he successively measured the four sides of the field and found, approximately, 300 m, 900 m, 610 m, 440 m. He's had a lot of trouble in finding the area. Can you help him by indicating how to do it?

The purpose of the first phase is to specify that there is a lack of data to establish the exact shape of the quadrilateral that Alphonse wants to know the area. In the second phase, the length of a diagonal (630 m), without specifying which one, is given to the students to discuss the shape of the quadrilateral: convex or non-convex, crossed or not. Finally, once the shape of the quadrilateral has been decided, the third phase is devoted to concluding the resolution of the task. In each of the three phases, after collecting the students' written productions, we proceeded to a pooling. These exchanges allow us to list all the methods used by the students on the blackboard in order to discuss them. In particular, the results obtained are compared in relation to the selected shape of the terrain and the methods used to calculate the area.

Several methods can be used to achieve this task and they involve different geometric paradigms depending on both curriculum expectations and the students' personal work. The resolution of this task requires a modelling approach and involves work on approximation issues related to measurement.

### **Collection and analysis of data**

To conduct our research we collected the students' written productions (45 productions for each phase, for a total of 135 productions). We also recorded and transcribed the exchanges that occurred during the pooling.

The students' written productions were studied by using a grid specifying the type of work done on the figure(s) and the types of discourses related to the area and its measurement. The analysis of the recordings of the pooling complements the analysis of productions and helps to refine student profiles. With the help of a statistical analysis software, we were able to identify five groups of students providing different forms of geometric work. Given the size of the present paper, we have limited ourselves to presenting briefly only four of these forms of work.

### **The expected geometrical work in master**

A teacher trainer was solicited to implement the task in her group. She solved the problem informally before giving it to her students and her reasoning is in line with the work expected from the students who want to pass the competitive examination which presupposes knowledge of the geometry taught at the College (Grade 6 -9). She immediately detects that a data is missing and asks

for a diagonal. Using the value of a diagonal (630 m), she starts by constructing the quadrilateral with ruler and compass (instrumental dimension) to get an idea of its shape (semiotic dimension). She draws the figure using a scale that matches 1 cm on the figure to 1 m on the field. Thus, her work begins in the [Sem-Ins] plane with the intention of using a configuration as a support for the reasoning. She then deduces two possible forms of the quadrilateral, one convex and the other not convex<sup>1</sup>. Then, she suggests to start by determining the area of the terrain with a convex shape. Her choice is motivated by the fact that, in this case, the task is easier for the students. To achieve the task, she makes a freehand drawing and decomposes the quadrilateral into two triangles. On the drawing, she specifies known and unknown values used to calculate the height of each triangle thanks to Pythagoras's theorem (discursive dimension). The work of proof is then based on algebraic computation with a strong numerical component. The freehand drawing becomes a support for algebraic calculation (semiotic dimension) by avoiding any direct measurement.

The mathematical work expected from preservice school students at the master's level can be summarized as follows:

1. The work is initiated in the [Sem-Ins] plane in order to distinguish the different cases that may exist. Work is guided by elements of the theoretical referent relating to the quadrilateral: various forms of the quadrilateral and importance of the diagonal to determine the nature of the quadrilateral.
2. The technique of triangulation is completely integrated in work (still in the [Sem-Ins] plane) but it is not explicitly mentioned. Perhaps this is due to the fact that the diagonal line shows the triangles directly and to the non-formal nature of the proof given by the teacher's trainer.
3. Work then moves to the [Sem-Dis] plane associated with algebraic work supported on the freehand figure and on Pythagoras's theorem used as a theoretical tool that switches work to algebraic work without any measurement on the drawing. The whole challenge of proof by calculation that is then made is to avoid any appeal to measurement.

This work is in accordance with the paradigm expected at this school level, which is a paradigm articulating GI and GII insofar as the reasoning is based on a particular figure without any generalization and on approximate values by prohibiting any effective measurement on the drawing.

### **Different forms of geometric work identified**

In the following, we only present the results of the first phase which, contrary to our initial expectations, followed an unexpected and longer course. Indeed, almost all the students did not identify the need for asking additional indications and they engaged in the search for the area by adding certain conditions such "The quadrilateral had to be particular" or "All the quadrilaterals with the same perimeter have the same area". These assumptions allow them, therefore, to reason on a particular figure. We present below four of the different forms of geometric work that we have identified at the end of this phase.

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<sup>1</sup> We leave aside the case of a diagonal outside the quadrilateral or the crossed quadrilaterals that appear, in this modelling exercise, as two triangular fields and not as a quadrilateral.

### **Form 1. An asserted work of measurement: Francis' work**

Francis begins his work with a scale construction with ruler and compass where he gets a trapezoid based on visual adjustment, guided by the idea that the figure should be particular (Sem-Ins).

Francis: I took the large 900 m base then from both ends with the compass, I made 410 on one side and 610 on each side and then with the ruler I tried to find the 300 with the two arcs of circle.

He then justifies why, it is indeed a trapezoid by relying on a geometric property of the theoretical reference frame, discursive dimension, (Sem-Dis).

Francis: I have drawn a perpendicular to the large base and this line was also perpendicular to the small one and since the two lines are perpendicular to the same line, they are parallel to each other so it makes a trapeze.

Finally, he completes his work in the [Ins-Dis] plane with calculations using formulas and by measuring the missing data on the drawing. He explains that he can do this because he has done his figure using a scale. His mathematical work is valid in the GI paradigm and moreover, as we will see again later on, this work is assumed by the student.

### **Form 2. A geometric work structured around the instrumental dimension: Ivana's work**

This work is representative of the work done by students who have used geometric drawing tools by attempting not to use measurement too explicitly. The mathematical work produced by Ivana was initiated in the [Sem-Ins] plane to construct a quadrilateral (ABCD) by means of geometric drawing tools and with the assumption that one of the angles of the quadrilateral was a right angle. This inclusion of an additional property is common among students who have chosen to use drawing tools.

Ivana : In fact I constructed the figure to have a right angle, and so I calculated the area of the first triangle by doing (...).

Then, she decomposes the resulting quadrilateral into two triangles (Sem), one of them ABC is general and the second ADC is right-angled in D so that she can apply the triangulation method (Dis) to determine the area of the quadrilateral as the sum of the areas of these two triangles. Thus, mathematical work is located in the [Sem-Dis] plane. The, she calculates the area of the rectangular triangle ADC in D, then she applies the Pythagoras theorem (Dis) in the same triangle to deduce the length of its hypotenuse [AC]. Finally, Ivana uses the graduated ruler to measure a length (the height of the triangle ABC from B) and she applies the formula of the area of a triangle to determine the area of the triangle ABC. The mathematical work thus ends in the [Ins-Dis] plane with an unspecified use of measurement. Therefore, we can say that the mathematical work is complete and that it is guided by the GI paradigm.

However, this work does not conform to "mathematical rules" since the student has no control over the results obtained. When the professor asked her about the validity of her work, she answered that "all the quadrilaterals have the same surface because they have the same perimeter" and that is why it is possible to reason on a particular figure. This "theorem in action" leads to the fact that her mathematical work is not valid or conform. With this work, we are faced with a fairly common

problem of interpretation about the identification of the exact paradigm in which the student works. Students, because of the very strong didactical contract at this level, generally avoid measurement and thus think they work in the GII paradigm. But, in fact and in part to avoid obstacles linked to the difficulty of giving a proof based solely on properties of the frame of reference, they surreptitiously move into the GI paradigm either, as Ivana, by using measurements not assumed or as we will see below by using data based on perception only.

**Form 3. A work without measurement and without theoretical control: Katia's work**

The following geometric work is one of the most frequent among students who did not want to use construction instruments. They forbid themselves measurements on the drawing and wish to work in GII but nevertheless they allow themselves a great semiotic freedom to obtain formulas or apply known area calculation formulas.

Let's illustrate this approach with Katia's production. She begins by drawing two freehand figures (Sem), suggesting a transformation from the first to the second that leaves the area of the figure, more or less, invariant.

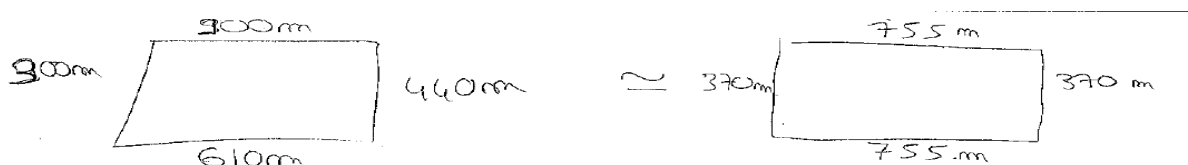


Figure 2: Katia's work

Katia then explained that one has to take the initial lengths and imagine that the terrain is a rectangle. To determine this area, she applies the technique giving this value by multiplying the lengths of the two sides (Ins). The lengths of each new sides are obtained as an arithmetic average of the opposite sides. Thus, the work is located in the plane [Sem-Ins] because it is oriented towards the use of a well-known area calculation formula by transforming the initial geometric figure. This work is produced without any control by means of material tools (geometric instruments) and theoretical tools (properties). Therefore, if this geometric work may be considered as partially valuable in GI it is not valid in any of the paradigms.

**Form 4. A geometric work initiated in GII but unfinished: Severine's work**

This form of work was rare in both groups and initiated by students with a good level in mathematics. In her work, Severine starts by producing a free-hand dissection of the quadrilateral into a trapezoid and two rectangle triangles (Sem). She explains her procedure in this way:

Séverine: I started from a general quadrilateral and then I drew two right-angled triangles (inside) in order to complete the quadrilateral.

Then, she explained that it would be necessary to find the heights of the triangles and also to use Pythagoras theorem to do the calculations (Dis). She intended to do this work but she did not perform it. However, at this stage, this can be considered normal because a data is missing, but she did not asked for it. She was thus blocked in the plane [Sem-Dis] and the teacher intervened by trying to motivate the claim of a diagonal.



Teacher: Can you comment on what Séverine suggests, do you have the necessary information to do the calculation

Students then request additional data on measurements but do not ask for the length of the diagonal. Francis intervened once again by indicating that it is possible to measure on the drawing because it was made with a scale. Séverine does not agree with him and she refuses to measure on the figure (GI paradigm) preferring to remain blocked and not complete the task.

## Conclusion

In this preliminary study, which is part of a more global research project on teacher training in geometry, we were able to identify some different students' forms of geometrical work when they have to achieve a geometrical task which relates to the estimation of the area of a field. The results we have obtained show blockages and rebounds among students who are trying to meet institutional expectations (a geometric work in GII). The work they produce is profoundly incomplete because it is often confined to a single plane or dimension of the MWS. Moreover, and for us, it is the most worrying from the teacher education viewpoint, these teachers students do not seem to have control tools on their productions other than perceptual and instrumental. When they have tried to use theoretical controls, there were generally based on false "theorems in action" that do not ensure the validity of mathematical work from an epistemological perspective. It should be pointed out, however, that the situation is destabilizing for the students because it does not follow the usual didactic contract, which implies that all the data needed to solve the problem have to be included in the task statement. Moreover, almost all the teacher students think that all the geometric shapes they are asked to study must have some particular shapes and properties. And, on their own initiative, they suppose that the quadrilateral has a specific property (right angle, parallel sides...) which allows them not to be blocked in a dimension or plane of the MWS but it results that their work cannot be considered to be valid. From this observation, we intend to develop didactical situations that ensures a first exploration work on drawing in order to question the construction and validation methods within a geometry I that promotes both modelling and controlled approximation work.

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