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In-service teachers' conceptions of parallelogram definitions

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The present research examines middle school mathematics teachers' conceptions of geometric definitions, and specifically whether they possess geometric conceptions at the van Hiele fourth level. Sixty two teachers participated in the research. We used an open-ended questionnaire for data collection. The questionnaire included three questions concerned with the equivalence between definitions, sufficient and necessary conditions, and deductive geometric reasoning related to definitions. We used deductive and inductive content analysis to analyse the data. We found that generally, the participating teachers did not possess the geometric conceptions at the van Hiele fourth level and had difficulties in understanding what the imperative and optional features of mathematical definitions are. In addition, the participants' conceptions of definitions have specific features that explain why they have not attained the van Hiele fourth level.

Keywords: Conceptions, parallelogram, definitions, in-service teachers

Introduction

Definitions are used as building blocks for the construction of mathematical theorems (Pimm, 1993), which makes them play a central role in building understanding of the meanings of mathematical concepts (Wilson, 1990). According to van Hiele and van Hiele's (1958) theory about geometric thinking development, at the informal deduction level, the learner understands the importance of precise definitions, how a particular attribute derives from another, and the inclusion relationship between groups of shapes; the learner can construct internal connections between the different properties of the same shape and connections between the properties of different shapes. At the subsequent level; that the formal deduction level, the learner understands the role of definitions and recognizes the features of a formal definition, such as necessary and sufficient attributes and equivalent definitions. At this more advanced level, the learner understands the functions of definitions, axioms and theorems as a deductive chain.

Zaslavsky and Shir (2005) distinguish between two features of the definition: those which are imperative and those which are optional. Some of the imperative features of definitions are their being non-contradicting, unambiguous and hierarchical, while some of the optional features of mathematical definitions is the requirement about their being minimal; in other words include a minimum of sufficient attributes of the concept. In addition to the aspect of imperative and optional features of definitions, van Dormolen and Zaslavsky (2003) mention four criteria for logical necessities of definitions; one of them is the criterion of equivalence. Other criteria are part of general culture but are not necessary from logical perspective, such as the criterion of minimality and the criterion of elegance, which we need when we want to choose between two equivalent definitions. Using these criteria, we take the 'nicer definition'; the one has less words and less

symbols and uses more general basic concepts. Empirical research shows that many mathematic educators prefer mathematical definitions to be minimal and elegant or parsimonious (Vinner, 1991; Van Dormolen & Zaslavsky, 2003; Leikin & Winicky-Landman, 2001). On the other hand, there are those who, in certain cases, also prefer a non-minimal definition (de Villiers, 1998; Pimm, 1993; van Dormolen & Zaslavsky, 2003; Zaslavsky & Shir, 2005). For example, it is possible to define two similar triangles as two triangles in which two angles in one triangle have equal angles in the second triangle. From the pedagogically perspective, adherence only to the minimal definition, may impair the understanding of the concept of similar triangles (Zaslavsky & Shir, 2005).

One feature of mathematical definitions appreciated by mathematicians and mathematics educators is the equivalence of definitions of the same concept (Harel, Selden & Selden, 2006). Leikin & Winicky-Landman (2001) studied mathematical definitions in a non-geometric context, and found that many high school teachers did not notice that a particular concept could be defined in a number of equivalent ways. When the learner attends to the equivalent definition, he/she performs in van Hiele informal deduction level. Haj Yahia, Hershkowitz & Dreyfus (2014) found that many senior school students rejected correct geometric proofs because they failed to notice that there might be more than one definition for a particular concept. When we attend to the equivalent geometrical definitions, however, we perform as expected in van Hiele & van Hiele (1958)'s informal level: we accept the notion that we can derive one attribute from the other/s. For example, if in a quadrilateral every two opposite sides are parallel, then we can deduce that every two opposite sides are equal and vice versa – and these two statements are equivalent.

Another important aspect of definitions is giving a definition that contains non-sufficient attributes (de Villiers, 1998). Many studies reported that when the students asked to define some quadrilaterals a part of the students gave brief definitions but not complete (de Villiers, 1998; Choi, Oh & Kyoung, 2008; Markovic & Romano, 2013). de Villiers (1998) reported that when the students asked to define the rhombus, a part of the students gave brief definitions but not complete. For example, a rhombus is a quadrilateral whose diagonals are perpendicular to each other. Markovic & Romano (2013) reported that some students define the square as "a geometric figure all four sides of which are equal".

Research rational and goals

Since the progress from one level to another on van Hiele and van Hiele's (1958) depends more on teaching than on age or biological maturity, different types of instruction can have a different effect on the progression from level to level. In order to prove and to be able to understand the role of proof, he/she must be at the formal deduction level. Moreover, teachers' conceptions are the basis for their mathematics teaching (Johnson, Blume, Shimizu, Graysay, & Konnova, 2014), especially their teaching of geometry. Thus investigating the teachers' conception about mathematical definition plays central issue in the teaching processes. The teachers are expected to be at the formal deductive level, where they understand the role of definitions and recognize the logical features of a formal definition, such as necessary and sufficient attributes and equivalent definitions. In addition, they are expected to know the imperative features and the optional features of definitions (Zaslavsky & Shir, 2005).

The present research intends to examine mathematics teachers' conceptions of geometric definitions. The results of the research would make us more knowledgeable regarding these conceptions, and in particular regarding the features of these definitions. This is especially needed because of the little research done on teachers' conceptions of a basic aspect of their ability to teach geometry; i.e geometric definitions.

Research question

What are the features of middle school teachers' conceptions of geometric definitions?

Methodology

Sixty-two in-service middle school mathematics teachers who constitute theoretical sample (Glaser & Strauss, 1967) participated in this study. These teachers studied for completing their M.Ed. (Master of Education) degree in mathematics education in an academic college for teachers. We used an open-ended questionnaire for data collection. The questionnaire included three questions that are concerned with the following topics related to definitions: the equivalence between definitions of the same concept, sufficient and necessary conditions of a concept, and finally deductive geometric reasoning related to definitions. To provoke teachers' thinking, we used prompts consisting of fictive or real contradicting statements of students that represent common conceptions of definitions of geometrical concepts. The participating pre-service teachers were asked to fill the questionnaire, by commenting on each statement; whether they agree with it or not, and to explain their answers.

For analysing the data, we used a qualitative coding method (Salanda, 2015) that is close to grounded theory (Glaser & Strauss, 1967). We used deductive codes derived from a theoretical perspective (Charmaz et al., 2007) and inductive codes for the themes not present in existing research about geometric education. Using the deductive codes, we characterized the answer of each participant according to its satisfaction of the aspects of definition: being minimal, being hierarchical, awareness of sufficient conditions, awareness of equivalent definitions, etc. Afterwards we looked at shortages of the participants' conceptions, for example being aware of the sufficient conditions but enforcing other conditions. Next we looked at the reasoning behind the answer, for example being not aware of equivalence definitions as a result of considering the lack of the 'edges' term.

The coding was conducted independently by the three authors. The inter-rater reliability of coding by Cohen's Kappa was 0.86, which shows and appropriate value of the interrater reliability.

Results

This section is splitting into four parts. We present the results concerning (a) the equivalence between definitions, (b) sufficient and necessary conditions, and finally (c) the deductive geometrical reasoning.

The equivalence between definitions of the same concept

The first item in the questionnaire related to the following question:

The mathematics teacher asked his students to define the "parallelogram".

Sami** defined the parallelogram as "Parallelogram is a quadrilateral in which every two opposite sides are parallel". **Rami** defined the parallelogram as "Parallelogram is a quadrilateral in which every two opposite angles are equal". **Rafi** defined the parallelogram as "Parallelogram is a quadrilateral in which the diagonals crossing each other" **Salim** defined the parallelogram as "Parallelogram is a quadrilateral in which every two opposite sides are equal". **For each of these claims you need to determine whether it is a definition and explain your response.

Answering the first item, 25.5 % (16) of the participants demonstrated knowledge of equivalent definitions. 14.5% of the participants, though they demonstrated knowledge of equivalent definitions, preferred some of these definitions over others. Specifically, these students preferred the definition of the parallelogram that included properties about the sides of the parallelogram, probably because the name indicates that the sides are parallel. Three of them preferred the definition that mentions the property of “pairs of opposite sides being parallel”.

27.5% (17) of the participants claimed that the definition should include conditions about sides of a parallelogram, but did not demonstrate knowledge of equivalent definitions. These students were not satisfied with the definition that includes just the term ‘angles’ or ‘diagonals’, but wanted the definition statement to include conditions about the sides. Eight of these students even wanted the statement to include conditions related to two components; sides and parallelism. One student wrote, “*This could not be a definition of the parallelogram, for it just talks about angles*”.

The previous results indicate that the need for a condition about the sides of the quadrilateral or/and the opposite sides being parallel was mentioned by the students who demonstrated knowledge of equivalent definitions and those who did not demonstrate such knowledge.

27.5% (17) of the participants differentiated between definitions and properties. Doing so, they did not accept the statements that describe the parallelogram in terms of its angles or diagonals as definition, saying that it is property and not a definition. Five of these students did not elaborate further. For example, one of them wrote, “*This is not a definition because it is a property*”. The rest of these students (twelve students) argued that the definition should talk about sides and not any other element of the parallelogram. One of them wrote, “*This is a property and not a definition of a parallelogram for it includes information about the angles and not the sides*”.

19.5% (12) of the participants did not accept a statement as a definition because of their little understanding of the sufficient conditions issue of the definition. These students did not accept the property “each pair of opposite sides are parallel” as sufficient for a quadrilateral to be a parallelogram. Instead, they suggested that the correct definition should include also the equality of the edges. One of these students wrote, “*the definition should be a parallelogram is a quadrilateral that each pair of its opposite sides are parallel and equal*”.

For the first item only and because of the various percentages in its results, we will summarize these results in Table 1 to make them more accessible.

Category	Percent
Demonstrating knowledge of equivalent definitions	25.5%
- Preferring definitions that included properties about the sides	
Not demonstrating knowledge of equivalent definitions	
- Preferring definitions that included properties about the sides	27.5%
- differentiating between definitions and properties	27.5%
- little understanding of the sufficient conditions	19.5%

Table 1: Categories of students' responses on the equivalence item and their percentages

Sufficient and Necessary conditions of a concept

The second item in the questionnaire related to the following question:

The mathematics teacher asked his students to define the "parallelogram".

Sami** defined the parallelogram as "Parallelogram is a quadrilateral in which every two opposite sides are parallel". **Rami** defined the parallelogram as "Parallelogram is a quadrilateral in which two opposite sides are parallel". **Rafi** defined the parallelogram as "Parallelogram is a quadrilateral in which every two opposite sides are parallel and equal" **For each of these claims you have to determine whether it is definition And to explain your response!

Answering the second item, 82% (51) of the participants did not consider correctly that the statement "The parallelogram is a quadrilateral in which two opposite sides are parallel" an accurate definition of the parallelogram. Twenty-four of them reasoned that the statement should include 'all opposite sides' and not just two opposite sides. The rest (twenty-seven students), in considering the statement not an accurate one, reasoned that the statement describes a trapezium and not a parallelogram. Eleven students wrote that the statement is an accurate statement because it talks about the opposite sides being parallel.

87% (54) of the participants considered correctly, "The parallelogram is a quadrilateral in which each pair of opposite sides are parallel" an accurate definition of the parallelogram. Eight students considered this definition not to include sufficient conditions because it should also include the equality between the sides as condition".

Almost all the participants considered the statement "The parallelogram is a quadrilateral in which each pair of opposite sides are parallel and equal" an accurate definition of the parallelogram. Eleven of these students wrote that one of the conditions 'equal sides' or 'parallel sides' is enough for a quadrilateral to be a parallelogram. In spite of this knowledge, not any one of them concluded that the definition is not accurate or that it violates the minimalism criterion of definition.

Deductive geometric reasoning

The third item in the questionnaire was related to Deductive geometric reasoning.

*The mathematics teacher defines "The trapezium is a quadrilateral in which two opposite sides are parallel" and "The Parallelogram is a quadrilateral in which two pairs of opposite sides are parallel". Rami claims that the teacher's definitions mean that the parallelogram is a trapezium, while Sami claims that Rami is wrong. **Which of these claims is right? Explain your response!***

Answering the third item, 56.5% (35) of the participants supported the claim of Rami that the parallelogram is a trapezium according to the teacher definition, and also supported the opposite claim of Sami that parallelogram is not a trapezium according to the right definition of the trapezium.

19.5% (12) of the participants argue that Rami's claim that parallelogram is trapezium is wrong while Sami's claim that the parallelogram is a not trapezium is right. They argued that the teacher defined the trapezium as having one pair of parallel opposite sides, but she have to define it as having only one pair of parallel opposite sides and the parallelogram have two pairs of parallel opposite sides. This argument also led them to consider only the second claim is right.

24% (15) of the participants wrote that Rami's claim is right while Sami's is wrong. Doing so, they argued that the claim "The parallelogram is a trapezium" is right because the teacher defined the trapezium as having one pair of opposite angles, while she defined the parallelogram to have two pairs of parallel opposite sides. The shape that has two pairs of parallel opposite sides satisfies the condition of having a pair of parallel opposite sides. This means that according to the teacher, the parallelogram is a trapezium.

Discussion and conclusions

The present research intended to investigate the features of middle school teachers' conceptions of geometric definitions. It could be argued according the research results that the participating teachers have difficulties regarding the understanding the structure of the geometrical definitions and their functions and meaning.

One feature of the participating teachers' conceptions of mathematical definitions is its relatedness to their understanding the sufficient conditions issue of the definition. Some of them did not accept the statement "The parallelogram is a quadrilateral in which each pair of opposite sides are parallel" as an accurate definition of the parallelogram, this statement sufficient in order to deduce the rest of concept attributes These conceptions of the sufficient conditions also led them to demand non-minimal definitions to be the accurate definitions. This feature agrees with other reports about teachers' conceptions of the definition this result is in agreement with the results from Leikin and Winicki-Landman (2000) and from Zazkis and Leikin (2008) where prospective secondary

mathematics teachers indicated a preference for the barely-not-minimal definition over a minimal definition.

A second feature of the participating teachers' conceptions of mathematical definitions is that they differentiate between a definition and a property. They did not accept the statements that describe the parallelogram in terms of its angles or diagonals as definition, saying that it is property and not a definition, although that these statements satisfy all the imperative features of mathematical definition (Zaslavsky & Shir, 2005). This misunderstanding of the relationship between a definition and a property leads to difficulty in understanding the equivalence between definitions. From these participants' perspective of mathematical or geometrical concepts, for every concept there is only one definition (Leikin & Winicky-Landman, 2001). The participating teachers were not aware of the logical necessities of equivalence (van Dormolen & Zaslavsky, 2003).

Third feature of the participating students' conceptions of mathematical definitions is that it takes into consideration the name of the geometric concept. Some of the students only accepted a definition of the parallelogram that included the term 'parallel' or at least sides, and did not accept a definition that is related to relations between its angles or diagonals. Thus, the name affected the participating teachers' acceptance of some statements as definitions and their rejection of other statements as thus. This result confirms the finding of Haj-Yahya, Hershkowitz & Deryfus (2014) who reported that the parallelogram name affects students' proving processes. It seems that this influence of the name on the participating students' conceptions of definitions hindered these students from considering some of the equivalence and alternative definitions of the parallelogram as accurate definitions.

The fourth feature of the participating students' conceptions of mathematical definitions is that it could include contradictions. More than half of the participating students accepted two statements that result in the trapezium being a parallelogram, as well as the opposite, which means that there is no difference between the shapes. The results from the third class of questions was about the deductive reasoning, the results shows that in many cases the students did not rely their reasoning on the concept definition, this confirm previous results of Edward & Ward (2004) about non-geometrical context.

To conclude, the present research examines mathematics teachers' conceptions of geometric definitions. We found that generally, the participating teachers do not possess the geometric conceptions at the van Hiele fourth level didn't understand what are the imperative features of mathematical definition (Zaslavsky & Shir, 2005). The research results directs the preparation of workshops for teachers, so it would be possible to target in these workshops the features of teachers' conceptions of definitions. These workshops would constitute a platform for discussing the impact of mathematics teachers' conceptions of definitions on their teaching. This would make these conceptions sounder and help these teachers take better decisions regarding how to teach definitions in the mathematics classroom. Research is needed here to verify the consequences of educational programs on teachers' conceptions of definitions in particular and on their teaching in general. In addition, future research can utilize theoretical frameworks as those suggested Fujita and colleagues (see for example Fujita (2008)), which will clarify other aspects of in-service teachers' conceptions

regarding definitions.

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