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Mathematical creativity and geometry: The influence of geometrical figure apprehension on the production of multiple solutions

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The aim of the study is to investigate the creativity demonstrated by students in multiple-solution tasks (MSTs) in geometry taking into account the role of geometrical figure. To this end, the test that was administered among 149 eleventh graders consisted of two types of problems: a. with and b. without the relevant figure. Findings revealed that fluency and flexibility were higher when the verbal description of the problem was accompanied by the relevant figure. The originality of students' solutions, though, did not differ in the two types of problems. The similarity analysis suggested distinct student profiles: (a) poor fluency and flexibility, (b) moderate fluency and flexibility and (c) high fluency and flexibility. Although the majority of students belong to the first two groups, qualitative results provide evidence that students managed to improve and gain high fluency and flexibility. Findings also indicated that geometrical figure apprehension is a prerequisite for high levels of creativity in geometry.

Keywords: Creativity, geometry, multiple-solution tasks (MSTs), geometrical figure apprehension.

Introduction and Theoretical Framework

Creativity has been proposed as one of the major components to be included in Mathematics education, since “the essence of Mathematics is thinking creatively” (Mann, 2006, p. 239). Torrance (1994) defines creativity as multidimensional; fluency, flexibility, originality (or novelty) and elaboration are aspects of creativity. Fluency is related to the flow of ideas, flexibility has to do with the ability to shift between different ideas, novelty is associated with the originality of the individual’s ideas or products, and elaboration is associated with the individual’s ability to describe, illuminate, and generalize those ideas. A number of researchers (e.g. Silver, 1997; Stupel & Ben-Chaim, 2017) suggest that linking mathematical ideas using multiple approaches to solve problems (or prove statements) is essential for the development of mathematical reasoning and fosters better comprehension and increased creativity in Mathematics. Following Torrance, Silver (1997) claimed that in order to develop creativity by means of problem solving it is necessary to advance three components of creativity. Fluency is promoted by raising multiple ideas for the solution of one problem. Flexibility is raised when after one solution is already in hand, the individual searches for more possible solutions. Originality is developed when an individual succeeds in generating a new solution, in addition to those known to him/her at the time.

Geometry provides opportunities for investigation and proving activities that resemble the work of mathematicians (Herbst, 2002). The usefulness of geometrical figures in problem solving in

geometry is beyond doubt as they provide an intuitive presentation of all constituent relations of a geometrical situation (Duval, 1995). Duval (1995) distinguishes four apprehensions for a geometrical figure: perceptual, sequential, discursive and operative. To function as a geometrical figure, a drawing must evoke perceptual apprehension and at least one of the other three. Each has its specific laws of organization and processing of the visual stimulus array. In particular, perceptual apprehension refers to the recognition of a shape in a plane or in depth. Perceptual apprehension indicates the ability to name figures and the ability to recognize several sub-figures in the perceived figure. Sequential apprehension is required whenever one must construct a figure or describe its construction. The organization of the elementary figural units does not depend on perceptual laws and cues, but on technical constraints and mathematical properties. Discursive apprehension is related to the fact that mathematical properties represented in a drawing cannot be determined through perceptual apprehension. In any geometrical representation the perceptual recognition of geometrical properties must remain under the control of statements (e.g. definitions). However, it is through operative apprehension that we can get an insight into a problem solution when looking at a figure. Operative apprehension depends on the various ways of modifying a given figure: the mereologic, the optic and the place way. The mereologic way refers to the division of the whole figure given into parts of various shapes and the combination of them in another figure or sub-figures (reconfiguration), the optic way is when the figure is made larger or narrower, or slant, while the place way refers to its position or orientation variation. Each of these different modifications can be performed mentally or physically through various operations. These operations constitute a specific figural processing which provides figures with a heuristic function. One or more of these operations can highlight a figural modification that gives an insight into the solution of a problem in geometry.

MSTs contain an explicit requirement for solving them in multiple ways. A number of studies (e.g. Levav-Waynberg & Leikin, 2009) used MSTs as an instructional tool for developing knowledge and creativity, as well as a tool for the evaluation of creativity in geometry. Moving a step forward, the present study aims to shed new light as far as MSTs are concerned on evaluating and developing creativity in geometry taking into account the role of geometrical figure. As there is a need for a multidimensional approach as far as teaching and learning in geometry is concerned, we address the following research questions (RQ): (a) what is the influence of geometrical figure apprehension on the production of multiple solutions in geometry problems? (b) what kind of creativity (fluency, flexibility, originality) do students demonstrate when working with MSTs in which the geometrical figure is either given or not in the wording of the problem? (c) what are students' profiles, if any, in terms of the number of solutions in MSTs in geometry?

Methodology

A written test was administered among 149 eleventh graders (77 boys and 72 girls) as follows: 147 of the students completed the test in usual classroom conditions and 2 students took part in an individual interview during the solution process of the test. The test consisted of four geometry problems that were content and face validated by two experienced Mathematics school teachers in secondary education and two professors of Mathematics Education. The students were asked explicitly to solve them in as many ways as possible. In problems 1 and 3 the wording of the

problem was accompanied by the relevant figure, while in problems 2 and 4 only the wording of the problems was given. The duration of the test was 110 minutes. It is worth mentioning that the problems (without the relevant figure) were piloted among a small number of students before the final administration of the test. It was confirmed that all the problems (without the relevant figure) had the same level of difficulty.

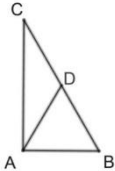
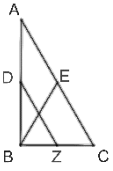
<p>Problem 1: Prove in as many ways you can that, that the median drawn to the hypotenuse of the right triangle equals half the hypotenuse.</p> 	<p>Problem 3: Let a rectangular triangle ABC ($B = 90^\circ$), D midpoint of segment AB, E the midpoint of segment AC and Z the midpoint of segment BC. Prove in as many ways you can that $DZ = EB$.</p> 
<p>Problem 2: AB is a diameter on a circle with center O. D and E are points on circle O so that $DO \parallel EB$. C is the intersection point of AD and BE. Prove in as many ways as you can that $CB = AB$.</p>	<p>Problem 4: Find the center of a circle in as many ways as you can, if you only know the circumference of the circle.</p>

Figure 1. The four geometry problems

The solutions provided by the students were classified in view of the strategy they used. Then, the aspects of creativity (fluency - flexibility - originality - total creativity) were calculated based on the scoring scheme that Leikin (2009) proposed for the evaluation of performance on MSTs. The construct of solution spaces that will be explained in the next section was also used to analyze the student problem-solving performance (Leikin, 2007). In order to answer the research questions, descriptive statistics and t-test at 95% significance level were used. Furthermore, a similarity statistical analysis was conducted using the computer software C.H.I.C. It is a method of analysis that determines the similarity connections of the variables (Gras, Suzuki, Guillet, & Spagnolo, 2008). On the similarity diagram we symbolize with P_i , $i=1, 2, 3, 4$ the problem referred to and symbolize with S_j , $j=0, 1, 2, 3, 4$ the number of solutions for each problem. The interviews with the two students and one of the authors were transcribed and examined qualitatively.

Results

In order to examine the influence of geometrical figure apprehension in the development of mathematical creativity, five solutions which were provided by students in problem 2 are presented in Figure 1. These solutions are grouped into three solution spaces based on the strategies and properties that students used. In the first solution space, the main strategy is that the straight line joining the middle of the two sides of a triangle is parallel and equal to half of the opposite side, and includes two solutions (1a, 1b). The first solution was found by 8.1% of the students while the second solution which requires drawing an auxiliary line was only found by the 2.7% of students; the originality of this solution space rated 10 (10.8% < 15%). In the second solution space, the main strategy is that equal corners were formed, and refers to two solutions (2a, 2b). These solutions

were provided by the 28.4% of students as it did not require any auxiliary construction. Its originality rated 0.1 (56.8% > 40%). The third solution space includes one solution (3) that refers to the construction and the use of the rhomb properties. This solution which requires constructing an auxiliary line was provided only by 1.4% of the students and the originality rated 10 (1.4% < 15%). As far as the fluency for problem 2 is concerned, 41.5% of the students were unable to solve the problem, while a similar percentage of students (49%) found only one way of solving it. The percentage of students who were able to find 2 or 3 ways was quite low (9.5%).

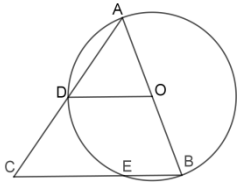
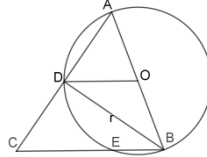
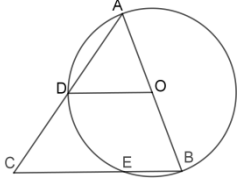
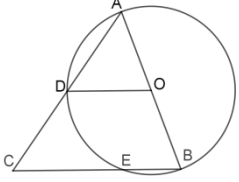
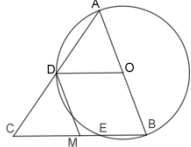
<p>1a Solution 8.1%</p> <p>DO=1/2AB, as equal radiuses in a circle. \RightarrowDO is a midline in triangle ABC, as parallel to BC and bisecting AB. \RightarrowDO=1/2AB=1/2BC \Rightarrow AB=BC.</p> 	<p>1b Solution 2.7%</p> <p>D is the mean of the AC, since DO joins the medium C with D and OD//BC. We construct the median DB of the triangle. The angle D1 goes to a semicircle so $D1 = 90^\circ$. Thus, DB is both median and altitude of the triangle ABC. Therefore, ABC is isosceles and AB = CB.</p> 	
<p>2a Solution 28.4%</p> <p>DO=AO, as equal radiuses in a circle. $\Rightarrow \angle ADO = \angle A$, as base angles in an isosceles triangle, $\angle ADO = \angle ACB$, as equal corresponding angles within parallel lines, $\angle ACB = \angle A$. \Rightarrow AB=BC, a triangle with 2 equal angles in isosceles.</p> 	<p>2b Solution 28.4%</p> <p>DO=AO, as equal radiuses in a circle. $\Rightarrow \angle AOD = \angle ABC$, as equal corresponding angles within parallel lines. $\Rightarrow \angle A = \angle A$, as shared angle $\Rightarrow \triangle AOD \sim \triangle ABC$ (2 equal angles) \Rightarrow AB=BC (a triangle similar to an isosceles triangle is also isosceles).</p> 	<p>3 Solution 1.4%</p> <p>We construct a parallel line from D to AB. Taking into account that it passes through the median D of the AC, it also passes from the median M of BC and $MD = AB/2$. $EB \parallel DO \Rightarrow MB \parallel DO$. Therefore, $MB = DO = BC/2$ and $OD = OB$ as the radius of the circle. Then, DOBM is a rhombus $\Rightarrow OB = MB \Rightarrow 1/2AB = 1/2BC \Rightarrow AB = BC$.</p> 

Figure 2. Collective solution spaces for problem 2

Students who produced solutions 1a, 2a and 2b possess at least three kinds of geometrical figure apprehension. In particular, the students apprehend sequentially the figure as they construct the figure from scratch by organizing its elementary units through constructive constraints and mathematical properties. Furthermore, the students have a perceptual apprehension, since they recognize the various subfigures in the circle and the inscribed triangle. They also have discursive apprehension, since they prove the solution of the problem based on theorems. In order to produce

solutions 1b and 3, except the other three types of geometrical figure apprehension, operative apprehension is required as well, since the students have to construct the auxiliary lines for the construction of the median of the triangle and the construction of the rhomb to find the solution of the problem. It is evident that a geometrical figure can be constructed in many ways (mereologic modification), each of which can lead the student to a different solution of the problem. Nevertheless, from the solutions given by the students we observed that some students construct the same auxiliary line but they solve the problem differently. In fact, students apprehend operatively the figure in the same way but have different discursive figure apprehension as for the same auxiliary structure they use different properties and theorems to find the solution.

On the basis of the results of the t-test, the number of solutions (fluency) that students give to problem 1 and 3 ($M = 2.16$, $SD = 1.42$), in which the problems are accompanied by the relevant figures, are statistically bigger ($t = 6.02$, $df = 146$, $p < 0.001$) than those that students give to problems 2 and 4 ($M = 1.44$, $SD = 1.21$) in which the figures are not given. Concerning flexibility of the solutions to the problems in which the figure is given ($M = 19.88$, $SD = 12.65$), and the problems without the corresponding figure ($M = 13.93$, $SD = 11.37$), significant differences are indicated with an apparently higher score in the problems in which the figure is given ($t = 5.57$, $df = 146$, $p < 0.001$). Even though according to the findings students' fluency and flexibility is higher in problems in which the figure is given, this is not the case with the originality of the solutions ($t = -1.87$, $df = 146$, $p > 0.05$) and total creativity ($t = -0.71$, $df = 146$, $p > 0.05$).

In fact, there are not any significant differences as far as originality and creativity are concerned in the two types of problems.

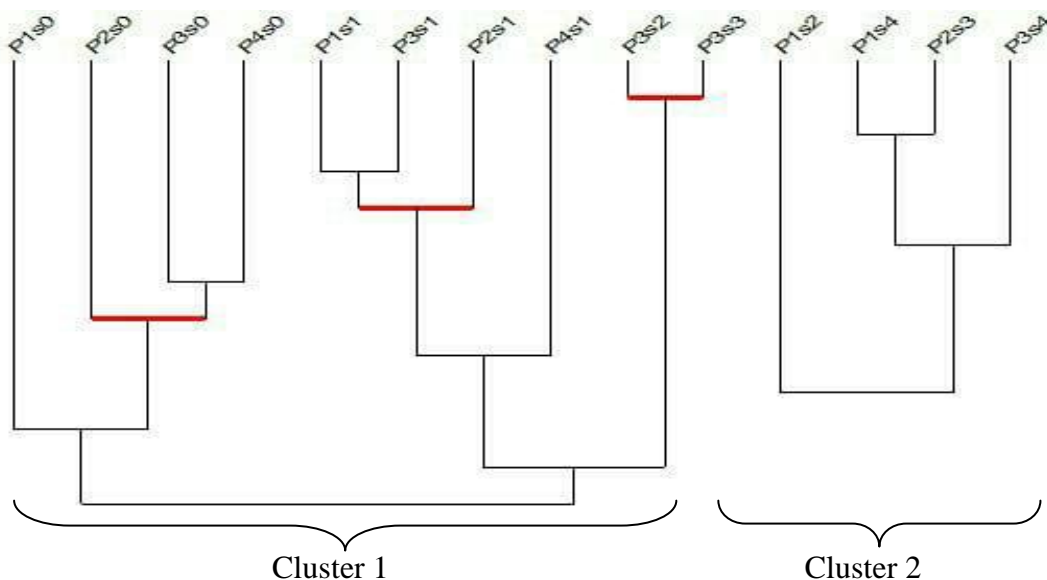


Figure 3. Similarity diagram of students' number of solutions

In the similarity diagram, which concerns students' ability to produce various solutions, there are two significant clusters. Cluster 1 consists of two sub-clusters. The first sub-cluster includes the variables P1s0, P2s0, P3s0, and P4s0. It refers to students who do not give any solution to any of the problems, regardless of the presence of figure or not. The second sub-cluster (P1s1, P2s1, P3s1, P4s1) is related to students who solve all problems in one way, regardless of whether the figure is

given or not. In the second sub-cluster the solutions belong to one solution space. This sub-cluster is associated with the variables P3s2, P3s3 that involve students who have solved the third problem in which the figure is given, in two or three ways, providing evidence that the figure affected its level of difficulty.

Cluster 2 contains the variables P1s2, P1s4, P2s3, and P3s4 that correspond to the students who are highly fluent and flexible in their solutions. On the one hand, these students give two, three or four solutions which mean they have the fluency to produce multiple solutions. On the other hand they are able to move from one solution to another, using different mathematical properties and representations as their solutions belong to at least two different solution spaces.

In fact, the results suggested that the participants of the study form three distinct student profiles:

- Profile 1: Students with poor fluency and flexibility (0 solutions)
- Profile 2: Students with moderate fluency (1-3 solutions) and flexibility (1 solution space)
- Profile 3: Students with high fluency(≥ 2 solutions) and flexibility(≥ 2 solution spaces)

The majority of the students have poor and moderate fluency and flexibility. In particular, fifty students (34%) have poor fluency and flexibility in at least one problem, sixty six students (45%) provide only one solution in at least one problem and twenty five students (17%) solve one of the problems in two different ways at least. Only six students (4%) have high fluency and flexibility in at least one problem.

The qualitative data provided further insight in to students' potential performance. During the solution process, changes concerning fluency and flexibility of student solutions were observed. The following extract from one of the student's interview (Student A) is indicative::

10. St: I'm not so good at geometry.

17. St: Better find another student who knows them better. I do not remember them at all.

19. St: Okay, I leave the first problem and go to the third problem.

(Having already found two ways in the first problem after solving the other three problems)

165. St: Let me think... I have an idea, half a minute to see something in the third problem.

166. St: Construct a parallel line from C to AB and from B to AC and create a rectangle for $A = 90^\circ$ (the student is drawing the new figure). From this I will say that the diagonals of the rectangle are equal and bisected. Therefore, $AD = 1/2BC$.

170. St: I thought of something else.

172. St: Could I construct a circle as we did in the forth problem? Because the right angle ends in a semicircle, let's say that BC will be a diameter and the center will be D. So DC, DB and DA will be radiuses, so they will be equal.

It is evident that, MSTs enhance flexibility in geometric knowledge and student thinking as they encourage connections between different ways of solving them. The student, as soon as he had solved the problems, moved from zero fluency and flexibility during the solution of the first problem to high fluency and flexibility of solutions at the end of the interview. During this process

he often made references to the previous problem solutions. It is worth mentioning, that the qualitative data strengthen quantitative results as the construction of auxiliary lines and reconfigurations that reflect the operative figure apprehension revealed to be the most important aspects for producing new ways of solving and achieving high levels of creativity in geometry.

Discussion

The present study explored students' creativity in MSTs using a multidimensional approach as the following dimensions are considered: (a) mathematical creativity; (b) geometrical figure apprehension; and (c) multiple-solution problem solving in mathematics education. Significant differences in the fluency and flexibility of student solutions were observed depending on whether or not the figure was given (RQb). In fact, students exhibited greater fluency and flexibility in the problems in which the verbal description of the problem was accompanied by the relevant figure. As far as the originality and the total creativity of students' solutions are concerned, they did not differ significantly in the two types of problems. On the one hand these findings are explained taking into account that even though flexibility and fluency are different aspects of creativity, they are strongly related to each other (e.g. Leikin, 2009). On the other hand, researchers (e.g. Levav-Waynberg & Leikin, 2012, Leikin, 2013) pointed out that originality is a more internal characteristic than fluency and flexibility. It is also more related to creativity but less dynamic.

The similarity analysis of the number of student solutions suggested three distinct profiles of students: (a) a poor fluency and flexibility group; (b) a moderate fluency and flexibility group; and (c) a group of high fluency and flexibility (RQc). The majority of students belong to the first two groups. However, qualitative results provide evidence that students managed to move from poor to high fluency and flexibility of solutions through references to previous problems they had solved. This is in line with Leikin's (2013) findings that fluency and flexibility are dynamic, whereas originality is a "gift". As our work deals with relative creativity (Leikin, 2009), we support that mathematical creativity should be developed in all students (Sheffield, 2009). As a matter of fact, creativity is influenced by the teaching, guidance and experiences of the individual (Silver, 1997). Thus, the systematic exploitation in the teaching of MSTs in geometry may enhance student fluency and flexibility.

Furthermore, emphasis should be given to activities that help students overcome the perceptual way of looking at a geometrical figure and moving to operative figure apprehension. The results suggest that the students who apprehend the figure operatively are able to undertake mereologic modifications that enable them to provide different proofs in geometrical problems (RQa). An interesting finding which highlights heuristic functioning of figures is that some students use the same auxiliary construction but their solutions are based on different properties and theorems. In other words, they have the same operative apprehension but different discursive apprehension of a figure. Indeed, operative apprehension of a figure is a source of creativity in geometry as it gives insight into the solution of the problem. Nevertheless, Duval (1995) indicated that the mathematical way of looking at figures only results from the coordination of separate processes of apprehension over a long time. In line with this assertion, our findings revealed that geometrical figure

apprehension is a prerequisite for students to provide a variety of solutions to the geometry problems and, thus, have high creativity.

Certainly, there is still need for further investigation into the teaching implications of the subject. In the future, it would be interesting and useful to examine the effects of intervention programs aiming at developing geometrical figure apprehension in view of MSTs. Furthermore, the validation of a theoretical model that examines the structure and relation of various components of geometrical figure apprehension, mathematical creativity and ability to prove in multiple ways would contribute to this domain of research too.

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