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The evolution of 9-year-old students' understanding of the relationships among geometrical shapes

Melania Bernabeu¹, Salvador Llinares² and Mar Moreno³

¹University of Alicante, Faculty of Education, Alicante, Spain; melania.bernabeu@ua.es

²University of Alicante, Faculty of Education, Alicante, Spain; sllinares@ua.es

³University of Alicante, Faculty of Education, Alicante, Spain; mmoreno@ua.es

We conducted a teaching experiment of ten lessons with third-grade students (9-years-olds) to analyse the evolving understanding of the relationships among geometrical shapes. In the teaching experiment, pupils recognized, described, represented, built and classified two-dimensional figures. Analysis of pre-/post assessments showed some changes in students reasoning about how they began making sense of the relationships among the geometrical shapes. However, the use of formal definitions of geometrical shapes constituted a challenge for students since they imposed linguistics conditions to how the relationships among figures could be established. The results show that understanding the relationships among geometrical shapes is a slow process and depends on the logical terms used in the formal definitions.

Keywords: Geometrical thinking, primary school, geometrical shapes, relationships among geometrical figures.

Introduction

Students develop the comprehension of geometrical figures in a progressive way, initially, they recognise the figures by perceptual similarity, continue with the recognition of attributes and finally they base their thinking on concept based on the definition (Satlow & Newcombe, 1998). Recognizing attributes and relating them to make classifications is considered a relevant aspect in the development of geometrical thinking (Clements, Swaminathan, Hannibal, & Sarama, 1999; Elia & Gagatsis, 2003; Levenson, Tirosh, & Psamir, 2011). Classification of geometrical shapes, links to definition and the analyse of attributes of different geometrical shapes to distinguish between critical and non-critical attributes. Classification process can be hierarchical or partitive (de Villiers, 1994; de Villiers, Govender, & Patterson, 2009).

By the term hierarchical classification is meant here the classification of a set of concepts in such a manner that more particular concepts form subsets of the more general concepts. [...] In contrast to a hierarchical classification there also exists the possibility of a partition classification of concepts. In such a classification however, the various subsets of concepts are considered to be disjoint from one another (de Villier, 1994, p. 11–12).

It has been shown that students have difficulties in the transition from the recognition of the attributes and distinguish critical and non-critical aspects to perform classifications, especially hierarchical classifications of the quadrilaterals (Bernabeu, Moreno, & Llinares, 2018; Erez & Yerushalmy, 2006; Gal & Linchevski, 2010; Jones 2000). To try to better understand this transition in students' primary school, we investigated third-grade students' evolving understanding of the relationships among geometrical shapes, especially to know how pupils understand geometrical

figures and establish relationships among these geometrical figures, that is, consider whether a geometrical figure belongs or no to a figure class.

Theoretical perspectives

The relationship between levels of geometrical thinking and figure comprehension

According to Van Hiele's theory (Clements & Battista, 1992; Battista, 2007), the development of students' geometrical thinking is divided into progressive levels. The first three levels describe the process up to the comprehension of relationships among geometrical shapes:

- Level 1: Visual. Students identify the figures in relation to their appearance. To recognize figures, they do so through familiar prototypes (e.g. saying that a rectangle "looks like a door"). They are not aware of the properties of geometrical figures. The reasoning of students is dominated by perception.
- Level 2: Analytic/Componential (Battista, 2007). Students identify shapes according to their properties and they can characterize the figures by their attributes. Battista (2007) refined this level proposing three sub-levels. He theorized that students begin with "visual-informal componential reasoning" where they begin to focus on parts of shapes and doing visual, informal and imprecise descriptions. Pupils advance to "informal and insufficient-formal componential reasoning" where they use formal terms and informal descriptions, but they make definitions with unnecessary attributes. Occasionally, students advance to "sufficient formal property-based reasoning" where they make descriptions with formal geometrical terms for all the properties of the geometrical figures. The change from insufficient to sufficient definitions is a gradual shift from seeing the figures as a whole to focusing on the parts of these to see the figure as a coherent structure (Pegg & Davey, 1998).
- Level 3: Abstract/Relational. Students identify relationships among shapes and give informal arguments to justify their classifications (e.g., a square is a special type of parallelogram). They can discover properties of kinds of figures by informal deduction. Therefore, it is considered that the hierarchical classification is at this level.

The dual nature of geometrical figures

Tall and Vinner (1981) developed the ideas of concept image and concept definition. Concept image is used to describe, "the total cognitive structure that is associated with the concept, which includes all the mental images, properties, and associated processes" (Tall & Vinner, 1981, p.152). Concept image may include images that are inappropriate and contradict the definition of the concept (Hershkowitz, Bruckheimer, & Vinner, 1987; Vinner, 1991; Levenson et al., 2011). On the other hand, concept definition is "a discursive description to specify that concept" (Tall & Vinner, 1981, page 152) accepted by the mathematical community. One relevant aspect in the relationship between concept image and concept definition is to become to recognize the attributes of the definition as a sufficient condition. This dual nature of geometrical shapes in which concept and image are interrelated is underlined by the notion of "figural concept" (Fischbein, 1993).

In the understanding of this dual nature of geometrical figures, the sufficient conditions to recognize or to represent examples of geometrical shapes can be revealed by the transformations among

various semiotic representations. Duval (1995a, 1999) indicates how his theory of registers of semiotic representation can be used as a tool to analyse the cognitive processes through which pupils develop the geometrical thinking. Duval (2017) indicates that we can analyse the students' productions in terms of registers regarding the coordination between different apprehensions: perceptual, discursive and operative. Perceptual apprehension is the ability to recognize a figure in a global way; the discursive apprehension is the ability to link mathematical properties with the parts of the figure; and operational apprehension is the ability to modify or construct a figure to solve the task (change orientation, decompose it). Duval (1995b, 2006) defined two of such transformations: Treatment and Conversion. Treatments are transformations of representations that happen within the same register: for example, completing a figure using perceptual criteria of connectivity or symmetry. Conversions consist of transformations of representations from one register to another (such as from the visual register to the discursive one), without changing the objects that denote. The association of a figure with mathematical statements, can be done in both directions: from the visual to the discursive (from the figure to the definition) and from the discursive to the visual (from the definition to recognize the geometrical figure) (Duval, 1995b). This translation is the one that allows linking the process of defining a geometrical figure giving a minimum set of relevant attributes. However, there are no descriptors of the cognitive processes through which students develop the geometrical thinking. Therefore, we propose to identify descriptors of the development of geometrical thinking about geometrical figures in 9-year-old pupils, and specifically,

- How students understand geometrical figures and,
- How reasoning about the relationships among them?

Method

Context

Participants were fifty-nine third-grade pupils (9-years-old) from an elementary school located in an area of average socioeconomic status in Spain. This study was conducted as a teaching experiment designed to allow pupils establish relationships among geometrical shapes. A researcher of the research group performed the role of the teacher in both classes during the instruction.

Teaching experiment

In each class, we carried out ten sessions, two per week from May to June during the academic year 2017-2018. Pupils were asked for: recognizing and analysing attributes of different shapes, identifying critical and non-critical attributes, establishing relationships between attributes of shapes. They also had to represent and/or construct the shapes according to the given conditions or classify attending to criteria. Contents used during all sessions are shown in Table 1.

First week	First session. Polygon attributes: closed, straight and non-intersecting sides. Polygons according to the number of sides	Second session. Diagonals. Concave and convex polygons
Second week	Third session. Symmetry axis. Symmetrical figures	Fourth session. Acute, right and obtuse angle

Third week	Fifth session. Types of triangles according to their sides: equilateral, isosceles and scalene triangle. Hierarchical relations: the equilateral is an isosceles	Sixth session. Types of triangles according to their angles: acute, right and obtuse triangle
Fourth week	Seventh session. Types of triangles according to their sides and angles	Eighth session. Types of lines: parallel and non-parallel. Types of quadrilaterals: parallelograms, trapezes and trapezoids
Fifth week	Ninth session. Types of parallelograms: square, rhombus, rectangle and rhomboid. Hierarchical relations: the square is a rhombus, and the square is a rectangle	Tenth session. Properties of parallelograms: equal opposite sides and angles; diagonals cut in half

Table 1. Conceptual characteristics of the sessions of the Teaching Experiment

During the sessions, pupils built different shapes with resources such as *mecano* or geoboard. For working the classification tasks, we used digital board, so students could share their resolution with the whole group. Next, students individually solved other tasks. We used hierarchical definitions. For example, we defined the isosceles triangle as a triangle with two equal sides. In this case, we consider that an equilateral triangle is an isosceles triangle since it has two equal sides. Furthermore, we defined a trapezium as a quadrilateral with one pair of parallel sides. In this case, a parallelogram is a trapezium (British version) (Fujita & Jones, 2007).

For this report, data come from the initial and the final test. The aim was to get evidences about pupils' understanding of geometrical figures. Students' answers were codified and agreed by the research group. For the analysis of the data and the creation of the scheme of emergent codes that we generate from the answers of the students, we rely on van Hiele levels (Clements & Battista, 1992; Battista, 2007); Duval's theory (1995a, 1995b, 1999) that considers the transformations of the registers and the coordination of the apprehensions, the relationships between image concept and definition concept (Tall & Vinner, 1981), and the way that geometrical figures were related (to classify a shape in different ways and label it with different names) (Erez & Yerushalmy, 2006).

Results

The code schemes

The emergent coding scheme changed to accommodate the features that pupils included in their work and regarding the specificity of the tasks (Table 2). We compared the responses to the initial and final test and developed the emergent coded schemes as follow:

- *Students identified geometrical figures by their appearance*: what that it means that the recognition of geometrical figures is purely perceptual. For example, concerning to task 5: "It is a normal and ordinary triangle"; "The same figure as the " figure a"; "All the figures are triangular"; and in the Task 6: "Yes, because it is the same".
- *Students recognized some attributes of geometrical figures without establishing relationships among attributes that define the general set*: that it, students could recognize

some attributes of geometrical figures as number of sides, names of figures or types of figures, but they still made mistakes when they related them to its definition and perform relationships among geometrical figures. The apprehensions are not still coordinated correctly. For example, in task 5: “It has three vertices”; “It is a triangle”; and in task 6: “It has four sides”; “It is a square”.

- *Students were able to establish relationships among figures by recognizing that a figure belonged to a general set.* That is it, pupils related the definition to the parts of plane shapes to realize relationships (hierarchical and partitive relations). They began to coordinate the apprehensions. For example, in the hierarchical relation, concerning to task 5: “Yes, because an equilateral is an isosceles” and in task 6: “Yes, because at least has 2 parallel sides”. Conversely, regarding partitive relation, in task 5 a pupil said: “No, because an equilateral is not an isosceles” and in task 6 another said: “No, because it has all parallel sides” (not including the parallelogram as a trapezium).

Initial and Final test: Progression of thinking about relationships among geometrical shapes

To report the progress of the understanding of plane geometrical figures, we present the answers to two tasks from the initial and final test. In both tasks we had a "Drawing Machine" that could make a set of figures with common conditions (Battista, 2012). In one task, we provided a set of isosceles triangles (a triangle with two sides equals) that the machine could make and a set of scalene triangles (triangles with the three sides different) that the machine could not make. The question was that if the Drawing Machine (Figure 1) could draw an equilateral triangle (visual example).

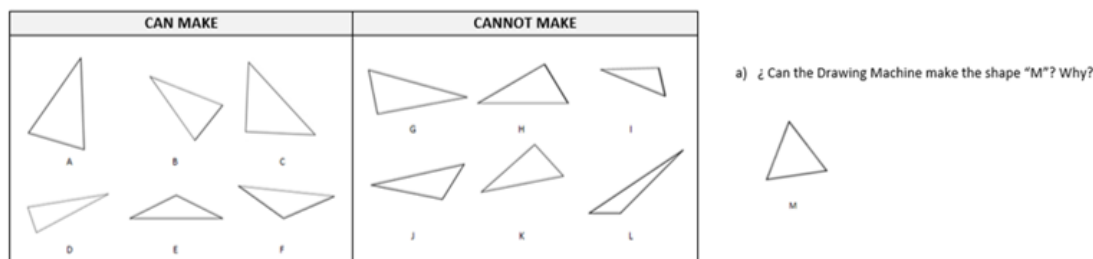


Figure 1. Figures of Triangles in the Task 5

In another task, a verbal description of the shape was given: The Drawing Machine can make “quadrilaterals (polygon with four-sided) with two parallel sides” and it was asked if the parallelogram (square) (visual example) could be made by the machine (Figure 2).

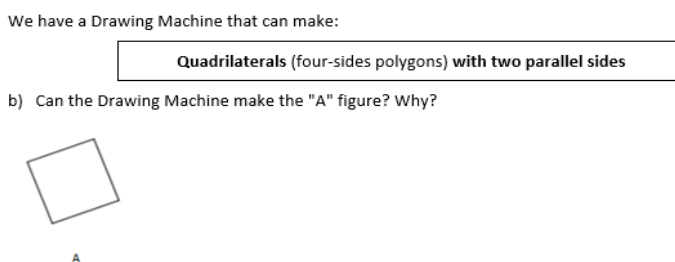


Figure 2. Figure of Quadrilateral in the Task 6

These tasks tried to show how students establish relationships among figures and use the logical relationships among the attributes identified in the figure. In this paper, we present the frequencies of codes of both tasks (118 answers, that is, 59 of each task) that reflect the progress from perceptual to analytic recognition, and the initial understanding of the relationships among figures (Table 2). The frequency of the students' answers has shown a development in the reasoning of elementary school pupils about geometrical figures and it has also shown how they reason about the relationships among these figures.

Codes (n=118)		Initial-Test	Final-Test
1. Students identify the figures in relation to their appearance		26 (22%)	12 (10%)
2. Recognize some attributes of geometrical figures (but without established relationships with attributes that define the general set)		28 (24%)	48 (41%)
3. The relationship among figures (recognize a figure belong to a general set)	3.1. Recognize the attribute defines the set (recognize hierarchical relationship)	10 (8%)	26 (22%)
	3.2. Recognize the only partitive relationship	2 (2%)	7 (6%)
4. Others (meaningless or blank answers)		52 (44%)	25 (21%)

Table 2. Frequency of codes

The development of the understanding of geometrical figures

Analysis of data has shown that students have modified their way of reasoning from the initial test to the final. First, the purely perceptual recognition decreased from 22% to 10% after the teaching experiment. Also, the recognition of attributes in the figures presented in both tests increased from 24% to 41%. On the other hand, concerning the relationships among geometrical figures, it has been possible to verify how in the final tests, these relationships have been accentuated, since in the hierarchical relations had 8% in the initial test, and this rose to 22% in the final test. Regarding partisan relationships, these results rose by 2% in the initial to 6% in the final test. On the other hand, another positive data is the decrease of the frequencies in the code of "others", in which we coded answers that did not make sense with what was demanded in the statement or this was blank. This code decreased by 44% in the initial test to 22% in the final. In conclusion, the frequencies of the final test, compared with the initial test, provide us with information to intuit that pupils begin to develop their understanding of geometrical figures.

Reasoning about the relationships among geometrical figures

Initially, 12% of the answers were linked to the relationships among geometrical figures to the general set provided in the tasks. However, after the teaching experiment this percentage increased to 33%, where it was intended to develop the recognition of the critical attributes of the figures so that they could make relationships among geometrical figures. Of this total, regarding the hierarchical relations, the answer increased from 8% to 22%. Regarding to partitive relations, it

increased from 2% to 6%. These data lead us to think that students begin to develop the relationships among geometrical figures.

Discussion

The aim of this study is to know how students understand geometrical figures and how they reason about the relationships among them. We used a teaching experiment to check how geometrical thinking evolved if we introduced vocabulary and definitions related to hierarchical relationships. As well as, we analysed two tasks, from an initial and a final test, to verify this development of the understanding of geometrical figures. Data has shown us that many pupils answered the tasks perceptually without considering the relationships of the figures or the definition of the concept, in the initial test. In contrast, after the teaching experiment, in the final test, most of pupils began to identify attributes to the plane geometrical figures and to reason about the relationships among them. These data lead us to think that the development of the understanding of geometrical figures is progressive (Pegg & Davey, 1998; Satlow & Newcombe, 1998).

From these data, we have been able to begin to comprise the progress of the understanding of the plane geometrical figures in relation to Duval's theory (1995a, 1995b, 1999). Because students at the beginning did not know how to coordinate the apprehensions and after the teaching experiment, they began to coordinate the apprehensions making the necessary transformations, the relationships among attributes and the use of minimum attributes of the definition for doing the relationships among geometrical figures. In this way, while developing an understanding of geometric figures and how to relate them. They began to develop the personal figural concept (Fujita & Jones, 2007). Thus, we can intuit that the realization of the hierarchical classifications depends on the language that is used in the definitions of the geometrical concepts, which can help identify the minimum attributes to recognize a type of geometrical figure.

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