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Shalom Eliahou

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An adaptive upper bound on the Ramsey numbers $R(3,\ldots,3)$

S. Eliahou

Abstract

Since 2002, the best known upper bound on the Ramsey numbers $R_n(3) = R(3,\ldots,3)$ is $R_n(3) \leq n!(e-1/6)+1$ for all $n \geq 4$. It is based on the current estimate $R_4(3) \leq 62$. We show here how any closing-in on $R_4(3)$ yields an improved upper bound on $R_n(3)$ for all $n \geq 4$. For instance, with our present adaptive bound, the conjectured value $R_4(3) = 51$ implies $R_n(3) \leq n!(e-5/8)+1$ for all $n \geq 4$.

1 Introduction

For $n \geq 1$, the n-color Ramsey number $R_n(3) = R(3,\ldots,3)$ denotes the smallest N such that, for any n-coloring of the edges of the complete graph K_N, there is a monochromatic triangle. See e.g. [4, 8, 11] for background on Ramsey theory. There is a well known recursive upper bound on $R_n(3)$ due to [5], namely

$$R_n(3) \leq n(R_{n-1}(3)-1)+2 \tag{1}$$

for all $n \geq 2$. Currently, the only exactly known values of $R_n(3)$ are $R_1(3) = 3$, $R_2(3) = 6$ and $R_3(3) = 17$. As for $n = 4$, the current state of knowledge is

$$51 \leq R_4(3) \leq 62.$$

The lower bound is due to [1] and the upper bound to [3], down from the preceding bound $R_4(3) \leq 64$ in [9]. Moreover, it is conjectured in [14] that

$$R_4(3) = 51.$$

Here is a brief summary of successive upper bounds on $R_n(3)$. In [5], the authors proved that

$$R_n(3) \leq n!e + 1$$

for all $n \geq 2$. Whitehead's results [13] led to

$$R_n(3) \leq n!(e - 1/24) + 1$$

for all $n \geq 2$, and Wan [12] further improved it to

$$R_n(3) \leq n!(e - e^{-1} + 3)/2 + 1.$$

The last improvement came in 2002, when it was proved in [15] that

$$R_n(3) \leq n!(e - 1/6) + 1$$

for all $n \geq 4$. That bound relies on the estimate $R_4(3) \leq 62$ by [3].

Because of the recurrence relation (1), any improved upper bound on $R_k(3)$ for some $k \geq 4$ will yield an improved upper bound on $R_n(3)$ for all $n \geq k$. Our purpose here is to make this automatic improvement explicit. For instance, combined with our adaptive upper bound, the above-mentioned conjecture $R_4(3) = 51$ implies

$$R_n(3) \leq n!(e - 5/8) + 1$$

for all $n \geq 4$. This would be a substantial improvement over the current upper bound $n!(e - 1/6) + 1$, since $e - 1/6 \approx 2.55$ while $e - 5/8 \approx 2.09$.

2 Main results

As reported in [7], it is proved in [15] that

$$R_n(3) \leq n!(e - 1/6) + 1$$

for all $n \geq 4$. But the latter paper is in Chinese and not easily accessible to English readers. In this section, we prove a somewhat more general statement. We shall need the formulas below.

2.1 Useful formulas

In proving $R_n(3) \leq n!e+1$, the authors of [5] used without comment the formula

$$\lfloor (n+1)!e \rfloor = (n+1)\lfloor n!e \rfloor + 1$$

for all $n \geq 1$. For convenience, we provide a proof here, as a direct consequence of the auxiliary formula below.

Proposition 2.1. *For all $n \geq 1$, we have $\lfloor n!e \rfloor = \sum_{i=0}^{n} n!/i!$.*

Proof. We have $e = 1/0! + 1/1! + \sum_{i=2}^{\infty} 1/i! = 2 + \sum_{i=2}^{\infty} 1/i!$. Since $e < 3$, it follows that $\sum_{i=2}^{\infty} 1/i! < 1$. Now $n!e = \sum_{i=0}^{n} n!/i! + \sum_{i=n+1}^{\infty} n!/i!$. The left-hand summand is an integer, while the right-hand one satisfies

$$\sum_{i=n+1}^{\infty} n!/i! = \sum_{j=1}^{\infty} \frac{1}{\Pi_{k=1}^{j}(n+k)} \leq \sum_{i=2}^{\infty} 1/i! < 1.$$

This concludes the proof. □

Corollary 2.2 ([5])**.** *For all $n \geq 1$, we have $\lfloor (n+1)!e \rfloor = (n+1)\lfloor n!e \rfloor + 1$.*

Proof. Applying Proposition 2.1 for $n+1$ and then for n, we have

$$\begin{aligned} \lfloor (n+1)!e \rfloor &= \sum_{i=0}^{n+1} (n+1)!/i! \\ &= (n+1)\sum_{i=0}^{n} n!/i! + (n+1)!/(n+1)! \\ &= (n+1)\lfloor n!e \rfloor + 1. \quad \square \end{aligned}$$

2.2 An optimal model

We now exhibit an optimal model for the recursion (1).

Proposition 2.3. *Given $q \in \mathbb{Q}$, let $f\colon \mathbb{N} \to \mathbb{Z}$ be defined by $f(n) = \lfloor n!(e-q) \rfloor + 1$ for $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$ such that $n!q \in \mathbb{Z}$, we have*

$$f(n+1) = (n+1)(f(n)-1) + 2. \tag{2}$$

Proof. We have

$$\begin{aligned} f(n+1) &= \lfloor (n+1)!(e-q) \rfloor + 1 \\ &= \lfloor (n+1)!e \rfloor - (n+1)!q + 1 \quad [\text{since } (n+1)!q \in \mathbb{Z}] \\ &= (n+1)\lfloor n!e \rfloor + 1 - (n+1)!q + 1 \quad [\text{by Corollary 2.2}] \\ &= (n+1)\lfloor n!(e-q) \rfloor + 2 \quad [\text{since } n!q \in \mathbb{Z}] \\ &= (n+1)(f(n)-1)+2. \quad \square \end{aligned}$$

2.3 An adaptive bound

Our adaptive upper bound on $R_n(3)$ is provided by the following statements.

Proposition 2.4. *Let $k \in \mathbb{N}$ and $q \in \mathbb{Q}$ satisfy $k \geq 2$, $R_k(3) \leq k!(e-q)+1$ and $k!q \in \mathbb{N}$. Then $R_n(3) \leq n!(e-q)+1$ for all $n \geq k$.*

Proof. As in Proposition 2.3, denote $f(n) = \lfloor n!(e-q) \rfloor + 1$ for $n \in \mathbb{N}$. By assumption, we have

$$R_k(3) \leq f(k) \tag{3}$$

and $k!q \in \mathbb{Z}$. It suffices to prove the claim for $n = k+1$, since if $k!q \in \mathbb{N}$ then $(k+1)!q \in \mathbb{N}$. By successive application of (1), (3) and (2), we have

$$\begin{aligned} R_{k+1}(3) &\leq (k+1)(R_k(3)-1)+2 \\ &\leq (k+1)(f(k)-1)+2 \\ &= f(k+1). \end{aligned}$$

Note that using (2) is allowed by Proposition 2.3 and the assumption $k!q \in \mathbb{N}$. $\square$

Theorem 2.5. *Let $k \geq 2$ be an integer. Let $a \in \mathbb{N}$ satisfy $a \leq \lfloor k!e \rfloor - R_k(3) + 1$, and let $q = a/k!$. Then $R_n(3) \leq n!(e-q)+1$ for all $n \geq k$.*

Proof. We have $a \leq k!e - R_k(3) + 1$, so $R_k(3) \leq k!e - a + 1 = k!(e-q)+1$. Moreover $k!q = a \in \mathbb{N}$. The conclusion follows from Proposition 2.4. $\square$

Remark 2.6. *Theorem 2.5 is the best possible application of Proposition 2.4. Indeed, with the value $a' = \lfloor k!e \rfloor - R_k(3) + 2$ and $q' = a'/k!$, it no longer holds that $R_k(3) \leq k!(e-q')+1$.*

2.4 The case $k = 4$

We now apply the above result to the case $k = 4$. We only know $51 \leq R_4(3) \leq 62$ so far. Note that by Proposition 2.1, we have

$$\lfloor 4!e \rfloor = \sum_{i=0}^{4} 4!/i! = 24 + 24 + 12 + 4 + 1 = 65. \tag{4}$$

Proposition 2.7. *Let $a \in \mathbb{N}$ satisfy $a \leq 66 - R_4(3)$. Then setting $q = a/24$, we have $R_n(3) \leq n!(e-q)+1$ for all $n \geq 4$.*

Proof. By (4), a satisfies the hypotheses of Theorem 2.5. The conclusion follows. □

When the exact value of $R_4(3)$ will be known, Proposition 2.7 will provide an adapted upper bound on $R_n(3)$ for all $n \geq 4$. In the meantime, here are three possible outcomes.

Corollary 2.8 ([15])**.** *$R_n(3) \leq n!(e-1/6)+1$ for all $n \geq 4$.*

Proof. Since $R_4(3) \leq 62$, we may take $a = 4$ in Proposition 2.7. The conclusion follows from that result with $q = a/4! = 1/6$. □

Note that the above bound dos not extend to $n = 3$, since $R_3(3) = 17$, whereas by Proposition 2.1, we have $\lfloor 3!(e-1/6) \rfloor + 1 = \lfloor 3!e \rfloor = 3! + 3! + 3 + 1 = 16$.

As mentioned earlier, it is conjectured in [14] that $R_4(3) = 51$. If true, Proposition 2.7 will yield the following improved upper bound.

Corollary 2.9. *If $R_4(3) = 51$, then $R_n(3) \leq n!(e-5/8)+1$ for all $n \geq 4$.*

Proof. By Proposition 2.7, with $a = 66 - 51 = 15$ and $q = 15/4! = 5/8$. □

As noted in the Introduction, this would be a substantial improvement over the current upper bound $n!(e-1/6)+1$, since $e - 1/6 \approx 2.55$ whereas $e - 5/8 \approx 2.09$.

An intermediate step would be, for instance, to show $R_4(3) \leq 54$ if at all true. This would yield the following weaker improvement.

Corollary 2.10. *If $R_4(3) \leq 54$, then $R_n(3) \leq n!(e-1/2)+1$ for all $n \geq 4$.*

Proof. By Proposition 2.7, with $a = 66 - 54 = 12$ and $q = a/4! = 1/2$. □

Remark 2.11. *The above three corollaries are best possible applications of Proposition 2.7, as in each case we took the largest admissible value for $a \in \mathbb{N}$.*

2.5 The case $k = 5$

Let us also briefly consider the case $k = 5$. At the time of writing, we only know $162 \leq R_5(3) \leq 307$. See [7].

Proposition 2.12. *Let $a \in \mathbb{N}$ satisfy $a \leq 327 - R_5(3)$. Then setting $q = a/120$, we have $R_n(3) \leq n!(e-q)+1$ for all $n \geq 5$.*

Proof. By Theorem 2.5 and the value $\lfloor 5!e \rfloor = 326$ given by Proposition 2.1. □

Here again are three possible outcomes. Knowing only $R_5(3) \leq 307$ does not allow to improve the current estimate $R_n(3) \leq n!(e-1/6)+1$. At the other extreme, if $R_5(3) = 162$ holds true, it would yield $R_n(3) \leq n!(e-11/8)+1$ for all $n \geq 5$. As an intermediate estimate, if $R_5(3) \leq 227$ holds true, it would imply $R_n(3) \leq n!(e-5/6)+1$ for all $n \geq 5$.

3 Concluding remarks

3.1 On $\lim_{n\to\infty} R_n(3)^{1/n}$

The adaptive upper bound on $R_n(3)$ given by Theorem 2.5 may still be quite far from reality, as the asymptotic behavior of $R_n(3)$ remains poorly understood. For instance, is there a constant c such that $R_{n+1}(3) \leq cR_n(3)$ for all n? Or, maybe, such that $R_n(3) \geq cn!$ for all n? The former would imply that $\lim_{n\to\infty} R_n(3)^{1/n}$, known by [2] to exist, is finite, whereas the latter would imply $\lim_{n\to\infty} R_n(3)^{1/n} = \infty$. At the time of writing, it is not known whether that limit is finite or infinite. See e.g. [6], where this question is discussed together with related problems.

3.2 Link with the Schur numbers

The Schur number $S(n)$ is defined as the largest integer N such that for any n-coloring of the integers $\{1,2,\ldots,N\}$, there is a monochromatic triple of integers $1 \leq x,y,z \leq N$ such that $x+y=z$. The existence of $S(n)$ was established by Schur in [10], an early manifestation of Ramsey theory. Still in [10], Schur proved the upper bound

$$S(n) \leq n!e - 1 \tag{5}$$

for all $n \geq 2$. The similarity with the upper bound $R_n(3) \leq n!e+1$ proved 40 years later in [5] is striking. In fact, there is a well known relationship between these

numbers, namely

$$S(n) \leq R_n(3) - 2. \tag{6}$$

Thus, via (6), our adaptive upper bound on $R_n(3)$ given by Theorem 2.5 also yields an upper bound on $S(n)$.

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Author's adress:
Shalom Eliahou, Univ. Littoral Côte d'Opale,
EA 2597 - LMPA - Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville,
F-62228 Calais, France,
CNRS, FR 2956, France.
E-mail: `eliahou@univ-littoral.fr`