

# An adaptive upper bound on the Ramsey numbers $R(3, \dots, 3)$

S. Eliahou

## Abstract

Since 2002, the best known upper bound on the Ramsey numbers  $R_n(3) = R(3, \dots, 3)$  is  $R_n(3) \leq n!(e - 1/6) + 1$  for all  $n \geq 4$ . It is based on the current estimate  $R_4(3) \leq 62$ . We show here how any closing-in on  $R_4(3)$  yields an improved upper bound on  $R_n(3)$  for all  $n \geq 4$ . For instance, with our present adaptive bound, the conjectured value  $R_4(3) = 51$  implies  $R_n(3) \leq n!(e - 5/8) + 1$  for all  $n \geq 4$ .

## 1 Introduction

For  $n \geq 1$ , the  $n$ -color Ramsey number  $R_n(3) = R(3, \dots, 3)$  denotes the smallest  $N$  such that, for any  $n$ -coloring of the edges of the complete graph  $K_N$ , there is a monochromatic triangle. See e.g. [4, 8, 11] for background on Ramsey theory. There is a well known recursive upper bound on  $R_n(3)$  due to [5], namely

$$R_n(3) \leq n(R_{n-1}(3) - 1) + 2 \tag{1}$$

for all  $n \geq 2$ . Currently, the only exactly known values of  $R_n(3)$  are  $R_1(3) = 3$ ,  $R_2(3) = 6$  and  $R_3(3) = 17$ . As for  $n = 4$ , the current state of knowledge is

$$51 \leq R_4(3) \leq 62.$$

The lower bound is due to [1] and the upper bound to [3], down from the preceding bound  $R_4(3) \leq 64$  in [9]. Moreover, it is conjectured in [14] that

$$R_4(3) = 51.$$

Here is a brief summary of successive upper bounds on  $R_n(3)$ . In [5], the authors proved that

$$R_n(3) \leq n!e + 1$$

for all  $n \geq 2$ . Whitehead's results [13] led to

$$R_n(3) \leq n!(e - 1/24) + 1$$

for all  $n \geq 2$ , and Wan [12] further improved it to

$$R_n(3) \leq n!(e - e^{-1} + 3)/2 + 1.$$

The last improvement came in 2002, when it was proved in [15] that

$$R_n(3) \leq n!(e - 1/6) + 1$$

for all  $n \geq 4$ . That bound relies on the estimate  $R_4(3) \leq 62$  by [3].

Because of the recurrence relation (1), any improved upper bound on  $R_k(3)$  for some  $k \geq 4$  will yield an improved upper bound on  $R_n(3)$  for all  $n \geq k$ . Our purpose here is to make this automatic improvement explicit. For instance, combined with our adaptive upper bound, the above-mentioned conjecture  $R_4(3) = 51$  implies

$$R_n(3) \leq n!(e - 5/8) + 1$$

for all  $n \geq 4$ . This would be a substantial improvement over the current upper bound  $n!(e - 1/6) + 1$ , since  $e - 1/6 \approx 2.55$  while  $e - 5/8 \approx 2.09$ .

## 2 Main results

As reported in [7], it is proved in [15] that

$$R_n(3) \leq n!(e - 1/6) + 1$$

for all  $n \geq 4$ . But the latter paper is in Chinese and not easily accessible to English readers. In this section, we prove a somewhat more general statement. We shall need the formulas below.

## 2.1 Useful formulas

In proving  $R_n(3) \leq n!e + 1$ , the authors of [5] used without comment the formula

$$\lfloor (n+1)!e \rfloor = (n+1)\lfloor n!e \rfloor + 1$$

for all  $n \geq 1$ . For convenience, we provide a proof here, as a direct consequence of the auxiliary formula below.

**Proposition 2.1.** *For all  $n \geq 1$ , we have  $\lfloor n!e \rfloor = \sum_{i=0}^n n!/i!$ .*

*Proof.* We have  $e = 1/0! + 1/1! + \sum_{i=2}^{\infty} 1/i! = 2 + \sum_{i=2}^{\infty} 1/i!$ . Since  $e < 3$ , it follows that  $\sum_{i=2}^{\infty} 1/i! < 1$ . Now  $n!e = \sum_{i=0}^n n!/i! + \sum_{i=n+1}^{\infty} n!/i!$ . The left-hand summand is an integer, while the right-hand one satisfies

$$\sum_{i=n+1}^{\infty} n!/i! = \sum_{j=1}^{\infty} \frac{1}{\prod_{k=1}^j (n+k)} \leq \sum_{i=2}^{\infty} 1/i! < 1.$$

This concludes the proof. □

**Corollary 2.2** ([5]). *For all  $n \geq 1$ , we have  $\lfloor (n+1)!e \rfloor = (n+1)\lfloor n!e \rfloor + 1$ .*

*Proof.* Applying Proposition 2.1 for  $n+1$  and then for  $n$ , we have

$$\begin{aligned} \lfloor (n+1)!e \rfloor &= \sum_{i=0}^{n+1} (n+1)!/i! \\ &= (n+1) \sum_{i=0}^n n!/i! + (n+1)!/(n+1)! \\ &= (n+1)\lfloor n!e \rfloor + 1. \quad \square \end{aligned}$$

## 2.2 An optimal model

We now exhibit an optimal model for the recursion (1).

**Proposition 2.3.** *Given  $q \in \mathbb{Q}$ , let  $f: \mathbb{N} \rightarrow \mathbb{Z}$  be defined by  $f(n) = \lfloor n!(e-q) \rfloor + 1$  for  $n \in \mathbb{N}$ . Then, for all  $n \in \mathbb{N}$  such that  $n!q \in \mathbb{Z}$ , we have*

$$f(n+1) = (n+1)(f(n) - 1) + 2. \tag{2}$$

*Proof.* We have

$$\begin{aligned}
f(n+1) &= \lfloor (n+1)!(e-q) \rfloor + 1 \\
&= \lfloor (n+1)!e \rfloor - (n+1)!q + 1 \quad [\text{since } (n+1)!q \in \mathbb{Z}] \\
&= (n+1)\lfloor n!e \rfloor + 1 - (n+1)!q + 1 \quad [\text{by Corollary 2.2}] \\
&= (n+1)\lfloor n!(e-q) \rfloor + 2 \quad [\text{since } n!q \in \mathbb{Z}] \\
&= (n+1)(f(n) - 1) + 2. \quad \square
\end{aligned}$$

### 2.3 An adaptive bound

Our adaptive upper bound on  $R_n(3)$  is provided by the following statements.

**Proposition 2.4.** *Let  $k \in \mathbb{N}$  and  $q \in \mathbb{Q}$  satisfy  $k \geq 2$ ,  $R_k(3) \leq k!(e-q) + 1$  and  $k!q \in \mathbb{N}$ . Then  $R_n(3) \leq n!(e-q) + 1$  for all  $n \geq k$ .*

*Proof.* As in Proposition 2.3, denote  $f(n) = \lfloor n!(e-q) \rfloor + 1$  for  $n \in \mathbb{N}$ . By assumption, we have

$$R_k(3) \leq f(k) \tag{3}$$

and  $k!q \in \mathbb{Z}$ . It suffices to prove the claim for  $n = k+1$ , since if  $k!q \in \mathbb{N}$  then  $(k+1)!q \in \mathbb{N}$ . By successive application of (1), (3) and (2), we have

$$\begin{aligned}
R_{k+1}(3) &\leq (k+1)(R_k(3) - 1) + 2 \\
&\leq (k+1)(f(k) - 1) + 2 \\
&= f(k+1).
\end{aligned}$$

Note that using (2) is allowed by Proposition 2.3 and the assumption  $k!q \in \mathbb{N}$ .  $\square$

**Theorem 2.5.** *Let  $k \geq 2$  be an integer. Let  $a \in \mathbb{N}$  satisfy  $a \leq \lfloor k!e \rfloor - R_k(3) + 1$ , and let  $q = a/k!$ . Then  $R_n(3) \leq n!(e-q) + 1$  for all  $n \geq k$ .*

*Proof.* We have  $a \leq k!e - R_k(3) + 1$ , so  $R_k(3) \leq k!e - a + 1 = k!(e-q) + 1$ . Moreover  $k!q = a \in \mathbb{N}$ . The conclusion follows from Proposition 2.4.  $\square$

**Remark 2.6.** *Theorem 2.5 is the best possible application of Proposition 2.4. Indeed, with the value  $a' = \lfloor k!e \rfloor - R_k(3) + 2$  and  $q' = a'/k!$ , it no longer holds that  $R_k(3) \leq k!(e-q') + 1$ .*

## 2.4 The case $k = 4$

We now apply the above result to the case  $k = 4$ . We only know  $51 \leq R_4(3) \leq 62$  so far. Note that by Proposition 2.1, we have

$$\lfloor 4!e \rfloor = \sum_{i=0}^4 4!/i! = 24 + 24 + 12 + 4 + 1 = 65. \quad (4)$$

**Proposition 2.7.** *Let  $a \in \mathbb{N}$  satisfy  $a \leq 66 - R_4(3)$ . Then setting  $q = a/24$ , we have  $R_n(3) \leq n!(e - q) + 1$  for all  $n \geq 4$ .*

*Proof.* By (4),  $a$  satisfies the hypotheses of Theorem 2.5. The conclusion follows.  $\square$

When the exact value of  $R_4(3)$  will be known, Proposition 2.7 will provide an adapted upper bound on  $R_n(3)$  for all  $n \geq 4$ . In the meantime, here are three possible outcomes.

**Corollary 2.8** ([15]).  *$R_n(3) \leq n!(e - 1/6) + 1$  for all  $n \geq 4$ .*

*Proof.* Since  $R_4(3) \leq 62$ , we may take  $a = 4$  in Proposition 2.7. The conclusion follows from that result with  $q = a/4! = 1/6$ .  $\square$

Note that the above bound does not extend to  $n = 3$ , since  $R_3(3) = 17$ , whereas by Proposition 2.1, we have  $\lfloor 3!(e - 1/6) \rfloor + 1 = \lfloor 3!e \rfloor = 3! + 3! + 3 + 1 = 16$ .

As mentioned earlier, it is conjectured in [14] that  $R_4(3) = 51$ . If true, Proposition 2.7 will yield the following improved upper bound.

**Corollary 2.9.** *If  $R_4(3) = 51$ , then  $R_n(3) \leq n!(e - 5/8) + 1$  for all  $n \geq 4$ .*

*Proof.* By Proposition 2.7, with  $a = 66 - 51 = 15$  and  $q = 15/4! = 5/8$ .  $\square$

As noted in the Introduction, this would be a substantial improvement over the current upper bound  $n!(e - 1/6) + 1$ , since  $e - 1/6 \approx 2.55$  whereas  $e - 5/8 \approx 2.09$ .

An intermediate step would be, for instance, to show  $R_4(3) \leq 54$  if at all true. This would yield the following weaker improvement.

**Corollary 2.10.** *If  $R_4(3) \leq 54$ , then  $R_n(3) \leq n!(e - 1/2) + 1$  for all  $n \geq 4$ .*

*Proof.* By Proposition 2.7, with  $a = 66 - 54 = 12$  and  $q = a/4! = 1/2$ .  $\square$

**Remark 2.11.** *The above three corollaries are best possible applications of Proposition 2.7, as in each case we took the largest admissible value for  $a \in \mathbb{N}$ .*

## 2.5 The case $k = 5$

Let us also briefly consider the case  $k = 5$ . At the time of writing, we only know  $162 \leq R_5(3) \leq 307$ . See [7].

**Proposition 2.12.** *Let  $a \in \mathbb{N}$  satisfy  $a \leq 327 - R_5(3)$ . Then setting  $q = a/120$ , we have  $R_n(3) \leq n!(e - q) + 1$  for all  $n \geq 5$ .*

*Proof.* By Theorem 2.5 and the value  $\lfloor 5!e \rfloor = 326$  given by Proposition 2.1.  $\square$

Here again are three possible outcomes. Knowing only  $R_5(3) \leq 307$  does not allow to improve the current estimate  $R_n(3) \leq n!(e - 1/6) + 1$ . At the other extreme, if  $R_5(3) = 162$  holds true, it would yield  $R_n(3) \leq n!(e - 11/8) + 1$  for all  $n \geq 5$ . As an intermediate estimate, if  $R_5(3) \leq 227$  holds true, it would imply  $R_n(3) \leq n!(e - 5/6) + 1$  for all  $n \geq 5$ .

## 3 Concluding remarks

### 3.1 On $\lim_{n \rightarrow \infty} R_n(3)^{1/n}$

The adaptive upper bound on  $R_n(3)$  given by Theorem 2.5 may still be quite far from reality, as the asymptotic behavior of  $R_n(3)$  remains poorly understood. For instance, is there a constant  $c$  such that  $R_{n+1}(3) \leq cR_n(3)$  for all  $n$ ? Or, maybe, such that  $R_n(3) \geq cn!$  for all  $n$ ? The former would imply that  $\lim_{n \rightarrow \infty} R_n(3)^{1/n}$ , known by [2] to exist, is finite, whereas the latter would imply  $\lim_{n \rightarrow \infty} R_n(3)^{1/n} = \infty$ . At the time of writing, it is not known whether that limit is finite or infinite. See e.g. [6], where this question is discussed together with related problems.

### 3.2 Link with the Schur numbers

The Schur number  $S(n)$  is defined as the largest integer  $N$  such that for any  $n$ -coloring of the integers  $\{1, 2, \dots, N\}$ , there is a monochromatic triple of integers  $1 \leq x, y, z \leq N$  such that  $x + y = z$ . The existence of  $S(n)$  was established by Schur in [10], an early manifestation of Ramsey theory. Still in [10], Schur proved the upper bound

$$S(n) \leq n!e - 1 \tag{5}$$

for all  $n \geq 2$ . The similarity with the upper bound  $R_n(3) \leq n!e + 1$  proved 40 years later in [5] is striking. In fact, there is a well known relationship between these

numbers, namely

$$S(n) \leq R_n(3) - 2. \quad (6)$$

Thus, via (6), our adaptive upper bound on  $R_n(3)$  given by Theorem 2.5 also yields an upper bound on  $S(n)$ .

**Acknowledgements.** The author wishes to thank L. Boza and S.P. Radziszowski for informal discussions during the preparation of this note and for their useful comments on a preliminary version of it.

## References

- [1] F.R.K. CHUNG, On The Ramsey Numbers  $N(3, 3, \dots, 3; 2)$ , *Discrete Mathematics* 5 (1973) 317–321.
- [2] F.R.K. CHUNG AND C. GRINSTEAD, A Survey of Bounds for Classical Ramsey Numbers, *J. Graph Theory* 7 (1983) 25–37.
- [3] S.E. FETTES, R.L. KRAMER AND S.P. RADZISZOWSKI, An upper bound of 62 on the classical Ramsey number  $R(3, 3, 3, 3)$ , *Ars Combin.* 72 (2004) 41–63.
- [4] R.L. GRAHAM, B.L. ROTHSCCHILD AND J.H. SPENCER, *Ramsey Theory*, John Wiley & Sons, 1990.
- [5] R.E. GREENWOOD AND A.M. GLEASON, Combinatorial relations and chromatic graphs, *Canadian J. Math* 7 (1955) 1–7.
- [6] M. LIANG, S.P. RADZISZOWSKI AND X. XU, On a diagonal conjecture for classical Ramsey numbers, *Discrete Appl. Math.* 267 (2019) 195–200.
- [7] S.P. RADZISZOWSKI, Small Ramsey numbers, *Electron. J. Combin.* 1 (1994), Dynamic Survey DS1, Revision #15 (2017), 104pp. (electronic).
- [8] F.P. RAMSEY, On a Problem of Formal Logic, *Proceedings of the London Mathematical Society* 30 (1930) 264–286.
- [9] A.T. SÁNCHEZ-FLORES, An improved upper bound for Ramsey number  $N(3, 3, 3, 3; 2)$ , *Discrete Math.* 140 (1995) 281–286.
- [10] I. SCHUR, Über die Kongruenz  $x^m + y^m \equiv z^m \pmod{p}$ , *Jahresbericht der Deutschen Mathematiker Vereinigung* 25 (1916) 114–117.
- [11] A. SOIFER, *The mathematical coloring book. Mathematics of coloring and the colorful life of its creators.* Springer, New York, 2009. ISBN: 978-0-387-74640-1.
- [12] H. WAN, Upper bounds for Ramsey numbers  $R(3, 3, \dots, 3)$  and Schur numbers, *J. Graph Theory* 26 (1997) 119–122.
- [13] E.G. WHITEHEAD, JR., Algebraic structure of chromatic graphs associated with the Ramsey number  $N(3, 3, 3; 2)$ , *Discrete Math.* 1 (1971/72) 113–114.

- [14] X. XU AND S.P. RADZISZOWSKI, On some open questions for Ramsey and Folkman numbers. Graph theory – favorite conjectures and open problems. 1, 43–62, Probl. Books in Math., Springer, [Cham], 2016.
- [15] X. XU, Z. XIE AND Z. CHEN, Upper bounds for Ramsey numbers  $R_n(3)$  and Schur numbers, Mathematics in Economics 19 (2002) 81–84.

**Author's adress:**

Shalom Eliahou, Univ. Littoral Côte d'Opale,  
EA 2597 - LMPA - Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville,  
F-62228 Calais, France,  
CNRS, FR 2956, France.  
E-mail: eliahou@univ-littoral.fr