



HAL
open science

An adaptive upper bound on the Ramsey numbers

$$R(3, \dots, 3)$$

Shalom Eliahou

► **To cite this version:**

Shalom Eliahou. An adaptive upper bound on the Ramsey numbers $R(3, \dots, 3)$. *Integers: Electronic Journal of Combinatorial Number Theory*, 2020, 20 (A54), pp.7. hal-02401197

HAL Id: hal-02401197

<https://hal.science/hal-02401197>

Submitted on 9 Dec 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

An adaptive upper bound on the Ramsey numbers $R(3, \dots, 3)$

S. Eliahou

Abstract

Since 2002, the best known upper bound on the Ramsey numbers $R_n(3) = R(3, \dots, 3)$ is $R_n(3) \leq n!(e - 1/6) + 1$ for all $n \geq 4$. It is based on the current estimate $R_4(3) \leq 62$. We show here how any closing-in on $R_4(3)$ yields an improved upper bound on $R_n(3)$ for all $n \geq 4$. For instance, with our present adaptive bound, the conjectured value $R_4(3) = 51$ implies $R_n(3) \leq n!(e - 5/8) + 1$ for all $n \geq 4$.

1 Introduction

For $n \geq 1$, the n -color Ramsey number $R_n(3) = R(3, \dots, 3)$ denotes the smallest N such that, for any n -coloring of the edges of the complete graph K_N , there is a monochromatic triangle. See e.g. [4, 8, 11] for background on Ramsey theory. There is a well known recursive upper bound on $R_n(3)$ due to [5], namely

$$R_n(3) \leq n(R_{n-1}(3) - 1) + 2 \tag{1}$$

for all $n \geq 2$. Currently, the only exactly known values of $R_n(3)$ are $R_1(3) = 3$, $R_2(3) = 6$ and $R_3(3) = 17$. As for $n = 4$, the current state of knowledge is

$$51 \leq R_4(3) \leq 62.$$

The lower bound is due to [1] and the upper bound to [3], down from the preceding bound $R_4(3) \leq 64$ in [9]. Moreover, it is conjectured in [14] that

$$R_4(3) = 51.$$

Here is a brief summary of successive upper bounds on $R_n(3)$. In [5], the authors proved that

$$R_n(3) \leq n!e + 1$$

for all $n \geq 2$. Whitehead's results [13] led to

$$R_n(3) \leq n!(e - 1/24) + 1$$

for all $n \geq 2$, and Wan [12] further improved it to

$$R_n(3) \leq n!(e - e^{-1} + 3)/2 + 1.$$

The last improvement came in 2002, when it was proved in [15] that

$$R_n(3) \leq n!(e - 1/6) + 1$$

for all $n \geq 4$. That bound relies on the estimate $R_4(3) \leq 62$ by [3].

Because of the recurrence relation (1), any improved upper bound on $R_k(3)$ for some $k \geq 4$ will yield an improved upper bound on $R_n(3)$ for all $n \geq k$. Our purpose here is to make this automatic improvement explicit. For instance, combined with our adaptive upper bound, the above-mentioned conjecture $R_4(3) = 51$ implies

$$R_n(3) \leq n!(e - 5/8) + 1$$

for all $n \geq 4$. This would be a substantial improvement over the current upper bound $n!(e - 1/6) + 1$, since $e - 1/6 \approx 2.55$ while $e - 5/8 \approx 2.09$.

2 Main results

As reported in [7], it is proved in [15] that

$$R_n(3) \leq n!(e - 1/6) + 1$$

for all $n \geq 4$. But the latter paper is in Chinese and not easily accessible to English readers. In this section, we prove a somewhat more general statement. We shall need the formulas below.

2.1 Useful formulas

In proving $R_n(3) \leq n!e + 1$, the authors of [5] used without comment the formula

$$\lfloor (n+1)!e \rfloor = (n+1)\lfloor n!e \rfloor + 1$$

for all $n \geq 1$. For convenience, we provide a proof here, as a direct consequence of the auxiliary formula below.

Proposition 2.1. *For all $n \geq 1$, we have $\lfloor n!e \rfloor = \sum_{i=0}^n n!/i!$.*

Proof. We have $e = 1/0! + 1/1! + \sum_{i=2}^{\infty} 1/i! = 2 + \sum_{i=2}^{\infty} 1/i!$. Since $e < 3$, it follows that $\sum_{i=2}^{\infty} 1/i! < 1$. Now $n!e = \sum_{i=0}^n n!/i! + \sum_{i=n+1}^{\infty} n!/i!$. The left-hand summand is an integer, while the right-hand one satisfies

$$\sum_{i=n+1}^{\infty} n!/i! = \sum_{j=1}^{\infty} \frac{1}{\prod_{k=1}^j (n+k)} \leq \sum_{i=2}^{\infty} 1/i! < 1.$$

This concludes the proof. □

Corollary 2.2 ([5]). *For all $n \geq 1$, we have $\lfloor (n+1)!e \rfloor = (n+1)\lfloor n!e \rfloor + 1$.*

Proof. Applying Proposition 2.1 for $n+1$ and then for n , we have

$$\begin{aligned} \lfloor (n+1)!e \rfloor &= \sum_{i=0}^{n+1} (n+1)!/i! \\ &= (n+1) \sum_{i=0}^n n!/i! + (n+1)!/(n+1)! \\ &= (n+1)\lfloor n!e \rfloor + 1. \quad \square \end{aligned}$$

2.2 An optimal model

We now exhibit an optimal model for the recursion (1).

Proposition 2.3. *Given $q \in \mathbb{Q}$, let $f: \mathbb{N} \rightarrow \mathbb{Z}$ be defined by $f(n) = \lfloor n!(e-q) \rfloor + 1$ for $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$ such that $n!q \in \mathbb{Z}$, we have*

$$f(n+1) = (n+1)(f(n) - 1) + 2. \tag{2}$$

Proof. We have

$$\begin{aligned}
f(n+1) &= \lfloor (n+1)!(e-q) \rfloor + 1 \\
&= \lfloor (n+1)!e \rfloor - (n+1)!q + 1 \quad [\text{since } (n+1)!q \in \mathbb{Z}] \\
&= (n+1)\lfloor n!e \rfloor + 1 - (n+1)!q + 1 \quad [\text{by Corollary 2.2}] \\
&= (n+1)\lfloor n!(e-q) \rfloor + 2 \quad [\text{since } n!q \in \mathbb{Z}] \\
&= (n+1)(f(n) - 1) + 2. \quad \square
\end{aligned}$$

2.3 An adaptive bound

Our adaptive upper bound on $R_n(3)$ is provided by the following statements.

Proposition 2.4. *Let $k \in \mathbb{N}$ and $q \in \mathbb{Q}$ satisfy $k \geq 2$, $R_k(3) \leq k!(e-q) + 1$ and $k!q \in \mathbb{N}$. Then $R_n(3) \leq n!(e-q) + 1$ for all $n \geq k$.*

Proof. As in Proposition 2.3, denote $f(n) = \lfloor n!(e-q) \rfloor + 1$ for $n \in \mathbb{N}$. By assumption, we have

$$R_k(3) \leq f(k) \tag{3}$$

and $k!q \in \mathbb{Z}$. It suffices to prove the claim for $n = k+1$, since if $k!q \in \mathbb{N}$ then $(k+1)!q \in \mathbb{N}$. By successive application of (1), (3) and (2), we have

$$\begin{aligned}
R_{k+1}(3) &\leq (k+1)(R_k(3) - 1) + 2 \\
&\leq (k+1)(f(k) - 1) + 2 \\
&= f(k+1).
\end{aligned}$$

Note that using (2) is allowed by Proposition 2.3 and the assumption $k!q \in \mathbb{N}$. \square

Theorem 2.5. *Let $k \geq 2$ be an integer. Let $a \in \mathbb{N}$ satisfy $a \leq \lfloor k!e \rfloor - R_k(3) + 1$, and let $q = a/k!$. Then $R_n(3) \leq n!(e-q) + 1$ for all $n \geq k$.*

Proof. We have $a \leq k!e - R_k(3) + 1$, so $R_k(3) \leq k!e - a + 1 = k!(e-q) + 1$. Moreover $k!q = a \in \mathbb{N}$. The conclusion follows from Proposition 2.4. \square

Remark 2.6. *Theorem 2.5 is the best possible application of Proposition 2.4. Indeed, with the value $a' = \lfloor k!e \rfloor - R_k(3) + 2$ and $q' = a'/k!$, it no longer holds that $R_k(3) \leq k!(e-q') + 1$.*

2.4 The case $k = 4$

We now apply the above result to the case $k = 4$. We only know $51 \leq R_4(3) \leq 62$ so far. Note that by Proposition 2.1, we have

$$\lfloor 4!e \rfloor = \sum_{i=0}^4 4!/i! = 24 + 24 + 12 + 4 + 1 = 65. \quad (4)$$

Proposition 2.7. *Let $a \in \mathbb{N}$ satisfy $a \leq 66 - R_4(3)$. Then setting $q = a/24$, we have $R_n(3) \leq n!(e - q) + 1$ for all $n \geq 4$.*

Proof. By (4), a satisfies the hypotheses of Theorem 2.5. The conclusion follows. \square

When the exact value of $R_4(3)$ will be known, Proposition 2.7 will provide an adapted upper bound on $R_n(3)$ for all $n \geq 4$. In the meantime, here are three possible outcomes.

Corollary 2.8 ([15]). *$R_n(3) \leq n!(e - 1/6) + 1$ for all $n \geq 4$.*

Proof. Since $R_4(3) \leq 62$, we may take $a = 4$ in Proposition 2.7. The conclusion follows from that result with $q = a/4! = 1/6$. \square

Note that the above bound does not extend to $n = 3$, since $R_3(3) = 17$, whereas by Proposition 2.1, we have $\lfloor 3!(e - 1/6) \rfloor + 1 = \lfloor 3!e \rfloor = 3! + 3! + 3 + 1 = 16$.

As mentioned earlier, it is conjectured in [14] that $R_4(3) = 51$. If true, Proposition 2.7 will yield the following improved upper bound.

Corollary 2.9. *If $R_4(3) = 51$, then $R_n(3) \leq n!(e - 5/8) + 1$ for all $n \geq 4$.*

Proof. By Proposition 2.7, with $a = 66 - 51 = 15$ and $q = 15/4! = 5/8$. \square

As noted in the Introduction, this would be a substantial improvement over the current upper bound $n!(e - 1/6) + 1$, since $e - 1/6 \approx 2.55$ whereas $e - 5/8 \approx 2.09$.

An intermediate step would be, for instance, to show $R_4(3) \leq 54$ if at all true. This would yield the following weaker improvement.

Corollary 2.10. *If $R_4(3) \leq 54$, then $R_n(3) \leq n!(e - 1/2) + 1$ for all $n \geq 4$.*

Proof. By Proposition 2.7, with $a = 66 - 54 = 12$ and $q = a/4! = 1/2$. \square

Remark 2.11. *The above three corollaries are best possible applications of Proposition 2.7, as in each case we took the largest admissible value for $a \in \mathbb{N}$.*

2.5 The case $k = 5$

Let us also briefly consider the case $k = 5$. At the time of writing, we only know $162 \leq R_5(3) \leq 307$. See [7].

Proposition 2.12. *Let $a \in \mathbb{N}$ satisfy $a \leq 327 - R_5(3)$. Then setting $q = a/120$, we have $R_n(3) \leq n!(e - q) + 1$ for all $n \geq 5$.*

Proof. By Theorem 2.5 and the value $\lfloor 5!e \rfloor = 326$ given by Proposition 2.1. \square

Here again are three possible outcomes. Knowing only $R_5(3) \leq 307$ does not allow to improve the current estimate $R_n(3) \leq n!(e - 1/6) + 1$. At the other extreme, if $R_5(3) = 162$ holds true, it would yield $R_n(3) \leq n!(e - 11/8) + 1$ for all $n \geq 5$. As an intermediate estimate, if $R_5(3) \leq 227$ holds true, it would imply $R_n(3) \leq n!(e - 5/6) + 1$ for all $n \geq 5$.

3 Concluding remarks

3.1 On $\lim_{n \rightarrow \infty} R_n(3)^{1/n}$

The adaptive upper bound on $R_n(3)$ given by Theorem 2.5 may still be quite far from reality, as the asymptotic behavior of $R_n(3)$ remains poorly understood. For instance, is there a constant c such that $R_{n+1}(3) \leq cR_n(3)$ for all n ? Or, maybe, such that $R_n(3) \geq cn!$ for all n ? The former would imply that $\lim_{n \rightarrow \infty} R_n(3)^{1/n}$, known by [2] to exist, is finite, whereas the latter would imply $\lim_{n \rightarrow \infty} R_n(3)^{1/n} = \infty$. At the time of writing, it is not known whether that limit is finite or infinite. See e.g. [6], where this question is discussed together with related problems.

3.2 Link with the Schur numbers

The Schur number $S(n)$ is defined as the largest integer N such that for any n -coloring of the integers $\{1, 2, \dots, N\}$, there is a monochromatic triple of integers $1 \leq x, y, z \leq N$ such that $x + y = z$. The existence of $S(n)$ was established by Schur in [10], an early manifestation of Ramsey theory. Still in [10], Schur proved the upper bound

$$S(n) \leq n!e - 1 \tag{5}$$

for all $n \geq 2$. The similarity with the upper bound $R_n(3) \leq n!e + 1$ proved 40 years later in [5] is striking. In fact, there is a well known relationship between these

numbers, namely

$$S(n) \leq R_n(3) - 2. \quad (6)$$

Thus, via (6), our adaptive upper bound on $R_n(3)$ given by Theorem 2.5 also yields an upper bound on $S(n)$.

Acknowledgements. The author wishes to thank L. Boza and S.P. Radziszowski for informal discussions during the preparation of this note and for their useful comments on a preliminary version of it.

References

- [1] F.R.K. CHUNG, On The Ramsey Numbers $N(3, 3, \dots, 3; 2)$, *Discrete Mathematics* 5 (1973) 317–321.
- [2] F.R.K. CHUNG AND C. GRINSTEAD, A Survey of Bounds for Classical Ramsey Numbers, *J. Graph Theory* 7 (1983) 25–37.
- [3] S.E. FETTES, R.L. KRAMER AND S.P. RADZISZOWSKI, An upper bound of 62 on the classical Ramsey number $R(3, 3, 3, 3)$, *Ars Combin.* 72 (2004) 41–63.
- [4] R.L. GRAHAM, B.L. ROTHSCCHILD AND J.H. SPENCER, *Ramsey Theory*, John Wiley & Sons, 1990.
- [5] R.E. GREENWOOD AND A.M. GLEASON, Combinatorial relations and chromatic graphs, *Canadian J. Math* 7 (1955) 1–7.
- [6] M. LIANG, S.P. RADZISZOWSKI AND X. XU, On a diagonal conjecture for classical Ramsey numbers, *Discrete Appl. Math.* 267 (2019) 195–200.
- [7] S.P. RADZISZOWSKI, Small Ramsey numbers, *Electron. J. Combin.* 1 (1994), Dynamic Survey DS1, Revision #15 (2017), 104pp. (electronic).
- [8] F.P. RAMSEY, On a Problem of Formal Logic, *Proceedings of the London Mathematical Society* 30 (1930) 264–286.
- [9] A.T. SÁNCHEZ-FLORES, An improved upper bound for Ramsey number $N(3, 3, 3, 3; 2)$, *Discrete Math.* 140 (1995) 281–286.
- [10] I. SCHUR, Über die Kongruenz $x^m + y^m \equiv z^m \pmod{p}$, *Jahresbericht der Deutschen Mathematiker Vereinigung* 25 (1916) 114–117.
- [11] A. SOIFER, *The mathematical coloring book. Mathematics of coloring and the colorful life of its creators.* Springer, New York, 2009. ISBN: 978-0-387-74640-1.
- [12] H. WAN, Upper bounds for Ramsey numbers $R(3, 3, \dots, 3)$ and Schur numbers, *J. Graph Theory* 26 (1997) 119–122.
- [13] E.G. WHITEHEAD, JR., Algebraic structure of chromatic graphs associated with the Ramsey number $N(3, 3, 3; 2)$, *Discrete Math.* 1 (1971/72) 113–114.

- [14] X. XU AND S.P. RADZISZOWSKI, On some open questions for Ramsey and Folkman numbers. Graph theory – favorite conjectures and open problems. 1, 43–62, Probl. Books in Math., Springer, [Cham], 2016.
- [15] X. XU, Z. XIE AND Z. CHEN, Upper bounds for Ramsey numbers $R_n(3)$ and Schur numbers, Mathematics in Economics 19 (2002) 81–84.

Author's adress:

Shalom Eliahou, Univ. Littoral Côte d'Opale,
EA 2597 - LMPA - Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville,
F-62228 Calais, France,
CNRS, FR 2956, France.
E-mail: eliahou@univ-littoral.fr