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# Rational number operations: What understandings do children demonstrate?

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*This qualitative and interpretative study aims to identify grade 5 students' knowledge of rational number operations in order to ascertain their rational number sense and operation sense before and after a teaching experiment. Data was gathered using two tests and four individual semi-structured interviews. The results show that, in the pre-test, children performed rational number operations based on their whole number knowledge. In the post-test, after the teaching experiment, they showed conceptual understanding of the operations, demonstrating rational number sense and operation sense.*

*Keywords: Rational numbers, fractions, fractions operations.*

## Introduction

Rational numbers are important mathematical entities, with an important place in the mathematics curriculum, particularly in early years. They involve complex ideas, which represent conceptual obstacles for children (Behr & Post, 1992). Mathematics education researchers have identified difficulties in children's work with these numbers, for example, regarding operations (Hansen, Drews, Dudgeon, Lawton, & Surtees, 2014). Teaching methods used in schools, typically oriented towards the application of rules, lead students to perform sequences of procedures without understanding their meaning (Braithwaite, Pyke, & Siegler, 2017), for example, obtain equal denominators and then add or subtract numerators; multiply numerators and denominators on fraction multiplication; and “invert and multiply” on fraction division. Consequently, children do not develop rational number sense (Lewis & Perry, 2014) nor operation sense (Huinker, 2002). Thus, this study aims to identify grade 5 students' knowledge of rational number operations in a pre-test, before a teaching experiment, and in a post-test, after that teaching experiment, in order to ascertain their rational number sense and operation sense.

## Rational number operations

### General aspects

Students often demonstrate not understanding what happens when they add, subtract, multiply or divide fractions because the teaching of this topic is typically based on the presentation of rules that they do not understand. Usually, they apply their knowledge of integers on rational number operation tasks, and they operate with rational numbers as if they were integers. Thus, children make mistakes that demonstrate misunderstandings in their conceptual understanding: “strategy errors”, for example, applying the operation to numerators and denominators, independently; and “execution errors”, for example, incorrect performing a calculation procedure or failing to change numerators or denominators when attempting to obtain equivalent fractions (Braithwaite et al., 2017).

Rational number understanding involves rational number sense, that is, understanding: (i) that a fraction is a number; (ii) partitioning fractions; (iii) the meaning of denominators; (iv) knowing what is the whole; (v) fraction size; and (vi) fractions can represent quantities greater than one (Lewis & Perry, 2014). Operation sense is another component of rational number sense. According to Huinker (2002), it includes the following understandings: (i) meanings and models for operations, e. g., rectangular, circular or area models, used to represent the operations; (ii) ability to recognize and describe real-world situations for specific operations; (iii) meaning of symbols and formal mathematical language; and (iv) knowledge the effects of operation on numbers. In the perspective of Brown and Quinn (2006), gaps in the conceptual understanding of rational number operations are the cause of poor student performance in these operations. Both number and operation sense represent a way of thinking about numbers and their operations. Problem solving allows children to explore and understand rich mathematical situations, in which operations arise. Thus, problems can more easily assign operations meaning and promote meaningful learning. The use of models can help children to understand the conceptual basis of each operation. As they establish a relationship between models and algorithms, children can understand what each step of an algorithm represent in practice (Behr & Post, 1992).

### **Addition and subtraction**

Fraction addition and subtraction are often taught based on the use of procedures, such as, “denominators have to be equal, and then you can add or subtract the numerators”. However, students often do not understand the need of common denominators or the role of equivalent fractions, which constitute fundamental understandings in these operations (Behr & Post, 1992). This procedure-based approach does not promote the conceptual understanding of fraction addition and subtraction. Together with the prior knowledge of whole numbers, it leads children to make common mistakes, because they confuse the procedures. In addition, children have more difficulty understanding questions involving subtraction than involving addition. Sometimes, they do not identify the correct operation (Behr & Post, 1992). Braithwaite et al. (2017) refer that most children’s difficulties are associated with the nature of denominators. When they are equal, children usually just add or subtract numerators; when they are different, they usually make independent whole number errors, that is, apply the operations to numerators and denominators (strategy errors). Behr and Post (1992) point out that the use of models can help children to understand the involved concepts when they add or subtract fractions.

### **Multiplication**

Multiplication is usually viewed as successive addition of “equal groups”. However, in the rational number field, this is an inappropriate idea because the product can be smaller than both factors and this is not compatible with an additive procedure (Greer, 1992). Tasks with a “multiplicative relation” meaning can be useful on learning rational number multiplication so that children overcome this misunderstanding. The use of models allows children to understand how the unit change during the problem and that the product is not always greater than each of the factors (Behr & Post, 1992). Children multiply both fractions but they do not understand how and why the algorithm works.

## **Division**

The division operation is often linked to sharing situations. In these situations, the goal is to get the unit value. However, learning division only based on this meaning leads students to the incorrect idea that division always decreases or that the dividend should be greater than the divisor (Greer, 1992). The meaning “measure” of division is another way to think about this operation, requesting the number of groups instead the unit value. This meaning allows a more robust understanding of division. The use of models makes it easier for children to understand this meaning of division and allows them to verify how many times the divisor goes into the dividend. The common denominator algorithm is an interesting strategy to help children understand this situation and how the algorithm works because it represents the way children reason about this meaning of division.

## **Research methodology**

This is a pre-test-intervention-post-test study (carried out in 2017/18). Participants are four grade 5 students from a public school in Portugal, from a disadvantaged socioeconomic environment. They were selected because of their ease in oral communication and the fact that their performances on tasks were representative of the remaining class, as they had different performance levels. A pre-test, applied before a teaching experiment, involved different rational number concepts. Data analysis in this paper focuses on fraction addition and subtraction because these are contents already learned in previous years, so, children already had this knowledge. A post-test was applied after that teaching experiment (in the following lesson) and also involved different rational number concepts. However, the focus was all four operations because multiplication and division operations were addressed during the teaching experiment since multiplication and division were new contents in this grade. The children solved the tests, individually, for 50 minutes, and then they were individually interviewed in order to explain their reasoning in each question. The interviews took place in the same day they solved tests and the following day.

The teaching experiment lasted six weeks, consisting of eighteen tasks (45 questions in total). The tasks involving all four operations were the last eleven ones. Addition, subtraction, multiplication and division, in problem solving situations, were addressed separately, in this specific order. These tasks involved 27 questions (13 questions involved the use of models). The teaching experiment followed an exploratory approach (Ponte & Quaresma, 2016), that is, tasks were presented to children and they worked individually, followed by a group discussion and synthesis. To solve the tasks, in an initial phase, the children were guided to use (circular or rectangular) models, provided in the task statements. The actions performed in the models were accompanied by the corresponding numerical sentence (numerical sentences changed as the models changed), so that the children understood the relationship between them. At a later stage, they could choose the solution strategy they preferred. The goal was to progress from the use of models to symbolic manipulation with understanding. On fraction addition, the children had two circular models to represent the quantities to be added. The denominators of fractions and the number of divisions in the models were equal, so, they just had to add the number of pieces. Later, the denominators of fractions and the number of divisions were different, so, they had to make divisions in the models because they could not add pieces with different sizes. Lastly, models had no division and the denominators of fractions were different so, the children had to make their own (equal) divisions to get the result. Subtraction of fractions followed the same procedure. Circular and rectangular models were used to

promote understanding about these operations, allowing to understand the need of equal denominators and the role of equivalent fractions.

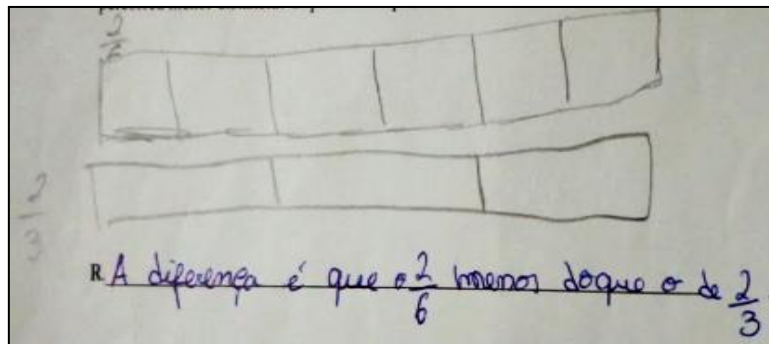
On fraction multiplication (meaning “multiplicative relation”), a rectangular model was used. This model allowed the children to understand the role of different fractions when they are multiplied, for example, the role of operator and the change of the reference unit throughout the problem. When children are asked to solve  $\frac{1}{4} \times \frac{1}{2}$ , they find a half because is the initial quantity and then they find  $\frac{1}{4}$  of that part. Fraction division focused on the “measure” meaning and the common denominator algorithm was addressed. The rectangular model was effective in understanding this meaning of division and allowed to conceptually understand the algorithm. For example, “Luísa had  $2\frac{1}{2}$  meters of tape to put in hats and each hat takes  $\frac{1}{4}$  meters. In how many hats can she put tape?” The children represented the dividend in models by shading and then they could verify how many  $\frac{1}{4}$  are in  $2\frac{1}{2}$ .

For data analysis, the interviews were all audiotaped, transcribed and analysed in depth, as well as the children solution strategies in both tests. These two methods of data collection allowed us to analyse students’ rational number sense and operation sense according to the Lewis and Perry (2014) and Huinker (2002) categorizations, respectively, before and after the teaching experiment. These methods also allowed to ascertain how they evolved from pre- to post-test, and the role of the models in the development of children conceptual understanding.

## **Results**

### **Pre-test**

A fraction addition operation was involved in the question “Pedro, João and Gonçalo started running a path with 6 km in a park near their school but none of them got to the end. Pedro stopped after  $\frac{2}{6}$ , João run only  $\frac{2}{3}$  and Gonçalo  $\frac{1}{2}$ . What fraction represents the travelled distance by the three friends together?” Although most children had identified the implicit operation in the story problem, their performance was quite weak. They often added numerators and denominators independently, to get the result (strategy error), and presented the result  $\frac{5}{11}$ . When they were asked if it would be possible to add fractions with different denominators, they demonstrated not understand the question. Just one child, Ana, tried to use a rectangular model to support her reasoning, but it was not correctly built and she left the question unanswered. In a fraction subtraction question, “What is the difference between the boy who ran the longest distance and the boy who ran the shortest distance?”, again, the children subtracted numerators and denominators independently, to get the result (strategy error). Ana (Figure 1), again, tried to use rectangular models to reason about the question, however, these were not correctly built. She could not identify which operation should use.



The difference is that the  $\frac{2}{6}$  smaller than the  $\frac{2}{3}$ .

Figure 1: Ana’s solution

**Post-test**

In the post-test, an addition question was posed to children “Clara, Maria and Gabriela started a path. Clara stopped to rest after  $\frac{2}{5}$  of the path, Maria stopped after  $\frac{4}{10}$  and Gabriela after  $\frac{3}{5}$ . What fraction represents the distance travelled by the three friends together until the moment they stopped to rest?” Most of children were successful both in identifying the operation and applied the correct algorithm. They demonstrated to understand the situation through the following statements “Here below not everything was the same size. I made it to be and I started here, with 10. [...] Then, I added all parts and left the bottom number which represents the size of the parts that are here!” (Ana), or “The whole course had five parts and all together made seven of these parts, seven bits of  $\frac{1}{5}$ !” (David). Just one child, Pedro, did not correctly add the numerators after obtaining equivalent fractions. He obtained the fraction  $\frac{10}{10}$  and said it was equal to one unit (Figure 2), making an execution error.

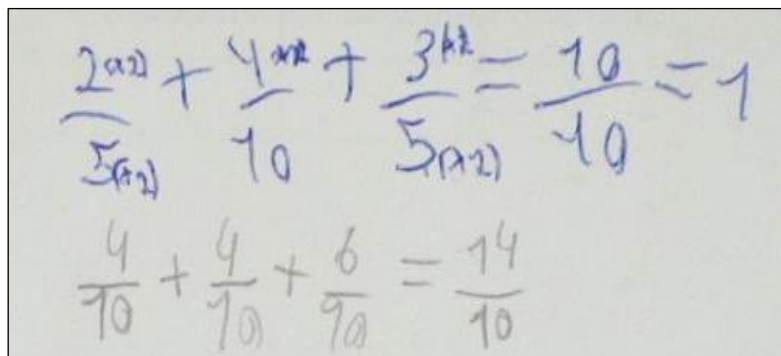
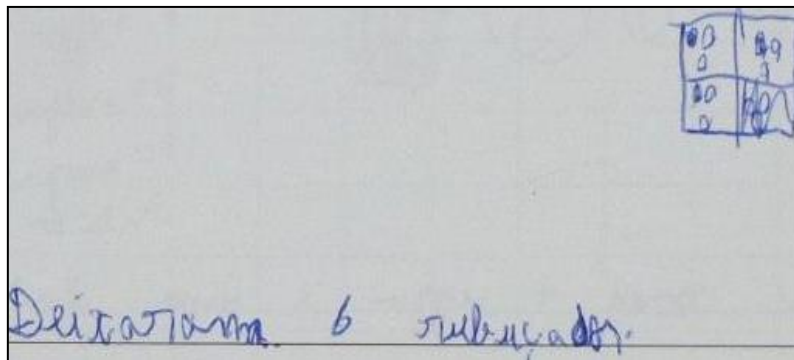


Figure 2: Pedro’s solution (i)

During the interview, Pedro identified and corrected his error and, after the new result  $\frac{14}{10}$ , he wrote that “together they did more than a whole path!” or “I think this is so because this boy made six. These only made four!” Fraction subtraction question was posed to children “In the end of the course, they decided to eat sweets from a bag with  $\frac{3}{4}$  kg. Together they ate  $\frac{1}{3}$  kg. How much did they leave in the bag?” Most children identified the correct operation and used it as solution strategy. They demonstrated understanding its meaning saying that “It’s a subtraction operation! If they eat  $\frac{1}{3}$  [Kg] from the whole bag, we have to get that part [...] but we can’t do three minus one because the parts are not equal and they have to be! We need equal fractions!” (David). Just one child, Pedro, tried to solve this question using a model (Figure 3).



**Figure 3: Pedro's solution (ii)**

He justified that “each friend ate  $\frac{1}{3}$ , so, since it is one of three, I counted one because it represents what they ate, and it is one of three because they were all the sweets they had!” Thus, he referred that six sweets were left, because it is the number of unshaded circles. Pedro considered the initial quantity as a continuous entity, as shown by the model that he tried to use, but considered the withdrawn part as a discrete entity. We do not know how much candy is in the bag but its weight. The child considered continuous and discrete quantities as measured with the same unit.

The division as measure was underlying in the following question: “Maria is making chocolate cakes for her sister’s birthday. She has  $2\frac{1}{4}$  kg of sugar and each recipe of chocolate cake takes  $\frac{1}{8}$  kg of this ingredient. How many chocolate cake recipes can she make?” Most of children had a good performance in this question. They identified the correct operation to be used in the problem, using the common denominator algorithm as solution strategy and demonstrated to understand its meaning “Here, I divided to see how many  $\frac{1}{8}$  fit in  $2\frac{1}{4}$  [...] she can make eighteen cakes!” (David). Only Ana did not identify the correct operation in this situation. She added both amounts and justified her choice saying that, “she wanted to make a cake with the whole amount, so, I made an addition!” This child also demonstrated difficulties regarding fraction simplification.

The following fraction multiplication question (multiplicative relation) was posed to children: “To decorate a cake, Maria put chocolate in  $\frac{5}{10}$  of the whole cake and coconut in  $\frac{2}{3}$  of the part that had chocolate. What fraction of the whole cake has both decorations?” Only David identified and used multiplication operation as solution strategy. He justified that “Maria put coconut over the chocolate part, so, I multiplied because it’s one part of another!  $\frac{2}{6}$  is the shaded parts twice!” When asked about the relationship between the operation and the model, he said “2 is the number of pieces and 6 is the size of the pieces. It’s just a little bit. [...] So, we have to multiply numerators and denominators!” The remaining children had a poor performance in this question. Nara subtracted both quantities, however, during the interview, she said that “ $\frac{5}{10}$  means there was half of the cake, but  $\frac{2}{3}$ ... well... it was in that part [ $\frac{5}{10}$ ]! I think it was not to subtract, it was to multiply!” So, she demonstrated some understanding about this meaning of multiplication. Pedro and Ana attempted to solve the problem using circular models but they demonstrated some difficulties in understanding this question. They also demonstrated misunderstandings related to divisions in the models. Horizontal and vertical divisions were made in the circular model. Pedro, after representing  $\frac{5}{10}$  on his model, said:

Pedro: Then it was  $\frac{2}{3}$  with coconut...

Researcher: Where could we shadow this amount ( $\frac{2}{3}$ )?

Pedro: Here [unshaded part]!

Researcher: Don't forget the coconut was in the chocolate part!

Pedro: Ah! I think I had to split this part [ $\frac{5}{10}$ ]... To put the coconut on top because it is "of"! Its  $\frac{2}{3}$  of  $\frac{5}{10}$ !

Although Pedro was unable to solve the problem correctly, during the interview he showed better understanding about the question as well as its relation with the fraction multiplication.

## Discussion and conclusion

The children who participated in this study had already learned fraction addition and subtraction in previous years. However, in the pre-test, their performance in questions involving these operations were very weak. They added and subtracted numerators and denominators to get the result, which constitutes a common mistake especially when denominators are different (Braithwaite et al., 2017). This solution strategy demonstrates lack of understanding that a fraction represents a single number. It also shows misunderstanding related to meaning of the denominator, since they added and subtracted fractions with different denominators, for example, on fraction addition ( $\frac{2}{6} + \frac{2}{3} + \frac{1}{2} = \frac{5}{11}$ ). These mistakes show that these children were unaware of the relationship between numerator and denominator, as in the study of Hansen et al. (2014), reflecting weaknesses in their rational number sense. Children's solutions also show that they did not conceptually understand these operations. Similar results were found in the study of Brown and Quinn (2006), which suggests that the poor performance is a consequence of gaps in the conceptual understanding of operations. These children did not understand the meaning of the operations and they have very limited knowledge of models to represent them, as shown in Figure 1. They also did not reason about the results obtained, which they did not understand to be inadequate in relation to the situation. Thus, their operation sense also had serious shortcomings. Their responses showed that they learned to solve operations applying rules (that they generalized in a wrong way from whole number operations) without understanding their meaning.

In the post-test, the children showed a better understanding of rational number addition and subtraction, compared to the pre-test. On these operations, they demonstrated to understand the use of equal denominators and what they represent, which was possible due to the use of models during the teaching experiment. The children know that it is not possible to add or subtract fractions with different denominators, as reported by Ana. That is, they understood that a fraction represents a number and not two independent numbers. This is an important understanding of student's rational number sense. Knowledge concerning to the meaning of the denominator, another component of rational number sense, was also observed, for example, "I kept the value down, which tells the size of the parts that are here!" (Ana). Justifications such as "The whole course had five parts and together they travelled seven of these parts. Seven bits of  $\frac{1}{5}$ !" (David), showed that these children know the fraction size and that a fraction can represent quantities larger than one. Regarding operation sense, they began to understand the different meanings of operations and to assign



meaning to the symbols. The reasoning “Here I divided to see how many times this  $[\frac{1}{8}]$  fits in  $2\frac{1}{4}$ ” (David), was possible due to the use of models. He showed that he conceptually understood this meaning of division. He reflected in the quantities and how they are related and also identified the appropriate operation. The children also demonstrated understanding the effect of operations on numbers when they analysed the reasonableness of results: “I think it is this way because this boy made six. These only made four!” (Pedro) or “ $\frac{2}{6}$  are the pieces shaded twice. [...] It’s just a little bit!” In general, these children demonstrated conceptual understanding of formal mathematical language. The division operation as measure and the multiplication operation as multiplicative relation, used in this teaching experiment, allowed a deep understanding of these operations. The children overcome the misconception that multiplication always increases and division always decreases. This was possible due to the use of models during the lessons, especially in an initial phase, as recommended by Behr and Post (1992). Therefore, we conclude that this teaching experiment based on the use of models, allowed the students to construct the algorithm from the models, and led them to develop some components of their rational number sense and operation sense. This study represents an alternative way of approaching rational number operations, promoting students’ conceptual understanding, which often does not occur.

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