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Sandra Gleissberg, Klaus-Peter Eichler

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Sandra Gleißberg[1] and Klaus-Peter Eichler[2]

[1]PH Schwäbisch Gmünd, Germany; sandra.gleissberg@ph-gmuend.de

[2]NORD University Bodø, Norway; klaus-peter.eichler@nord.no

In this paper, we present an analysis of four teaching materials of grades 2 and 3, which are widely used in Germany. We analyzed the tasks in these textbooks and workbooks with a focus on developing and consolidating the operational understanding of multiplication. For this purpose, we present a framework developed by us. Its foundations are the levels of representation according to Bruner and the fact that mathematical terms are concepts. The analysis shows a homogeneous picture across all four textbooks and workbooks: Already in class 2, the vast majority of tasks require working on the non-verbal–symbolic level alone. Only a small part of the tasks promotes connections to the iconic, to the enactive or to the verbal-symbolic level and challenges students to intermodal translations from one level of representation to another. In grade 3, compared to grade 2, an even larger proportion of tasks are limited to the non-verbal–symbolic level.

Keywords: Elementary school mathematics, multiplication, textbook analysis.

Context of the study.

Multiplication is a core content of teaching mathematics in primary school. The learning objectives in regard to multiplication are that students firstly should acquire a conceptual understanding of the operation multiplication, secondly, should be able to solve any task using different strategies and finally they should be able to retrieve more and more basic multiplication facts directly from memory (e.g., Kling & Bay-Williams, 2015; CCSSI, 2010 p. 23; KMK, 2004, p. 9).

> There seems to be an international consensus nowadays that students should both acquire a sound conceptual understanding of multiplication and eventually solve all basic tasks accurately and effortlessly (Gaidoschik, 2017, p. 2).

However, three problems can very often be observed up to the secondary school level. *First*, there are deficits in the mastery of the basic facts[1], i.e. in their retrieval as fact knowledge. Often students have forgotten basic multiplication facts they have once memorized and are unable to reconstruct this knowledge. This can be seen, for example, if they are unable to reduce fractions in secondary school because they cannot identify common factors.

Second, while students are often able to reproduce the basic task equations from memory, they are unable to solve multiplication tasks with a factor greater than 10.

Third, we repeatedly observe difficulties with the conceptual understanding of multiplication: students know that $5 \times 4 = 20$, but they are unable to justify this because they do not know the meaning of this term. Often, they just argue that 5×4 is a shortened notation for the addition task $4 + 4 + 4 + 4 + 4$.

[1] Basic fact means in this paper an equation of multiplication in which both factors are less than 10 or equal 10.

All three problems have the same reason: A lack of the conceptual understanding of multiplication. In light of these observations, it is necessary to investigate the causes and to provide information to improve the teaching and its results. We are sure that no teacher works with the intention of bad results. There are obviously factors with a negative influence on teaching and its results. Such a factor is, for example, practicing with spontaneously chosen tasks, in which subjectively important, especially familiar or widespread tasks dominate, and the exercise then does not meet the need. In our opinion, teachers are obviously not aware of the negative influence of such factors.

Undoubtedly, the success of teaching depends on the selection of tasks and the way teachers work with them (cf. Fanghänel, 2000). One essential source of such tasks is the teaching material[2]. Therefore, the impact of teaching materials such as textbooks and workbooks is given. That is why we investigated four teaching materials of grades 2 and 3 which are widely used in Germany. The objective of our analysis is the suitability of the tasks contained in the materials in regard to promote understanding of the operation multiplication.

About the matter: Understanding multiplication.

Terms like 3×4 or 6×8 are non-verbal–symbolic representations of *concepts*. They are mental reflections of objective reality and describe classes of situations. In this sense, the product $m \times n$ can be represented both by the union of m equal and pairwise disjunctive sets with the cardinality n or by a combinatorial representation:

Def.: If $a, b \in \mathbf{N}$ and $a = |A|$, $b = |B|$, than is $a \times b = |A \times B|$ the product of a and b[3].

Def.: If $a, b \in \mathbf{N}$ and $|B_1| = |B_2| = |B_3| = \ldots = |B_a| = b$ and $\forall$ i, k mit $i \neq k$: $B_i \cap B_k = \varnothing$,
than is $a \times b = | B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_a |$ the product of a and b.

Both possibilities of representation can be spatial-simultaneous as well as temporal-successive.

This is consistent with the findings of Anghileri and Johnson (1992). They identify six key aspects of multiplication (and division): equal grouping, allocation/rate, number line, array, scale factor, Cartesian product. Regarding the so-called "axiomatic" definition of multiplication as a repeated addition, Park and Nunes (2001, p. 771) emphasize "that the origin of children's understanding of multiplicative relations is in their schema of one-to-many correspondence." and that the „repeated addition is only a procedure for solving multiplication problems, not its conceptual basis" (ibid.).

Content-related understanding of a concept means to be able to identify, realize and systematize it. When identifying a term like 3×4, an appropriate example is found in an enactive, an iconic, or a verbal-symbolic representation. Realizing a term such as 3×4 means to create an enactive representation (e.g., an action), an iconic representation (e.g., a picture) or a verbal-symbolic representation (e.g., a suitable story, a word problem), which is consistent with this term. This is illustrated in Figure 1. When students systemize terms, they find relationships between two terms.

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[2] The terms "teaching material" and "program" mean in this paper the unity of textbook and workbook.

[3] |M| means the cardinality of the set M.

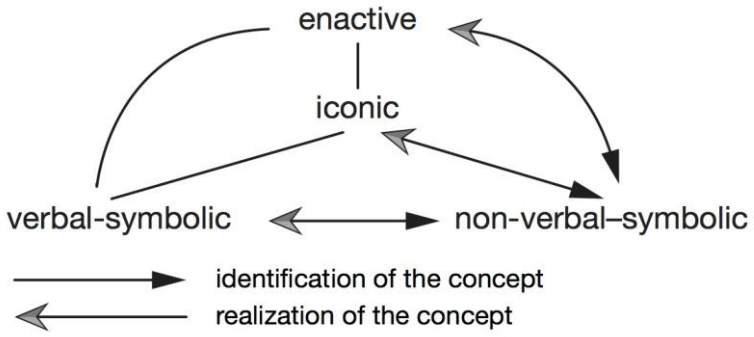

Figure 1: Levels of representation according Bruner (1964)

This can be referenced with an enactive or an iconic representation. Often this relationship is created without any references only on the non-verbal–symbolic level.

The importance of the conceptual understanding of the operation multiplication consists on the one hand in the fact that students can apply this knowledge to describe reality, e.g., to solve word problems. On the other hand, it enables students to solve every multiplication task, especially those with factors greater than 10, by drawing e.g., dots and skip counting, laying e.g., cubes and skip counting and deriving answers using reasoning strategies based on known facts. (Baroody, 2006, Kling & Bay-Williams, 2015).

Mastery of the basic facts of multiplication must be distinguished from fluently multiplying: Fluently multiplying means to solve tasks using strategies, properties of the operation, means determine the value of a term by calculating at skill level (Kling & Bay-Williams, 2015, p. 550). Mastery of all the basic tasks of multiplication means to be able to retrieve from memory all products of two one-digit numbers. In our opinion, mastery is indispensable at the latest in middle of grade 3 because then this knowledge is needed as a tool for solving tasks with factors greater than 10.

Even if the final goal here is to retrieve the facts from memory, the ability to solve the basic tasks of multiplication is the basis for this and must consequently be secured before memorizing the basic tasks of multiplication. Only if students have a conceptual understanding of the operation multiplication and are able to determine the value of the terms, they can reconstruct forgotten basic facts. The reconstruction of forgotten facts can be done by using connections to other facts, e.g., by using properties of multiplication, by activities with manipulatives or by using mental representations.

Framework for the classification of tasks for the consolidation of arithmetic operations.

Because a suitable work with tasks in the sense of Fanghänel (cf. 2000) is essential for the quality of teaching and its results, we examined this work with tasks and the selection of tasks in more detail. The choice of tasks undoubtedly depends on the range of tasks offered by published programs (textbooks, workbooks etc.). To assess this offer of tasks, we analyzed four widely used programs[4]. We decided to analyse textbooks and workbooks for both grade 2 and grade 3 because

[4] There were the programs „Fredo Mathematik“ and Super M (both Cornelsen Schulbuchverlag) as well as „Denken und Rechnen“ and „Welt der Zahl“ (both Bildungshaus Schulbuchverlage).

teachers usually use these materials anyway. Supplementary materials such as software or online resources were therefore not analyzed.

Many frameworks exist for analyzing textbooks regarding different aspects. A short overview gives for example O'Keeffe (2013). A specific framework which focusses the appropriation of arithmetic operations and takes in consideration that terms like 3 × 4 are concepts was not available until now. We developed the following framework to classify tasks for the appropriation of multiplication. It is generally usable for all arithmetic operations. This framework is not limited to analyzing printed materials. It can also be used to analyze educational software etc. The understanding of the operation multiplication must be promoted. Therefore, we were looking for tasks which emphasize the identification, realization, and systematization of concepts, e.g., of terms. In ggrades 2 and 3, mastering the basic number sentences by memory should be supported. Finally, we determined the number of tasks which aim to determine the value of the term. So, we classified the tasks in the programs as follows:

ICE: Tasks for the **I**dentification of the **C**oncept starting from the **E**nactive level of representation. These are tasks such as 'Place five times three cubes and write a matching task.' (The word 'task' here means a term in the mathematical sense of the word.)

ICI: Tasks for the **I**dentification of the **C**oncept starting from the **I**conic level of representation. In these tasks, a picture is given and the students should give a task that matches the picture. (The word 'task' here means a term in the mathematical sense of the word.)

ICV: Tasks for the **I**dentification of the **C**oncept starting from the **V**erbal–symbolic level of representation. These are tasks such as 'Three rows of five knights each come out of the castle. Write a suitable task' or 'Clown August has five trousers and three jackets. How many different possibilities does he have to be dressed?'

RCE: Tasks for the **R**ealization of **C**oncepts to the **E**nactive level of representation. These are tasks such as 'Lay with cubes matching 4 × 3'.

RCI: Tasks for the **R**ealization of **C**oncepts to the **I**conic representation level. These are tasks such as 'Draw a picture matching 4 × 3'.

RCV: Tasks for the **R**ealization of **C**oncepts to the **V**erbal–symbolic level of representation. These are tasks like 'Write a story matching 4 × 3.'

SCE: Tasks for **S**ystematizing **C**oncepts with reference to the **E**nactive representation level. These are tasks such as 'How do you go from 5 × 4 to 6 × 4? Lay with cubes and explain'.

SCI: Tasks for **S**ystematizing **C**oncepts with reference to the **I**conic representation level. These are tasks such as 'How do you go from 5 × 4 to 6 × 4? Draw and explain'.

SCW: Tasks for **S**ystematizing **C**oncepts **W**ithout reference to the enactive or to the iconic level of representation. Such tasks are, for example, 'You know 10 × 8. How can you then calculate 9 × 8?' Tasks like those shown below are also tasks for systematization without a reference to the enactive or to the iconic level of representation:

1	0	×	7	=				5	×	8	=				2	×	3	=				5	×	7	=		
	1	×	7	=				2	×	8	=				4	×	3	=				7	×	5	=		
	9	×	7	=				7	×	8	=				8	×	3	=									

VT: All tasks where the **V**alue of a **T**erm must be determined without any reference.

Our framework offers the possibility to validly classify all tasks in teaching materials, as we implemented all possible forms of representation as well as all steps of the process of learning concepts, as described above.

We are well aware that the determination of term values is indispensable in math lessons. Similarly, it is essential to establish relations between equations without always referring them in enactive or iconic mode. Mastering both requirements - determining the value of a term and establishing the connection between two terms - at skill level is crucial for successful work in the next grades. We use the term skill level as described in the IDMT assessment framework (cf. Brendefur et al., 2016, pp. 177). Nevertheless, these skills can only be used meaningfully if students can represent non-verbal–symbolic expressions at the enactive or the iconic level. The present study, therefore, focuses on those tasks that challenge students to an intermodal translation between the levels of representation.

Findings.

The findings of our analysis are presented in more detail in table 1 below:

		program 1		**program 2**		**program 3**		**program 4**	
		grade 2	**grade 3**	**grade 2**	**grade 3**	**grade 2**	**grade 3**	**grade 2**	**grade 3**
	tasks	1368	368	1421	585	1147	430	1499	731
1	**ICE**	57	0	0	0	21	3	9	4
2	**ICI**	175	21	164	0	104	3	175	8
3	**ICV**	31	3	35	30	94	10	51	12
4	**RCE**	5	0	0	0	16	10	6	0
5	**RCI**	62	3	181	0	20	0	54	0
6	**RCV**	0	0	0	0	7	1	3	10
7	**SCE**	0	0	0	0	0	0	0	0
8	**SCI**	0	0	0	0	3	0	0	0
sum (l-8)		330	27	380	30	265	27	298	34
ratio		**24.1%**	**7.3%**	**26.7%**	**5.1%**	**23.1%**	**6.3%**	**19.9%**	**4.7%**
9	**SCW**	279	66	82	58	212	57	318	128
10	**VT**	759	275	959	497	670	346	883	569
(9) + (10)		1038	341	1041	555	882	403	1201	697
ratio		**75.9%**	**92.7%**	**73.3%**	**94.9%**	**76.9%**	**93.7%**	**80.1%**	**95.3%**

Table 1: Tasks of multiplication offered in four different programs (textbooks and workbooks)

Table 1 shows the results of our analysis. It shows that the offer of tasks across all four analyzed programs is approximately equal, one-sided and qualitatively unsatisfactory.

Findings of our analysis of second grade programs.

The result of our analysis shows that tasks requiring translations between the levels of representation are significantly under-represented in all four programs. The only tasks to be found

are those in which the students are challenged to draw a matching picture for a given term (RCI). Not every program contains tasks that require the students to lay or build something with material suitable to a given term (RCE). In three of the programs, there are 5, 16, and 6 tasks. The total number of these tasks is far too small.

Tasks that challenge students to represent a term verbal-symbolically (RCV) are extremely rare. Only two of the programs offer such tasks at all and this only seven respectively three times. This finding explains the well-known shortcomings of the students in solving word problems.

The four programs contained 0, 0, 3, 0 tasks for systematization with reference at the iconic level. Not one of the four programs meets our expectations. The systematization of the basic number sentences on a purely non-verbal–symbolic level plays a major role in all four programs: 279, 82, 212 and 318 tasks are represented here. The tasks of systematization on a purely non-verbal–symbolic level are wrapped in different manners. According to the mathematical core, they are tasks like the ones shown below.

$3 \times 8 = 24 \Rightarrow 6 \times 8 = 48$ (doubling),
$5 \times 8 + 1 \times 8 = 6 \times 8$ or $10 \times 7 - 1 \times 7 = 9 \times 7$ (so-called neighbourhood tasks),
$5 \times 8 = 8 \times 5$ (so-called commutativity tasks).

Such tasks are undoubtedly essential and indispensable. In solving these tasks, however, the students do not have to think about the meaning of the operation. They can undoubtedly use these tasks to acquire networked knowledge, the meaning of which they do not necessarily have to grasp: New equations are developed from equations. For quite a few students, this is exactly the picture of mathematics lessons: lessons in which new strings are created from existing strings. Hence it is not surprising that some students consider multiplication as a trick of mathematicians to shorten long equations of addition.

Findings of our analysis of third grade programs.

First of all, it is apparent that the range of tasks offered for multiplication in third grade is considerably less than in second grade: In total there are 368, 585, 430, and 731 tasks for multiplication. This is not even half of the amount that is offered in second grade. The rate of tasks that promote the understanding of the content of the operation by establishing references to the enactive, iconic or verbal–symbolic level drops markedly: the four programs contain only 27 (7.3%), 30 (5.1%), 27 (6.3%) and 34 (4.7%) of such tasks. Tasks that support an active construction of meaning are thus almost completely missing.

The emphasis is on the determination of term values. With regard to networked memorization, it should be noted that, as in second grade, the systematization of equations is always carried out only at the non-verbal–symbolic level. Tasks that promote the development of memorizing techniques, thematize such techniques, etc. are in fact not included: In all four programs there are altogether only 4 (!) such tasks.

Discussion.

The operation multiplication is acquired in grade 2. Here, the focus is first of all on the conceptual understanding of multiplication. This is usually developed on the basis of contexts. Therefore, it is

to be expected that the programs of this grade will offer a qualitatively and quantitatively wide range of tasks with reference to the enactive or iconic level, to ensure the understanding of the meaning of the operation. We expected, that also the grade 3 programs would contain a variety of tasks to systematize and to consolidate the meaning of the operation. We hoped to find a lot of tasks for identification, realization or systematization with reference to the enactive or iconic level.

Accordingly, there are – as expected – in grade 2 many tasks in all the programs in which students have to *identify* a suitable term, particularly to pictures and – more rarely – to actions.

Because the cognitive process that takes place in mathematics lessons only finds a preliminary, relative completion when applying the acquired knowledge, we expected to see many tasks in which the students have to *realize* the term. We expected that students would be challenged to draw a picture that represents a term, to represent the term using manipulatives or to tell a story fitting the term. The evaluation of the four programs in grade 2 shows that in the programs are only 67 (5%), 181 (12%), 43 (4%), and 63(4%) such tasks and that the tasks are mostly from the type ICI.

The range of tasks that support a conceptual understanding is insufficient in both grades, while the dominance of tasks which aim determining the value of a term is remarkable. It starts in second grade and dominates in third grade. In each of the investigated programs, the teacher must use additional tasks that she/he chooses from other sources or that she/he independently develops.

Right from the development of the first basic facts onwards, it makes sense to network these and build up connected, systematic knowledge. Therefore, we expected already in second grade many tasks that promote the systematization of the basic number sentences with reference to images (iconic level - SCE) or actions (enactive level - SCI). The analysis of the programs shows that such systematization, in fact, did not take place. Not one of the four programs contained a task with a suggestion for systematization with an enactive reference.

If the teacher does not supplement the range of tasks offered by the programs, the students have too little opportunity to grasp the meaning of multiplication and to consolidate this knowledge.

The analysis of the grade 3 programs shows that if a child in grade 2 has not understood the meaning of the operation multiplication, the materials in grade 3 hardly offer any chance for building up this understanding.

Based on our findings we cannot draw conclusions regarding classroom practice. But, we can assume that the examined materials are a basis for learning multiplication in math classrooms. The teacher always can compensate deficits of the materials. Therefore, further research is needed, especially the teachers' choice of tasks has to be analyzed.

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