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# Teaching and learning decimal numbers: the role of numeration units

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*We focus on the teaching and learning of decimal numbers at primary level (grades 4 and 5) in France. We are especially interested in the role played by registers of semiotic representation. Three classroom episodes are analyzed by considering the role played by the discursive register of numeration units in conceptualizing decimal numbers.*

*Keywords: Decimal numbers, registers of semiotic representation, numeration units.*

This paper focuses on the teaching and learning of decimal numbers at primary school (grades 4 and 5) in France. Many authors underline errors specific to decimal numbers (Baturu, 2000; Durkin & Rittle-Johnson, 2015; Steinle & Stacey, 2003). We rely on the theory of registers of semiotic representation by considering a crucial role played by registers (discursive, symbolic, iconic or material) in teaching and learning processes (Duval, 1995, 2017). We grant particular importance to the register of numeration units (tens, units...) or numeration-units-numbers. Most studies (Chambris, 2015; Houdement & Tempier, 2015; Van de Walle, 2010) have highlighted the potential of this register for teaching and learning whole numbers. In our study, we question the extension of these results to decimal numbers. We aim to tackle the role of this register of numeration units (tenths, hundredths...) for teaching and learning decimal numbers. The focus is on case studies based on classroom episodes that illustrate both difficulties and levers in teaching and learning decimal numbers with respect to the role played by this register.

## **Theoretical frame: registers of semiotic representation of decimal numbers**

In Duval's framework, a given mathematical object has multiple representations depending on registers of semiotic representation (discursive, symbolic, iconic or material) that express specific properties of the object while never embodying the object. These registers open the possibility of two fundamental cognitive activities: the activity of treatment (a transformation inside a given register), the activity of conversion (mobilization from one register into another). Duval (1995, 2017) discusses the double designation of mathematical object inside registers, that is, representations of a mathematical object that may be associated and have different functionalities. Duval considers this process a fundamental part of mathematics. Using this approach, we have identified different registers of semiotic representation of decimal numbers. These registers may be used in various teaching situations, sometimes simultaneously: symbolic (fractional notation, in particular those of the type  $\frac{1}{10}$ ,  $\frac{1}{100}$ ,  $\frac{1}{1000}$ , decimal notation and the sums associated to these symbolic writings); discursive (where numeration units - tenths, hundredths, thousandths - and numeration-units-numbers are involved which can be verbalized or written); mixed (in notations of the form 1 unit + 3 tenths + 2 hundredths or in place value chart); iconic, figurative, or material (such as a square made of 100 tiles 25 of which are grey). These registers may be used both verbally and on paper. For instance, we can "verbalize" symbolic decimal fractional notation as numeration units

( $\frac{1}{10}$  as one tenth or  $\frac{15}{10}$  as fifteen tenths). We grant particular importance to the representation register of numeration units. This discursive register of numeration units allows conversions between different types of numeration units (Chambris, 2015; Houdement & Tempier, 2015) or reunitizing strategies (Baturu, 2000) possible by grouping units such as “10 tenths gives 1 unit” or “10 hundredths gives 1 tenth” or partitioning units “partitioning 1 unit into ten equal parts gives 1 tenth” or “partitioning 1 tenth into ten equal parts gives 1 hundredth”. We will name these conversions unit-conversions to distinguish them with conversions in Duval’s meaning. These unit-conversions provide various justifications of comparison and computation with decimal and whole numbers while shining light on decimal and positional principles (Ross, 1989). We hypothesize that the register of representation related to numeration units has specificities with respect to decimal numbers that may provide potential levers or difficulties in teaching and learning.

## Methodology

The research study takes place in a French primary school, with a voluntary group of 4 experienced and reflexive teachers involved in a collaborative research effort about the teaching of decimal numbers at primary school. A total of 50 decimal numbers lessons taught in 9 classes of grade 4 or 5 were recorded during 4 years. In this collaborative study, most of the time, teachers have been responsible for the design of tasks for students and researchers have been in charge of the analysis of the mathematical activities of both students and teachers. Even if this methodology differs from engineering design, we have used classical methodological tools such as confrontation between *a priori* analyses of tasks and *a posteriori* analysis of data collected (Artigue, 1992; Hodgson, Kuzniak, & Lagrange, 2016). According to our theoretical point of view we have focused on classroom episodes in which register of numeration units or numeration-units-numbers was specifically used. We focus on both the activity of treatment inside the register of numeration units and the activity of conversion from this register to others (Duval, 2017). Particular classroom episodes allow us to illustrate three types of phenomena related to the use of semiotic register of representation (summarized in the Table below). These episodes are also about different steps in the teaching decimal numbers and symbolic notations, related to the French curriculum.

episodes	tasks	phenomena	indicators
1 (grade 4)	place decimal fractions on number line	difficulties in invoking relations between a unit and its subdivisions	registers: fractional notations, numeration units unit-conversions
2 (grade 5)	compare decimal numbers	difficulties in coming back to the register of numeration units in order to justify comparison techniques	registers: decimal notations, numeration unit unit-conversions
3 (grade 4)	convert sums of decimal fractions (length measurements) to decimal notations.	potential for bringing together numeration units and measurement units	registers: decimal and fractional notations, numeration units, measurement units

**Table 1: Tasks, phenomena and indicators used in the analysis process**

### **First classroom episode: unit-conversions and decimal numbers**

According to the French curriculum, students of grade 4 begin to learn decimal numbers as decimal fractions. In this first episode, students have made a decimal number line marked in hundredths. Hundredths have already been introduced as follow: if we divide a unit into one hundred equal parts, one hundredth is one part of these parts. Similarly, tenths arise from partitioning one unit into ten equal parts. Students are able to place five hundredths, both used verbally (five hundredths) and on paper ( $\frac{5}{100}$ ) on this number line. They should place “ $\frac{15}{10}$  – fifteen tenths” on the number line marked in hundredths. Such a mathematical task seems to be difficult for most students. In the transcript below, a student solves the problem on the board. An incorrect response is first proposed as students have mistaken hundredths for tenths (count 15 hundredths instead of 15 tenths).

Teacher: What can we try to plot?

Student: Fifteen tenths [*teacher writes  $\frac{15}{10}$  on the whiteboard*].

Teacher: How did you do?

Student: We count on fifteen.

Teacher: You count on fifteen? [...] Ok, if I count on fifteen, you said, the number you stop at fifteen hundredths [*teacher writes  $\frac{15}{100}$  on the blackboard*] But we want fifteen tenths. How do we do?

Then, students use unit-conversions, in grouping point of view (such as ten tenths is equivalent to one unit and a hundred of hundredths is equivalent to one unit): fifteen tenths are equivalent to one unit and five tenths. A student notices “it only needs five”, allowing students to focus on five tenths. A brief exchange between the teacher and students points out that five tenths is equivalent to fifty hundredths. Other students seem to be confused about this unit-conversion, both in symbolic register of fractional notation and verbal register of numeration units.

Student: Five tenths are equivalent to fifty hundredths. We add fifty hundredths and get fifteen tenths.

Teacher: OK [*she is writing  $\frac{5}{10} = \frac{50}{100}$  on the whiteboard*] What do you think about it? Is that true?

Student: It is the same thing. Ten hundredths are equivalent to one tenth.

Teacher: Show me five tenths [...]

Teacher: Good to you. How many hundredths in one tenth? [...]

Teacher: OK. To make five tenths, we need fifty hundredths.

Students have difficulties with unit-conversions involved in this task. This task relies on understanding relationship between tenths and hundredths (ten hundredths is one tenth). Such a relationship relies on relationship between tenths and units (hundredths and units) built in *grouping way*, that is, not in the way (*partitioning way*) they have been introduced. In that way, decimal numbers differ from whole numbers in regard to numeration units. For whole numbers, numeration units are necessarily introduced in grouping way and partitioning way and grouping units for

reunitizing (Baturu, 2000) can be seen as more evident for students. For decimal numbers, as numeration units (tenths, hundredths...) are introduced in a *partitioning way*, it is necessary to teach explicitly the grouping of such numeration units in order to allow treatments as reunitizing or units-conversions inside this register. Students seem not to be able to (re)build it by themselves.

## Second classroom episode: unit-conversion and comparison of decimal numbers

This second episode takes place in a class of grade 5: students have already been introduced to decimal notation and are used to work with it. In this episode (see the transcript and the Figure 1 below), students have to place decimal numbers in ascending order. Decimal numbers are represented in the semiotic register of symbolic writing, using decimal notation. The teacher tries to lead students to justify how they compare decimal numbers. More precisely, she asks a student why he added zeroes ending “1,9” to compare this number to “1,589”. Such a strategy of “similar writing” can be seen as treatment inside the symbolic register of decimal notations. It is possible to justify this treatment by a discourse about unit-conversion in the verbal register of numeration unit: “nine tenths” is converted to “nine thousandth” because “one tenth is the same as a hundred of thousandths”.

Student: So for example here [*E points the decimal notation 1,589 at the interactive whiteboard*] it is not because there are more digits that it is greater than one comma nine.

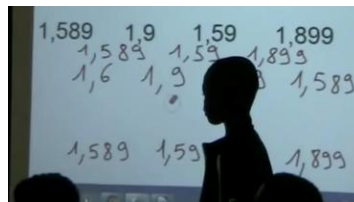


Figure 1: Compare decimal numbers

[...] because the tenths, it is what we are looking at first in order to know if it is less than. For example / Here we see [*Student points the digit “5” of the decimal notation “1,589”*] five tenths and here nine.

Teacher: How could you justify to someone who believes that five hundred eighty nine is greater than nine? Because five hundred eighty nine thousands is greater than...

Student: Nine is nine tenths.

Teacher: How could we see it? What could prove us it is the greater?

Student: We add two zeroes [*by drawing with his finger at interactive whiteboard*] if for nine we can put two zeroes / and then he will understand that nine is greater than one comma five hundred eighty nine [*Another student: Yes I agree*].

Teacher: And what is it useful for? Putting these zeroes as he did? For example?

Another student: To do the same... the same numbers at each side/ to make easier the//

During this episode, discursive register of numeration units seems to be used by a student to justify the comparison of “1,9” and “1,589” (nine tenths and five tenths). Nevertheless his mathematical explanation remains incomplete and related to place values. Then, the student is not able to justify why it is possible to put two ending “0” at the decimal notation “1,9” by using the unit-conversion

from tenths to thousandths. After this classroom episode, most of the students have taken the action of “adding zeroes” to decimal notations in order to achieve symbolic writings with similar numbers of digits. However, such a transformation of symbolic writings seems to be deprived of mathematical possible reasons. In order to specially focus on the use of unit-conversions, with the help of the researchers, the teacher designed a set of tasks, as follow. Such tasks make various symbolic and discursive semiotic registers appear:

Exercise 1: Compare 45 tenths and 440 thousandths

Exercise 2: Compare  $2 + \frac{34}{100}$  and 2,034

Exercise 3: Compare 0,17 and 0,2

Exercise 4: Compare  $3 + \frac{4}{10}$  and  $4 - \frac{7}{10}$

Exercise 5: Compare  $5 + \frac{37}{100}$  and  $\frac{54}{10}$

First, most of the students convert given representations of numbers to decimal notations (converting 45 tenth to 4,5). The former session that was devoted to decimal notations and the fact that such tasks (with diverse representations of decimal numbers) may be unusual for French students highlight this point. Furthermore, some of treatments in symbolic register of decimal notations are invalid. For example, a group of students concludes that “ $2 + \frac{34}{100}$ ” is equal to “2,034” after adding a “0” in front of the “3” of “2,34”. This confirms students’ difficulties in treatments related to comparison of numbers, in symbolic register of decimal notation. It also seems difficult for students to formulate justifications in the register of numeration units even if the teacher or the researchers try to seek explanations about their strategy of comparison. The transcript and Figure 2 below (related to interactions between a researcher and a small group of students) illustrate it is not so easy for students to explain why they can add an ending “0” at the decimal notation “5,4” to compare this number to “5,37”. At the end, students manage to formulate the unit-conversion in register of numeration units, saying that four tenth is the same as forty hundredths but it is tough to lead them to this point.

Student 1: The greater would be that one [*by pointing 5,4 at the whiteboard*].

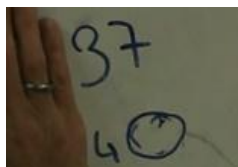
Student 2: Because here, you add a zero [*by adding a zero / writing 5,40 at the white board*].

Researcher: But what does it mean that zero that you are adding?

Student 1: Actually, it is the hundredths. It is zero thousandth here and seven hundredths here.

Student 2: It is to get the same number of digits in the decimal part.

Researcher: Wait, I look at the end [...] when I hide that [*the researcher hides the integer part and the comma*] what do you do when you are adding a zero?



**Figure 2: Hide the integer part and the comma**

- Student 1: Then it makes forty and thirty seven. Actually we can't add the thing [*by pointing the zero*].
- Student 2: Yes you can as it doesn't add anything. No...
- Researcher: But why does it add nothing?
- Student 2: So it is zero. It is... ha [...] because it is as if it was zero hundredth more [...]
- Student 1: But it doesn't mean anything at all, zero hundredth more! [...]
- Researcher: You see. If I want to explain to a friend why I am allowed to add a zero.
- Student 1: The zero it does not add anything.
- Researcher: Yes but what can I say? The zero that adds nothing, you agree that it is not so, as you earlier explained it.
- Student 1: Yes! Because the four is the same as forty hundredths [Researcher: The four what?].
- Student 1: Four tenths is the same as forty hundredths [Researcher: And why?].
- Student 1: Because... ten hundredths, it makes one, one tenth.

### Third classroom episode: numeration units and measurement units

This third episode takes place in grade 4: students have to build relations between decimal fractional notation and decimal notation. The teacher has introduced a mixed numeration place value and length measurements chart. The columns of this chart are fixed and related to numeration units (hundreds, tens, units, tenths and hundredths) and the units of length measurement (decimeters, centimeters and millimeters) move inside these columns. For example, the centimeters can be fixed in the unit column if a length measurement is given in centimeters or in the tenths columns if the measurement is given in decimeters. This mixed chart has already been used by students to convert given decimal notations of length measurements (as for example “18,2 cm”) to length measurements units values (as for example “1 dm – 8 cm – 2 mm”) and *vice versa*. During this classroom episode, students use the chart to convert sums of decimal fractions (corresponding to length measurements) to decimal notations. This trajectory is not so easy: under this innovative scenario, the conversion of fractional notations to decimal notations has not been taught before. First students convert “ $3 + \frac{5}{10}$  cm” to “3,5 cm”. The transcript and Figure 3 illustrate how the mixed chart numeration place value and length measurements is then used by students to convert the sum “ $1 + \frac{2}{10} + \frac{8}{100}$  dm” to “1,28 dm”.



Figure 3: mixed numeration place value and length measurements chart

- Teacher: You can indicate here, measurement units, one is.
- Student: It is decimeters.

- Teacher: One is decimeters [*Student is adding “DM” over the “1” in the column of units of the place value chart*]. Two you told me. Two tenths of decimeters. It is [*Student: Centimeters*] Centimeters [*Student is adding “CM” over the “2” in the column of tenths of the place value chart*]. And hundredths remain.
- Student: Millimeters [*Student is adding “MM” over the “8” in the column of hundredths of the place value chart*].
- Teacher: I agree. So we would have one centimeter, one decimeter, plus two tenths of decimeters, it is the same as two centimeters, plus eight hundredth of decimeters. OK and where is comma, yes, just after units [...] thus it would make one?
- Student: One comma twenty eight [...] Centimeters.

Such an innovative scenario of decimal notations teaching reveals that the role of the comma “,” in symbolic registers of decimal notations as numbers or as length measurements may be different. In the case of measurement units, the comma of a decimal notation indicates the measurement unit that is the point of reference. This reference can change (being centimeters or decimeters, etc.). The mobile place of measurement units in the columns of the place value chart highlights this property. At the opposite, in the case of numbers and numeration the comma always refers to the “same” numeration unit, *the unit*. It is the positional mark of the place of *the unit* and then, of the other numeration units (as hundreds, tens, tenths, hundredths, etc.) in decimal notations. So if numeration and measurements units have similarities, they also have differences that it is important to consider.

## Conclusion and discussion

Our work unravels the conditions under which results obtained regarding the register of numeration units on whole numbers (Chambris, 2015; Houdement & Tempier, 2015; Van de Walle, 2010) can be extended to decimal numbers. First, the semiotic register of numeration units is linked to the earlier introduction of decimal fractions defined as parts of a whole unit. However, unit-conversions in this register remain difficult. Such treatments inside this register rely on a deep conceptual understanding of relationship between unit, tenths, hundredths... To allow unit-conversions and reunitizing (Baturu, 2000), a more explicit teaching of grouping meaning is needed, especially in the case of reunitizing – these numeration units being built in a sharing meaning in French curriculum. Second, it seems difficult for students to associate the numeration units with the decimal notation. Once this symbolic representation has been introduced, it is commonly verbalized as a couple of integers. For example, the decimal notation “2,58” is mostly verbalized as “two comma fifty eight” and not “two units and fifty eight tenths” by students. Then, the support of the discursive register of numeration units is difficult to make appear in order to justify strategies of decimal numbers comparison. The verbalization of “writing gestures” related to transformations of decimal notations takes precedence in students’ discourse. Those “writing gestures” are also deprived of justifications and coming back to unit-conversions is not so easy. It seems to be important that discursive register remains in use and that they are not replaced by a verbalization of decimal notation with couples of integers. From this point of view, conversions between symbolic register of decimal notation and register of numeration units must be reinforced in technique justification (comparison, calculus...). Third, even if it is possible to think about closer ties between symbolic registers of decimal notations as numbers and as length measurements, we have to be careful with the asymmetric role played by comma for either, or both these two registers. New



perspectives to bring measurements units and numeration units closer in the teaching of decimal numbers can be outlined: with the design of specific tasks particularly related to the use of an unusual numeration place value chart mixing the use of numeration units and the units of length measurement.

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