Method of moments: a general framework for frequency and time-domain numerical methods

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Method of Moments: A General Framework for Frequency- and Time-domain Numerical Methods

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***Abstract*—Many numerical methods have been derived and developed in the past five decades for solving electromagnetic structure problems. They can be categorized into frequency- and time-domain methods. Among them are frequency- and time-domain spectral domain methods, finite-difference based methods, finite element FEM) methods and integral equation methods. These methods have been studied extensively and become mature enough such that a number of software packages based on them have been developed commercially. However, all these methods appear to have been unrelated and formulated on different mathematical bases. In this paper, we show that all these methods can be derived directly with the Method of Moments (MoM). The differences among them are (1) the equations and quantities to be solved and (2) the expansion and weighting functions used. The work presented here opens a new horizon for developing other new methods under a unifying framework of MoM without considering which methods they belong to.**

***Index Terms*—Method of Moments, frequency domain, time-domain, spectral domain method, finite-difference method, finite-element method, integral equation method, and pseudo-spectral time-domain method.**

I. INTRODUCTION

To SOLVE electromagnetic structure problems, two types of numerical methods have been developed so far: frequency-domain and time-domain methods. Thanks to significant advancements in high-speed computer technology, both types of the methods become increasingly effective in computing complex electromagnetic structures used in electronic circuits, communication systems, and sensor networks and applications. While frequency-domain methods are efficient in simulation of narrow band signals, time-domain methods becomes increasingly popular due to their capability of solving wide-band signals. Typically among the frequency-domain methods are the finite-difference frequency-domain methods [1], frequency-domain finite-element method [2], spectral domain method [2], mode matching [2] and transverse-resonance method [2]. Among the time-domain techniques are the time-domain finite-element methods [3], time-domain integral-equation methods [2][4] and the finite-difference time-domain based methods (e.g. FDTD and TLM) [5][6]. As the results of research advances on these methods, simulators have been developed worldwide and have been made commercially available (e.g. HFSS by Ansoft [7] and XFDTD by Remcom [8]).

In the numerical methods, the continuous system of Maxwell's equations is replaced with a discrete system or a system of approximations whose solutions are accurate enough with certain parametric constraints. So far, different numerical methods have been developed through different mathematical processes. They appear to be unrelated to each other. For instance, the finite-element (FEM) method was based on numerical solutions or functional minimization for a generalized wave equation [2]. The integral equation (IE) methods came about with applying the method of moments (MoM) to an integral equation in spatial domain and finite-difference in time domain if needed [4]. The widely used finite-difference frequency-domain and time-domain methods were formulated by direct finite-differencing the frequency-domain and time-domain Maxwell's equations, respectively [1][5]. The TLM method was derived with the utilization of transmission line networks [6]. Even the classical mode matching was simply application of the boundary conditions with mode expansions.

In this paper, all these numerical methods are shown to be derivable from the well-known Method of Moment (MoM) [9]. In other words, all these numerical methods can be obtained by first expanding solutions in terms of sets of basis functions and then minimizing the associated residual errors with sets of weighting functions. Differences among the different methods are simply the uses of different expansion and weighting functions as well as solutions of different equations. Consequently, different numerical methods are generalized under a unifying framework. Numerical issues such as numerical stability and dispersions can be directly related to the convergence of the expansions and the measure of the errors in frequency and spectral domain. *In other words, the work presented in this paper provides a general theoretical framework under which a common procedure for developing a numerical technique (including a potentially new one) is defined.*

The paper is organized in the following manner. First, a general MoM is reviewed. Then the MoM derivations of the frequency- and time-domain finite-difference based methods, spectral domain methods, finite-element methods, and integral equation method are derived. Finally, conclusions and discussions are made.

II. THE METHOD OF MOMENTS

The Method of Moments (MoM) was developed and reported decades ago; [9] is an excellent classical reference on the method. In the method, the general equation to be solved is expressed as:

$$Lf - g = 0 \tag{1}$$

where L is a mathematical operator that can be of either differential, integral or mixed differential and integral; it can be in space, in time or in mixed time and space. f is the unknown function to be solved. G is the known source function.

The MoM solution involves two computing steps: solution expansion and residual (error) minimization.

In the solution expansion step, a pre-selected set of known basis functions in both space and time is first chosen and then used to expand f. Denote the basis functions as ϕ_j and j as the index for each basis function (which has to be independent of each other). Then, f is approximated by a trial function $\overline{f}$ in terms of weighted sum of the basis functions:

$$f \approx \overline{f} = \sum_{m=1}^{N} a_j \phi_j \tag{2}$$

where a_j is the expansion coefficient that are to be found.

In the second step of MoM, the residual error of placing (2) in (1) is evaluated as:

$$R = L\overline{f} - g \tag{3}$$

Forcing R to be exact zero in the whole solution domain is equivalent to finding exact solutions, a difficult task in general. In the context of MoM, it is made to be zero in a weighted and averaged sense with a pre-selected set of known weighting functions, denoted as W_i . In a mathematical term, this means that R is forced to be null in terms of its inner product with the weighting functions:

$$< R, W_i >= 0 \qquad i=1,2,\ldots M \tag{4}$$

The inner product usually refers to an integral over the solution domain defined in a particular function space, for instance the Hilbert space [10]. Both the basis and weighting functions have to be members of a complete set defined in the function space. In addition, the trial function $\overline{f}$ has to satisfy the essential boundary condition.

When the weighting functions are chosen to be the same as the expansion functions, $W_i = \Phi_i$ the MoM becomes the Galerkin's method [9]. It has been shown that Galerkin's method is equivalent to minimizing inner product of R to itself, i.e. $<R,R>$.

(4) is essentially a system of equations for the unknown expansion coefficient a_j . It is normally solvable, some times in a recursive fashion. Once a_j is found, the approximating solution $\overline{f}$ is obtained.

III. DERIVATIONS OF THE FINITE-DIFFERENCE BASED METHODS WITH MoM

The finite-differenced based methods are perhaps most straightforward methods where differential operators in electromagnetic equations are simply replaced with their finite-difference counterparts. The resulting equations are usually algebraic formulations that can be solved relatively easily. The time-domain finite-difference methods have become popular these days while the frequency-domain finite-difference methods are still effective and efficient for narrow band computation.

The time-domain finite-difference based methods include finite-difference time-domain (FDTD) method, transmission-line-matrix (TLM) method, multi-resolution time-domain (MRTD) method, Crank-Nicolson FDTD, alternating-direction-implicit (ADI) FDTD and unconditionally stable FDTD using weighted Laguerre polynomials. Their MoM derivations were presented in [11]. Therefore, the derivations are not repeated here.

In the following paragraphs, we focus on the MoM derivation of the frequency-domain finite-difference method (FDFD). Before the derivation, a rooftop function is introduced below:

$$T(\xi, \xi_o, \Delta\xi) = \begin{cases} 1 - \dfrac{|\xi - \xi_o|}{\Delta\xi} & when \quad |\xi - \xi_o| \le \Delta\xi \\ 0 & when \quad |\xi - \xi_o| > \Delta\xi \end{cases} \tag{5}$$

Its graphical representation is shown in Fig. 1.

Then, consider one of the frequency-domain Maxwell's equations:

$$j\omega\varepsilon_o\varepsilon_r E_x = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial x} - J_x \tag{6}$$

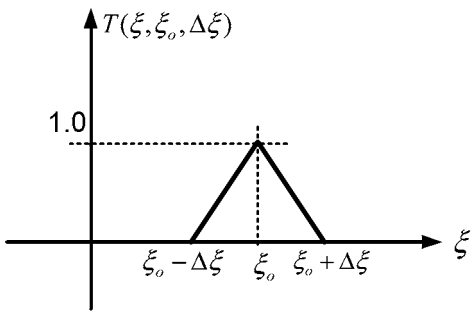

Fig. 1 Illustration of function $T(\xi,\xi_o,\Delta\xi)$

With the rooftop function defined, the field components are expanded as follows:

$$
\begin{aligned}
E_x &\approx \bar{E}_x = \sum_{i_x,i_y,i_z} E_x|_{i_x+\frac{1}{2},i_y,i_z}\, T[x,(i_x+\frac{1}{2})\Delta x,\Delta x]T(y,i_y\Delta y,\Delta y)T(z,i_z\Delta z,\Delta z)\\
H_y &\approx \bar{H}_y = \sum_{i_x,i_y,i_z} H_y|_{i_x+\frac{1}{2},i_y,i_z+\frac{1}{2}}\, T[x,(i_x+\frac{1}{2})\Delta x,\Delta x]T(y,i_y\Delta y,\Delta y)T[z,(i_z+\frac{1}{2})\Delta z,\Delta z)\\
H_z &\approx \bar{H}_z = \sum_{i_x,i_y,i_z} H_z|_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z}\, T[x,(i_x+\frac{1}{2})\Delta x,\Delta x]T[y,(i_y+\frac{1}{2})\Delta y,\Delta y]T(z,i_z\Delta z,\Delta z)
\end{aligned}
\tag{7}
$$

They are substituted into (6) and the residual errors are minimized with

$$
\int_{z=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty}(.....)\delta[x-(i_x+\frac{1}{2})\Delta x]\delta(y-i_y\Delta y)\delta(z-i_z\Delta z)dxdydz
\tag{8}
$$

The result is:

$$
\begin{aligned}
j\omega\varepsilon_o\varepsilon_r\Delta y\Delta z E_x|^{n+1}_{i_x+\frac{1}{2},i_y,i_z} &- \Delta z H_z|_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z} + \Delta z H_z|_{i_x+\frac{1}{2},i_y-\frac{1}{2},i_z}\\
+\Delta y H_y|_{i_x+\frac{1}{2},i_y,i_z+\frac{1}{2}} &- \Delta z H_z|_{i_x+\frac{1}{2},i_y,i_z+\frac{1}{2}} = -J_x|_{i_x+\frac{1}{2},i_y,i_z}
\end{aligned}
\tag{9}
$$

This is exactly the same as the finite-difference frequency-domain formulations derived from Maxwell's equations by applying the finite differences directly.

IV. DERIVATIONS OF THE FREQUENCY-DOMAIN AND TIME-DOMAIN SPECTRAL DOMAIN METHOD WITH MoM

The frequency-domain spectral-domain method developed so far is a specialized numerical method designed effectively for the transmission line structures of planar types, such as microstrip lines [2]. In it, based on the frequency-domain Maxwell's equations, an integral equation was first derived where current densities or charges on the metal strips are the unknown solutions to be found. Then, the MoM technique is applied and a system of linear equations is obtained for the expansion coefficients. The core of the spectral-domain method is that the elements of the coefficient matrix of the system of the linear equations are efficiently found with the use of the Green's functions in the spectral (or spatial frequency) domain (rather than directly in the spatial domain). An excellent MoM derivation of the frequency-domain spectral domain method is presented in [12].

The time-domain spectral domain method that has been more comprehensively developed so far is the pseudo-spectral time-domain (PSTD) method [13]. In the original development of PSTD, the Fourier transforms (or other transforms) were applied to field components in spatial domain. However, in the context of MoM, this process can be shown to amount to expansion and residual minimization as described below. First, field quantities are expanded as:

$$
\begin{aligned}
E_\alpha &= \frac{1}{N_x N_y N_z}\sum_{l_x,l_y,l_z,n}\Big[\sum_{i_x,i_y,i_z} E_\alpha|^n_{i_x,i_y,i_z}\, e^{-j(\frac{2\pi}{N_x}l_x i_x+\frac{2\pi}{N_y}l_y i_y+\frac{2\pi}{N_z}l_z i_z)}\Big] e^{j(\frac{2\pi}{N_x\Delta x}l_x x+\frac{2\pi}{N_y\Delta y}l_y y+\frac{2\pi}{N_z\Delta z}l_z z)}\bullet\\
&\quad T(t,n\Delta t,\Delta t)\\
H_\alpha &= \frac{1}{N_x N_y N_z}\sum_{l_x,l_y,l_z,n}\Big[\sum_{i_x,i_y,i_z} H_\alpha|^{n+\frac{1}{2}}_{i_x,i_y,i_z}\, e^{-j(\frac{2\pi}{N_x}l_x i_x+\frac{2\pi}{N_y}l_y i_y+\frac{2\pi}{N_z}l_z i_z)}\Big] e^{j(\frac{2\pi}{N_x\Delta x}l_x x+\frac{2\pi}{N_y\Delta y}l_y y+\frac{2\pi}{N_z\Delta z}l_z z)}\bullet\\
&\quad T[t,(n+\frac{1}{2})\Delta t,\Delta t]\\
&\alpha = x,y,z
\end{aligned}
\tag{10}
$$

Substitution of (10) into Maxwell's equation in the Cartesian coordinates, and residual minimization of the resulting equations

$$
\int_{t=-\infty}^{\infty}\int_{z=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}(\quad)\delta(x-i_x\Delta x)
$$

with

$$
\delta(y-i_y\Delta y)\delta(z-i_z\Delta z)\delta[t-(n+\frac{1}{2})\Delta t]dxdydzdt
$$

lead to, for instance,

$$
\begin{aligned}
&\frac{E_x|^{n+1}_{i_x,i_y,i_z} - E_x|^n_{i_x,i_y,i_z}}{\Delta t} -\\
&FFT^{-1}[jk_y FFT(H_z^{n+\frac{1}{2}})] - FFT^{-1}[jk_z FFT(H_y^{n+\frac{1}{2}})]\\
&\qquad + J_x|^{n+\frac{1}{2}}_{i_x,i_y,i_z} = 0
\end{aligned}
\tag{11}
$$

(11) is exactly the same as the original PSTD equation presented in [13]. Therefore, the PSTD formulations can be derived by applying the MoM procedures with particular sets of expansion functions that related to the Fourier transforms.

III. DERIVATIONS OF THE FREQUENCY-DOMAIN AND TIME-DOMAIN FEM WITH MoM

In the frequency-domain FEM formulations known so far, either an integral equation or a functional (that correspond to weak forms of the wave equations) was first established [2][3]. Then the function expansion and residual minimization were applied in *spatial* domain. Or, the expansion was performed in spatial domain and the expansion coefficients were obtained through the residual minimization. In the *time*-domain FEM method, the expansion and residual minimization were carried out exactly in the same way as that for the frequency-domain FEM *in the spatial domain*. In the *time domain*, however, the finite-differencing was applied to replace temporal derivatives. For instance, as described in [3], the second-order derivative in *time* is replaced with the central finite difference operator (i.e. equation (26) of [3]). However, by careful examinations, it can be shown that such a replacement actually amounts to expansion and residual weighting with a high-order *temporal* function and a Dirac impulse function, respectively.

III. DERIVATIONS OF INTEGRAL EQUATION METHODS WITH MoM

Integral equations methods have been of interest to electromagnetic modeling community for more than 30 years. Many different integral equations have been developed from Maxwell's equations such as the electric field integral equation (EFIE) [4]. They have been solved by following the solution steps of MoM, expansion of E and H and minimization of residual errors, in both frequency- and time-domains. For example, in [4], the *expansion* function used *in space* is the divergence-conforming basis functions on curvilinear triangles and *expansion* function used *in time* is the Knab's bandlimited interpolation function [4]:

$$T(t) = \frac{\sin(s\omega_o t)}{s\omega_o t} \frac{\sin\left[a\sqrt{(\frac{t}{N\Delta t})^2 - 1} \right]}{\sinh(a)\sqrt{(\frac{t}{N\Delta t})^2 - 1}} \qquad (12)$$

The use of (12) is to control late-time instability that has crippled the wide applications of time-domain integral equation methods.

In all, integral equation techniques have been derived so far with an application of the MoM in space and in time (if needed) for electromagnetic problems. They naturally fall within the frame work of MoM.

V. DISCUSSIONS AND CONCLUSIONS

In this paper, frequency- and time-domain numerical methods, such as frequency- and time-domain finite-difference methods, frequency- and time-domain spectral domain methods, frequency- and time-domain finite-element methods, and frequency- and time-domain integral equation methods have been derived with MoM. The differences among the different methods are shown to be the uses of different expansion and weighting functions as well as the equations and quantities to be solved. In other words, numerical methods can be generalized under the framework of MoM. New methods, particularly effective and efficient for specific structures, can now be developed with the MoM framework. One of such applications was reported in [14] where an unconditionally stable time-domain integral equations was developed by using the weighted Laguerre polynomials as the expansion and weighting functions in time.

In our other publication such as [11], we have shown numerical issues associated with numerical methods can be easily understood with the MoM framework. For instance, numerical stability of a time-domain numerical methods is associated with the convergence of the expansion coefficients as $t \to \infty$, while numerical dispersion is a measure of the residual errors in the spectral domain due to the expansion.

REFERENCES

[1] T. Weiland, "Three dimensional resonator mode computation by finite-difference method," *IEEE Trans. Magnetics*, Vol. 21, No. 6, Nov 1985 pp. 2340 - 2343

[2] T. Itoh, Ed., *Numerical Techniques for Microwave and Millimeter-wave Passive Structures*, John Wiley & Sons, 1989

[3] J. Lee et al., "Time-domain finite-element methods," *IEEE Trans. Antennas Propagat.*, No. 3, Mar. 1997, pp. 430-442

[4] G. Pisharody and D. Weile, "Robust solution of time-domain integral equations using loop-tree decomposition and band limited extrapolation," *IEEE Trans. Ant. Propagat.*, Vol. 53, No. 6, June 2005, pp. 2089-2098

[5] A. Taflove and S. C. Hagness, Ed., *Computational Electrodynamics: The Finite-Difference Time-Domain Method*, 2nd ed., Norwood, MA: Artech House, 2000

[6] W. J. R. Hoefer, "The transmission-line-matrix method – theory and applications," *IEEE Trans. Microwave Theo. Tech.*, Vol.33, No. 10, Oct. 1985, pp.882–893

[7] Ansoft Inc., http://www.ansoft.com/products/hf/hfss/

[8] Remcom Inc., http://www.remcom.com/xfdtd6/

[9] R. Harrington, *Field Computations by Moment Methods*, New York, Macmillan, 1968

[10] *L.* Máté, *Hilbert Space Methods in Science and Engineering*, Bristol ; New York : A. Hilger, c1989.

[11] Z. Chen and S. Luo, "Generalization of the finite-difference based time-domain methods using the method of moments", *IEEE Trans. Ant. Propagat.,* Vol. 54, No. 9, Sept., 2006, pp. 2515-2524

[12] D. B. Davison and J. T. Aberle, "An introduction to spectral domain method-of-moments formulations," *IEEE Trans. Ant. Propagat.* Vol. 46, No. 3, June 2004, pp: 11 – 19

[13] Q. Liu, "The pseudospectral time-domain (PSTD) method: a new algorithm for solutions of Maxwell's equations," *Digest of 1997 IEEE Antennas and Propagation Symposium*, Vol. 1, pp.122 – 125, 13-18 July 1997

[14] Y. Chung, T. Sarkar et. al., "Solution of time domain electric field integral equation using the Laguerre polynomials," *IEEE Trans. Ant. Propagat.*, No. 9, Sept. 2004, pp. 2319-2328