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Yves Benoist

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# Positive harmonic functions on the Heisenberg group II

Yves Benoist

## Abstract

We describe the extremal positive harmonic functions for finitely supported measures on the discrete Heisenberg group : they are either harmonic characters or induced of harmonic characters.

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*Key words* Harmonic function, Martin boundary, Random walk, Nilpotent group

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# 1 Introduction

In this paper, we present the classification of the positive harmonic functions on the discrete Heisenberg group  $G = H_3(\mathbb{Z})$ .

## 1.1 Positive harmonic functions

Let  $\mu = \sum_{s \in S} \mu_s \delta_s$  be a positive measure on  $G$  with finite support  $S \subset G$ . We recall that a function  $h$  on  $G$  is said to be  $\mu$ -harmonic if it satisfies the equality  $h = P_\mu h$  where  $P_\mu h(g) := \sum_{s \in S} \mu_s h(sg)$  for all  $g$  in  $G$ . We want to describe the cone  $\mathcal{H}_\mu^+$  of positive  $\mu$ -harmonic functions  $h$  on  $G$ . By Choquet Theorem, it is enough to describe its extremal rays.

The main aim of this paper is to prove that the extremal positive  $\mu$ -harmonic functions on  $G$  are proportional either to a character of  $G$  or to a translate of a function which is induced of a character on an abelian subgroup (Theorem 1.1).

The special case where  $\mu$  is the *southwest measure* was handled in the introductory paper [2]. This case was striking because the classical partition function  $h(x, y, z) := p_y(z)$  with

$$p_y(z) := \text{number of partition of } z \text{ by } y \text{ non-negative integers}$$

occurs as one of these extremal positive harmonic functions. This partition function  $p_y(z)$  is the simplest instance of a “harmonic function induced from the character of an abelian subgroup” that we will introduce in this paper.

## 1.2 Construction of harmonic functions

The simplest examples of  $\mu$ -harmonic functions are  $\mu$ -harmonic characters. Those are the characters  $\chi : G \rightarrow \mathbb{R}_{>0}$  such that  $\sum_{s \in S} \mu_s \chi(s) = 1$ . Such a function  $h = \chi$  is an extremal positive  $\mu$ -harmonic function on  $G$  which is invariant by the center  $Z$  of  $G$ , see Lemma 2.1.

We now introduce another construction of extremal positive  $\mu$ -harmonic functions by inducing harmonic characters. Let  $S_0 \subset S$  be a non-empty abelian subset and  $G_0$  be the subgroup of  $G$  generated by  $S_0$ . Denote by  $\mu_0 := \sum_{s \in G_0} \mu_s \delta_s$  the measure restriction of  $\mu$  to  $G_0$ . Let  $\chi_0$  be a  $\mu_0$ -harmonic character of  $G_0$ . We extend  $\chi_0$  as a function on  $G$ , still denoted

$\chi_0$ , which is 0 outside  $G_0$ . This function  $\chi_0$  is  $\mu$ -subharmonic, so that the sequence  $P_\mu^n \chi_0$  is increasing. We set

$$h_{G_0, \chi_0} = \lim_{n \rightarrow \infty} P_\mu^n \chi_0.$$

We will tell exactly for which pairs  $(G_0, \chi_0)$  the function  $h_{G_0, \chi_0}$  is finite, in Lemma 3.6 and in Propositions 5.1, 5.4 and 5.5. When it is finite, the function  $h_{G_0, \chi_0}$  is an extremal positive  $\mu$ -harmonic function on  $G$ , see Lemma 3.1. We will call  $h_{G_0, \chi_0}$  the *harmonic function on  $G$  induced of the  $\mu_0$ -harmonic character  $\chi_0$  of  $G_0$* .

For  $g$  in  $G$ , we denote by  $\rho_g : g' \mapsto g'g$  the right translation by  $g$  on  $G$ . Whenever a function  $h$  is  $\mu$ -harmonic, the function  $h_g := h \circ \rho_g$  is also  $\mu$ -harmonic.

### 1.3 Main results

Our main theorem tells us that conversely these three constructions are the only possible ones.

**Theorem 1.1.** *Let  $G = H_3(\mathbb{Z})$  be the discrete Heisenberg group and  $\mu$  be a positive measure on  $G$  whose support  $S$  is finite and generates the group  $G$ . Then every extremal positive  $\mu$ -harmonic function  $h$  on  $G$  is proportional either to a character  $\chi$  of  $G$  or to a translate  $h_{G_0, \chi_0} \circ \rho_g$  of a function induced of a harmonic character of an abelian subgroup.*

*Remark 1.2.* Theorem 1.1 can not be extended to all nilpotent groups  $G$ . Indeed, the conclusion of Theorem 1.1 is not always valid for a probability measure  $\mu$  on the nilpotent group  $G$  of rank 4 with cyclic center.

Theorem 1.1 has been announced in [2]. It will be proven in Chapter 4. Indeed it is a direct consequence of Propositions 4.6 and 4.8. We will give a more precise description of the extremal positive  $\mu$ -harmonic functions  $h$  in Theorem 5.10. In particular, we will say exactly when and how many of these new examples occur. Here are two corollaries of Theorem 5.10 that we will prove in Section 5.4. The first corollary tells us that these new examples always vanish somewhere.

**Corollary 1.3.** *Same notation. Let  $h$  be an extremal positive  $\mu$ -harmonic function on  $G$  which does not vanish. Then  $h$  is a character of  $G$ .*

The second corollary tells us exactly when no new example occurs. We denote by  $G_\mu^+$  the semigroup generated by  $S$ .

**Corollary 1.4.** *Same notation with  $\mu(G) = 1$ . The following are equivalent:*  
 (i) *Every extremal positive  $\mu$ -harmonic function  $h$  on  $G$  is a character of  $G$ .*  
 (ii)  *$G_\mu^+$  contains two non-central elements whose product is in  $Z \setminus \{0\}$ .*

This corollary 1.4 is illustrated in the figures of Section 5.4.

## 1.4 Previous results

The study of harmonic functions on groups has a very long history. I will just point out the part of it which is relevant for our purposes.

As a general motivation, let us recall that the extremal bounded  $\mu$ -harmonic functions on a group  $G$  form the *Poisson boundary* of  $(G, \mu)$ . They are used to study random walks on  $G$ -spaces. The extremal positive  $\mu$ -harmonic functions on  $G$  are related to the Martin boundary of  $(G, \mu)$ . They are used to study more precisely the behavior of these random walks, see [1], [9], [13] or [15].

### 1.4.1 Abelian groups

This part of the history begins with the Choquet–Deny Theorem in [5] :

*Let  $G$  be a finitely generated abelian group and  $\mu$  be a positive finite measure on  $G$  whose support generates  $G$  as a group. Then every extremal positive  $\mu$ -harmonic function  $h$  on  $G$  is proportional to a character.*

Indeed the proof of this theorem is very short : one notices that the harmonicity equation (2.1) is a decomposition of  $h$  as a sum of positive harmonic functions and hence all the terms in this sum are proportional to  $h$ .

### 1.4.2 Bounded harmonic functions

The Choquet-Deny theorem has been extended to nilpotent groups when  $\mu$  has mass 1 and  $h$  is bounded. This is due to Dynkin and Maljutov in [7] :

*Let  $G$  be a finitely generated nilpotent group and  $\mu$  be a probability measure on  $G$  whose support generates  $G$  as a group. Then every bounded  $\mu$ -harmonic function on  $G$  is constant.*

### 1.4.3 When $S$ generates $G$ as a semigroup

The Choquet-Deny theorem has also been extended to nilpotent groups for  $h$  unbounded under an extra assumption. This is due to Margulis in [12] :

*Let  $G$  be a finitely generated nilpotent group and  $\mu$  be a positive measure on  $G$  whose support generates  $G$  AS A SEMIGROUP. Then every extremal positive  $\mu$ -harmonic function on  $G$  is proportional to a character.*

### 1.4.4 The Heisenberg group

The main significance of our Theorem 1.1 is that even though Choquet-Deny theorem can not be extended to finitely generated nilpotent groups without this extra assumption. However, for the Heisenberg group, one can describe all the positive harmonic functions. Note that, because of Margulis theorem, most of our paper will deal with a positive measure whose support generates  $G$  as a group but does not necessarily generate  $G$  as a semigroup.

Many recent works focus on the random walks on the discrete Heisenberg group  $G$  as in [3], [6] and [8], or on nilpotent groups as in [4] and [10], or on the geometry of words in  $G$  as in [11] and [14]. We mention these related results even though we will not use them.

## 1.5 Strategy of proof

We now explain the strategy of proof of Theorem 1.1 and the organization of the paper.

In Chapter 2, we recall well-known facts on positive harmonic functions and notations for the discrete Heisenberg group  $G$  and its positive measures  $\mu$  with a finite support  $S$ .

In Chapter 3, we begin the proof of Theorem 1.1. When  $h$  is an extremal  $\mu$ -harmonic function on  $G$ , we focus on the equality  $h(g) = P_\mu^n h(g)$  where the right-hand side is written as a weighted sum of values  $h(wg)$  for words  $w$  of length  $n$  in  $S$ , as in Equation (2.2). In Lemmas 3.1 and 3.2, we check that when the contribution in this sum of the words  $w$  whose letters are in a proper subgroup of  $G$ , is not negligible, then  $h$  is an induced harmonic function. In Lemma 3.8, we prove a useful generalization: we allow  $w$  to be a concatenation of  $k$  subwords whose letters are in a proper subgroup with  $k \geq 1$  fixed. The proofs are very general and do not assume  $G$  to be nilpotent.

In Chapter 4, we assume that  $h$  is not induced, and we want to prove that  $h$  is invariant by the center  $Z$  of  $G$ . The main idea is to construct a symmetric relation  $\mathcal{R}_n$  among the words in  $S^n$  such that two related words  $w$  and  $w'$  have same weight and their image  $\dot{w}$  and  $\dot{w}'$  in  $G$  differ by a non-trivial element  $z$  of  $Z$ . A key point is to be able to compare the number of words related to  $w$  and the number of words related to  $w'$ , see Lemma 4.2. This allows us to prove that  $h$  is proportional to one of its translate  $h_z$ , see Proposition 4.1. The last step is to prove that  $h$  is indeed equal to its translate  $h_z$ . This is done in Propositions 4.6 and 4.8. The key point there, Lemma 4.9 is based on a counting argument that again involves the partition function.

In Chapter 5, we give a complete classification of the extremal  $\mu$ -harmonic functions that are induced of a character, see Theorem 5.10. Their existence is an important new feature of this article. The proof of this classification in Propositions 5.1, 5.4 and 5.5 uses a transience property for random walks on  $\mathbb{Z}$  similar to the large deviation inequality, see Lemma 5.3.

## 2 Notation and preliminary results

We introduce in this chapter notations that will be used all over this article.

### 2.1 The cone of $\mu$ -harmonic functions

We first recall classical facts on positive  $\mu$ -harmonic functions.

Let  $G$  be a finitely generated group and  $\mu$  be a positive measure with finite support  $S \subset G$ . We denote by  $G_\mu^+$  the subsemigroup of  $G$  generated by  $S$  and by  $G_\mu$  the subgroup of  $G$  generated by  $S$ .

A positive function  $h : G \rightarrow [0, \infty[$  is said to be  $\mu$ -harmonic if it satisfies the equality

$$h = P_\mu h \quad \text{where} \quad P_\mu h : g \rightarrow \sum_{s \in S} \mu_s h(sg). \quad (2.1)$$

A positive  $\mu$ -harmonic function is said to be *extremal* or  $\mu$ -*extremal* if every smaller positive  $\mu$ -harmonic function  $h' \leq h$  is a multiple of  $h$ .

A function  $h$  is said to be  $\mu$ -*superharmonic*, respectively  $\mu$ -*subharmonic*, if it satisfies the inequality  $h \geq P_\mu h$ , respectively  $h \leq P_\mu h$ .



We will often write the  $n^{\text{th}}$  power of the operator  $P_\mu$  under the form

$$P_\mu^n h(g) = \sum_{w \in S^n} \mu_w h(\dot{w}g), \quad (2.2)$$

where, for a word  $w = s_1 \dots s_n \in S^n$  of length  $\ell_w = n$ , the constant  $\mu_w > 0$  is the product  $\mu_w := \mu_{s_1} \cdots \mu_{s_n} > 0$  and where the element  $\dot{w} \in G$  is the product  $\dot{w} := s_1 \cdots s_n$  in  $G$ .

Let  $\mathcal{H}_\mu^+$  be the convex cone of positive  $\mu$ -harmonic functions  $h$  on  $G$  and  $\mathcal{E}$  be the set of  $\mu$ -extremal ones. We endow  $\mathcal{H}_\mu^+$  with the topology of the pointwise convergence. When  $G_\mu^+ = G$  the cone  $\mathcal{H}_\mu^+$  has a compact basis, this means that there exists a compact subset of  $\mathcal{H}_\mu^+$  that meets all rays of  $\mathcal{H}_\mu^+$ . In general, the cone  $\mathcal{H}_\mu^+$  might not have a compact basis but it is *well-capped*, this means that it is a union of closed convex subcones  $\mathcal{H}_{\mu,i}^+$  with compact basis such that  $\mathcal{H}_\mu^+ \setminus \mathcal{H}_{\mu,i}^+$  is also convex. This cone  $\mathcal{H}_\mu^+$  is also *reticulated*, this means that every two positive  $\mu$ -harmonic functions  $h_1$  and  $h_2$  admit a maximal  $\mu$ -harmonic lower bound  $h_m$  and also a minimal  $\mu$ -harmonic upper bound  $h_M$ . Indeed one has

$$\begin{aligned} h_m &= \lim_{n \rightarrow \infty} P_\mu^n(\min(h_1, h_2)) \geq 0; \quad \text{and} \\ h_M &= \lim_{n \rightarrow \infty} P_\mu^n(\max(h_1, h_2)) \leq h_1 + h_2 < \infty. \end{aligned}$$

By Choquet Theorem, it is enough to describe the extremal rays of this cone  $\mathcal{H}_\mu^+$ . Indeed, every positive  $\mu$ -harmonic function  $h$  can be written as an average of non-proportional extremal  $\mu$ -harmonic functions :  $h = \int_{\mathcal{E}} f d\alpha(f)$ , and this writing is unique in the sense that the class of the image of this probability measure  $\alpha$  on the set  $\mathbb{P}(\mathcal{E})$  of extremal rays is unique.

In this paper a *character* will always mean a multiplicative morphism  $\chi : G \mapsto \mathbb{R}_{>0}$ . A character  $\chi$  is  $\mu$ -harmonic if and only if it satisfies the equation  $\sum_{s \in S} \mu_s \chi(s) = 1$ .

## 2.2 Harmonic characters

We discuss here harmonic characters on nilpotent groups.

Let  $G$  be a nilpotent finitely generated group and  $\mu$  be a positive finite measure on  $G$  with finite support generating  $G$ .

**Lemma 2.1.** *Every  $\mu$ -harmonic character of  $G$  is an extremal positive  $\mu$ -harmonic function.*

*Proof of Lemma 2.1.* Assume that  $\chi = h' + h''$  with both  $h'$  and  $h''$  positive and  $\mu$ -harmonic. We want to prove that the function  $\tilde{h}' := \chi^{-1}h'$  is constant. We notice that the measure  $\tilde{\mu} := \chi\mu$  on  $G$  is a probability measure and the function  $\tilde{h}'$  is a bounded  $\tilde{\mu}$ -harmonic function. Therefore by Dynkin–Maljutov theorem, see Section 1.4, the function  $\tilde{h}'$  is constant.  $\square$

## 2.3 The Heisenberg group

We gather here notation that we will use in this article for the discrete Heisenberg group.

Recall that the discrete Heisenberg group  $G := H_3(\mathbb{Z})$  is the set  $\mathbb{Z}^3$  of triples seen as matrices  $(x, y, z) := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ . It is endowed with the product

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy'). \quad (2.3)$$

We will denote by  $0 := (0, 0, 0)$  the identity element of  $G$ , and by  $z_0$  the generator  $z_0 := (0, 0, 1)$  of the center  $Z$  of  $G$ .

For two elements  $g = (x, y, z)$ ,  $g' = (x', y', z')$  of  $G$ , we will denote by  $c_{g,g'}$  the integer  $c_{g,g'} := xy' - yx'$  so that

$$gg'g^{-1}g'^{-1} = z_0^{c_{g,g'}}. \quad (2.4)$$

Let  $\bar{G} := G/Z \simeq \mathbb{Z}^2$  be the abelianization of  $G$  that we embed in the real vector space  $V := \bar{G} \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^2$ .

Let  $\mu$  be a positive measure on  $G$  with finite support  $S$ . We denote by  $\bar{\mu}$  the image of  $\mu$  in  $\bar{G}$  and by  $\bar{S}$  its support.

We denote by  $V_\mu$  the vector subspace of  $G$  generated by  $\bar{S}$  and by  $V_\mu^+$  the smallest convex cone of  $V$  containing  $\bar{S}$ . Note that, when  $G_\mu = G$ , one always has  $V_\mu = V$ , and, when  $G_\mu^+ = G$ , one always has  $V_\mu^+ = V$ .

The description of  $\mathcal{H}_\mu^+$ , when  $G_\mu = G$  will heavily depend on the shape of  $V_\mu^+$ . We will often distinguish the three cases :

$$V_\mu^+ = \text{the plane, a half-plane, or a properly convex cone.} \quad (2.5)$$

## 3 Induced harmonic functions

In this chapter we present general facts on  $\mu$ -harmonic functions on a finitely generated group  $G$ . These facts will be particularly useful when  $G$  is the Heisenberg group.

### 3.1 Construction of induced harmonic functions

The following lemma gives us a method to construct  $\mu$ -harmonic functions starting from a harmonic function for a smaller measure  $\mu_0$ . This lemma will be mainly useful when  $\mu_0$  is the restriction of  $\mu$  to a proper subgroup  $G_0$ .

Let  $G$  be a finitely generated group and  $\mu$  and  $\mu_0$  be positive measures on  $G$  with finite support such that  $\mu_0 < \mu$ , i.e. such that  $\mu_1 := \mu - \mu_0$  is also a positive measure.

**Lemma 3.1.** *Let  $h_0$  be a positive  $\mu_0$ -harmonic function on  $G$  such that the function  $h := \sup_{n \geq 1} P_\mu^n h_0$  is finite.*

- (i) *Then one has  $h = \lim_{n \rightarrow \infty} P_\mu^n h_0$  and  $h$  is a positive  $\mu$ -harmonic function.*
- (ii) *One can recover  $h_0$  from  $h$  as  $h_0 = \lim_{n \rightarrow \infty} P_{\mu_0}^n h$ .*
- (iii) *Moreover when  $h_0$  is  $\mu_0$ -extremal then  $h$  is  $\mu$ -extremal too.*

*Proof of Lemma 3.1.* (i) We first notice that, since  $h_0 = P_{\mu_0} h_0 \leq P_\mu h_0$ , the sequence  $P_\mu^n h_0$  is increasing. Hence, when this sequence is bounded it converges to a  $\mu$ -harmonic function.

(ii) Since  $h = P_\mu h \geq P_{\mu_0} h \geq P_{\mu_0} h_0 = h_0$ , the sequence  $P_{\mu_0}^n h$  is decreasing and converges to a  $\mu_0$ -harmonic function  $h'_0 := \lim_{n \rightarrow \infty} P_{\mu_0}^n h$  such that  $h'_0 \geq h_0$ . We want to prove that the function  $h''_0 := h'_0 - h_0$  is zero. Since  $h_0 \leq h'_0 \leq h$ , one also has  $P_\mu^n h_0 \leq P_\mu^n h'_0 \leq h$ . Therefore one also has  $\lim_{n \rightarrow \infty} P_\mu^n h'_0 = h$  and hence  $\lim_{n \rightarrow \infty} P_\mu^n h''_0 = 0$ . Since  $h''_0$  is  $\mu_0$ -harmonic, this last sequence is increasing and hence one has  $h''_0 = 0$ .

(iii) Assume now that  $h_0$  is  $\mu_0$ -extremal and assume that  $h$  is the sum of two positive  $\mu$ -harmonic functions  $h = h' + h''$ . We want to prove that  $h$  and  $h'$  are proportional. By (ii), the functions  $h'_0 = \lim_{n \rightarrow \infty} P_{\mu_0}^n h'$  and  $h''_0 = \lim_{n \rightarrow \infty} P_{\mu_0}^n h''$  are  $\mu_0$ -harmonic and give a decomposition  $h_0 = h'_0 + h''_0$ . Therefore, one has  $h'_0 = \lambda' h_0$  and  $h''_0 = \lambda'' h_0$  for positive constants  $\lambda'$  and  $\lambda''$  with  $\lambda' + \lambda'' = 1$ . One has the inequalities  $h' \geq \lim_{n \rightarrow \infty} P_\mu^n h'_0 = \lambda' h$  and  $h'' \geq \lim_{n \rightarrow \infty} P_\mu^n h''_0 = \lambda'' h$ . Since  $h = h' + h''$ , these inequalities are equalities: one has  $h' = \lambda' h$  and  $h'' = \lambda'' h$ . This proves that the function  $h$  is  $\mu$ -extremal.  $\square$

## 3.2 Recognizing induced harmonic functions

The following lemma is a converse of Lemma 3.1. It tells us how to recognize a  $\mu$ -harmonic function that is induced of a  $\mu_0$ -harmonic function.

Let  $G$  be a finitely generated group and  $\mu_0 < \mu$  be positive measures on  $G$  with finite support.

**Lemma 3.2.** *Let  $h$  be a positive  $\mu$ -harmonic function on  $G$  such that the function  $h_0 := \inf_{n \geq 1} P_{\mu_0}^n h$  is non-zero.*

- (i) *Then one has  $h_0 = \lim_{n \rightarrow \infty} P_{\mu_0}^n h$  and  $h_0$  is a positive  $\mu_0$ -harmonic function.*
- (ii) *One has the inequality  $h \geq \lim_{n \rightarrow \infty} P_{\mu}^n h_0$ .*
- (iii) *Moreover when  $h$  is  $\mu$ -extremal, one has the equality  $h = \lim_{n \rightarrow \infty} P_{\mu}^n h_0$  and  $h_0$  is  $\mu_0$ -extremal too.*

*Proof of Lemma 3.2.* The argument is very similar to Lemma 3.1

(i) Since the function  $h$  is positive and  $\mu$ -harmonic, the sequence  $P_{\mu_0}^n h$  is decreasing. Hence it has a limit  $h_0$  which is  $\mu_0$ -harmonic.

(ii) By assumption, this limit  $h_0$  is non-zero. By construction, one has the inequality  $h \geq h_0$ . Since  $h$  is  $\mu$ -harmonic, the sequence  $P_{\mu}^n h_0$  is bounded by  $h$  and, by Lemma 3.1, the limit  $h' := \lim_{n \rightarrow \infty} P_{\mu}^n h_0$  exists, is  $\mu$ -harmonic and is bounded by  $h$ .

(iii) Assume now that  $h$  is  $\mu$ -extremal. Then one has  $h' = \lambda' h$  for some constant  $\lambda' \geq 0$ . Again by Lemma 3.1, one also has

$$h_0 = \lim_{n \rightarrow \infty} P_{\mu_0}^n h' = \lambda' \lim_{n \rightarrow \infty} P_{\mu_0}^n h = \lambda' h_0. \quad (3.1)$$

Therefore one has  $\lambda' = 1$ .

It remains to check that  $h_0$  is  $\mu_0$ -extremal. Assume that  $h_0 = h'_0 + h''_0$  with both  $h'_0$  and  $h''_0$  positive  $\mu_0$ -harmonic. The limit  $h' := \lim_{n \rightarrow \infty} P_{\mu}^n h'_0$  is a  $\mu$ -harmonic function bounded by  $h$ . Hence one has  $h' = \lambda' h$  and by the same computation as (3.1), one gets  $h'_0 = \lambda' h_0$ . This proves that  $h_0$  is extremal.  $\square$

The following definition relies on the previous lemmas :

**Definition 3.3.** A  $\mu$ -harmonic function  $h$  on  $G$  is said to be *induced from a subgroup  $G_0$*  if

$$\lim_{n \rightarrow \infty} P_{\mu_0}^n h \neq 0.$$

where  $\mu_0$  is the restriction of  $\mu$  to  $G_0$ .

When  $h$  is  $\mu$ -extremal this means that  $h = h_{G_0, h_0} \circ \rho_g$  where  $g$  is in  $G$  and  $h_0$  is an extremal  $\mu_0$ -harmonic function on  $G_0$ , and  $h_{G_0, h_0} := \lim_{n \rightarrow \infty} P_\mu^n h_0$ .

A  $\mu$ -harmonic function is said to be induced, if there exists a subgroup  $G_0$  of infinite index in  $G$  such that  $h$  is induced from  $G_0$ . It is said to be *non-induced* otherwise.

**Corollary 3.4.** *Let  $G$  be a finitely generated group and  $\mu$  a positive measure on  $G$  with finite support. A  $\mu$ -harmonic character  $\chi$  of  $G$  is never induced.*

*Proof.* Let  $\mu_0 < \mu$ . Since  $P_\mu \chi = \chi$ , one has  $P_{\mu_0} \chi = \alpha \chi$  with some constant  $\alpha < 1$ . Therefore, one has  $\lim_{n \rightarrow \infty} P_{\mu_0}^n \chi = 0$ , and, by Lemma 3.1, the  $\mu$ -harmonic function  $\chi$  is not induced of a  $\mu_0$ -harmonic function  $h_0$ .  $\square$

### 3.3 Double induction

The following lemma tells us that two successive inductions of a positive harmonic function is equivalent to a direct induction.

Let  $G$  be a finitely generated group.

**Lemma 3.5.** *Let  $\mu_0 < \mu'_0 < \mu$  be positive measures on  $G$  with finite support. Let  $h_0$  be a positive  $\mu_0$ -harmonic function on  $G$ . The following are equivalent:*

(i) *the function  $h := \lim_{n \rightarrow \infty} P_\mu^n h_0$  is finite.*

(ii) *the functions  $h'_0 := \lim_{n \rightarrow \infty} P_{\mu'_0}^n h_0$  and  $h' := \lim_{n \rightarrow \infty} P_\mu^n h'_0$  are finite.*

*In this case, the two induced  $\mu$  harmonic functions are equal  $h = h'$ .*

*Proof of Lemma 3.5.* (i)  $\implies$  (ii) Since  $h_0 \leq h$ , one has the inequalities  $P_{\mu'_0}^n h_0 \leq P_{\mu'_0}^n h \leq P_\mu^n h \leq h$  and  $h'_0 \leq h$ . Therefore, one also has the inequalities  $P_\mu^n h'_0 \leq P_\mu^n h \leq h$  and  $h' \leq h$ .

(ii)  $\implies$  (i) Since  $h_0 \leq h'_0$ , one has  $P_\mu^n h_0 \leq P_\mu^n h'_0$  and  $h \leq h'$ .  $\square$

### 3.4 Induction of characters

We give now a few conditions that have to be satisfied in order for the induction of a harmonic character to be a finite function.

Let  $G$  be a finitely generated group and  $\mu$  be positive measures on  $G$  with finite support  $S$  such that  $G = G_\mu$ . We write  $\mu = \mu_0 + \mu_1$  as a sum of two positive measures and set  $G_0 := G_{\mu_0}$ . Let  $\chi_0$  be a  $\mu_0$ -harmonic character of  $G_0$  that we extend by 0 as a function on  $G$ . We denote by

$$Z_G(G_0) := \{g \in G \mid gg_0 = g_0g \text{ for all } g_0 \text{ in } G_0\}$$

the centralizer of  $G_0$  in  $G$ , and by

$$N_G(G_0, \chi_0) := \{g \in G \mid gg_0g^{-1} \in G_0 \text{ and } \chi_0(gg_0g^{-1}) = \chi_0(g_0) \text{ for all } g_0 \text{ in } G_0\}$$

the normalizer of  $(G_0, \chi_0)$  in  $G$ .

**Lemma 3.6.** *If the induced  $\mu$ -harmonic function  $h_{G_0, \chi_0}$  is finite, then :*

- (i) *The measure  $\mu_0$  is the restriction of  $\mu$  to  $G_0$  and  $S_0 = S \cap G_0$ .*
- (ii) *The subgroup  $G_0$  has infinite index in  $G$ .*
- (iii) *One has  $G_{\mu_1}^+ \cap G_0 = \emptyset$ .*
- (iv) *One has  $G_{\mu_1}^+ \cap Z_G(G_0) = \emptyset$ .*
- (v) *One has  $G_{\mu_1}^+ \cap N_G(G_0, \chi_0) = \emptyset$ .*

*Remark 3.7.* - In particular, the supports  $S_0$  of  $\mu_0$  and  $S_1$  of  $\mu_1$  are disjoint and the semigroup  $G_{\mu_1}^+$  does not meet the center  $Z$  of  $G$ .

- Note also that if one wants  $h_{G_0, \chi_0}$  to be  $\mu$ -extremal, the group  $G_0$  must be generated by  $S_0$ .

- The above conditions are not the only necessary conditions, as we will see in Chapter 5.

*Proof of Lemma 3.6.* (i) This means that  $\mu_1(G_0) = 0$  and follows from (iii).

(ii) This follows from (iii). Indeed pick an element  $s_1$  in the support of  $\mu_1$ , if the index were finite, there would exist a positive power  $s_1^d$  belonging to  $G_0$ .

(iii) This follows from (v) because  $G_0 \subset N_G(G_0, \chi_0)$ .

(iv) This follows from (v) because  $Z_G(G_0) \subset N_G(G_0, \chi_0)$ .

(v) This point is the main content of Lemma 3.6. We proceed by contradiction. Let  $S_1$  be the support of  $\mu_1$  and  $w_1 = s_1 \dots s_\ell \in S_1^\ell$ , with  $\ell \geq 1$  be a word such that  $w_1$  belongs to  $N_G(G_0, \chi_0)$ .

The proof relies on a cautious analysis of the words that occur in Equality (2.2). We recall the notation  $\mu_{1, w_1} := \mu_{1, s_1} \cdots \mu_{1, s_\ell} > 0$ . We will denote  $P_{w_1}$

for the operator of left translation by  $\dot{w}_1 := s_1 \cdots s_\ell \in G$ ; it is given by  $P_{w_1} h(g) = h(\dot{w}_1 g)$  for all function  $h$  on  $G$  and all  $g$  in  $G$ . One computes

$$\begin{aligned}
P_\mu^{n+\ell} \chi_0(\dot{w}_1^{-1}) &\geq \sum_{1 \leq i \leq n} \mu_{1, w_1} P_{\mu_0}^i P_{w_1} P_{\mu_0}^{n-i} \chi_0(\dot{w}_1^{-1}) \\
&= \sum_{1 \leq i \leq n} \mu_{1, w_1} P_{\mu_0}^i P_{w_1} \chi_0(\dot{w}_1^{-1}) \quad \text{because } \chi_0 \text{ is } \mu_0\text{-harmonic} \\
&= \sum_{1 \leq i \leq n} \mu_{1, w_1} \sum_{w_0 \in S_0^i} \mu_{0, w_0} \chi_0(\dot{w}_1 \dot{w}_0 \dot{w}_1^{-1}) \quad \text{by definition of } P_{\mu_0} \\
&= \sum_{1 \leq i \leq n} \mu_{1, w_1} \sum_{w_0 \in S_0^i} \mu_{0, w_0} \chi_0(\dot{w}_0) \quad \text{because } \dot{w}_1 \text{ normalizes } \chi_0 \\
&= \sum_{1 \leq i \leq n} \mu_{1, w_1} \chi_0(0) = n \mu_{1, w_1} \quad \text{because } \chi_0 \text{ is } \mu_0\text{-harmonic.}
\end{aligned}$$

This goes to infinity with  $n$ , and the induced function is not finite.  $\square$

### 3.5 Negligeable trajectories

We now discuss a lemma on non-induced extremal positive  $\mu$ -harmonic functions. This lemma will be useful for the proof of the  $Z$ -semiinvariance of these functions on the Heisenberg group.

Let  $G$  be a finitely generated group and  $\mu$  be a positive measure on  $G$  with finite support  $S$  generating  $G$ .

For every word  $w = s_1 \dots s_n \in S^n$ , we define  $k_w \geq 0$  to be the smallest integer  $k$  for which we can write  $w = w_0 \dots w_k$  as a concatenation of *non-generating* subwords  $w_j$ . *Non-generating* means that there exists an infinite index subgroup  $G_j$  of  $G$  containing all the letters  $s_i$  occurring in the subword  $w_j$ . The following lemma tells us that the words with  $k_w$  bounded are negligible in the sum (2.2) for a non-induced  $\mu$ -harmonic function.

**Lemma 3.8.** *Let  $h$  be a non-induced positive  $\mu$ -harmonic function on  $G$ . Then, for all  $k \geq 0$ , and  $g$  in  $G$ , the partial sums*

$$I_{n,k}(g) := \sum_{w \in S^n, k_w \leq k} \mu_w h(\dot{w}g). \quad (3.2)$$

*converge to 0 when  $n \rightarrow \infty$ .*

*Proof of Lemma 3.8.* We argue by induction on  $k$ . Since  $h$  is non-induced, the claim (3.2) is true for  $k = 0$ .

Now let  $k \geq 1$  and  $g$  in  $G$ . For  $w$  in  $S^n$  we introduce the maximal non-generating suffix  $\sigma$  of  $w$ . Suffix means that one can write  $w = w'\sigma$ . We denote by  $S_{0,w}$  the set of letters of  $\sigma$  and by  $\ell_{0,w}$  the length of  $\sigma$ . Since there are only finitely many non-generating subsets  $S_0$  of  $S$ , we can write  $I_{n,k}(g)$  as a finite sum  $I_{n,k}(g) = \sum I_{n,k,S_0}(g)$  where  $I_{n,k,S_0}(g)$  involves the words  $w$  for which  $S_{0,w} = S_0$ .

Fix  $\varepsilon_0 > 0$ . We choose  $\ell_0$  such that, one has  $P_{\mu_0}^{\ell_0} h(g) \leq \varepsilon_0$  for all restriction  $\mu_0$  of  $\mu$  to a non-generating subset  $S_0$  of  $S$ . We decompose the sum  $I_{n,k,S_0}(g)$  as a sum of two terms

$$I_{n,k,S_0,\ell_0}(g) = I'_{n,k,S_0,\ell_0}(g) + I''_{n,k,S_0,\ell_0}(g)$$

where  $I'_{n,k,S_0,\ell_0}(g)$  involves the words  $w$  for which  $\ell_{0,w} \geq \ell_0$  and  $I''_{n,k,S_0,\ell_0}(g)$  involves the words  $w$  for which  $\ell_{0,w} < \ell_0$ .

Bounding  $I'_n$ . One computes, using the  $\mu$ -harmonicity of  $h$ ,

$$\begin{aligned} I'_{n,k,S_0,\ell}(g) &\leq \sum_{w_0 \in S_0^{\ell_0}} \mu_{w_0} \sum_{w_1 \in S^{n-\ell_0}} \mu_{w_1} h(\dot{w}_1 \dot{w}_0 g) \\ &\leq \sum_{w_0 \in S_0^{\ell_0}} \mu_{w_0} h(\dot{w}_0 g) \leq \varepsilon_0 \end{aligned}$$

Bounding  $I''_n$ . One decomposes  $I''_{n,k,S_0,\ell}$  as a finite sum

$$I''_{n,k,S_0,\ell}(g) = \sum_{\sigma} \mu_{\sigma} I''_{n,k,\sigma}(g)$$

over the finitely many words  $\sigma$  of length  $\ell < \ell_0$  where

$$\begin{aligned} I''_{n,k,\sigma}(g) &\leq \sum_{w' \in S^{n-\ell}, k_{w'} \leq k-1} \mu_{w'} h(\dot{w}' \dot{\sigma} g) \\ &\leq I_{n-\ell,k-1}(\dot{\sigma} g). \end{aligned}$$

Therefore by the induction hypothesis one has  $\lim_{n \rightarrow \infty} I''_{n,k,\sigma}(g) = 0$ . Since  $\varepsilon_0$  can be chosen arbitrarily small, one deduces that  $\lim_{n \rightarrow \infty} I_{n,k}(g) = 0$ .  $\square$

### 3.6 When $G_{\mu}^+$ meets the center

There is a simple case where the semiinvariance of  $\mu$ -harmonic functions is easy to prove, namely when  $G_{\mu}^+$  meets the center.



Let  $G$  be a finitely generated group,  $Z$  be the center of  $G$  and  $\mu$  be a finite positive measure on  $G$ .

**Lemma 3.9.** *Assume that an element  $z$  of  $Z$  belongs to the semigroup  $G_\mu^+$ . Then, for every extremal positive  $\mu$ -harmonic function  $h$  on  $G$  there exists a constant  $q > 0$  such that  $h_z = qh$ .*

We recall that  $h_z$  is the function  $g \mapsto h(gz)$ .

*Proof of Lemma 3.9.* This is a slight generalization of the Choquet–Deny Theorem. Let  $n \geq 1$  be an integer such that  $z$  is in the support of  $\mu^{*n}$ . The equality  $h = P_\mu^n h$  is of the form  $h = \alpha h_z + h'$  where  $\alpha > 0$  and  $h'$  is a positive function. Since the function  $h_z$  is also  $\mu$ -harmonic, the extremality of  $h$  implies that  $h_z$  is proportional to  $h$ .  $\square$

## 4 $Z$ -Invariance of harmonic functions

In all this chapter we keep the following notation :

$$\begin{aligned} G &= H_3(\mathbb{Z}) \text{ is the Heisenberg group, } Z \text{ is the center of } G, \\ \mu &\text{ is a positive measure with finite support } S \text{ such that } G_\mu = G, \\ h &\text{ is a positive } \mu\text{-harmonic function on } G. \end{aligned} \quad (4.1)$$

In this chapter we will mainly focus on non-induced  $\mu$ -harmonic functions, and we will prove that they are  $Z$ -invariant, see Definition 3.3. Note that, in the Heisenberg group, the subgroups of infinite index are exactly the abelian subgroup. Therefore  $h$  is non-induced, if and only if  $\lim_{n \rightarrow \infty} P_{\mu_0}^n h = 0$ , for all restriction  $\mu_0$  of  $\mu$  to an abelian subset  $S_0$  of  $S$ .

### 4.1 Semiinvariance of harmonic functions

In this section we prove that  $h$  is semiinvariant by one central element.

**Proposition 4.1.** *Keep notation (4.1) and assume that  $h$  is  $\mu$ -extremal and non-induced. Then there exist  $z \neq 0$  in  $Z$  and  $q > 0$  such that  $h_z = qh$ .*

*Proof of Proposition 4.1.* By Lemma 3.9, we can assume  $S \cap Z = \emptyset$ .

For  $n \geq 2$ , we introduce a symmetric relation on  $S^n$  given by

$$\mathcal{R}_n := \{(w, w') \in S^n \times S^n \mid w = w_0 s s' w'_0 \text{ and } w' = w_0 s' s w'_0 \text{ where} \\ w_0 \in S^i, w'_0 \in S^{n-i-2}, s \in S, s' \in S \text{ with } s s' \neq s' s\}.$$

This means that  $w$  and  $w'$  are obtained from one another by switching two consecutive non-commuting letters. For a word  $w \in S^n$  we let

$k_w$  = the number of pairs of consecutive non-commuting letters in  $w$ .

Since  $G$  is the Heisenberg group  $H_3(\mathbb{Z})$  and since  $S \cap Z = \emptyset$ , this number  $k_w$  is the same as the one occurring in Lemma 3.8. Indeed, there exists a unique partition  $S = S_0 \cup \dots \cup S_\ell$  of  $G$  such that two elements  $s, s'$  of  $S$  commute if and only if they belong to the same  $S_i$ . To go on the proof of Proposition 4.1, we will need the following two lemmas.  $\square$

We denote by  $p_0 := \max_{s, s' \in S} |c_{s, s'}|$

**Lemma 4.2.** For  $(w, w') \in \mathcal{R}_n$ , one has

- (i)  $\dot{w} = \dot{w}' z_0^p$  for some integer  $p$  with  $0 < |p| \leq p_0$ ,
- (ii)  $\mu_{w'} = \mu_w$  and
- (iii)  $|k_{w'} - k_w| \leq 2$ .

*Proof of Lemma 4.2.* (i) This follows from the equality  $ss' = s's z_0^{c_{s, s'}}$ .

(ii) The same letters occur in  $w$  and  $w'$ .

(iii) The pairs of adjacent letters in  $w$  and  $w'$  are the same except for at most two of them.  $\square$

**Lemma 4.3.** For  $g$  in  $G$ , one has  $h(g) \leq \sum_{0 < |p| \leq p_0} h(z_0^p g)$ .

*Proof of Lemma 4.3.* Replacing  $h$  by its translate  $h_g$ , we can assume that  $g = 0$ . We want to prove that the following difference is non-positive :

$$D := h(0) - \sum_{0 < |p| \leq p_0} h(z_0^p) \leq 0.$$

Using notations (2.2), we compute  $D$  as

$$D = \sum_{w \in S^n} \mu_w h(\dot{w}) - \sum_{0 < |p| \leq p_0} \sum_{w' \in S^n} \mu_{w'} h(\dot{w}' z_0^p).$$

We fix  $\varepsilon_0 > 0$ . By Lemma 3.8, one can find  $k_0 \geq 2 + 2\varepsilon_0^{-1}$  such that the first sum limited at the trajectories  $w$  for which  $k_w < k_0$  is bounded by  $\varepsilon_0$ . Using the fact that the fibers of the maps  $\mathcal{R}_n \mapsto S^n$ ;  $(w, w') \mapsto w$  have cardinality  $k_w$ , one gets

$$D \leq \varepsilon_0 + \sum_{\substack{(w, w') \in \mathcal{R}_n, \\ k_w \geq k_0}} \left( \frac{\mu_w}{k_w} h(\dot{w}) - \frac{\mu_{w'}}{k_{w'}} \sum_{0 < |p| \leq p_0} h(\dot{w}' z_0^p) \right).$$

By Lemma 4.2, the element  $\dot{w}$  is equal to at least one of those  $\dot{w}' z_0^p$ , therefore one gets

$$D \leq \varepsilon_0 + \sum_{\substack{(w, w') \in \mathcal{R}_n, \\ k_w \geq k_0}} \mu_w \frac{k_{w'} - k_w}{k_w k_{w'}} h(\dot{w}).$$

By Lemma 4.2, one has  $|k_{w'} - k_w| \leq 2$ , and  $2/k_{w'} \leq 2/(k_0 - 2) \leq \varepsilon_0$ , and

$$D \leq \varepsilon_0 + \varepsilon_0 \sum_{(w, w') \in \mathcal{R}_n} \frac{\mu_w}{k_w} h(\dot{w}).$$

Using again that  $k_w$  is the cardinality of the fiber and using the harmonicity of  $h$ , one gets

$$D \leq \varepsilon_0 + \varepsilon_0 \sum_{w \in S^n} \mu_w h(\dot{w}) = \varepsilon_0 + \varepsilon_0 h(0).$$

Since  $\varepsilon_0$  can be chosen arbitrarily small, this gives  $D \leq 0$  as expected.  $\square$

*End of proof of Proposition 4.1.* Lemma 4.3 tells us that the positive harmonic function  $h$  is bounded by a finite sum of  $\mu$ -harmonic functions  $h_z$  with  $z \in Z$  non zero. Since all the positive  $\mu$ -harmonic functions  $h$  and  $h_z$  are extremal, and since the cone  $\mathcal{H}_\mu^+$  is reticulatated,  $h$  has to be proportional to one of these translates  $h_z$ .  $\square$

*Remark 4.4.* We now want to deduce from the semi-invariance of  $h$  proven in Proposition 4.1, the  $Z$ -invariance of  $h$ . This is not a general fact. Indeed, the harmonic function  $h$  in Case 3.b) of Theorem 5.10 can be  $Z$ -semiinvariant but is not  $Z$ -invariant. Hence, we have to use once more the assumption that  $h$  is not induced. One technical difficulty comes from the fact that, when  $G_\mu^+ \neq G$ , the cone  $\mathcal{H}_\mu^+$  often does not have a compact basis. This prevent us from using the same arguments as in [12].

## 4.2 $z$ -invariance and $Z$ -invariance

We first notice that in order to prove the  $Z$ -invariance of a positive  $\mu$ -harmonic function  $h$  on the Heisenberg group  $G$ , it is enough to check that it is invariant under one non trivial element of  $Z$ .

**Lemma 4.5.** *Keep notation (4.1) and assume that there exists  $z \neq 0$  in  $Z$  such that  $h_z = h$ . Then  $h$  is  $Z$ -invariant. In particular, if  $h$  is  $\mu$ -extremal, it is proportional to a  $\mu$ -harmonic character of  $G$ .*

*Proof of Lemma 4.5.* We write  $z = z_0^p$ . We can assume that  $p$  is the smallest positive integer for which  $h_z = h$ . We can also assume that  $h$  is extremal in the convex cone of  $z$ -invariant positive  $\mu$ -harmonic functions. Therefore the functions  $h_{z_0^i}$ , for  $i = 1, \dots, p$ , are non-proportional functions which are extremal in this cone, and the function  $f := h_{z_0} + \dots + h_{z_0^p}$  is  $\mu$ -harmonic,  $Z$ -invariant and it is extremal among the  $\mu$ -harmonic functions on  $G/Z$ . Since  $G/Z$  is abelian, by the Choquet–Deny Theorem, this function  $f$  is a  $\mu$ -harmonic character of  $G$ . Therefore, by Lemma 2.1, this function  $f$  is  $\mu$ -extremal and one has  $p = 1$ . This means that  $h$  is  $Z$ -invariant.  $\square$

## 4.3 $Z$ -invariance when $V_\mu^+$ contains a line

In this section, we finish the proof of our main Theorem 1.1 when the cone  $V_\mu^+$  is the plane or a half-plane, see (2.5).

**Proposition 4.6.** *Keep notation (4.1), assume that  $V_\mu^+$  contains a line and that the  $\mu$ -harmonic function  $h$  is not induced. Then  $h$  is  $Z$ -invariant.*

*Proof of Proposition 4.6.* We can assume that  $h$  is  $\mu$ -extremal and apply Proposition 4.1. Then our claim follows from the following slightly stronger Proposition 4.7. This stronger version will also be useful in Chapter 5.  $\square$

**Proposition 4.7.** *Keep notation (4.1) and assume that the cone  $V_\mu^+$  contains a line. Assume also that there exists  $z \neq 0$  in  $Z$  and  $q > 0$  such that  $h_z = qh$ . Then the function  $h$  is  $Z$ -invariant.*

*Proof of Proposition 4.7.* According to Lemma 4.5, it is enough to prove that  $q = 1$ . Replacing  $h$  by a multiple of a suitable translate, we can assume that

$h(0) = 1$ . Since the cone  $V_\mu^+$  contains a line, there exists two words  $w_0$  in  $S^{n_0}$  and  $w'_0$  in  $S^{n'_0}$  whose product is in the center:

$$\dot{w}_0 \dot{w}'_0 = z^a \text{ for some } a \text{ in } \mathbb{Z}.$$

Since the cone  $V_\mu^+$  is not a line, there exists also a word  $w_1$  in  $S^{n_1}$  such that

$$\dot{w}_0 \dot{w}_1 \dot{w}_0^{-1} \dot{w}_1^{-1} = z^b \text{ for some } b \geq 1.$$

Assume, by contradiction that  $q > 1$ . Choose an integer  $\ell \geq 1$  such that  $C := \mu_{w_0} \mu_{w'_0} q^{a+b\ell} > 1$ . Notice the equality, for all  $k \geq 1$ ,

$$\dot{w}_0^k \dot{w}_1^\ell \dot{w}'_0{}^k \dot{w}_1^{-\ell} = z^{ak+b\ell k}.$$

And compute with  $n := kn_0 + \ell n_1 + kn'_0$ ,

$$\begin{aligned} h(\dot{w}_1^{-\ell}) &= P_\mu^n h(\dot{w}_1^{-\ell}) \\ &\geq \mu_{w_0}^k \mu_{w_1}^\ell \mu_{w'_0}^k h(\dot{w}_0^k \dot{w}_1^\ell \dot{w}'_0{}^k \dot{w}_1^{-\ell}) \\ &\geq \mu_{w_0}^k \mu_{w_1}^\ell \mu_{w'_0}^k q^{ak+b\ell k} = \mu_{w_1}^\ell C^k \end{aligned}$$

Since  $C > 1$  and since this inequality is valid for all integer  $k \geq 1$  one gets a contradiction. This proves that  $q \leq 1$ .

Working similarly with the equality  $\dot{w}'_0{}^k \dot{w}_1^\ell \dot{w}_0^k \dot{w}_1^{-\ell} = z^{ak-b\ell k}$ , one proves that  $q \geq 1$ . And hence one has  $q = 1$ .  $\square$

#### 4.4 $Z$ -invariance when $V_\mu^+$ contains no line

In this section, we finish the proof of our main Theorem 1.1 when the cone  $V_\mu^+$  is properly convex, see (2.5).

**Proposition 4.8.** *Keep notation (4.1), assume that  $V_\mu^+$  contains no line and that the  $\mu$ -harmonic function  $h$  is not induced. Then  $h$  is  $Z$ -invariant.*

*Beginning of proof of Proposition 4.8.* The proof is by induction on the cardinality of the support  $S$  of  $\mu$ . We will use the induction hypothesis inside the proof of Lemma 4.10.

**First step** We begin the proof by a few reduction steps.

We can assume that  $h$  is  $\mu$ -extremal, so that, by Proposition 4.1, there exist  $z = z_0^p \neq 0$  in  $Z$  and  $q > 0$  such that  $h_z = qh$ . According to Lemma 4.5, it is enough to prove that  $q = 1$ .

**a. We can assume  $z = z_0$ .** Because we can replace  $h$  by the function  $f := q_0^{-1}h_{z_0} + \cdots + q_0^{-p}h_{z_0^p}$  where  $q_0 > 0$  is chosen so that  $q_0^p = q$ . This function  $f$  is  $\mu$ -harmonic and  $Z$ -semiinvariant. It might not be  $\mu$ -extremal, but this property will not be used in the argument below.

**b. We can assume  $S \cap Z = \emptyset$ .** Indeed, by **a**, if  $\mu_Z$  is the restriction of  $\mu$  to the center, one has  $P_{\mu_Z}h = \lambda h$  for a constant  $0 \leq \lambda < 1$ . But then the function  $h$  is harmonic for the measure  $(1 - \lambda)^{-1}(\mu - \mu_Z)$ . It might not be extremal for this measure, but, as we just said, this is not important.

**c. We can assume  $h(0) = 1$ .** Because we can replace  $h$  by a multiple of a suitable translate.

**d. We can assume  $q < 1$ .** Because we can replace the generator  $z_0$  by its inverse. We are now looking for a contradiction.

**Second step** We now can enter the key argument of the proof. Since the cone  $V_\mu^+$  is properly convex and since  $S \cap Z = \emptyset$ , we can find a partition of the support of  $\mu$  in two non-empty subsets

$$S = S_1 \cup S_2,$$

such that

$$c_{s_1, s_2} \geq 1 \quad \text{for all } s_1 \text{ in } S_1 \text{ and } s_2 \text{ in } S_2. \quad (4.2)$$

We will use the decomposition  $\mu = \mu_1 + \mu_2$  where  $\mu_1 := \mathbf{1}_{S_1} \mu$  and  $\mu_2 := \mathbf{1}_{S_2} \mu$ . The proof again starts with the equality (2.2) which tells us that, for all  $n \geq 1$ ,

$$1 = h(0) = \sum_{w \in S^n} \mu_w h(w). \quad (4.3)$$

We will cut this sum into pieces parametrized by pairs  $(w_1, w_2) \in S_1^{n_1} \times S_2^{n_2}$ , with  $n_1 + n_2 = n$ . We define

$$B_{w_1, w_2} = \{w \in S^n \text{ containing } w_1 \text{ and } w_2 \text{ as subwords}\}$$

For instance when  $w_1 = 11$  and  $w_2 = 23$ , one has

$$B_{w_1, w_2} = \{1123, 1213, 1231, 2113, 2131, 2311\}.$$

This allows us to write the above sum (4.3) as

$$1 = \sum_{n_1+n_2=n} \sum_{w_1 \in S_1^{n_1}} \sum_{w_2 \in S_2^{n_2}} \sum_{w \in B_{w_1, w_2}} \mu_w h(\dot{w}). \quad (4.4)$$

For every  $w$  in  $B_{w_1, w_2}$ , we write

$$\dot{w} = \dot{w}_2 \dot{w}_1 z_0^{n_w} \text{ for some integer } n_w \geq 1. \quad (4.5)$$

Then Equality (4.4) becomes

$$1 = \sum_{n_1+n_2=n} \sum_{w_1 \in S_1^{n_1}} \sum_{w_2 \in S_2^{n_2}} \mu_{w_1} \mu_{w_2} h(\dot{w}_2 \dot{w}_1) \left( \sum_{w \in B_{w_1, w_2}} q^{n_w} \right). \quad (4.6)$$

To pursue our analysis, we will need the following lemma which bounds this last sum.  $\square$

**Lemma 4.9.** *For all  $w_1$  in  $S_1^{n_1}$  and  $w_2$  in  $S_2^{n_2}$ , one has*

$$\sum_{w \in B_{w_1, w_2}} q^{n_w} \leq \eta(q)^{-1} < \infty. \quad (4.7)$$

where  $\eta(q) := \prod_{i \geq 1} (1 - q^i) > 0$ .

Note that this upper bound does not depend on  $(w_1, w_2)$ .

*Proof of Lemma 4.9.* For each word  $w = s_1 \dots s_n$  in  $B_{w_1, w_2}$ , we set

$$m_w := |\{(i, j) \mid 1 \leq i < j \leq n \text{ and } s_i \in S_1, s_j \in S_2\}|.$$

Condition (4.2) implies that

$$m_w \leq n_w \text{ for all } w \text{ in } B_{w_1, w_2}$$

A word  $w = s_1 \dots s_n$  in  $B_{w_1, w_2}$  is determined by the increasing sequence  $1 \leq i_1 < i_2 < \dots < i_{n_2} \leq n$  of places  $i$  where  $s_i$  belongs to  $S_2$ , and  $m_w$  is given by

$$m_w = (i_{n_2} - n_2) + \dots + (i_2 - 2) + (i_1 - 1).$$

Therefore, for all  $m \geq 1$ , the number

$$p(n_1, n_2, m) := |\{w \in B_{w_1, w_2} \mid m_w = m\}|$$

is equal to the number of partitions of  $m$  by  $n_2$  non-increasing integers  $a_1, \dots, a_{n_2}$  bounded by  $n_1$  :

$$p(n_1, n_2, m) = |\{n_1 \geq a_1 \geq \dots \geq a_{n_2} \geq 0 \text{ and } m = a_1 + \dots + a_{n_2}\}|.$$

This quantity is bounded by the partition function

$$p(m) = |\{a_1 \geq \dots \geq a_k \geq \dots \geq 0 \text{ and } m = a_1 + \dots + a_k + \dots\}|.$$

The generating function of the partition function is

$$\sum_{m \geq 0} p(m)q^m = \prod_{i > 0} (1 + q^i + q^{2i} + \dots) = \prod_{i > 0} (1 - q^i)^{-1} = \eta(q)^{-1}.$$

We now collect the sequence of inequalities we have just proven

$$\sum_{w \in B_{w_1, w_2}} q^{n_w} \leq \sum_{w \in B_{w_1, w_2}} q^{m_w} = \sum_{m \geq 0} p(n_1, n_2, m)q^m \leq \sum_{m \geq 0} p(m)q^m = \eta(q)^{-1}$$

and we obtain the bound (4.7) we were looking for.  $\square$

*End of proof of Proposition 4.8.* We plug Inequality (4.7) in Formula (4.6) and we obtain, for all  $n \geq 1$

$$\sum_{n_1 + n_2 = n} P_{\mu_1}^{n_1} P_{\mu_2}^{n_2} h(0) \geq \eta(q) > 0. \quad (4.8)$$

This contradicts the following Lemma 4.10  $\square$

**Lemma 4.10.** *With the same notation. In particular  $\mu = \mu_1 + \mu_2$  with  $S_1$  and  $S_2$  disjoint, and  $h$  is a non-induced  $\mu$ -harmonic function on  $G$ .*

a) *One has  $\lim_{n \rightarrow \infty} P_{\mu_1}^n h = 0$  and  $\lim_{n \rightarrow \infty} P_{\mu_2}^n h = 0$ .*

b) *One also has*

$$\sum_{n_1 + n_2 = n} P_{\mu_1}^{n_1} P_{\mu_2}^{n_2} h = 0. \quad (4.9)$$

*Proof of Lemma 4.10.* a) Let us prove it for  $\mu_1$ .

If  $S_1$  is abelian, this follows from the assumption that  $h$  is non-induced.

If  $S_1$  is not abelian, we will use our induction hypothesis. Assume by contradiction that the  $\mu_1$ -harmonic function  $h' := \lim_{n \rightarrow \infty} P_{\mu_1}^n h$  is non-zero. By Lemma 3.2, this function  $h'$  is  $\mu_1$ -extremal and satisfies

$$\lim_{n \rightarrow \infty} P_{\mu}^n h' = h. \quad (4.10)$$



By Lemma 3.5, this  $\mu_1$ -harmonic function  $h'$  is not induced and, since  $S_1$  is smaller than  $S$ , the function  $h'$  is a  $\mu_1$ -harmonic character of the group  $G_{\mu_1}$ . Since this group  $G_{\mu_1}$  has finite index in the group  $G_\mu$ , Lemma 3.6.ii tells us that the function  $\lim_{n \rightarrow \infty} P_\mu^n h'$  is not finite. This contradicts (4.10).

b) The argument is the same as for Lemma 3.8, but is simpler. We fix  $g$  in  $G$  and  $\varepsilon_0 > 0$ . According to point a), there exists  $N_1 \geq 1$  such that  $P_{\mu_1}^{N_1} h(g) \leq \varepsilon_0$ . Let  $I_n$  be the left-hand side of (4.9). We decompose  $I_n(g)$  as the sum of two terms

$$I_n(g) = I'_n(g) + I''_n(g)$$

where  $I'_n(g)$  involves the terms with  $n_1 \geq N_1$  and  $I''_n(g)$  involves the terms with  $n_1 < N_1$

Bounding  $I'_n$ . One computes, using the  $\mu$ -harmonicity of  $h$ ,

$$\begin{aligned} I'_n(g) &= \sum_{n'_1 + n_2 = n - N_1} P_{\mu_1}^{N_1} P_{\mu_1}^{n'_1} P_{\mu_2}^{n_2} h(g) \\ &\leq P_{\mu_1}^{N_1} P_\mu^{n - N_1} h(g) = P_{\mu_1}^{N_1} h(g) \leq \varepsilon_0. \end{aligned}$$

Bounding  $I''_n$ . One decomposes  $I''_n(g)$  as a finite sum

$$I''_n(g) = \sum_{n_1 < N_1} \sum_{w_1 \in S_1^{n_1}} \mu_{w_1} P_{\mu_2}^{n - n_1} (\dot{w}_1 g)$$

over the finitely many words  $w_1$  of length  $n_1 < N_1$ . By point a), all terms of the sum go to 0 so that one has  $\lim_{n \rightarrow \infty} I''_n(g) = 0$ .

Since  $\varepsilon_0$  can be chosen arbitrarily small, one deduces  $\lim_{n \rightarrow \infty} I_n(g) = 0$ .  $\square$

This ends the proof of Proposition 4.8.

## 5 Existence of induced harmonic functions

In this chapter we keep the following notations :

$$\begin{aligned} G &= H_3(\mathbb{Z}) \text{ is the Heisenberg group, } Z \text{ is the center of } G, \\ \mu &\text{ is a positive measure with finite support } S \text{ such that } G_\mu = G, \\ S_0 &\subset S \text{ is a maximal abelian subset, } \mu_0 := \mathbf{1}_{S_0} \mu, G_0 := G_{\mu_0}, \\ \chi_0 &\text{ is a } \mu_0\text{-harmonic character of } G_0 \text{ and } \mu_1 := \mu - \mu_0. \end{aligned} \quad (5.1)$$

We will give in this chapter a necessary and sufficient condition for the induced  $\mu$ -harmonic function  $h_{G_0, \chi_0}$  to be finite.

In Lemma 3.6.iv, we have already found that the following condition is necessary :

$$S_1 \cap Z = \emptyset \quad \text{and} \quad V_{\mu_1}^+ \cap V_{\mu_0} = \{0\}. \quad (5.2)$$

We will assume that it is satisfied.

We distinguish two cases according to the rank of the abelian group  $G_0$ .

### 5.1 Induction of characters when $\text{rank } G_0 = 1$

In this section we give the necessary and sufficient condition for the induced function  $h_{G_0, \chi_0}$  to be finite when  $\text{rank } G_0 = 1$  or equivalently when  $G_0 \cap Z = \{0\}$ .

**Proposition 5.1.** *Keep notation (5.1). Assume (5.2) and  $\text{rank } G_0 = 1$ . Then the induced harmonic function  $h := h_{G_0, \chi_0}$  is finite if and only if the probability measure  $\tilde{\mu}_0 := \chi_0 \mu_0$  on  $G_0$  is not centered.*

*Remark 5.2.* - The measure  $\tilde{\mu}_0 = \chi_0 \mu_0$  is a probability measure because  $\chi_0$  is a  $\mu_0$ -harmonic character.

- The condition  $\tilde{\mu}_0$  centered means, as usual, that  $\sum_{s \in S_0} \tilde{\mu}_{0,s} \bar{s} = 0$  in  $V$ , where  $\bar{s}$  is the image of  $s$  in  $V$ .

- This condition  $\tilde{\mu}_0$  non-centered is always satisfied when  $V_{\mu}^+$  contains no line.

*Proof of Proposition 5.1.* We can assume that

$$S_0 \subset \{(x, 0, 0) \mid x \in \mathbb{Z}\} \quad \text{and} \quad S_1 \subset \{(x, y, z) \in G \mid y \geq 1\}.$$

Let  $\tau : G_0 \mapsto \mathbb{Z}$  be the morphism given by  $\tau(g_0) = x$  for  $g_0 = (x, 0, 0)$ .

**First case** When  $\tilde{\mu}_0$  is centered.

We fix  $s_1$  in  $S_1$  and we compute, as in Lemma 3.6, for  $n \geq 1$ ,

$$\begin{aligned} h(s_1^{-1}) &\geq P^{n+1} \chi_0(s_1^{-1}) \\ &\geq \mu_{s_1} \sum_{k \leq n} P_{\mu_0}^k P_{s_1} P_{\mu_0}^{n-k} \chi_0(s_1^{-1}) \\ &= \mu_{s_1} \sum_{k \leq n} P_{\mu_0}^k P_{s_1} \chi_0(s_1^{-1}) \\ &= \mu_{s_1} \sum_{k \leq n} \sum_{w \in S_0^k} \mu_{0,w} \chi_0(s_1 w s_1^{-1}) \end{aligned} \quad (5.3)$$

The words  $w$  that contribute to this sum must satisfy  $s_1 \dot{w} s_1^{-1} \dot{w}^{-1} \in G_0$ , i.e.  $\tau(\dot{w}) = 0$ . Hence letting  $n$  go to  $\infty$ , one gets

$$h(s_1^{-1}) \geq \mu_{s_1} \sum_{k \geq 0} \sum_{w \in S_0^k} \tilde{\mu}_{0,w} \mathbf{1}_{\{\tau(\dot{w})=0\}}.$$

If we write  $w = s_1 \dots s_n$  and  $x_i := \tau(s_i)$ , and if we think of these letters  $s_i$  as independent random variables with same law  $\tilde{\mu}_0$ , this inequality can be rewritten as

$$h(s_1^{-1}) \geq \mu_{s_1} \sum_{k \geq 0} \mathbb{P}(x_1 + \dots + x_k = 0).$$

But since the random variables  $x_i \in \mathbb{Z}$  are centered, the expected number of passage at 0 of the walk  $x_1 + \dots + x_k$  is infinite, and the function  $h$  is not finite.

**Second case** When  $\tilde{\mu}_0$  is not centered.

The computation is similar but more involved since we want to prove finiteness of  $h(g)$  at every point  $g$  in  $G$ .

We want a uniform upperbound for

$$P_\mu^n \chi_0(g) = \sum_{w \in S^n} \mu_w \chi_0(\dot{w}g).$$

The only words  $w$  that contribute to this sum are those for which  $\dot{w}g$  is in  $G_0$ . By assumption (5.2), if we extract from  $w$  the maximal subword  $\sigma = s_1 \dots s_\ell$  whose letters are in  $S_1$ , the length  $\ell$  of  $\sigma$  is uniformly bounded by an integer  $\ell_0$ . Therefore we can split the above sum into a finite sum

$$P_\mu^n \chi_0(g) = \sum_{\ell \leq \ell_0} \sum_{\sigma \in S_1^\ell} \mu_\sigma Q_{\sigma, n-\ell} \chi_0(g),$$

where

$$\begin{aligned} Q_{\sigma, n} \chi_0(g) &= \sum_{k_0 + \dots + k_\ell = n} P_{\mu_0}^{k_\ell} P_{s_\ell} \dots P_{\mu_0}^{k_1} P_{s_1} P_{\mu_0}^{k_0} \chi_0(g) \\ &= \sum_{k_1 + \dots + k_\ell \leq n} P_{\mu_0}^{k_\ell} P_{s_\ell} \dots P_{\mu_0}^{k_1} P_{s_1} \chi_0(g) \end{aligned} \quad (5.4)$$

We want to bound the limit

$$\begin{aligned} Q_\infty(g) &:= \lim_{n \rightarrow \infty} Q_{\sigma, n} \chi_0(g) \\ &= \sum_{k_1 \geq 0} \dots \sum_{k_\ell \geq 0} \sum_{w_1 \in S_0^{k_1}} \dots \sum_{w_\ell \in S_0^{k_\ell}} \mu_{w_1} \dots \mu_{w_\ell} \chi_0(s_1 \dot{w}_1 \dots s_\ell \dot{w}_\ell g). \end{aligned} \quad (5.5)$$

For  $i \leq \ell$ , let  $\sigma_i := s_1 \cdots s_i \in G$  and  $b_i \geq 1$  be the integer given by

$$\sigma_i g_0 \sigma_i^{-1} g_0^{-1} = z_0^{-b_i \tau(g_0)} \quad \text{for all } g_0 \text{ in } G_0,$$

so that one has

$$s_1 \dot{w}_1 \cdots s_\ell \dot{w}_\ell g = \dot{w}_1 \cdots \dot{w}_\ell \sigma_\ell g z_0^{-b_1 \tau(\dot{w}_1) - \cdots - b_\ell \tau(\dot{w}_\ell)}. \quad (5.6)$$

Writing  $\sigma_\ell g = g_0 z_0^c$  with  $g_0$  in  $G_0$  and  $c$  in  $\mathbb{Z}$  one gets

$$Q_\infty(g) = \chi_0(g_0) \sum_{k_1 \geq 0} \cdots \sum_{k_\ell \geq 0} \sum_{w_1 \in S_0^{k_1}} \cdots \sum_{w_\ell \in S_0^{k_\ell}} \tilde{\mu}_{0, w_1} \cdots \tilde{\mu}_{0, w_\ell} \mathbf{1}_{\{b_1 \tau(\dot{w}_1) + \cdots + b_\ell \tau(\dot{w}_\ell) = c\}}.$$

If we think of all the letters occuring in one of the words  $w_1, \dots, w_\ell$  as independent random variables with same law  $\tilde{\mu}_0$ , this equality can be written as

$$Q_\infty(g) = \chi_0(g_0) \sum_{k_1 \geq 0} \cdots \sum_{k_\ell \geq 0} \mathbb{P}(b_1 S_{1, k_1} + \cdots + b_\ell S_{\ell, k_\ell} = c)$$

where  $S_{i, k_i} := \tau(\dot{w}_i)$ . Then the finiteness of  $Q_\infty(g)$  follows from the following Lemma 5.3.  $\square$

**Lemma 5.3.** *Let  $(X_{i, k})_{i \leq \ell, k \geq 1}$ , be independent real variables with same law. Assume this law has finite support and is not centered. Let  $(b_i)_{i \leq \ell}$  be positive numbers and  $c$  be a real number. Set  $S_{i, k} := X_{i, 1} + \cdots + X_{i, k}$ . Then one has*

$$\sum_{k_1 \geq 0} \cdots \sum_{k_\ell \geq 0} \mathbb{P}(b_1 S_{1, k_1} + \cdots + b_\ell S_{\ell, k_\ell} = c) < \infty. \quad (5.7)$$

*Proof of Lemma 5.3.* We adapt the classical proof of the large deviation inequality. We set  $X = X_{1, 1}$ . Assume for instance that  $\mathbb{E}(X) > 0$ . One can choose  $\varepsilon > 0$  so that all the expectations

$$\alpha_i := \mathbb{E}(e^{-\varepsilon b_i X})$$

are smaller than 1. Then one computes

$$\begin{aligned} \mathbb{P}(b_1 S_{1, k_1} + \cdots + b_\ell S_{\ell, k_\ell} = c) &\leq \mathbb{E}(e^{\varepsilon(c - b_1 S_{1, k_1} - \cdots - b_\ell S_{\ell, k_\ell})}) \\ &= e^{\varepsilon c} \mathbb{E}(e^{-\varepsilon b_1 X})^{k_1} \cdots \mathbb{E}(e^{-\varepsilon b_\ell X})^{k_\ell} \\ &= e^{\varepsilon c} \alpha_1^{k_1} \cdots \alpha_\ell^{k_\ell} \end{aligned}$$

and therefore, summing all these inequalities, we find the following upper-bound for the left-hand side  $L$  of (5.7)

$$L \leq e^{\varepsilon c} (1 - \alpha_1)^{-1} \cdots (1 - \alpha_\ell)^{-1} < \infty.$$

This ends the proof of the lemma and of Proposition 5.1.  $\square$

## 5.2 Induction of characters when $\text{rank } G_0 = 2$

In this section we give the necessary and sufficient condition for the induced function  $h_{G_0, \chi_0}$  to be finite when  $\text{rank } G_0 = 2$  or equivalently when  $G_0 \cap Z \neq \{0\}$ .

We split the statement into two cases depending on the shape of the convex cone  $V_\mu^+$ .

**Proposition 5.4.** *Keep notation (5.1). Assume (5.2) and  $\text{rank } G_0 = 2$ . Assume moreover that the cone  $V_\mu^+$  contains a line. Then the induced harmonic function  $h := h_{G_0, \chi_0}$  is not finite.*

*Proof of Proposition 5.4.* This follows from Proposition 4.7. Indeed, let  $z$  be a non-zero element of  $G_0 \cap Z$  and  $q := \chi_0(z)$ . Assume by contradiction that the function  $h$  is finite. By Lemma 3.1, this function  $h$  is  $\mu$ -extremal, By construction this function  $h$  is semiinvariant : one has  $h_z = qh$ . Hence by Proposition 4.7, one has  $q = 1$  and by Lemma 4.5 the  $\mu$ -harmonic function  $h$  is  $Z$ -invariant. Therefore, by the Choquet–Deny Theorem, this function  $h$  is a  $\mu$ -harmonic character of  $G$ . But by Corollary 3.4, a  $\mu$ -harmonic character is never induced. Contradiction.  $\square$

**Proposition 5.5.** *Keep notation (5.1). Assume (5.2) and  $\text{rank } G_0 = 2$ . Assume moreover that the cone  $V_\mu^+$  contains no line. Then the induced harmonic function  $h := h_{G_0, \chi_0}$  is finite if and only if*

$$\text{there exist } s_0 \text{ in } S_0 \text{ and } s_1 \text{ in } S_1 \text{ such that } \chi_0(s_0 s_1 s_0^{-1} s_1^{-1}) > 1. \quad (5.8)$$

*Remark 5.6.* Since Assumption (5.2) is satisfied and since the cone  $V_\mu^+$  is properly convex, this condition (5.8) is equivalent to

$$\text{for all } s_0 \text{ in } S_0 \setminus Z \text{ and } s_1 \text{ in } S_1 \text{ one has } \chi_0(s_0 s_1 s_0^{-1} s_1^{-1}) > 1. \quad (5.9)$$

*Proof of Proposition 5.5.* The calculation is the same as for Proposition 5.1, but the interpretation is different. We can assume that

$$S_0 \subset \{(x, 0, z) \in G \mid x \geq 0\} \text{ and } S_1 \subset \{(x, y, z) \in G \mid y \geq 1\}.$$

Let  $\tau : G_0 \mapsto \mathbb{Z}$  be the morphism given by  $\tau(g_0) = x$  for  $g_0 = (x, 0, z)$ .

**Proof of  $\implies$**  We already know that the half-line  $V_{\mu_0}^+$  is extremal in the convex cone  $V_\mu^+$ . Assume by contraposition, that for all  $s_0$  in  $S_0$  and  $s_1$  in

$S_1$  one has  $\chi_0(s_0 s_1 s_0^{-1} s_1^{-1}) \leq 1$ . We fix  $s_1$  in  $S_1$  and we compute, for  $n \geq 1$ , as in (5.3),

$$\begin{aligned} h(s_1^{-1}) &\geq \mu_{s_1} \sum_{k \leq n} \sum_{w \in S_0^k} \mu_{0,w} \chi_0(s_1 w s_1^{-1}) \\ &\geq \mu_{s_1} \sum_{k \leq n} \sum_{w \in S_0^k} \mu_{0,w} \chi_0(w) \\ &\geq \mu_{s_1} \sum_{k \leq n} \chi_0(0) \geq n \mu_{s_1} \end{aligned}$$

Letting  $n$  go to  $\infty$ , one gets  $h(s_1^{-1}) = \infty$ .

**Proof of  $\Leftarrow$**  As for Proposition 5.1, one can find  $\ell_0$  and split  $P_\mu^n \chi_0(g)$  as a sum parametrized by words  $\sigma = s_1 \dots s_\ell$  with letters in  $S_1$  and  $\ell \leq \ell_0$  :

$$P_\mu^n \chi_0(g) = \sum_{\ell \leq \ell_0} \sum_{\sigma \in S_1^\ell} \mu_\sigma Q_{\sigma, n-\ell} \chi_0(g), \text{ where, as in (5.4),}$$

$$Q_{\sigma, n} \chi_0(g) = \sum_{k_1 + \dots + k_\ell \leq n} P_{\mu_0}^{k_\ell} P_{s_\ell} \cdots P_{\mu_0}^{k_1} P_{s_1} \chi_0(g).$$

As in (5.5) , we want to bound the limit

$$Q_\infty(g) := \lim_{n \rightarrow \infty} Q_{\sigma, n} \chi_0(g).$$

The only words  $w$  that contribute to this sum are those for which  $wg$  is in  $G_0$ . For  $i \leq \ell$ , let  $\sigma_i := s_1 \cdots s_i \in G$  and  $b_i \geq 1$  be the integer given by

$$\sigma_i g_0 \sigma_i^{-1} g_0^{-1} = z_0^{-b_i \tau(g_0)} \text{ for all } g_0 \text{ in } G_0.$$

Using Equality (5.6), one gets

$$Q_\infty(g) = \mu_\sigma \chi_0(\sigma_\ell g) F_1 \cdots F_\ell$$

where, for all  $i \leq \ell$ ,

$$F_i := \sum_{k \geq 0} \sum_{w \in S_0^k} \mu_{0,w} \chi_0(\sigma_i w \sigma_i^{-1}).$$

We want to prove that the sums  $F_i$  are finite. We will denote by  $q_0 > 0$  the real number such that for all  $i$  in  $\mathbb{Z}$  such that  $z_0^i$  is in  $G_0$ , one has  $\chi_0(z_0^i) = q_0^i$ . By assumption, one has  $q_0 > 1$ . One computes then

$$F_i = \sum_{k \geq 0} \sum_{w \in S_0^k} \tilde{\mu}_{0,w} q_0^{-b_i \tau(w)},$$

where, as before,  $\tilde{\mu}_0$  is the probability measure  $\chi_0\mu_0$ . Let  $p_w$  be the number of letters of  $w$  that belong to  $S_0 \setminus Z$  and  $\alpha := \tilde{\mu}_0(S_0 \setminus Z) < 1$ . One goes on :

$$\begin{aligned} F_i &\leq \sum_{k \geq 0} \sum_{w \in S_0^k} \tilde{\mu}_{0,w} q_0^{-p_w} \\ &= \sum_{k \geq 0} \sum_{j \leq k} \binom{j}{k} \alpha^j (1 - \alpha)^{k-j} q_0^{-j} \\ &= \sum_{k \geq 0} (1 - \alpha + \alpha q_0^{-1})^k = \alpha^{-1} (1 - q_0^{-1})^{-1} < \infty. \end{aligned}$$

This proves the finiteness of  $F_i$ , of  $\mathcal{Q}_\infty(g)$  and of the function  $h_{G_0, \chi_0}$ .  $\square$

### 5.3 Existence of harmonic characters

We recall in this section when harmonic  $\mu$ -characters on abelian groups do exist.

Let  $G_0 = \mathbb{Z}^d$  and  $\mu_0$  be a positive measure with finite support  $S_0$  generating  $G_0$  as a group. For a character  $\chi_0$  of  $G_0$  we set

$$\mathbb{E}(\chi_0) := \sum_{s \in S_0} \mu_{0,s} \chi_0(s).$$

The map  $\chi_0 \rightarrow \mathbb{E}(\chi_0)$  is the Laplace transform of  $\mu_0$ . We denote by

$$\lambda(\mu_0) := \inf_{\chi_0} \mathbb{E}(\chi_0) \tag{5.10}$$

the minimum of this Laplace transform. Here is an example where it is easy to compute  $\lambda(\mu_0)$ .

*Remark 5.7.* If  $S_0$  is included in a properly convex set, one has  $\lambda(\mu_0) = \mu_0(0)$ . More generally, if  $S_0$  is included in a half-space bounded by a hyperplane  $H_0$ , one has  $\lambda(\mu_0) = \lambda(\mu_0|_{H_0})$ .

**Lemma 5.8.** *There exists a  $\mu_0$ -harmonic character if and only if  $\lambda(\mu_0) \leq 1$ . We can choose it so that  $\tilde{\mu}_0 := \chi_0\mu_0$  is not centered if and only if  $\lambda(\mu_0) < 1$ .*

*Proof.* Lemma 5.8 follows from the following three remarks:

- A character  $\chi_0$  is  $\mu$ -harmonic if and only if  $\mathbb{E}(\chi_0) = 1$ .
- The group of characters is isomorphic to  $\mathbb{R}^d$ , hence it is connected.
- Since  $S_0$  contains non-zero elements one has  $\sup_{\chi_0} \mathbb{E}(\chi_0) = \infty$ .  $\square$

- Corollary 5.9.** a) If  $\mu_0(S_0) \leq 1$ ,  $\mu_0$ -harmonic characters exist.  
b) If  $\mu_0(S_0) < 1$ , we can choose it so that  $\tilde{\mu}_0 := \chi_0 \mu_0$  is not centered.  
c) If  $\mu_0(S_0) > 1$  and  $\mu_0$  is centered,  $\mu_0$ -harmonic characters do not exist.

*Proof.* This follows from Lemma 5.8 and the inequality  $\lambda(\mu_0) \leq \mu_0(G_0)$ .  $\square$

## 5.4 Conclusion

We sum up in the following theorem the main results we have obtained in this paper.

Let  $G = H_3(\mathbb{Z})$  be the Heisenberg group,  $Z$  be the center of  $G$ ,  $\mu$  be a positive measure on  $G$  with finite support  $S$  such that  $G_\mu = G$ . We use notation of Section 2.3.

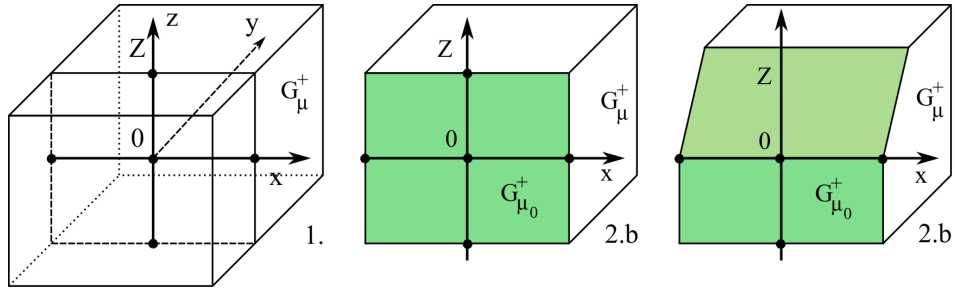
**Theorem 5.10.** *The extremal positive  $\mu$ -harmonic functions on  $G$  are proportional either to a character of  $G$  or to a translate of a function  $h$  induced of a character on an abelian subgroup. Here is the list when  $\mu(G) = 1$ .*

- 1) **When  $V_\mu^+$  is the plane.** *There is no induced  $\mu$ -harmonic function.*
- 2) **When  $V_\mu^+$  is a half-plane.** *Let  $V_0$  be the boundary line of  $V_\mu^+$  and  $G_0 \subset G$  be the subgroup generated by the elements of  $S$  above  $V_0$  and  $\mu_0 := \mu|_{G_0}$ . Then  $h$  is equal to a function  $h_{G_0, \chi_0}$  induced of a  $\mu_0$ -harmonic character  $\chi_0$  of  $G_0$ .*
  - a) *If  $G_0 \cap Z = \{0\}$  there are exactly two such  $h_{G_0, \chi_0}$ .*
  - b) *If  $G_0 \cap Z \neq \{0\}$  there is no such  $h_{G_0, \chi_0}$ .*
- 3) **When  $V_\mu^+$  is properly convex.** *Let  $V_i^+$ ,  $i = 0, 1$ , be the two extremal rays of  $V_\mu^+$ , let  $G_i \subset G$ , be the two subgroups generated by the elements of  $S$  above  $V_i^+$  and  $\mu_i := \mu|_{G_i}$ . Then  $h$  is equal to a function  $h_{G_i, \chi_i}$  induced of a  $\mu_i$ -harmonic character  $\chi_i$  of  $G_i$ ,  $i = 0$  or  $1$ .*
  - a) *If  $G_i \cap Z = \{0\}$  there is exactly one such  $h_{G_i, \chi_i}$ .*
  - b) *If  $G_i \cap Z \neq \{0\}$  there are uncountably many such  $h_{G_i, \chi_i}$ .*

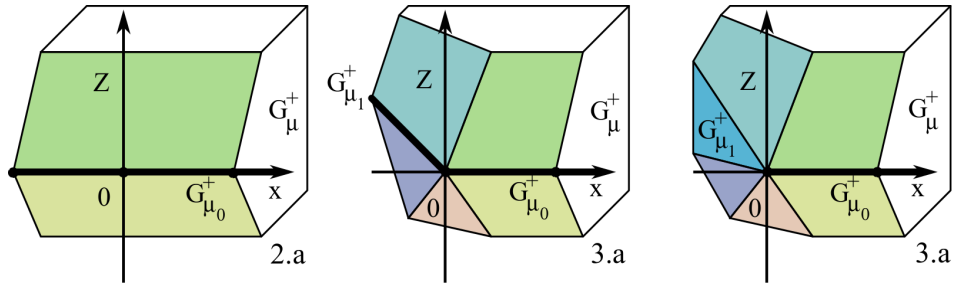
*Remark 5.11.* In the nine figures, we have drawn a rough approximation of the shape of the semigroup  $G_\mu^+ \subset G$  and its subsemigroups  $G_{\mu_0}^+$  and  $G_{\mu_1}^+$ , in order to illustrate the different cases that occur in Theorem 5.10. In these pictures the center is the vertical axis.

*Proof of Theorem 5.10.* The first claim follows from Proposition 4.6 and Proposition 4.8. Moreover, Case 1) and the first claims of Cases 2) and 3) follow from Lemma 3.6.iv.

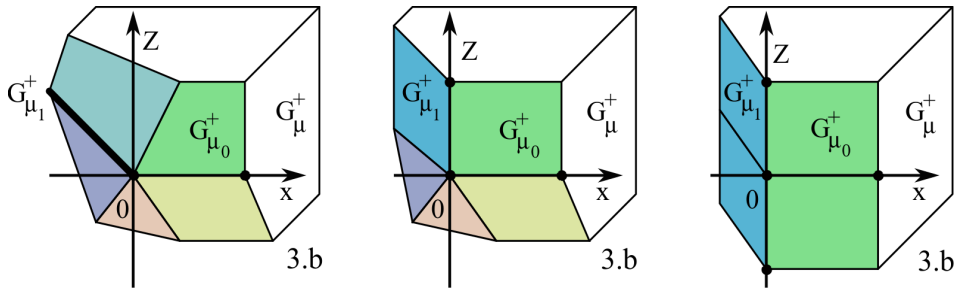




In Case 1 and in Case 2.b, no harmonic function is induced of a character of an abelian subgroup  $G_0$ .



In Case 2.a, exactly two harmonic functions are induced of a character of  $G_0 = G_{\mu_0}$ , and no other. In Case 3.a, only one harmonic function is induced of a character of  $G_0 = G_{\mu_0}$ , and one or infinitely many are induced of a character of  $G_1 = G_{\mu_1}$ .



In case 3.b, infinitely many harmonic functions are induced of a character of  $G_0 = G_{\mu_0}$  and one or infinitely many are induced of a character of  $G_1 = G_{\mu_1}$ .

- **Case 2.a)** Since  $\text{rank } G_0 = 1$ , by Proposition 5.1,  $\chi_0$  must be a  $\mu_0$ -harmonic character of  $G_0$  with  $\chi_0\mu_0$  non centered. Since  $\mu_0(G_0) < 1$  and since  $\mu_0$  is not supported by a half-line, there are exactly two such  $\chi_0$ .
- **Case 2.b)** Since  $\text{rank } G_0 = 2$ , this follows from Proposition 5.4.
- **Case 3.a)** Since  $\text{rank } G_i = 1$ , by Proposition 5.1,  $\chi_i$  must be a  $\mu_i$ -harmonic character of  $G_i$  with  $\chi_i\mu_i$  non centered. Since  $\mu_i(G_i) < 1$  and since  $\mu_i$  is supported by a half-line, there is exactly one such  $\chi_i$ .
- **Case 3.b)** Since  $\text{rank } G_i = 2$ , by Proposition 5.5,  $\chi_i$  must be a  $\mu_i$ -harmonic character of  $G_i$  satisfying (5.8). Since  $\mu(G_i) < 1$ , and since  $\mu_i$  is supported by a half-space delimited by  $Z$ , there are uncountably many such  $\chi_i$ .  $\square$

*Remark 5.12.* - When  $\mu$  is not assumed to be a probability measure, the formulation of Theorem 5.10 has to be modified. Indeed, if  $\mu(\{0\}) \geq 1$ , positive  $\mu$ -harmonic functions cannot exist. More precisely, each of the three cases 2.a), 3.a) and 3.b) has to be split into two subcases :

- 2.a') If  $G_0 \cap Z = \{0\}$  and  $\lambda(\mu_0) < 1$ , there are exactly two such  $h_{G_0, \chi_0}$ .
  - 2.a'') If  $G_0 \cap Z = \{0\}$  and  $\lambda(\mu_0) \geq 1$ , there are no such  $h_{G_0, \chi_0}$ .
  - 3.a') If  $G_i \cap Z = \{0\}$  and  $\mu(\{0\}) < 1$  there is exactly one such  $h_{G_i, \chi_i}$ .
  - 3.a'') If  $G_i \cap Z = \{0\}$  and  $\mu(\{0\}) \geq 1$  there are no such  $h_{G_i, \chi_i}$ .
  - 3.b') If  $G_i \cap Z \neq \{0\}$  and  $\lambda(\mu_z)_i < 1$ , there are uncountably many such  $h_{G_i, \chi_i}$ .
  - 3.b'') If  $G_i \cap Z \neq \{0\}$  and  $\lambda(\mu_z)_i \geq 1$ , there are no such  $h_{G_i, \chi_i}$ .
- Note that Cases 2.a''), 3.a'') and 3.b'') do not occur when  $\mu$  is a probability measure.

- Here is the definition, motivated by Condition (5.8), of the constants  $\lambda(\mu_z)_i$ . For instance, for  $i = 0$ , we choose  $s_j \in \text{supp}(\mu_j) \setminus Z$ ,  $j = 0$  or  $1$ , and set  $\lambda(\mu_z)_0 := \inf\{\sum_{s \in Z} \mu_s \chi_z(s) \mid \chi_z \text{ character of } Z, \chi_z(s_0 s_1 s_0^{-1} s_1^{-1}) > 1\}$ .

We now can deduce from Theorem 5.10 the corollaries in the introduction :

*Proof of Corollary 1.3.* The support of  $h_{G_0, \chi_0}$  is the semigroup generated by  $G_0$  and  $S^{-1}$ . In the cases where a  $\mu$ -harmonic function  $h_{G_0, \chi_0}$  induced of a character is finite, by Lemma 3.6.iv, one has  $V_{\mu_1}^+ \cap V_{\mu_0} = \{0\}$ , and this semigroup is never equal to  $G$ .  $\square$

*Proof of Corollary 1.4.* Both conditions (i), (ii) are true in Cases 1) and 2.b). Both conditions are not true in Cases 2.b), 3.a) and 3.b).  $\square$

One also has the following variation of Corollary 1.4.

**Corollary 5.13.** *Same notation and  $\mu(G) = 1$ . The following are equivalent:*  
 (i) *Extremal positive  $\mu$ -harmonic functions are  $Z^p$ -semiinvariant for a  $p \geq 1$ .*  
 (ii)  $G_\mu^+ \cap Z \not\subset \{0\}$ .

Condition (i) means that there exist  $q > 0$  and a non-zero element  $z$  in  $Z$  such that  $h_z = qh$ .

*Proof of Corollary 5.13.* Both conditions (i), (ii) are true in Cases 1), 2.b) and 3.b). Both conditions are not true in Cases 2.a) and 3.a).  $\square$

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