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► **To cite this version:**

Mohamad Houda, Paul Lescot. TWO COMPUTATIONS CONCERNING THE ISOVECTORS OF THE BACKWARD HEAT EQUATION WITH QUADRATIC POTENTIAL. *Methods and Applications of Analysis*, In press. hal-02399987

HAL Id: hal-02399987

<https://hal.science/hal-02399987>

Submitted on 9 Dec 2019

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TWO COMPUTATIONS CONCERNING THE ISOVECTORS OF THE BACKWARD HEAT EQUATION WITH QUADRATIC POTENTIAL

MOHAMAD HOUDA & PAUL LESCOT

ABSTRACT. We determine the isovectors of the backward heat equation with quadratic potential term in the space variable. This generalizes the calculations of Lescot-Zambrini (cf.[4]). These results first appeared in the first author's PhD thesis (Rouen, 2013).

1. BACKWARD EQUATION

The fundamental equation of Euclidean Quantum Mechanics of Zambrini is the backward heat equation with potential $V(t, q)$:

$$\begin{cases} \theta^2 \frac{\partial \eta_u^V}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta_u^V}{\partial q^2} + V(t, q) \eta_u^V & (C_1^{(V)}) \\ \eta_u^V(0, q) = u(q) \end{cases}$$

where t represents the time variable, q the space variable, and θ is a real parameter, strictly positive (in physics, $\theta = \sqrt{\hbar}$).

This equation is not well-posed in general, but existence and uniqueness of a solution are insured whenever the initial condition u belongs to the set of **analytic vectors** for the operator appearing on the right-hand side of the equation (see *e.g.* [1], Lemma 4, p. 429).

We shall denote by $\eta_u(t, q) = \eta_u^0(t, q)$ the solution of the backward heat equation with null potential :

$$\begin{cases} \theta^2 \frac{\partial \eta_u}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta_u}{\partial q^2} & (C_1^{(0)}) \\ \eta_u(0, q) = \eta_u^0(0, q) = u(q). \end{cases}$$

The authors wish to thank the referees for constructive comments.

1.1. Generalization of results due to Lescot-Zambrini (cf.[4]).

Theorem 1.1. *Let $a(t), b(t)$ and $c(t)$ be continuous functions and $V(t, q) = a(t)q^2 + b(t)q + c(t)$.*

For all initial conditions u , the solution $\eta_u^V(t, q)$ of:

$$\theta^2 \frac{\partial \eta_u^V}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta_u^V}{\partial q^2} + V(t, q) \eta_u^V \quad (C_1^{(V)})$$

such that $\eta_u^V(0, q) = u(q)$ is given by:

$$(1.1) \quad \eta_u^V(t, q) = \varphi_1(t, q) \eta_u(\varphi_2(t, q), \varphi_3(t, q)),$$

Date: 01 November 2019.

where: $\eta_u(t, q) = \eta_u^0(t, q)$ and $\varphi_1(t, q)$, $\varphi_2(t, q)$, $\varphi_3(t, q)$ only depend on $a(t)$, $b(t)$ and $c(t)$ (via formulas (1.7), (1.8), (1.9), (1.11), (1.12), (1.13) and (1.14)).

Proof. We shall make formula 1.1 explicit with the following initial conditions:

$$\begin{cases} \varphi_1(0, q) = 1 \\ \varphi_2(0, q) = 0 \\ \varphi_3(0, q) = q. \end{cases}$$

In this case :

$$\eta_u^V(0, q) = \eta_u(0, q) = u(q).$$

We now differentiate formula 1.1 with respect to t and q ;
 η_u and its derivatives will be taken in $(\varphi_2(t, q), \varphi_3(t, q))$.

$$\frac{\partial \eta_u^V}{\partial t} = \frac{\partial \varphi_1}{\partial t} \eta_u + \varphi_1 \left(\frac{\partial \eta_u}{\partial t} \frac{\partial \varphi_2}{\partial t} + \frac{\partial \eta_u}{\partial q} \frac{\partial \varphi_3}{\partial t} \right),$$

$$\frac{\partial \eta_u^V}{\partial q} = \frac{\partial \varphi_1}{\partial q} \eta_u + \varphi_1 \left(\frac{\partial \eta_u}{\partial t} \frac{\partial \varphi_2}{\partial q} + \frac{\partial \eta_u}{\partial q} \frac{\partial \varphi_3}{\partial q} \right),$$

$$\begin{aligned} \frac{\partial^2 \eta_u^V}{\partial q^2} &= \frac{\partial^2 \varphi_1}{\partial q^2} \eta_u + 2 \frac{\partial \varphi_1}{\partial q} \left(\frac{\partial \eta_u}{\partial t} \frac{\partial \varphi_2}{\partial q} + \frac{\partial \eta_u}{\partial q} \frac{\partial \varphi_3}{\partial q} \right) \\ &+ \varphi_1 \left[\frac{\partial^2 \eta_u}{\partial t^2} \left(\frac{\partial \varphi_2}{\partial q} \right)^2 + \frac{\partial^2 \eta_u}{\partial q^2} \left(\frac{\partial \varphi_3}{\partial q} \right)^2 + 2 \frac{\partial^2 \eta_u}{\partial t \partial q} \frac{\partial \varphi_2}{\partial q} \frac{\partial \varphi_3}{\partial q} + \frac{\partial \eta_u}{\partial t} \frac{\partial^2 \varphi_2}{\partial q^2} + \frac{\partial \eta_u}{\partial q} \frac{\partial^2 \varphi_3}{\partial q^2} \right] \end{aligned}$$

and

$$V(t, q) \eta_u^V = a(t) q^2 \varphi_1 \eta_u + b(t) q \varphi_1 \eta_u + c(t) \varphi_1 \eta_u.$$

So, it's enough to have:

$$(1.2) \quad \theta^2 \varphi_1 \frac{\partial \varphi_2}{\partial t} - \theta^2 \varphi_1 \left(\frac{\partial \varphi_3}{\partial q} \right)^2 + \frac{\theta^4}{2} \left(2 \frac{\partial \varphi_1}{\partial q} \frac{\partial \varphi_2}{\partial q} + \frac{\partial^2 \varphi_2}{\partial q^2} \right) = 0$$

$$(1.3) \quad \frac{\theta^4}{2} \varphi_1 \left(\frac{\partial \varphi_2}{\partial q} \right)^2 = 0$$

$$(1.4) \quad \theta^2 \varphi_1 \frac{\partial \varphi_3}{\partial t} + \frac{\theta^4}{2} \left(2 \frac{\partial \varphi_1}{\partial q} \frac{\partial \varphi_3}{\partial q} + \varphi_1 \frac{\partial^2 \varphi_3}{\partial q^2} \right) = 0$$

$$(1.5) \quad \theta^4 \frac{\partial \varphi_2}{\partial q} \frac{\partial \varphi_3}{\partial q} = 0$$

$$(1.6) \quad \theta^2 \frac{\partial \varphi_1}{\partial t} + \frac{\theta^4}{2} \frac{\partial^2 \varphi_1}{\partial q^2} - \varphi_1 (a(t) q^2 + b(t) q + c(t)) = 0.$$

As $\varphi_1(0, q) = 1$, the equation (1.3) gives us:
 $(\frac{\partial \varphi_2}{\partial q})^2 = 0$, then

$$\varphi_2 = \varphi_2(t)$$

and (1.5) is then automatically satisfied.

So, the equation (1.2) implies:

$\varphi_1(t, q)(\frac{\partial \varphi_2}{\partial t} - (\frac{\partial \varphi_3}{\partial q})^2) = 0$, then for all (t, q) :

$$\begin{aligned} \frac{\partial \varphi_2}{\partial t} &= (\frac{\partial \varphi_3}{\partial q})^2 \\ \frac{\partial \varphi_3}{\partial q} &= A(t) \end{aligned}$$

$$(1.7) \quad \varphi_3(t, q) = A(t)q + B(t)$$

and

$$(1.8) \quad \varphi_2(t) = \int_0^t A^2(s) ds,$$

then the equation (1.4) gives :

$$\begin{aligned} \varphi_1 \frac{\partial \varphi_3}{\partial t} + \theta^2 \frac{\partial \varphi_1}{\partial q} \frac{\partial \varphi_3}{\partial q} &= 0 \\ \varphi_1 (\dot{A}(t)q + \dot{B}(t)) + \theta^2 A(t) \frac{\partial \varphi_1}{\partial q} &= 0 \end{aligned}$$

$$(1.9) \quad \varphi_1(t, q) = k(t) e^{-\frac{1}{\theta^2 A(t)} (\frac{\dot{A}(t)}{2} q^2 + \dot{B}(t)q)},$$

therefore :

$$\frac{\partial \varphi_1}{\partial t} = \left\{ \dot{k}(t) + \frac{k(t)}{\theta^2 A(t)} \left(\frac{1}{2} (\frac{\dot{A}^2(t)}{A(t)} - \ddot{A}(t)) q^2 + (\frac{\dot{A}(t)}{A(t)} \dot{B}(t) - \ddot{B}(t)) q \right) \right\} e^{-\frac{1}{\theta^2 A(t)} (\frac{\dot{A}(t)}{2} q^2 + \dot{B}(t)q)},$$

$$\frac{\partial \varphi_1}{\partial q} = \frac{-1}{\theta^2} \frac{\varphi_1}{A(t)} (\dot{A}(t)q + \dot{B}(t))$$

and

$$\frac{\partial^2 \varphi_1}{\partial q^2} = \frac{1}{\theta^4} \frac{\varphi_1}{A^2(t)} (\dot{A}^2(t)q^2 + 2\dot{A}(t)\dot{B}(t)q - \theta^2 \dot{A}(t)A(t) + \dot{B}^2(t)),$$

then the equation (1.6) becomes :

$$\begin{aligned} k(t) \left(-\frac{1}{2} \frac{\ddot{A}(t)}{A(t)} + \frac{\dot{A}^2(t)}{A^2(t)} - a(t) \right) q^2 + k(t) \left(-\frac{\ddot{B}(t)}{A(t)} + 2\dot{B}(t) \frac{\dot{A}(t)}{A^2(t)} - b(t) \right) q \\ + \theta^2 \dot{k}(t) - k(t)c(t) + \frac{k(t)}{2} \frac{\dot{B}^2(t)}{A^2(t)} - \theta^2 \frac{k(t)}{2} \frac{\dot{A}(t)}{A(t)} = 0. \end{aligned}$$

As $k(t) \neq 0$, the equation (1.6) is equivalent to the following system :

$$(1.10) \quad \begin{cases} -\frac{1}{2} \frac{\ddot{A}(t)}{A(t)} + \frac{\dot{A}^2(t)}{A^2(t)} - a(t) = 0 \\ -\frac{\ddot{B}(t)}{A(t)} + 2\dot{B}(t) \frac{\dot{A}(t)}{A^2(t)} - b(t) = 0 \\ \theta^2 \dot{k}(t) - k(t)c(t) + \frac{k(t)}{2} \frac{\dot{B}^2(t)}{A^2(t)} - \theta^2 \frac{k(t)}{2} \frac{\dot{A}(t)}{A(t)} = 0 \end{cases}$$

knowing that :

$$\begin{cases} A(0) = 1 \\ B(0) = 0 \\ k(0) = 1 \\ \dot{A}(0) = 0 \\ \dot{B}(0) = 0 \end{cases}$$

This is due to initial conditions at time t on the functions $\varphi_1(t, q)$, $\varphi_2(t, q)$, and $\varphi_3(t, q)$.

Let

$$(1.11) \quad A(t) = e^{\int_0^t \rho(s) ds},$$

then

$$\rho(t) = \frac{\dot{A}(t)}{A(t)}$$

and

$$\dot{\rho}(t) + \rho^2(t) = \frac{\ddot{A}(t)}{A(t)},$$

therefore the first equation of (1.10) is equivalent to a Riccati equation :

$$(1.12) \quad \begin{cases} \rho^2(t) - \dot{\rho}(t) = 2a(t) \\ \rho(0) = \frac{\dot{A}(0)}{A(0)} = 0. \end{cases}$$

The second equation of (1.10) gives: $2 \frac{\dot{A}(t)}{A(t)} \dot{B}(t) - \ddot{B}(t) = A(t)b(t)$,

then

$$\begin{aligned} \left(A^{-2}(t) \dot{B}(t) \right)' &= -2A^{-3}(t) \dot{A}(t) \dot{B}(t) + A^{-2}(t) \ddot{B}(t) \\ &= -A^{-2}(t) \left(2 \frac{\dot{A}(t)}{A(t)} \dot{B}(t) - \ddot{B}(t) \right) \\ &= -A^{-2}(t) (A(t)b(t)) \\ &= -\frac{b(t)}{A(t)}, \end{aligned}$$

therefore $B(t)$ is the solution of :

$$(1.13) \quad \begin{cases} \dot{B}(t) = -A^2(t) \int_0^t \frac{b(s)}{A(s)} ds \\ B(0) = 0. \end{cases}$$

Let $\Delta(t) = \int_0^t \frac{b(s)}{A(s)} ds$ and define k by:

$$(1.14) \quad \begin{cases} \dot{k}(t) &= \frac{k(t)}{\theta^2} \left(c(t) - \frac{1}{2}A^2(t)\Delta^2(t) + \frac{\theta^2}{2}\rho(t) \right) \\ k(0) &= 1. \end{cases}$$

this makes the first equation of (1.7) satisfied. \square

Remark 1.2. If $a(t), b(t)$ and $c(t)$ were constants, then we could apply the method of Rosencrans (cf. [6]).

1.2. Particular cases.

i) $\mathbf{V}(\mathbf{t}, \mathbf{q}) = \mathbf{a}(\mathbf{t})\mathbf{q}^2$ (semiclassical case).

Then

$$B(t) = 0, \Delta(t) = 0,$$

$$\begin{aligned} k(t) &= k(0) e^{\frac{1}{\theta^2} \int_0^t \left(c(s) - \frac{1}{2}A^2(s)\Delta^2(s) + \frac{\theta^2}{2}\rho(s) \right) ds} \\ &= e^{\frac{1}{2} \int_0^t \frac{\dot{A}(s)}{A(s)} ds} \\ &= \sqrt{A(t)} \end{aligned}$$

and

$$\eta_u^V(t, q) = \sqrt{A(t)} e^{-\frac{q^2}{2\theta^2}\rho(t)} \eta_u(\varphi_2(t), A(t)q)$$

is the solution of $(C_1^{(V)})$ such that $\eta_u^V(0, q) = \eta_u(0, q) = u(q)$.

In this case, as in the next one, the formulas really involve only $A(t)$.

i - a) $\mathbf{V}(\mathbf{t}, \mathbf{q}) = \frac{\omega^2}{2}\mathbf{q}^2$.

Then

$$\rho(t) = -\omega \tanh(\omega t), A(t) = \frac{1}{\cosh(\omega t)}, \varphi_2(t, q) = \frac{1}{\omega} \tanh(\omega t)$$

and according to **i)**

$$\eta_u^V(t, q) = \frac{1}{\sqrt{\cosh(\omega t)}} e^{\frac{\omega}{2\theta^2} \tanh(\omega t) q^2} \eta_u\left(\frac{1}{\omega} \tanh(\omega t), \frac{q}{\cosh(\omega t)}\right)$$

is the solution of $(C_1^{(V)})$.

This result is due to Zambrini (cf. [7], p.227, (51)).

The existence of such a formula was mentioned in [5], p. 96, and the similar result for the Schrödinger equation with quadratic potential (with $\cos(\omega t)$ taking the place of $\cosh(\omega t)$) appeared in [5], p. 83.

ii) $\mathbf{V}(\mathbf{t}, \mathbf{q}) = \lambda(\mathbf{t})\mathbf{q}$.

Then $\rho(t) = 0$, $A(t) = 1$, $\dot{B}(t) = -\int_0^t \lambda(s) ds$, $B(t) = \int_0^t \dot{B}(u) du = -\int_0^t (\int_0^u \lambda(s) ds) du$
and

$$\eta_u^V(t, q) = k(t) e^{-\frac{1}{\theta^2} \dot{B}(t) q} \eta_u(t, q + B(t))$$

is the solution of $(C_1^{(V)})$, such that $\eta_u^V(0, q) = \eta_u(0, q) = u(q)$.

ii – a) $\mathbf{V}(t, \mathbf{q}) = \lambda \mathbf{q}$.

Then $A(t) = 1, \Delta(t) = \lambda t, \dot{B}(t) = -\lambda t, B(t) = -\frac{\lambda}{2}t^2, k(t) = e^{-\frac{\lambda^2}{6\theta^2}t^3}$
and

$$\eta_u^V(t, q) = e^{-\frac{1}{\theta^2}(\frac{\lambda^2}{6}t^3 - \lambda tq)} \eta_u(t, q - \frac{\lambda t^2}{2})$$

is the solution of (C_1^V) .

We hereby recover a result of Lescot and Zambrini (cf.[4], p.219, (1)).

2. ISOVECTORS

Let \mathcal{G}_V defined by :

$$\mathcal{G}_V = \{N \mid \mathcal{L}_N(I) \subset I\}$$

and \mathcal{H}_V by :

$$\mathcal{H}_V = \{N \in \mathcal{G}_V \mid \frac{\partial N^S}{\partial S} = 0\}.$$

These definitions are taken from Lescot-Zambrini (cf.[4]).

We remind the reader that a \mathcal{G}_V is a Lie algebra for the usual Lie bracket of vectors fields, and \mathcal{H}_V is a Lie subalgebra of \mathcal{G}_V (cf.[4], p.210-211).

According to Lescot-Zambrini (cf.[4], p.214-215, 3.9^o- 3.29), for all $N \in \mathcal{H}_V$, we can write :

$$N^q: = \frac{1}{2}\dot{T}_N(t)q + \ell_N(t)$$

$$N^t: = T_N(t)$$

$$N^S: = h(q, t, S),$$

for $\sigma(t), l(t)$ and $T_N(t)$ satisfying :

$$(2.1) -\frac{1}{4}\ddot{T}_N q^2 - \ddot{\ell}q + \dot{\sigma} + T_N \frac{\partial V}{\partial t} + \frac{1}{2}\dot{T}_N q \frac{\partial V}{\partial q} + \ell \frac{\partial V}{\partial q} + \dot{T}_N V - \frac{\theta^2}{4}\ddot{T}_N = 0.$$

Let us set $l_N = l$.

Proposition 2.1. *Let \mathcal{K}_V defined by :*

$$\mathcal{K}_V = \{N \in \mathcal{H}_V / \ell_N = 0\},$$

then \mathcal{K}_V is a Lie subalgebra of \mathcal{H}_V .

Proof. Let $N_1, N_2 \in \mathcal{K}_V$; one has

$$\begin{aligned}
[N_1, N_2]^q &= [N_1, N_2](q) \\
&= N_1(N_2^q) - N_2(N_1^q) \\
&= N_1^q \frac{\partial N_2^q}{\partial q} + N_1^t \frac{\partial N_2^q}{\partial t} - (N_2^q \frac{\partial N_1^q}{\partial q} + N_2^t \frac{\partial N_1^q}{\partial t}) \\
&= \frac{1}{2} \dot{T}_{N_2} (\frac{1}{2} \dot{T}_{N_1} q + \ell_{N_1}(t)) + T_{N_1} (\frac{1}{2} \ddot{T}_{N_2} q + \dot{\ell}_{N_2}) - \frac{1}{2} \dot{T}_{N_1} (\frac{1}{2} \dot{T}_{N_2} q + \ell_{N_2}(t)) \\
&\quad - T_{N_2}(t) (\frac{1}{2} \ddot{T}_{N_1} q + \dot{\ell}_{N_1}) \\
&= \left(T_{N_1}(t) \dot{\ell}_{N_2} - T_{N_2}(t) \dot{\ell}_{N_1} + \frac{1}{2} (\ell_{N_1}(t) \dot{T}_{N_2} - \ell_{N_2}(t) \dot{T}_{N_1}) \right) \\
&\quad + \frac{1}{2} (T_{N_1}(t) \ddot{T}_{N_2}(t) - T_{N_2}(t) \ddot{T}_{N_1}(t)) q
\end{aligned}$$

and

$$\ell_{[N_1, N_2]} = T_{N_1}(t) \dot{\ell}_{N_2} - T_{N_2}(t) \dot{\ell}_{N_1} + \frac{1}{2} (\ell_{N_1}(t) \dot{T}_{N_2} - \ell_{N_2}(t) \dot{T}_{N_1}) = 0.$$

Then \mathcal{K}_V is stable by the Lie bracket, therefore \mathcal{K}_V is a Lie subalgebra of \mathcal{H}_V . \square

Lemma 2.2. *If $V_1(t, q) = V_2(t, q) + \frac{C}{q^2}$ (for C a constant), then $\mathcal{K}_{V_1} = \mathcal{K}_{V_2}$.*

Proof. Replacing $V_1(t, q)$ by its value in equation (2.1),

we get:

$$\begin{aligned}
&- \frac{1}{4} \ddot{T}_N q^2 - \ddot{\ell} q + \dot{\sigma} + T_N \frac{\partial V_2}{\partial t} + \frac{1}{2} \dot{T}_N q \left(\frac{\partial V_2}{\partial q} - 2 \frac{C}{q^3} \right) + \ell \left(\frac{\partial V_2}{\partial q} - 2 \frac{C}{q^3} \right) + \dot{T}_N \left(V_2 + \frac{C}{q^2} \right) \\
&- \frac{\theta^2}{4} \ddot{T}_N = 0
\end{aligned}$$

\Leftrightarrow

$$- \frac{1}{4} \ddot{T}_N q^2 - \ddot{\ell} q + \dot{\sigma} + T_N \frac{\partial V_2}{\partial t} + \frac{1}{2} \dot{T}_N q \frac{\partial V_2}{\partial q} + \ell \frac{\partial V_2}{\partial q} - \frac{2\ell C}{q^3} + \dot{T}_N V_2 - \frac{\theta^2}{4} \ddot{T}_N = 0.$$

Therefore, if $N \in \mathcal{K}_{V_2}$, as $l_N = 0$, we see that n satisfies (2.1) for $V = V_1$. Then V_1 et V_2 give the same isovectors such as $\ell_N = 0$.

Therefore :

$$\mathcal{K}_{V_1} = \mathcal{K}_{V_2}.$$

\square

Corollary 2.3. *If $V_1 = \frac{C}{q^2}$ ($C \neq 0$) and $V_2 = 0$, then $\mathcal{H}_{V_1} = \mathcal{K}_{V_1} = \mathcal{K}_{V_2}$ and therefore: $\mathcal{H}_{V_1} \subset \mathcal{H}_{V_2}$.*

Proof. Let $N \in \mathcal{H}_{V_1}$, the equation (2.1) becomes :

$$- \frac{1}{4} \ddot{T}_N q^2 - \ddot{\ell} q + \dot{\sigma} + \frac{1}{2} \dot{T}_N q \left(-2 \frac{C}{q^3} \right) + \ell \left(-2 \frac{C}{q^3} \right) + \dot{T}_N \left(\frac{C}{q^2} \right) - \frac{\theta^2}{4} \ddot{T}_N = 0,$$

that is

$$- \frac{1}{4} \ddot{T}_N q^2 - \ddot{\ell} q + \dot{\sigma} - \frac{\theta^2}{4} \ddot{T}_N - 2 \frac{\ell C}{q^3} = 0.$$

As T_N , ℓ and σ depend only on t , the system is equivalent to:

$$(2.2) \quad \left\{ \begin{array}{l} 2C\ell = 0 \\ \ddot{\ell} = 0 \\ \dot{\sigma} = \frac{\theta^2}{4} \ddot{T}_N \\ \ddot{T}_N = 0. \end{array} \right.$$

But $C \neq 0$, therefore $\ell_N = 0$ and

$$\mathcal{H}_{V_1} = \mathcal{K}_{V_1} \stackrel{(\text{Lemma 2.2})}{=} \mathcal{K}_{V_2} \subset \mathcal{H}_{V_2}.$$

□

Remark 2.4. Thus, we recover another result of Lescot and Zambrini(cf.[4], p.220, (3)).

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