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► **To cite this version:**

Louis-Clément Lefèvre. Deformations of representations of fundamental groups of non-compact complex varieties. 2019. hal-02399676v1

HAL Id: hal-02399676

<https://hal.science/hal-02399676v1>

Preprint submitted on 9 Dec 2019 (v1), last revised 11 Apr 2021 (v2)

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DEFORMATIONS OF REPRESENTATIONS OF FUNDAMENTAL GROUPS OF NON-COMPACT COMPLEX VARIETIES

LOUIS-CLÉMENT LEFÈVRE

ABSTRACT. We describe locally the representation varieties of fundamental groups for smooth complex manifolds admitting a compactification into a Kähler manifold, at representations coming from the monodromy of a variation of mixed Hodge structure. Given such a manifold X and such a linear representation ρ of its fundamental group $\pi_1(X, x)$, we use the theory of Goldman-Millson and pursue our previous work that combines mixed Hodge theory with derived deformation theory to construct a mixed Hodge structure on the formal local ring $\widehat{\mathcal{O}}_\rho$ to the representation variety of $\pi_1(X, x)$ at ρ . Then we show how a weighted-homogeneous presentation of $\widehat{\mathcal{O}}_\rho$ is induced directly from a splitting of the weight filtration of its mixed Hodge structure. In this way we recover and generalize theorems of Eyssidieux-Simpson (X compact) and of Kapovich-Millson (ρ finite).

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1. INTRODUCTION

1.1. Topology of non-compact varieties. Our objects of study will be complex manifolds X admitting a compactification into a compact Kähler manifold \overline{X} such that the complement is a simple normal crossing divisor. This includes the case of smooth quasi-projective varieties, since smooth projective varieties are examples of compact Kähler manifolds.

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2010 *Mathematics Subject Classification.* 14D07; 14C30; 18D50.

Key words and phrases. Representation varieties, Fundamental groups, Deformation Theory, Variations of Hodge structure, Mixed Hodge modules, L_∞ algebras.

The cohomology groups of X then come equipped with *mixed Hodge structures* (MHS), defined and constructed first by Deligne [Del71]. Then mixed Hodge structures have been constructed on other topological invariants of such varieties, in particular on the rational homotopy groups by Morgan [Mor78] and Hain [Hai87]. They can be used to give concrete restrictions on the possible fundamental groups of these varieties.

In previous work [Lef19] we constructed MHS on certain invariants associated to linear representations of the fundamental group $\pi_1(X, x)$. We want to pursue this work and analyze the restrictions that this gives on the representations.

1.2. Deformations of representations of fundamental groups. For X as above, $\pi_1(X, x)$ is always a finitely presentable group. Hence the set of representations of $\pi_1(X, x)$ into some fixed linear algebraic group G , defined over some field $\mathbf{k} \subset \mathbb{R}$ or $\mathbf{k} = \mathbb{C}$, has a natural structure of affine scheme that we denote by $\mathrm{Hom}(\pi_1(X, x), G)$. Given such a representation ρ , our main object of study is the formal completion of the local ring to $\mathrm{Hom}(\pi_1(X, x), G)$ at its \mathbf{k} -point ρ . We denote it by $\widehat{\mathcal{O}}_\rho$. Its associated deformation functor, a functor from the category $\mathbf{Art}_{\mathbf{k}}$ of local Artin \mathbf{k} -algebras with residue field \mathbf{k} to the category of sets, is given by

$$(1.1) \quad \mathrm{Def}_{\widehat{\mathcal{O}}_\rho}(A) := \mathrm{Hom}(\widehat{\mathcal{O}}_\rho, A), \quad A \in \mathbf{Art}_{\mathbf{k}}$$

and is canonically isomorphic to the functor of formal deformations of ρ

$$(1.2) \quad \mathrm{Def}_\rho(A) := \{\tilde{\rho} : \pi_1(X, x) \rightarrow G(A) \mid \tilde{\rho} = \rho \text{ over } G(\mathbf{k})\}.$$

Thus, if one can give a concrete presentation of $\widehat{\mathcal{O}}_\rho$ as

$$(1.3) \quad \widehat{\mathcal{O}}_\rho \simeq \mathbf{k}[[X_1, \dots, X_n]] / (P_1, \dots, P_r)$$

where P_1, \dots, P_r are polynomials then one can describe completely the deformation theory of ρ : deformations over A are given by elements x_1, \dots, x_n of A satisfying the equations P_1, \dots, P_r . In particular, if an order N is bigger than the maximal of the degrees of P_1, \dots, P_r then there are no obstructions for lifting deformations of order N to deformations of order $N + 1$.

This was described first by Goldman and Millson in [GM88] in the case where X is a compact Kähler manifold and ρ is the monodromy of a polarizable variation of Hodge structure over X : they show that $\widehat{\mathcal{O}}_\rho$ has a quadratic presentation, i.e. a presentation with P_1, \dots, P_r of degree at most 2.

A presentation of $\widehat{\mathcal{O}}_\rho$ was also obtained by Kapovich-Millson [KM98] when X may be non-compact and ρ is a representation with finite image: they show that there is a presentation with weights 1, 2 on the variables X_1, \dots, X_n and polynomials P_1, \dots, P_r that are homogeneous of possible degrees 2, 3, 4 with respects to these weights on the variables.

These two results have concrete applications to exhibit classes of groups that cannot be fundamental groups of smooth varieties. Our motivation is to generalize these results and unify them, and make more explicit the use of mixed Hodge structures.

Theorem 1.1. *Let ρ be the monodromy representation of a variation of mixed Hodge structure over X that is admissible in \overline{X} , and unipotent at infinity. Then there is a mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$, functorial in X and ρ , depending explicitly on the base-point x . Furthermore, one obtains a weighted-homogeneous presentation of $\widehat{\mathcal{O}}_\rho$ by splitting the weight filtration.*

In our previous work, we constructed this MHS on $\widehat{\mathcal{O}}_\rho$ only in the case treated by Goldman-Millson (recovering a result of Eyssidieux-Simpson [ES11]) and in the case of finite image treated by Kapovich-Millson. This already provided some unification of these two theories that is suitable for further generalizations, enlightening the role of mixed Hodge theory, but we were not able to fully recover the weighted-homogeneous presentation of Kapovich-Millson. It will now appear clearly that the weights 1, 2 on generators come from the MHS of $H^1(X)$ and the weights 2, 3, 4 on the relations come from the MHS of $H^2(X)$.

1.3. Deformations via DG Lie algebras. Let ρ be a representation of $\pi_1(X, x)$ into a linear algebraic group G with Lie algebra \mathfrak{g} , as above, at a fixed base-point. To this is associated a flat principal bundle P with associated bundle $\mathrm{ad}(\rho) := \mathrm{ad}(P) = P \times_G \mathfrak{g}$: this is a local system of Lie algebras over X . Let L be the algebra of \mathcal{C}^∞ differential forms over X with values in $\mathrm{ad}(\rho)$. This has a natural structure of differential graded (DG) Lie algebra.

The theory introduced by Goldman-Millson states that it is possible to associate to L a deformation functor Def_L , again from \mathbf{Art}_k to sets, that will be isomorphic to Def_ρ and that a quasi-isomorphism of DG Lie algebras $L \rightarrow M$ induces an isomorphism of deformation functors $\mathrm{Def}_L \rightarrow \mathrm{Def}_M$. So the local ring $\widehat{\mathcal{O}}_\rho$ is determined by the data of L up to quasi-isomorphism, and the obstruction theory (i.e. the presentation (1.3)) can be understood in terms of L only.

The conclusion in the compact cases that they treat follows quite easily when proved that in these cases, using the particular properties of harmonic analysis on Kähler manifolds, L is *formal*, i.e. quasi-isomorphic as DG Lie algebra to its cohomology $H(L)$, which is finite-dimensional and with zero differential. So $\widehat{\mathcal{O}}_\rho$ is essentially isomorphic to the formal local ring at 0 to the equation $[x, x] = 0$ in $H^1(L)$ with values in $H^2(L)$, this is a quadratic presentation.

In order to pursue this method to the non-formal case, we argued in our previous work [Lef19, § 4] that the right tool to use is L_∞ algebras. Namely on $H(L)$ there exists a sequence of multilinear *higher operations* (for each $n \geq 1$, a multilinear operation in n variables on $H(L)$, which for $n = 2$ is the Lie bracket, for $n = 1$ the differential which is zero here) arranged in a certain algebraic and combinatorial structure called *L_∞ algebra* such that L and $H(L)$ become quasi-isomorphic as *L_∞ algebras* and such that Def_L can be written in terms of $H(L)$ with its L_∞ structure only. So, we can again understand Def_L via linear algebra in finite-dimensional vector spaces but at the cost of working with more operations. A similar argumentation is central in a recent work of Budur-Rubió [BR18].

1.4. Hodge theory. In several special cases, the cohomology of X with coefficients in a local system \mathbb{V} also carries a MHS. If ρ is the monodromy representation of \mathbb{V} then the associated DG Lie algebra L as above also carries a MHS on cohomology and the Lie bracket is a morphism of MHS. What we want to exploit is precisely the interaction between the MHS on $H^\bullet(L)$ and the deformation functor of L .

A very interesting class of such local systems is provided by the *variations of (mixed) Hodge structure* (VHS, VMHS). The fact that $H^\bullet(X, \mathbb{V})$ carries a MHS was conjectured by Deligne and proved first in the case where X is compact; this then motivated a long development of the theory of VHS, VMHS, and their asymptotic behavior around the simple normal crossing divisor $D = \bar{X} \setminus X$. The MHS was constructed in the following cases:

- (1) When X is compact, due to Deligne and written by Zucker [Zuc79, § 1–2] for the case of pure VHS. This also gives straightforwardly the case of VMHS.
- (2) When \mathbb{V} is a pure VHS that extends to \overline{X} . After a theorem of Griffiths, this is the same as requiring that the monodromy of \mathbb{V} is trivial around D .
- (3) When X is one-dimensional and \mathbb{V} is a pure VHS, due to Zucker [Zuc79, § 13].
- (4) When X is one-dimensional and \mathbb{V} is a VMHS, due to Steenbrink-Zucker [SZ85]. The VMHS has to satisfy the *admissibility condition*, that they introduce, relatively to \overline{X} .
- (5) In the most general case, when \mathbb{V} is a VMHS on X admissible in \overline{X} , due to Saito [Sai86] using his theory of *mixed Hodge modules*.

The MHS on the cohomology of L actually always comes from a pre-existing structure at the level of L called *mixed Hodge complex*. This is the notion introduced by Deligne in [Del74], consisting of two complexes: $L_{\mathbb{Q}}$ over \mathbb{Q} carrying the weight filtration, and $L_{\mathbb{C}}$ over \mathbb{C} carrying the Hodge and weight filtrations, with a quasi-isomorphism $L_{\mathbb{Q}} \otimes \mathbb{C} \approx L_{\mathbb{C}}$ and axioms ensuring that the weight-graded pieces of cohomology glue to pure Hodge structures, defining a MHS on each $H^{\bullet}(L)$.

In our previous work [Lef19, § 7–9] we explained that the only input of geometry that we need in order to put a MHS on $\widehat{\mathcal{O}}_{\rho}$ is to have such an L which is at the same time a DG Lie algebra and a mixed Hodge complex. This is not so obvious because the naive way of constructing $L_{\mathbb{Q}}$, via sheaf theory or via singular cochains, does not provide a DG Lie algebra: the cup-product is not commutative at the level of cochains. The global picture is quite similar to the work of Morgan: in order to construct his MHS on homotopy groups, it is necessary to have an object A which is at the same time a mixed Hodge complex (thus containing all the information on the MHS on $H^{\bullet}(X)$) and a commutative DG algebra (by rational homotopy theory, this up to quasi-isomorphism contains all the information about the rational homotopy type of X). So he constructs $A_{\mathbb{Q}}$ as an algebra of rational polynomial differential forms on X with logarithmic poles along D . Similarly, in our work $L_{\mathbb{Q}}$ is an algebra of rational polynomial differential forms with coefficients in a rational local system. Fortunately, the Thom-Whitney functors, introduced by Navarro Aznar [Nav87] when reviewing this work of Morgan, allow us to give a straightforward and functorial construction of this $L_{\mathbb{Q}}$.

The data of such a structure on L is usually not unique (it depends on the chosen compactification \overline{X}) but it is unique up to quasi-isomorphism. The construction of our MHS on $\widehat{\mathcal{O}}_{\rho}$ from L is directly invariant under quasi-isomorphisms (of simultaneous structures of DG Lie algebra and mixed Hodge complexes) of L . Furthermore it is functorial in X, ρ and the base-point x , as soon as L can be constructed functorially in X, x, ρ .

1.5. Results. Our method divides the construction of a MHS on $\widehat{\mathcal{O}}_{\rho}$ into two well-distinct parts. In the first part, we construct L which is at the same time a DG Lie algebra and a mixed Hodge complex. This strongly depends on the “geometric situation” that we fix (a situation for X and ρ), hence through § 4–7 we will devote one section to each of the cases listed in § 1.4. The second part is purely algebraic and refers to L only and not any more to geometry. We treat this only in § 8.

In our previous work, we constructed this structure only for the situations where X is compact and ρ is the monodromy of a polarizable VHS, or when X may be non-compact and ρ has finite image. Our result in the most general case can now be stated as follows.

Theorem 1.2. *Let ρ be the monodromy representation of a graded-polarizable variation of mixed Hodge structure over $X \subset \overline{X}$, admissible in \overline{X} , unipotent at infinity. Then one can construct an object L which is at the same time a DG Lie algebra and a mixed Hodge complex which computes the cohomology of X with coefficients in the local system $\text{ad}(\rho)$. It is functorial in X, x, ρ up to quasi-isomorphism.*

By the main result of [Lef19] this is enough to construct the MHS on $\widehat{\mathcal{O}}_\rho$, functorial in X, x, ρ .

To put our work in a slightly more general context, that could allow to construct mixed Hodge complexes with many other possible algebraic structures, we actually construct a mixed Hodge complex $\text{MHC}(X, \overline{X}, \mathbb{V})$ attached to the VMHS \mathbb{V} , functorial in (X, \overline{X}) and in \mathbb{V} , equipped with canonical maps for two VMHS $\mathbb{V}_1, \mathbb{V}_2$

$$(1.4) \quad \text{MHC}(X, \overline{X}, \mathbb{V}_1) \otimes \text{MHC}(X, \overline{X}, \mathbb{V}_2) \longrightarrow \text{MHC}(X, \overline{X}, \mathbb{V}_1 \otimes \mathbb{V}_2)$$

(together with an identity and associativity condition) that are compatible with the interchange map of \mathbb{V}_1 and \mathbb{V}_2 . In particular if \mathbb{V} alone is equipped with a Lie bracket $[-, -]$ then there is an induced Lie bracket on $\text{MHC}(X, \overline{X}, \mathbb{V})$ via the composition

$$(1.5) \quad \text{MHC}(X, \overline{X}, \mathbb{V}) \otimes \text{MHC}(X, \overline{X}, \mathbb{V}) \longrightarrow \text{MHC}(X, \overline{X}, \mathbb{V} \otimes \mathbb{V}) \xrightarrow{[-, -]} \text{MHC}(X, \overline{X}, \mathbb{V}).$$

Typically $\text{MHC}(X, \overline{X}, \mathbb{V})$ is called a lax symmetric monoidal functor (§ 3.1) in the data of \mathbb{V} , and by this same argument inherits every kind of algebraic structure that can be described on \mathbb{V} by multilinear maps (i.e. by a linear operad, though we will not need to work with this language).

Now, we study the interaction between the deformation functor of L , that can be written in $H(L)$ with higher operations of L_∞ algebras, and the existing MHS on $H(L)$. What we show in § 8 is that there is a splitting of the weight filtration on $H(L)$ that is compatible with the higher operations of L_∞ algebra. Because the higher operations are multilinear and respect the weights, but the weights of $H(L)$ in each degree are limited, this forces many higher operations to vanish and the remaining ones give the equations for the deformation functor. The most general result, in its purely algebraic version, is:

Theorem 1.3. *Assume that the MHS on $H^1(L)$ has only strictly positive weights. Then $\widehat{\mathcal{O}}_\rho$ has a weighted-homogeneous presentation as in (1.3) with weights of generators coming from weights of $H^1(L)$ and weights of relations limited to the weights of $H^2(L)$.*

For example in the case of Kapovich-Millson, the MHS on $H^1(L)$ is directly related to the MHS on $H^1(X)$ and has only weights 1, 2, and the MHS on $H^2(L)$ is directly related to the one of $H^2(X)$ and has only weights 2, 3, 4. Our method allows us to perform such a detailed analysis of the weights from the weights of $H^\bullet(L)$. We sum this up in our last section 8.6.

In particular the hypothesis on the weights of $H^1(L)$ is satisfied when L comes from a pure VHS, because then the adjoint Lie algebra is a VHS of weight zero. So we get:

Theorem 1.4. *If ρ is the monodromy representation of a polarizable pure VHS over $X \subset \overline{X}$ with unipotent monodromy at infinity, then $\widehat{\mathcal{O}}_\rho$ has a weighted-homogeneous presentation.*

The precise understanding of the weights depends on the behavior of the VHS near infinity and this is easier to see in the case of curves. So once again we find it interesting to treat these cases separately. For VMHS, our method does not allow us to show that the presentation is finite, but it still allows to write down homogeneous equations.

1.6. Acknowledgements. I thank notably Yohan Brunebarbe, Nero Budur, Joana Cirici, Brad Drew, Clément Dupont, Konstantin Jakob, Marcel Rubi o, for useful conversations all along the preparation of this work. I thank particularly Philippe Eyssidieux and Jochen Heinloth for conversations as well as for comments on the final version of this work.

2. REMINDER ON MIXED HODGE THEORY

We will denote by \mathbf{k} a fixed sub-field of \mathbb{R} . We always denote increasing filtrations by a lower index and decreasing filtrations by an upper index: W_\bullet, F^\bullet . Filtrations of cochain complexes are always assumed to be biregular, i.e. they restrict to a finite filtration in each degree.

2.1. Mixed Hodge structure. Recall briefly that a *pure Hodge structure* (HS) of weight k over \mathbf{k} is given by a finite-dimensional vector space $H_{\mathbf{k}}$ over \mathbf{k} with a decomposition $H_{\mathbb{C}} := H_{\mathbf{k}} \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$ with $\overline{H^{p,q}} = H^{q,p}$, complex conjugation being taken with respect to $H_{\mathbf{k}} \subset H_{\mathbb{C}}$. In this case the Hodge filtration F^\bullet is defined as $F^p H_{\mathbb{C}} := \bigoplus_{r \geq p} H^{r,q}$.

A *mixed Hodge structure* (MHS) is given by a finite-dimensional vector space $H_{\mathbf{k}}$ over \mathbf{k} equipped with an increasing filtration W_\bullet and a decreasing filtration F^\bullet of $H_{\mathbb{C}}$, such that each term $\mathrm{Gr}_k^W H_{\mathbf{k}} := W_k H_{\mathbf{k}} / W_{k-1} H_{\mathbf{k}}$ with the induced filtration F^\bullet over \mathbb{C} is a pure Hodge structure of weight k , whose bigraded terms are again denoted by $H^{p,q}$.

For a pure Hodge structure H of weight k , a *polarization* is given by a bilinear form $Q : H_{\mathbf{k}} \otimes H_{\mathbf{k}} \rightarrow \mathbf{k}$ which is $(-1)^k$ -symmetric and satisfies the two relations

- (1) $Q(H^{p,q}, H^{r,s}) = 0$ if $(p,q) \neq (r,s)$,
- (2) $i^{p-q} Q(v, \bar{v}) > 0$ for $0 \neq v \in H^{p,q}$.

Mixed Hodge structures form an abelian category with tensor product and internal Hom. Hence one can consider various kinds of algebras internally to MHS: for example a Lie bracket $[-, -]$ on the MHS H will have to be defined over \mathbf{k} and satisfy $[W_k H_{\mathbf{k}}, W_\ell H_{\mathbf{k}}] \subset W_{k+\ell} H_{\mathbf{k}}$ and $[F^p H_{\mathbb{C}}, F^r H_{\mathbb{C}}] \subset F^{p+r} H_{\mathbb{C}}$.

A direct sum of pure HS of various weights is automatically a MHS with an obvious weight filtration. Such MHS are said to be *split*. In general however, MHS coming from the cohomology of complex varieties are not split. Nevertheless, there exists a canonical way of splitting a MHS H over \mathbb{C} by defining subspaces $A^{p,q} \subset H_{\mathbb{C}}$ that projects to $H^{p,q}$. This splitting is functorial and compatible with

tensor products. In this work, after having constructed a MHS, we will be mainly interested in describing such a splitting.

2.2. Variations of mixed Hodge structure. Let X be any complex manifold. We will always use the letter Ω_X^\bullet to denote the complex of holomorphic differential forms on X .

Definition 2.1. A *variation of Hodge structure* (VHS) of weight k over X is the data of a local system \mathbb{V} of finite-dimensional \mathbf{k} -vector spaces, to which is associated a flat connection ∇ on the holomorphic vector bundle $\mathcal{V} := \mathbb{V} \otimes \mathcal{O}_X$, and a filtration of \mathcal{V} by holomorphic sub-vector bundle \mathcal{F}^\bullet , such that at each point $x \in X$ the data $(\mathcal{V}_x, \mathcal{F}_x^\bullet)$ forms a pure Hodge structure of weight k , and furthermore ∇ satisfies $\nabla(\mathcal{F}^p) \subset \Omega_X^1 \otimes \mathcal{F}^{p-1}$.

A *polarization* for the VHS \mathbb{V} is a flat bilinear pairing defined over \mathbf{k} , $Q : \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbf{k}$, such that each $(\mathcal{V}_x, \mathcal{F}_x^\bullet, Q_x)$ is a polarized Hodge structure.

In the following we will work only with polarizable variations of Hodge structure.

Definition 2.2 ([SZ85, 3.4]). A *variation of mixed Hodge structure* (VMHS) over X is the data of a local system \mathbb{V} of finite-dimensional vector spaces over \mathbf{k} and a filtration W_\bullet of \mathbb{V} by sub-local systems \mathbb{W}_\bullet , together with a filtration as above of $\mathcal{V} = \mathbb{V} \otimes \mathcal{O}_x$ by sub-vector bundles \mathcal{F}^\bullet , such that the induced flat connection again satisfies $\nabla(\mathcal{F}^p) \subset \Omega_X^1 \otimes \mathcal{F}^{p-1}$ and furthermore each $\mathrm{Gr}_k^W \mathbb{V}$ with the induced \mathcal{F}^\bullet forms a variation of Hodge structure of weight k .

It is said to be *graded-polarizable* if each term $\mathrm{Gr}_k^W \mathbb{V}$ is a polarizable VHS.

2.3. Mixed Hodge complexes. Mixed Hodge structures exist only at the level of the cohomology of a variety and are not so easy to construct. The structure that naturally comes from geometry, and whose cohomology carries a MHS, has been introduced by Deligne.

Definition 2.3. A *pure Hodge complex of weight k* is the data of a bounded-below complex $K_{\mathbf{k}}^\bullet$ over \mathbf{k} with each $H^i(K_{\mathbf{k}})$ finite-dimensional, a bounded-below filtered complex $(K_{\mathbb{C}}^\bullet, F^\bullet)$ over \mathbb{C} , and a chain of quasi-isomorphisms $K_{\mathbf{k}} \otimes \mathbb{C} \approx K_{\mathbb{C}}$, such that

- (1) The differential d of $K_{\mathbb{C}}$ is strictly compatible with F ,
- (2) The filtration F defines a pure Hodge structure of weight $i + k$ on $H^i(K_{\mathbb{C}})$, with form over \mathbf{k} coming from $H^i(H_{\mathbf{k}})$.

A *mixed Hodge complex* (MHC) is the data of a filtered bounded below-complex $(K_{\mathbf{k}}^\bullet, W_\bullet)$ over \mathbf{k} with each $H^i(K_{\mathbf{k}})$ finite-dimensional, a bifiltered bounded-below complex $(K_{\mathbb{C}}^\bullet, W_\bullet, F^\bullet)$ over \mathbb{C} , a filtered quasi-isomorphism $(K_{\mathbf{k}}, W) \otimes \mathbb{C} \approx (K_{\mathbb{C}}, W)$, such for all k the data $\mathrm{Gr}_k^W(K_{\mathbf{k}})$, $(\mathrm{Gr}_k^W(K_{\mathbb{C}}), F)$ forms a pure Hodge complex of weight k .

The main theorem of Deligne is that for a mixed Hodge complex K the W -spectral sequence of K degenerates at E_2 and these are the weight-graded pieces of a MHS on each $H^i(K)$ with the induced filtration F and the shifted filtration $W[i]$ with $W[i]_k := W_{i+k}$.

For the needs of rational homotopy theory, Morgan introduced the notion of *mixed Hodge diagram* which is simply a mixed Hodge complex K whose both components $K_{\mathbf{k}}^\bullet$, $K_{\mathbb{C}}^\bullet$ are commutative DG algebras (non-negatively graded, and compatibly with the filtrations) and the quasi-isomorphisms relating them are quasi-isomorphisms of commutative DG algebras. Though being algebraically a simple

variant of the notion of mixed Hodge complexes, the construction of such mixed Hodge diagrams is more evolved because the commutative DG algebra $K_{\mathbf{k}}^{\bullet}$ does not come from differential forms nor from singular cochains. Similarly we introduced:

Definition 2.4 ([Lef19, § 8]). A *mixed Hodge diagram of Lie algebras* is a mixed Hodge complex L whose both components $L_{\mathbf{k}}, L_{\mathbb{C}}$ are also DG Lie algebras and the quasi-isomorphisms between them are quasi-isomorphisms of DG Lie algebras.

The notion of pure and mixed Hodge complex also exist at the level of sheaves, and this will be more practical for us.

Definition 2.5. Let X be a topological space. A *pure Hodge complex of sheaves* of weight k over X is the data of a bounded-below complex of sheaves $\mathcal{K}_{\mathbf{k}}^{\bullet}$ over \mathbf{k} , a bounded-below filtered complex of sheaves $(\mathcal{K}_{\mathbb{C}}^{\bullet}, F^{\bullet})$ over \mathbb{C} , and a quasi-isomorphism $\mathcal{K}_{\mathbf{k}} \otimes \mathbb{C} \approx \mathcal{K}_{\mathbb{C}}$, such that applying $R\Gamma(X, -)$ to all this data (i.e. to the complexes, the filtrations, and the quasi-isomorphisms between them) gives a pure Hodge complex of weight k .

Similarly, a *mixed Hodge complex of sheaves* over X is the data of a filtered bounded-below complex of sheaves $(\mathcal{K}_{\mathbf{k}}^{\bullet}, W_{\bullet})$ over \mathbf{k} , a bifiltered bounded-below complex of sheaves $(\mathcal{K}_{\mathbb{C}}^{\bullet}, W_{\bullet}, F^{\bullet})$ over \mathbb{C} , and a filtered quasi-isomorphism $(\mathcal{K}_{\mathbf{k}}, W) \otimes \mathbb{C} \approx (\mathcal{K}_{\mathbb{C}}, W)$ such that applying Gr_k^W to all this data gives a pure Hodge complex of sheaves of weight k ; in this case applying $R\Gamma(X, -)$ gives a mixed Hodge complex.

Remark 2.6. Mixed Hodge structures and mixed Hodge complexes can also be defined over the base field $\mathbf{k} = \mathbb{C}$ but this requires an appropriate modification of the definitions, see for example [ES11, § 1]. Essentially we drop the complex conjugation but we introduce a third filtration, denoted by \overline{G}^{\bullet} , playing the role of the conjugate \overline{F} of F . The properties of \overline{F} that would have followed simply by conjugation from F need to be included as axioms for \overline{G} . In any MHC over \mathbf{k} , keeping only its component over \mathbb{C} and the three filtrations W, F, \overline{F} defines a MHC over \mathbb{C} . This notion matters because we ultimately care about a splitting of the weight filtration which is already interesting if defined over \mathbb{C} , and MHS over \mathbb{C} are simpler.

3. THOM-WHITNEY FUNCTORS

3.1. A few words on lax symmetric monoidal functors. Instead of writing purely categorical diagrams, let us explain briefly the following classical situation. For any sheaf \mathcal{F} of abelian groups over a topological space X , there is a canonical Godement resolution $G(\mathcal{F})$ which is a complex of sheaves with a quasi-isomorphism $\mathcal{F} \rightarrow G(\mathcal{F})$. The Godement resolution has the advantage of being functorial. If \mathcal{F} comes equipped with a multiplication $\mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F}$ then one can define a multiplication on $G(\mathcal{F})$ using the Eilenberg-Zilber theory. However if the multiplication on \mathcal{F} is commutative then the multiplication on $G(\mathcal{F})$ will usually not be. For example, with \mathcal{F} the locally constant sheaf \mathbb{Q}_X , the multiplication on $G(\mathbb{Q}_X)$ looks like a cup-product formula, which is known to be non-commutative.

The categorical nature of this phenomenon is that the functor G is a *lax monoidal functor*, that is, for sheaves \mathcal{F}, \mathcal{G} there are canonical maps

$$(3.1) \quad G(\mathcal{F}) \otimes G(\mathcal{G}) \longrightarrow G(\mathcal{F} \otimes \mathcal{G})$$

satisfying natural identity and associativity conditions, but there is no commutative diagram

$$(3.2) \quad \begin{array}{ccc} G(\mathcal{F}) \otimes G(\mathcal{G}) & \longrightarrow & G(\mathcal{F} \otimes \mathcal{G}) \\ \wr \uparrow & & \uparrow \wr \\ G(\mathcal{G}) \otimes G(\mathcal{F}) & \longrightarrow & G(\mathcal{G} \otimes \mathcal{F}) \end{array}$$

where the vertical maps are the natural isomorphisms coming from the symmetry of the tensor product.

If the above diagram were commutative, then G would be a *lax symmetric monoidal functor*. In this case any commutative algebra structure (for example Lie algebra) on \mathcal{F} induces a commutative algebra structure on $G(\mathcal{F})$.

3.2. Thom-Whitney resolutions. In this section \mathbf{k} can be any field of characteristic zero. Since the construction of these functors is based on some differential forms with rational coefficients, it really matters that \mathbf{k} be both a field and that it has characteristic zero. The reference is [Nav87, § 1–6].

For a continuous map $f : X \rightarrow Y$ between topological spaces and for a bounded-below complex of sheaves of \mathbf{k} -vector spaces \mathcal{F} on X is defined the *Thom-Whitney direct image* $R_{\text{TW}}f_*\mathcal{F}$. This is again a bounded-below complex of sheaves which is quasi-isomorphic to the classical $Rf_*\mathcal{F}$ but the advantage is that it is lax symmetric monoidal. In particular if \mathcal{F} is a sheaf of commutative DG algebras then so is $R_{\text{TW}}f_*\mathcal{F}$. For f the constant map to a point, $R_{\text{TW}}f_*\mathcal{F}$ is also denoted by $R_{\text{TW}}\Gamma(X, \mathcal{F})$ (the *Thom-Whitney global sections*). For f the identity what we get is just the *Thom-Whitney resolution* of \mathcal{F} , denoted by $\text{TW}^*\mathcal{F}$.

Let us recall everything that we will need.

Proposition 3.1. *The Thom-Whitney functors have the following properties:*

- (1) $R_{\text{TW}}f_*\mathcal{F}$ is functorial in \mathcal{F} .
- (2) $R_{\text{TW}}f_*$ is naturally homotopy-equivalent to the classical Rf_* defined as the derived functor of f_* using the Godement resolution.
- (3) $R_{\text{TW}}f_*$ is a lax symmetric monoidal functor.
- (4) $\mathcal{F} \rightarrow \text{TW}^*\mathcal{F}$ is a soft resolution of \mathcal{F} .
- (5) $R_{\text{TW}}f_*$ transforms quasi-isomorphisms of complexes of sheaves into quasi-isomorphisms.
- (6) The Thom-Whitney resolution also provides filtered resolutions. If \mathcal{F} is equipped with a filtration, then $R_{\text{TW}}f_*\mathcal{F}$ gets an induced filtration, and filtered quasi-isomorphism of complexes of sheaves are sent to filtered quasi-isomorphisms.
- (7) If $g : Y \rightarrow Z$ is a second continuous map of topological spaces the the functors $R_{\text{TW}}(g \circ f)_*$ and $R_{\text{TW}}g_* \circ R_{\text{TW}}f_*$ are naturally quasi-isomorphic. Similarly, for filtered complexes of sheaves, the functors are filtered quasi-isomorphic.

For example, for the locally constant sheaf \mathbb{Q}_X , $R_{\text{TW}}\Gamma(X, \mathbb{Q}_X)$ looks like an algebra of polynomial differential forms over X whereas $R\Gamma(X, \mathbb{Q}_X)$ looks like a cochain complex with a non-commutative cup-product.

3.3. Application to Hodge theory. From the point of view of Hodge theory, this has the following consequence. Let $X \subset \bar{X}$ be an open subset of a compact

Kähler manifold whose complement is a SNC divisor. Let $j : X \hookrightarrow \overline{X}$. Navarro Aznar ([Nav87, § 7–8]) shows that on $R_{\text{TW}}j_*\mathbb{Q}_X$ there is a canonical filtration τ such that

$$(3.3) \quad ((R_{\text{TW}}j_*\mathbb{Q}_X, \tau), (\Omega_{\overline{X}}^\bullet(\log D), W, F))$$

is a mixed Hodge diagram of sheaves, whose components are sheaves of commutative DG algebras. From the axioms on Thom-Whitney resolutions that we wrote above, it follows that applying $R_{\text{TW}}\Gamma(\overline{X}, -)$ to the above diagrams (i.e. to each component and to the filtered quasi-isomorphisms between them) gives a mixed Hodge diagram of commutative DG algebras, whose cohomology computes the cohomology of X and puts a MHS on it. Furthermore such a diagram is functorial for morphisms, up to natural quasi-isomorphisms.

In our cases of interest, we deal with the cohomology with local coefficients.

Corollary 3.2. *If \mathcal{K} is a mixed Hodge complex of sheaves over a manifold \overline{X} , computing the cohomology of X with local coefficients and lax symmetric monoidal in the data of the local system, then $R_{\text{TW}}\Gamma(\overline{X}, \mathcal{K})$ is again a mixed Hodge complex that computes the cohomology with local coefficients and is lax symmetric monoidal in the data of the local system.*

For brevity we will simply say that our MHC, that will always be used to compute cohomology with local coefficients in a variation of (mixed) Hodge structure, are lax symmetric, when they are lax symmetric monoidal functors in the data of the VMHS. So if the VMHS has a Lie bracket, then we will get mixed Hodge diagrams of Lie algebras.

Also, because of this corollary, we will focus exclusively on constructing mixed Hodge complexes at the level of sheaves, since one then only has to apply the Thom-Whitney global section to get the mixed Hodge complex we care about.

4. VHS AND VMHS OVER A COMPACT BASE

We start with the simplest case where the base of the VHS is compact, that we already worked out in our previous article, but that will be useful to introduce the ideas. In this section $X = \overline{X}$ is a compact Kähler manifold. We fix again any field $\mathbf{k} \subset \mathbb{R}$ (everything could also work with MHS over \mathbb{C}).

4.1. Pure case. Assume first that \mathbb{V} is a pure polarizable VHS of some weight $w \in \mathbb{Z}$ defined over \mathbf{k} on X . Then there is a pure HS of weight $w + i$ on $H^i(X, \mathbb{V})$, for all i , constructed by Deligne-Zucker. Define a filtration F^\bullet on $\Omega_X^\bullet(\mathbb{V}) := \Omega_X^\bullet \otimes \mathbb{V}$ by

$$(4.1) \quad F^p(\Omega_X^\bullet(\mathbb{V})) = \bigoplus_{r+e=p} F^r \Omega_X \otimes F^e \mathbb{V}_{\mathbb{C}}.$$

In other words the classes of Hodge type (p, q) in $H^i(X, \mathbb{V})$ will come from differential forms of type (r, s) with values in $\mathcal{V}^{e, f}$ (the \mathcal{C}^∞ vector bundle of Hodge type (e, f)) for $(r + e, s + f) = (p, q)$. The statement is then:

Theorem 4.1 ([Zuc79, Thm. 2.9, Cor. 2.11]). *If X is compact Kähler and \mathbb{V} is a pure polarizable VHS of weight w then*

$$(4.2) \quad (\mathbb{V}, (\Omega_X^\bullet(\mathbb{V}), F))$$

is a pure Hodge complex of sheaves of weight w , which is lax symmetric monoidal in \mathbb{V} .

Proof. The cited theory of Deligne-Zucker already shows that this is a mixed Hodge complex. For two polarizable VHS $\mathbb{V}_1, \mathbb{V}_2$ there is a diagram

$$(4.3) \quad \begin{array}{ccc} \Omega_X^\bullet(\mathbb{V}_1) \otimes \Omega_X^\bullet(\mathbb{V}_2) & \longrightarrow & \Omega_X^\bullet(\mathbb{V}_1 \otimes \mathbb{V}_2) \\ \approx \uparrow & & \uparrow \approx \\ (\mathbb{V}_1 \otimes \mathbb{C}) \otimes (\mathbb{V}_2 \otimes \mathbb{C}) & \longrightarrow & (\mathbb{V}_1 \otimes \mathbb{V}_2) \otimes \mathbb{C} \end{array}$$

where the two vertical morphisms are quasi-isomorphisms (this is simply the holomorphic Poincaré lemma extended to local systems) and the top morphism is induced by the product in Ω_X^\bullet . This diagram is compatible with the symmetry exchanging \mathbb{V}_1 and \mathbb{V}_2 . Whence the lax symmetric condition. \square

So, we get with the Thom-Whitney functors:

Corollary 4.2. *The data*

$$(4.4) \quad (R_{\text{TW}}\Gamma(X, \mathbb{V}), (R_{\text{TW}}\Gamma(X, \Omega_X^\bullet(\mathbb{V}), F))$$

forms a pure Hodge complex of weight w which is lax symmetric monoidal in \mathbb{V} .

Remark 4.3. This is just a re-writing of [Lef19, § 10] using only $R_{\text{TW}}\Gamma$ instead of the more classical resolution by \mathcal{C}^∞ differential forms with values in \mathbb{V} .

4.2. Mixed case. Assume now that \mathbb{V} is a graded-polarizable VMHS. Since the base is compact Kähler there are no extra conditions on the behavior at infinity of the VMHS. The filtration F is defined exactly as in the previous case but \mathbb{V} also carries a filtration by sub-local systems \mathbb{W}_\bullet . Define a filtration W_\bullet on $\Omega_X^\bullet(\mathbb{V})$ simply as

$$(4.5) \quad W_k(\Omega_X^\bullet(\mathbb{V})) := \Omega_X^\bullet(\mathbb{W}_k).$$

Then by definition

$$(4.6) \quad \text{Gr}_k^W(\Omega_X^\bullet(\mathbb{V})) = \Omega_X^\bullet(\text{Gr}_k^W(\mathbb{V}))$$

but the terms $\text{Gr}_k^W(\mathbb{V})$ are polarizable VHS of weight k to which the previous section applies. So we get immediately the following:

Theorem 4.4 (See [SZ85, Rem. 4.18.iii]). *If X is compact Kähler and \mathbb{V} is a graded-polarizable VMHS then*

$$(4.7) \quad ((\mathbb{V}, W), (\Omega_X^\bullet(\mathbb{V}), W, F))$$

is a mixed Hodge complex of sheaves, which is lax symmetric monoidal.

Again the lax symmetric monoidal condition is obvious and we can apply $R_{\text{TW}}\Gamma(X, -)$ to get our lax symmetric MHC.

5. VHS EXTENDABLE TO A COMPACTIFICATION

Now we turn to the situation where the base X may be non-compact, but is compactified in a Kähler manifold $j : X \hookrightarrow \overline{X}$ by a SNC divisor D . But we first make the simpler but interesting hypothesis that \mathbb{V} is a pure polarizable VHS of some weight $w \in \mathbb{Z}$ over X that extends as VHS to \overline{X} , again defined over some field $\mathbf{k} \subset \mathbb{R}$ (or $\mathbf{k} = \mathbb{C}$). After a theorem of Griffiths, this is the same as requiring the monodromy of \mathbb{V} to be trivial locally around D : then the local system underlying \mathbb{V} extends across D and Griffiths' theorem [Gri70, Theorem 9.5] shows that the holomorphic bundles \mathcal{F}^\bullet also extend so as to define a VHS on \overline{X} .

Over X we can introduce the logarithmic complex $\Omega_{\overline{X}}^\bullet(\log D)$ which carries two filtrations W, F as usual and we can form the logarithmic complex with local coefficients $\Omega_{\overline{X}}^\bullet(\log D, \mathbb{V}) := \Omega_{\overline{X}}^\bullet(\log D) \otimes \mathbb{V}$. It again carries two filtrations F, W and since \mathbb{V} is pure W is simply the shift by w of the weight filtration of $\Omega_{\overline{X}}^\bullet(\log D)$. This will be the complex part of the MHC of sheaves.

It is easy to see that we again have a diagram

$$(5.1) \quad \begin{array}{ccc} \Omega_{\overline{X}}^\bullet(\log D, \mathbb{V}_1) \otimes \Omega_{\overline{X}}^\bullet(\log D, \mathbb{V}_2) & \longrightarrow & \Omega_{\overline{X}}^\bullet(\log D, \mathbb{V}_1 \otimes \mathbb{V}_2) \\ \approx \uparrow & & \uparrow \approx \\ (\mathbb{V}_1 \otimes \mathbb{C}) \otimes (\mathbb{V}_2 \otimes \mathbb{C}) & \longrightarrow & (\mathbb{V}_1 \otimes \mathbb{V}_2) \otimes \mathbb{C} \end{array}$$

where the vertical maps are quasi-isomorphisms and the top horizontal one is the product of differential forms.

For the component over \mathbf{k} we take $R_{\text{TW}j_*}\mathbb{V}$. The filtration is obtained for the canonical filtration τ . Recall that for any complex K^\bullet the canonical filtration τ is defined by $\tau_k K^n = 0$ if $k < n$, $\tau_k K^n = \text{Ker}(d)$ if $k = n$ and $\tau_k K^n = K^n$ for $k > n$. Any quasi-isomorphism is then automatically a filtered quasi-isomorphism with respect to τ . On $R_{\text{TW}j_*}\mathbb{V}$ we take for W the shifted filtration $\tau[w]$, with $\tau[w]_k := \tau_{w+k}$.

Proposition 5.1 (Compare [Nav87, Prop. 8.4]). *There is a canonical chain of filtered quasi-isomorphisms*

$$(5.2) \quad (R_{\text{TW}j_*}\mathbb{V}, W) \otimes \mathbb{C} \xleftarrow{\approx} (\Omega_{\overline{X}}^\bullet(\log D, \mathbb{V}), W).$$

Proof. First, the holomorphic Poincaré lemma with local coefficients implies that there is a quasi-isomorphism of sheaves over X

$$(5.3) \quad \mathbb{V} \otimes \mathbb{C} \xrightarrow{\approx} \Omega_X^\bullet(\mathbb{V})$$

that we compose with the Thom-Whitney resolution

$$(5.4) \quad \Omega_X^\bullet(\mathbb{V}) \xrightarrow{\approx} \text{TW}^* \Omega_X^\bullet(\mathbb{V})$$

to get the chain

$$(5.5) \quad R_{\text{TW}j_*}\mathbb{V} \otimes \mathbb{C} = R_{\text{TW}j_*}(\mathbb{V} \otimes \mathbb{C}) \xrightarrow{\approx} R_{\text{TW}j_*} \Omega_X^\bullet(\mathbb{V}) \xrightarrow{\approx} R_{\text{TW}j_*}(\text{TW}^* \Omega_X^\bullet(\mathbb{V})).$$

These are automatically filtered quasi-isomorphisms for τ as well as for $\tau[w]$. On the other hand there is a canonical morphisms over \overline{X}

$$(5.6) \quad \Omega_{\overline{X}}^\bullet(\log D, \mathbb{V}) \longrightarrow j_* \Omega_X^\bullet(\mathbb{V})$$

which, composed with $j_*(\Omega_X^\bullet(\mathbb{V})) \rightarrow j_*(\mathrm{TW}^*\Omega_X^\bullet(\mathbb{V}))$, induces a quasi-isomorphism

$$(5.7) \quad \Omega_X^\bullet(\log D, \mathbb{V}) \xrightarrow{\sim} j_*(\mathrm{TW}^*\Omega_X^\bullet(\mathbb{V})).$$

Then since $\mathrm{TW}^*\Omega_X^\bullet(\mathbb{V})$ is a soft sheaf we get

$$(5.8) \quad j_*(\mathrm{TW}^*\Omega_X^\bullet(\mathbb{V})) \xrightarrow{\sim} R_{\mathrm{TW}}j_*(\mathrm{TW}^*\Omega_X^\bullet(\mathbb{V})).$$

To sum up this gives

$$(5.9) \quad \Omega_X^\bullet(\log D, \mathbb{V}) \xrightarrow{\sim} j_*(\mathrm{TW}^*\Omega_X^\bullet(\mathbb{V})) \xrightarrow{\sim} R_{\mathrm{TW}}j_*(\mathrm{TW}^*\Omega_X^\bullet(\mathbb{V}))$$

where again these quasi-isomorphisms are filtered quasi-isomorphisms for τ and for $\tau[w]$.

Finally the classical filtered quasi-isomorphism

$$(5.10) \quad (\Omega_X^\bullet(\log D), \tau) \xrightarrow{\sim} (\Omega_X^\bullet(\log D), W)$$

also induces with local coefficients

$$(5.11) \quad (\Omega_X^\bullet(\log D), \tau) \otimes \mathbb{V} \xrightarrow{\sim} (\Omega_X^\bullet(\log D), W) \otimes \mathbb{V}.$$

By our choice of $W = \tau[w]$ over \mathbf{k} this gives

$$(5.12) \quad (\Omega_X^\bullet(\log D, \mathbb{V}), \tau[w]) \xrightarrow{\sim} (\Omega_X^\bullet(\log D, \mathbb{V}), W).$$

□

The classical theory of residues of forms with logarithmic poles is also easily extended to the case of local coefficients in our context. Recall that for this we decompose the SNC divisor into its components $D = \bigcup_I D_i$ (I a finite set), for each set $J \subset I$ we write $D_J := \bigcap_{i \in J} D_i$ and we let $\tilde{D}^k := \bigcup_{|J|=k} D_J$ with its canonical inclusion i_k into \overline{X} . For $k = 0$ this is \overline{X} . The classical residue is a morphism

$$(5.13) \quad \mathrm{Res} : W_k \Omega_X^n(\log D) \longrightarrow (i_k)_* \Omega_{\tilde{D}^k}^{n-k}$$

which induces a quasi-isomorphism

$$(5.14) \quad \mathrm{Gr}_k^W \Omega_X^n(\log D) \xrightarrow{\sim} (i_k)_* \Omega_{\tilde{D}^k}^{n-k}.$$

It is this fact that allows to verify the axioms of mixed Hodge complex by computing the Gr^W terms, since on the right-hand side the terms \tilde{D}^k are compact Kähler manifolds. Namely the terms $\mathrm{Gr}_k^W \Omega_X^n(\log D)$ comes equipped with pure Hodge structures of weight $k + n$ on their cohomology.

We easily define a residue morphism with local coefficients

$$(5.15) \quad \mathrm{Res} \otimes \mathrm{id}_{\mathbb{V}} : W_{k+w} \Omega_X^n(\log D, \mathbb{V}) \longrightarrow (i_k)_* \Omega_{\tilde{D}^k}^{n-k}(\mathbb{V})$$

and this allows us to conclude, combining with the case of compact Kähler manifolds:

Theorem 5.2. *For a polarizable VHS \mathbb{V} defined over X and extending as VHS to the compactification $j : X \hookrightarrow \overline{X}$, the data*

$$(5.16) \quad ((R_{\mathrm{TW}}j_* \mathbb{V}, W), (\Omega_X^\bullet(\log D, \mathbb{V}), W, F))$$

is a mixed Hodge complex of sheaves over X that computes $H^\bullet(X, \mathbb{V})$ and that is lax symmetric in \mathbb{V} .

Proof. Via the map (5.15), the terms $\mathrm{Gr}_{k+w}^W \Omega_X^n(\log D)$ carry pure Hodge structures of weight $n + k + w$, because on the right-hand side we get cohomology groups of compact Kähler manifolds with coefficients in the polarizable VHS \mathbb{V} of weight w . \square

From the construction we see that the Gr_w^W part of this MHC is the pure Hodge complex of weight w computing $H^\bullet(\overline{X}, \mathbb{V})$. Hence for all i , $H^i(X, \mathbb{V})$ has weights between $i + w$ and $2i + w$, with $\mathrm{Gr}_{w+i}^W H^i(X, \mathbb{V}) = H^i(\overline{X}, \mathbb{V})$.

6. DEGENERATING VMHS AND THE CASE OF CURVES

We now want to deal with VMHS, including the case of pure VHS, over X which degenerate when approaching to the boundary D . The preliminary technical step, that we will also need in the next section, is to understand the extension of local systems across D . After, we will specialize to the case where the base is a curve since this situation is simpler to understand.

6.1. Preliminaries on extensions of local systems. In this section X is still allowed to be any open subset of compact Kähler manifold \overline{X} whose complement is a SNC divisor D .

Let \mathbb{V} be a local system over X of vector spaces over \mathbb{C} . Deligne has shown ([Del70, § II.5], see also [HTT08, § 5]) that its associated flat vector bundle (\mathcal{V}, ∇) extends as a meromorphic bundle \mathcal{M} to \overline{X} . For each half-open interval I of length 1, there is a unique vector bundle with connection $(\overline{\mathcal{V}}^I, \nabla)$ over \overline{X} , seen as a lattice inside \mathcal{M} , extending (\mathcal{V}, ∇) such that ∇ has logarithmic poles on D and such that the residue of ∇ along any component D_i of D

$$(6.1) \quad \mathrm{Res}_{D_i}(\nabla) \in \Omega_{D_i}^1(\mathrm{End}(\overline{\mathcal{V}}))$$

has eigenvalues of real part contained in I .

For two such flat bundles $(\mathcal{V}_1, \nabla_1)$, $(\mathcal{V}_2, \nabla_2)$ with associated meromorphic extensions $\mathcal{M}_1, \mathcal{M}_2$ the meromorphic extension of $\mathcal{V}_1 \otimes \mathcal{V}_2$, equipped with the connection

$$(6.2) \quad \nabla = \nabla_1 \otimes \mathrm{id}_2 + \mathrm{id}_1 \otimes \nabla_2,$$

is exactly $\mathcal{M}_1 \otimes \mathcal{M}_2$. However the construction of $\overline{\mathcal{V}}^I$ is not compatible with this tensor product.

Hence we will make the assumption that our local systems have *unipotent* monodromy around each component D_i . In this case we can take for $(\overline{\mathcal{V}}, \nabla)$ the unique extension for which the residues are nilpotent (i.e. have 0 as only eigenvalue) and call it the *canonical extension* of (\mathcal{V}, ∇) . Note that the residue of a tensor product $(\mathcal{V}, \nabla) = (\mathcal{V}_1, \nabla_1) \otimes (\mathcal{V}_2, \nabla_2)$ is

$$(6.3) \quad \mathrm{Res}_{D_i}(\nabla) = \mathrm{Res}_{D_i}(\nabla_1) \otimes \mathrm{id}_2 + \mathrm{id}_1 \otimes \mathrm{Res}_{D_i}(\nabla_2)$$

which is again nilpotent if both residues are.

Classically, there is a quasi-isomorphism $Rj_* \mathbb{V}_{\mathbb{C}} \approx \Omega_{\overline{X}}^\bullet(\log D, \overline{\mathcal{V}})$ over \overline{X} and we want first to replace $Rj_* \mathbb{V}$ by $R_{\mathrm{TW}} j_* \mathbb{V}$. So we will need the following proposition.

Proposition 6.1. *For the local system \mathbb{V} over X with unipotent monodromy around D and with canonical extension $\overline{\mathcal{V}}$ as vector bundle over $j : X \hookrightarrow \overline{X}$, there are two chains of quasi-isomorphisms*

$$(6.4) \quad R_{\mathrm{TW}} j_* \mathbb{V} \xrightarrow{\sim} R_{\mathrm{TW}} j_* \Omega_X^\bullet(\mathbb{V}) \xrightarrow{\sim} R_{\mathrm{TW}} j_* (\mathrm{TW}^* \Omega_X^\bullet(\mathbb{V}))$$

and

$$(6.5) \quad \Omega_{\overline{X}}^{\bullet}(\log D, \overline{\mathcal{V}}) \xrightarrow{\sim} j_*(\mathrm{TW}^*\Omega_X^{\bullet}(\mathbb{V})) \xrightarrow{\sim} R_{\mathrm{TW}}j_*(\mathrm{TW}^*\Omega_X^{\bullet}(\mathbb{V})).$$

Proof. Comparing to the previous section, only the part

$$(6.6) \quad \Omega_{\overline{X}}^{\bullet}(\log D, \overline{\mathcal{V}}) \longrightarrow j_*(\mathrm{TW}^*\Omega_X^{\bullet}(\mathbb{V}))$$

is different. But since $\mathrm{TW}^*\Omega_X^{\bullet}(\mathbb{V})$ is a soft resolution of $\Omega_X^{\bullet}(\mathbb{V})$ which itself is quasi-isomorphic to \mathbb{V} , then $j_*(\mathrm{TW}^*\Omega_X^{\bullet}(\mathbb{V}))$ represents $Rj_*\mathbb{V}$ and the fact that this is a quasi-isomorphism is also classical (see [HTT08, Theorem 5.2.24]). \square

Again, the above chain is lax symmetric monoidal in \mathbb{V} because we restricted to local systems with unipotent monodromy.

6.2. Degenerations of VMHS. We continue with an open manifold $j : X \hookrightarrow \overline{X}$. First, it is known that the local monodromy of a polarizable VHS on X around D which is defined over a number field \mathbf{k} is automatically quasi-unipotent ([Sch73, Lemma 4.5]). Hence there is some finite cover of \overline{X} ramified along D over which it is unipotent. This justifies that we will only work with unipotent VHS.

We then specialize to the case where \overline{X} is a curve, so D is a finite set of points. In his study of degenerations of pure polarizable VHS, Schmid [Sch73] has shown that the holomorphic bundles \mathcal{F}^{\bullet} of a polarizable VHS \mathbb{V} of weight w on X can be extended to \overline{X} as holomorphic sub-vector bundles of $\overline{\mathcal{V}}$. Around $x \in D$ the residue operator is a nilpotent operator $N(x)$ of the fiber $\overline{\mathcal{V}}_x$ to which is associated a monodromy filtration:

Definition 6.2 ([SZ85, 2.1]). The *monodromy filtration* associated to a finite-dimensional vector space V and a nilpotent endomorphism N , centered at $w \in \mathbb{Z}$, is the unique increasing filtration M_{\bullet} such that for all $i \in \mathbb{Z}$:

- (1) $N(M_i) \subset M_{i-2}$,
- (2) N^i induces an isomorphism $\mathrm{Gr}_M^{w+i} V \xrightarrow{\simeq} \mathrm{Gr}_M^{w-i} V$.

Schmid then shows that the this monodromy filtration centered at w together with $\overline{\mathcal{F}}_x^{\bullet}$ gives a MHS on $\overline{\mathcal{V}}_x$.

In this situation Zucker [Zuc79, § 13] constructed, first a pure HS of weight $i+w$ on each $H^i(\overline{X}, j_*\mathbb{V})$, and then a MHS on $H^i(X, \mathbb{V})$ whose part of lowest weight is exactly $H^i(\overline{X}, j_*\mathbb{V})$. The Hodge filtration is defined on $\Omega_{\overline{X}}^{\bullet}(\log D, \overline{\mathcal{V}})$ as usual using the bundles $\overline{\mathcal{F}}^{\bullet} \subset \overline{\mathcal{V}}$. The description of the weight filtration uses the previous monodromy filtration; in particular the upper bound for the weights of $H^i(X, \mathbb{V})$ depends on the order of nilpotency of $N(x)$. The MHS constructed by Zucker comes indeed from a MHC and is defined over the field $\mathbf{k} \subset \mathbb{R}$ if \mathbb{V} is.

Suppose now that X is still a curve but that \mathbb{V} is a graded-polarizable VMHS. The local systems $\mathbb{W}_{\bullet} \subset \mathbb{V}$ also extend as vector bundles to $\overline{\mathcal{W}}_{\bullet} \subset \overline{\mathcal{V}}$. The problem of getting a similar good theory of degenerations is studied in detail in the paper of Steenbrink-Zucker. There is introduced the notion of *admissible VMHS* ([SZ85, 3.13]). For this is required graded-polarizability, that the filtration $\mathcal{F}^{\bullet} \subset \mathcal{V}$ also extend to $\overline{\mathcal{F}}^{\bullet} \subset \overline{\mathcal{V}}$, and that the nilpotent residue $N(x)$ around x behaves well with respect to the weight filtration of each fiber $\overline{\mathcal{V}}_x$, i.e. there exists the relative monodromy filtration:

Definition 6.3 ([SZ85, 2.5]). The *relative monodromy filtration* associated to a finite-dimensional vector space \mathbb{V} with an increasing filtration W_\bullet and a nilpotent endomorphism N respecting W_\bullet is a filtration M_\bullet of V such that:

- (1) For all $i \in \mathbb{Z}$, $N(M_i) \subset M_{i-2}$,
- (2) M induces on each $\mathrm{Gr}_W^k V$ the monodromy filtration of endomorphism induced by N , centered at k .

It is unique when it exists. Pure polarizable VHS are then admissible because of the results of Schmid. Also, VMHS of geometric origin are shown to be admissible in their paper (§ 5). For admissible VMHS, they are then able to construct a MHS on $H^i(X, \mathbb{V})$, again coming from the cohomology of a MHC and defined over $\mathbf{k} \subset \mathbb{R}$ if \mathbb{V} is.

Also, since we work on tensor product, we will need to know:

Proposition 6.4 ([SZ85, Appendix]). *For two finite-dimensional vector spaces V_1, V_2 with nilpotent endomorphisms N_1, N_2 and filtrations W_\bullet^1, W_\bullet^2 , having relative monodromy filtrations M_\bullet^1, M_\bullet^2 , then the monodromy filtration of $V_1 \otimes V_2$ with respect to the endomorphism $N = N_1 \otimes \mathrm{id}_{V_2} + \mathrm{id}_{V_1} \otimes N_2$ and the filtration $W^1 \otimes W^2$ exists and is $M^1 \otimes M^2$. Consequently, the category of admissible VMHS on a curve is closed under tensor product.*

6.3. Construction for admissible VMHS. First we follow entirely the construction of Steenbrink-Zucker.

Theorem 6.5 ([SZ85, § 4]). *For an admissible VMHS \mathbb{V} over a curve $j : X \hookrightarrow \bar{X}$ with unipotent monodromy around the singularities, there exists a filtration \mathfrak{W}_\bullet on all terms of Theorem 6.1 such that the data*

$$(6.7) \quad ((R_{\mathrm{TW}j_*} \mathbb{V}, \mathfrak{W}), (\Omega_{\bar{X}}^\bullet(\log D, \bar{\mathbb{V}}), \mathfrak{W}, F))$$

forms a mixed Hodge complex of sheaves computing $H^\bullet(X, \mathbb{V})$, defined over $\mathbf{k} \subset \mathbb{R}$ if \mathbb{V} is defined over \mathbf{k} .

Construction. At each point x of D , on the fiber $\bar{\mathbb{V}}_x$, is defined a filtration $\bar{\mathcal{W}}_\bullet(x)$ (coming from the canonical extensions $\bar{\mathcal{W}}_\bullet$ of the local systems \mathcal{W}_\bullet with associated flat bundles $(\mathcal{W}_\bullet, \nabla)$), a nilpotent endomorphism $N(x)$ (residue of ∇), and the relative monodromy filtration $M_\bullet(x)$ whose existence is part of the admissibility conditions. Then is defined a filtration

$$(6.8) \quad Z_k(x) := N(x)(\bar{\mathcal{W}}_k(x)) + M_{k-1}(x) \cap \bar{\mathcal{W}}_{k-1}(x).$$

This defines uniquely a filtration \mathfrak{W}_\bullet of $\Omega_{\bar{X}}^\bullet(\log D, \bar{\mathbb{V}})$ by sub-complexes, where $\mathfrak{W}_k \Omega_{\bar{X}}^\bullet(\log D, \bar{\mathbb{V}})$ is the sub-complex of $\Omega_{\bar{X}}^\bullet(\log D, \bar{\mathcal{W}}_k)$ formed by $\bar{\mathcal{W}}_k$ in degree 0 and whose part in degree 1 at x lies between $\mathrm{Im}(\nabla)$ and $\Omega_{\bar{X}}^1(\log D, \bar{\mathcal{W}}_k)$ and is sent to $Z_k(x)$ via the residue

$$(6.9) \quad \mathrm{Res}_x : \Omega_{\bar{X}}^1(\log D, \bar{\mathcal{W}}_k) \longrightarrow \bar{\mathcal{W}}_k(x).$$

For a local coordinate t at x , with $Z_k(x)$ extending to a local system \mathcal{Z}_k and a sub-bundle $\bar{\mathcal{Z}}_k$ of $\bar{\mathcal{W}}_k$, the complex $\mathfrak{W}_k \Omega_{\bar{X}}^\bullet(\log D, \bar{\mathbb{V}})$ is

$$(6.10) \quad \bar{\mathcal{W}}_k \xrightarrow{d} \frac{dt}{t} \otimes (\bar{\mathcal{Z}}_k + t\bar{\mathcal{W}}_k).$$

On $Rj_* \mathbb{V}$, for any functorial resolution Rj_* , this corresponds to the following construction: $\mathfrak{W}_k Rj_* \mathbb{V}$ is the sub-complex of $Rj_* \mathbb{V}$ formed by

- (1) First consider $Rj_*(\mathbb{W}_k) \subset Rj_*\mathbb{V}$.
 (2) Then truncate: $(\tau_{\leq 1}Rj_*(\mathbb{W}_k))^0 = Rj_*(\mathbb{W}_k)^0$,
- $$(6.11) \quad (\tau_{\leq 1}Rj_*(\mathbb{W}_k))^1 = \text{Ker}(d : Rj_*(\mathbb{W}_k)^1 \longrightarrow Rj_*(\mathbb{W}_k)^2)$$
- and $(\tau_{\leq 1}Rj_*(\mathbb{W}_k))^i = 0$ for $i > 1$. It has the same cohomology as $Rj_*(\mathbb{W}_k)$ since on a curve $R^i j_*(\mathbb{W}_k) = 0$ for $i > 1$.
 (3) Finally $(\mathfrak{W}_k Rj_*\mathbb{V})^0 := (Rj_*(\mathbb{W}_k))^0$ and
- $$(6.12) \quad (\mathfrak{W}_k Rj_*\mathbb{V})^1 := d(Rj_*(\mathbb{W}_k))^0 + (\tau_{\leq 1}Rj_*(\mathbb{W}_k))^1 \cap Rj_*(\mathbb{Z}_k)^1.$$

In our situation, we apply this with Rj_* replaced by any of the functors appearing in Proposition (6.1), that is, with R_{TW} , $R_{\text{TW}}j_*\Omega_X^\bullet$, $R_{\text{TW}}j_*(\text{TW}^*\Omega_X^\bullet)$, $j_*(\text{TW}^*\Omega_X^\bullet)$. When \mathbb{V} is defined over the field \mathbf{k} , then so is $Z_\bullet(x)$, hence \mathfrak{W}_\bullet on $R_{\text{TW}}j_*\mathbb{V}$ is defined over \mathbf{k} .

The computations of $\text{Gr}_\bullet^{\mathfrak{W}}$ done in their article indeed show that this gives a mixed Hodge complex of sheaves. \square

Proposition 6.6. *The above mixed Hodge complex is lax symmetric monoidal in \mathbb{V} .*

Proof. It will obviously be symmetric once it is lax monoidal because we used the Thom-Whitney functors. Given two VMHS $\mathbb{V}_1, \mathbb{V}_2$ and a fixed base point x , with associated residues $N_1(x), N_2(x)$, monodromy filtrations $M_\bullet^1(x), M_\bullet^2(x)$, filtrations $Z_\bullet^1(x), Z_\bullet^2(x)$ etc, the maps

$$(6.13) \quad R_{\text{TW}}j_*\mathbb{V}_1 \otimes R_{\text{TW}}j_*\mathbb{V}_2 \longrightarrow R_{\text{TW}}j_*(\mathbb{V}_1 \otimes \mathbb{V}_2)$$

and

$$(6.14) \quad \Omega_{\overline{X}}^\bullet(\log D, \overline{\mathcal{V}}_1) \otimes \Omega_{\overline{X}}^\bullet(\log D, \overline{\mathcal{V}}_2) \longrightarrow \Omega_{\overline{X}}^\bullet(\log D, \overline{\mathcal{V}}_1 \otimes \overline{\mathcal{V}}_2)$$

are the obvious ones (recall that $\overline{\mathcal{V}}_1 \otimes \overline{\mathcal{V}}_2 = \overline{\mathcal{V}}_1 \otimes \overline{\mathcal{V}}_2$ since we work with unipotent monodromy around D). The compatibility with the filtrations F^\bullet is also obvious.

So what it remains is to check the compatibility with the weight filtration. Since our complexes are concentrated in degrees 0 and 1 the only non-trivial multiplication is the one between degrees 0 and 1. If we denote by $M_\bullet(x)$ the relative monodromy filtration of $\mathbb{V}_1 \otimes \mathbb{V}_2$ at x , for the residue $N(x)$ with its associated $Z_\bullet(x)$, all we have to do (explicit for example in equation (6.10)) is to check that there is a natural inclusion

$$(6.15) \quad \overline{W}_k^1(x) \otimes Z_\ell^2(x) \subset Z_{k+\ell}(x)$$

for all k, ℓ .

This is linear algebra; to compute this let us drop the letter x and write W_\bullet for \overline{W}_\bullet . Recall that $N = N_1 \otimes \text{id}_2 + \text{id}_1 \otimes N_2$, $M = M^1 \otimes M^2$, $W = W^1 \otimes W^2$ and $Z_\ell = N(W_\ell) + M_{\ell-1} \cap W_{\ell-1}$. But it will be more practical to have another expression for Z_\bullet as proved in [Kas86, § 3.4]:

$$(6.16) \quad Z_\ell = N(W_\ell) + \bigcap_{j \geq 0} (N^j)^{-1}(W_{\ell-j}).$$

So, to compute $W_k^1 \otimes Z_\ell^2$, for one term of the sum

$$(6.17) \quad W_k^1 \otimes N_2(W_\ell^2) \subset (N_1 \otimes \text{id}_2 + \text{id}_1 \otimes N_2)(W_k^1 \otimes W_\ell^2) \subset N(W_{k+\ell}) \subset Z_{k+\ell}$$

and for the other term

$$\begin{aligned}
 (6.18) \quad W_k^1 \otimes \bigcap_{j \geq 0} (N_2^j)^{-1} (W_{\ell-j}^2) \\
 \subset \bigcap_{j \geq 0} ((N_1 \otimes \text{id}_2 + \text{id}_1 \otimes N_2)^j)^{-1} (W_k^1 \otimes W_{\ell-j}^2) \\
 \subset \bigcap_{j \geq 0} (N^j)^{-1} (W_{k+\ell-j}) \subset Z_{k+\ell}.
 \end{aligned}$$

This concludes. \square

In this construction, if \mathbb{V} has weights greater than w then $H^i(X, \mathbb{V})$ has weights greater than $w + i$ for all i .

Remark 6.7. If \mathbb{V} is a pure VHS of weight w , then in the above construction the filtration W_\bullet is concentrated in weight w and M_\bullet is simply the monodromy filtration of N , so the formula for Z is

$$(6.19) \quad Z_k(x) = N(x)(\overline{\mathbb{V}}_x) + M_{k-1}(x), \quad w \leq k \leq w + q + 1$$

(where q is the order of nilpotency of $N(x)$), with $Z_k(x) = 0$ or $\overline{\mathbb{V}}_x$ outside of this range. So the above construction reduces to the one of [Zuc79, § 13], modified for using $R_{\text{TW}}j_*$ instead of Rj_* .

Remark 6.8. If furthermore there is a point x of D over which \mathbb{V} is non-degenerate (i.e. extends as VHS, equivalently has trivial monodromy) then the following happens: the monodromy is trivial, the canonical extension of \mathbb{V} is \mathbb{V} itself, the operator $N(x)$ is zero, the monodromy filtration is trivial and the weight filtration is concentrated in w . One can then check line by line that everything reduces to the case of our section 5.

7. MIXED HODGE MODULES

The theory of mixed Hodge modules of Saito is the most general one in order to understand admissible variations of mixed Hodge structure over a general base manifold and to construct mixed Hodge structures on various related objects. It is contained in two articles [Sai88], [Sai86]. See also the survey of Schnell [Sch14]. Since we focus only on the construction of MHS on cohomology with local coefficients, the survey of Arapura [Ara10] as well as the short note [Sai89] are very useful.

The notion of admissible VMHS is studied by Kashiwara in [Kas86]. The definition is given by testing the admissibility of Steenbrink-Zucker on pull-backs to curves; the admissibility condition for a VMHS on X depends on the choice of a compactification \overline{X} but is invariant under birationally equivalent compactifications, hence it is independent of \overline{X} if X is quasi-projective.

In this section we restrict to VMHS over the field \mathbb{Q} in order to cite directly the literature, but it seems that this also works with a field $\mathbf{k} \subset \mathbb{R}$. It is however not obvious at all if there exists a notion of mixed Hodge module for VMHS over \mathbb{C} .

7.1. Brief reminder. For any complex manifold X of some dimension N is defined a category $\text{MF}(X)$ of *filtered \mathcal{D}_X -modules with \mathbb{Q} -structure* whose objects are triples $(P, \mathcal{M}_\bullet, \alpha)$ where P is a perverse sheaf over \mathbb{Q} on X , \mathcal{M} a regular holonomic \mathcal{D}_X -module with a good filtration \mathcal{F}^\bullet , and α is a quasi-isomorphism $\text{DR}(\mathcal{M}) \simeq P \otimes \mathbb{C}$ where

$$(7.1) \quad \text{DR}(\mathcal{M}) := \{\mathcal{M} \longrightarrow \Omega_X^1 \otimes \mathcal{M} \longrightarrow \cdots \longrightarrow \Omega_X^N \otimes \mathcal{M}\} [N].$$

Any VHS $(\mathbb{V}, \mathcal{F}^\bullet, \nabla)$ over X defines an object of $\text{MF}(X)$ where P is $\mathbb{V}[N]$ (shifting the local system over \mathbb{Q} by the dimension of X to make it a perverse sheaf on X) and \mathcal{M} is the \mathcal{D}_X -module coming from the flat connection $(\mathbb{V} \otimes \mathcal{O}_X, \nabla)$. Griffiths' transversality is then precisely the statement that \mathcal{F}^\bullet forms a filtration of \mathcal{M} as \mathcal{D}_X -module.

For each $w \in \mathbb{Z}$, the category of *pure Hodge modules of weight w* is a full subcategory $\text{MH}(X, w) \subset \text{MF}(X)$. The axioms are quite evolved; they imply that each element of $\text{MH}(X, w)$ has a *decomposition by strict support* running over the irreducible closed subvarieties $Z \subset X$, and that such an element with strict support Z is a VHS of weight $w - \dim(Z)$ over some dense open subset of Z .

Inside $\text{MH}(X, w)$ there is also the full sub-category of *polarizable Hodge modules*, corresponding to polarizable VHS. One important theorem of Saito is that any (polarizable) variation of Hodge structure over $X \subset \overline{X}$ of weight w defines a pure (polarizable) Hodge module of weight $w + N$ over \overline{X} with full support.

There is also a category $\text{MFW}(X)$ whose objects are the objects of $\text{MF}(X)$ equipped with an additional increasing filtration W_\bullet (so W_\bullet is defined on P as well as on \mathcal{M} and α is a filtered quasi-isomorphism). Any VMHS (without additional assumption) over X defines on object of $\text{MFW}(X)$ similarly as in the pure case. Then the category of *mixed Hodge modules* $\text{MHM}(X)$ is a full subcategory of $\text{MFW}(X)$, whose Gr_W^w of the objects are in $\text{MH}(X, w)$, with additional axioms on the way these graded parts are related.

Theorem 7.1 ([Sai86], [Sai89, 3.3]). *Let \mathbb{V} be a graded-polarizable VMHS over X , admissible in $j : X \hookrightarrow \overline{X}$, with unipotent monodromy around $D = \overline{X} \setminus X$. Then there are filtrations $\mathfrak{W}_\bullet, F^\bullet$ such that*

$$(7.2) \quad ((Rj_* \mathbb{V}, \mathfrak{W}), (\Omega_{\overline{X}}^\bullet(\log D, \overline{\mathbb{V}}), \mathfrak{W}, F))$$

is a mixed Hodge complex of sheaves over \overline{X} .

Sketch of construction. Consider on one side \mathcal{M} the meromorphic extension of the flat connection (\mathcal{V}, ∇) across D . This is the $\mathcal{D}_{\overline{X}}$ -module corresponding to the perverse sheaf $P := Rj_* \mathbb{V}[N]$ under the functor DR , containing $\overline{\mathbb{V}}$ as canonical lattice.

Then Saito proves that this defines a mixed Hodge module M on \overline{X} ([Sai86, Theorem 3.27]). The Hodge filtration is defined via the sheaves $j_* \mathcal{F}^\bullet$. There is a procedure to define a weight weight filtration (see details below) on P as well as on \mathcal{M} . And this defines the corresponding filtrations on $\Omega_{\overline{X}}^\bullet(\log D, \overline{\mathbb{V}})$ via the inclusion

$$(7.3) \quad (\Omega_{\overline{X}}^\bullet(\log D, \overline{\mathbb{V}}), \mathfrak{W}_\bullet[N], F_\bullet)[N] \subset (\text{DR}(\mathcal{M}), \mathfrak{W}_\bullet, F_\bullet)$$

(the shift is necessary because of the convention that a VHS of weight k extends to a Hodge module of weight $k + N$) which becomes a bifiltered quasi-isomorphism. In general if the monodromy is only quasi-unipotent we must use the extension $\overline{\mathbb{V}}^{(0,1)}$

where the residues have eigenvalues of real parts in $[0, 1)$. The weight filtration is constructed in such a way that the terms $\mathrm{Gr}_k^{\mathrm{wt}} M$ are pure polarizable Hodge modules of weight k .

It remains to check the axioms of MHC for $K := R\Gamma(\overline{X}, M)$, the derived push-forward to a point. This follows from the theorems of Saito ([Sai88, Theorem 5.3.1]) on pure Hodge modules: on the $\mathrm{Gr}_k^{\mathrm{wt}} K$, the filtration F is strict and the cohomology carries pure Hodge structures. This is written by Saito for push-forward via projective morphisms but for the push-forward to a point this also works for compact Kähler manifolds (see [Sai88, 5.3.8.2]), since this relies only on the analytic theory ([CKS87], and in dimension 1 this is just the work of Zucker). \square

What we will need now is to have a better understanding of the weight filtration and its multiplicativity.

Remark 7.2. There is no tensor product at the level of mixed Hodge modules, but only in their derived category. So part of the complications is that we have to study directly the functor from VMHS to MHC, and not the functor from VMHS to MHM, which is not so often treated in the literature. Also, there is no tensor product at the level of perverse sheaves. So we choose to drop the rational part and work only with the complex part now. This will be enough to define a \mathbb{C} -mixed Hodge complex which will be lax symmetric monoidal. We do not know if it is possible to introduce the Thom-Whitney functors here and to develop a commutative theory of mixed Hodge modules (using for example $R_{\mathrm{TW}}f_*$ for a projective morphism f , and the Thom-Whitney vanishing nearby and vanishing cycles defined by Navarro Aznar [Nav87, § 14-15]). We do not know if this even makes sense.

So let us explain the construction of the weight filtration. We reduce to \overline{X} being a polydisk of dimension N with coordinates x_1, \dots, x_N and X is the complement of a normal crossing divisor D with components $D_i = \{x_i = 0\}$. We assume that the admissible VMHS \mathbb{V} has unipotent monodromy around D and is extended to the $\mathcal{D}_{\overline{X}}$ -module \mathcal{M} . Along each axis D_i is defined an increasing V -filtration $V(i)$ of \mathcal{M} indexed by \mathbb{Z} (this is simpler with unipotent monodromy). It satisfies that $\mathrm{Gr}_0^{V(i)} \mathcal{M}$ is the vanishing cycles $\psi_{x_i} \mathcal{M}$ of \mathcal{M} for the function x_i and $\mathrm{Gr}_{-1}^{V(i)} \mathcal{M}$ is the nearby cycles $\varphi_{x_i} \mathcal{M}$. There are canonical maps $\mathrm{can}_i : \psi_{x_i} \mathcal{M} \rightarrow \varphi_{x_i} \mathcal{M}$ given by $-\partial_i$, $\mathrm{var}_i : \varphi_{x_i} \mathcal{M} \rightarrow \psi_{x_i} \mathcal{M}$ given by multiplication by x_i , and N_i the endomorphism of $\psi_{x_i} \mathcal{M}$ given by the logarithm of the monodromy around D_i .

For a subset of indices $I \subset \{1, \dots, N\}$, define $\Psi_I \mathcal{M}$ as the composition of the N functors φ_{x_i} (if $i \in I$) and ψ_{x_i} (if $i \notin I$) (the order actually doesn't matter). There are various maps between these induced by can and var : for $i \notin I$, can_i goes from $\Psi_{I \cup \{i\}} \mathcal{M}$ to $\Psi_I \mathcal{M}$, and var_i goes in the opposite direction. The important fact is that the filtered $\mathcal{D}_{\overline{X}}$ -module \mathcal{M} can be entirely reconstructed from this data ([GGM85], see also [Ara10, § 4.5], [Sai89]). In our case of unipotent monodromy, all $\Psi_I \mathcal{M}$ are the vector space $\overline{\mathbb{V}}_x$, the canonical fiber at $x = 0$ (if the monodromy is not unipotent, then there are other vector spaces corresponding to the different eigenvalues of the monodromy). Each of it is equipped with the weight filtration W_\bullet and the nilpotent endomorphisms N_i commuting between them and compatible with W_\bullet .

Then what we have to do is to replace W_\bullet on $\Psi_I \mathcal{M}$ by another filtration $W_{I,\bullet}$. If $I = \{i\}$ then $W_{I,k}$ is given, as in the case of curves, by the filtration $Z_i(W)_k := N_i(W_k) + M_{k-1} \cap W_{k-1}$ where M is the relative monodromy filtration for N_i . In

general we iterate this process: if $I = \{i_1, \dots, i_r\}$ then

$$(7.4) \quad W_{I,k} := Z_{i_1}(Z_{i_2}(\dots Z_{i_r}(W)))_k.$$

Again the order of application of Z does not matter.

The diagram of filtered vector spaces and maps between them that we get, and use to re-construct \mathcal{M} , gives us the filtered $\mathcal{D}_{\overline{X}}$ -module underlying the mixed Hodge module extending the VMHS \mathbb{V} to \overline{X} .

Proposition 7.3. *In the above MHC, the part $(\Omega_{\overline{X}}^\bullet(\log D, \overline{\mathbb{V}}), \mathfrak{W}_\bullet, F^\bullet)$ is lax symmetric monoidal in \mathbb{V} , giving a \mathbb{C} -mixed Hodge complex which is lax symmetric.*

Again the difficult step is to prove the lax monoidal condition. The symmetry will be obvious since we drop the rational part.

Proof. With the previous description, we see that this is a generalization of the case of curves. Given the two VMHS on X $\mathbb{V}_1, \mathbb{V}_2$, under the hypothesis of the previous theorem, and extended to mixed Hodge modules M_1, M_2 on \overline{X} with underlying $\mathcal{D}_{\overline{X}}$ -module $\mathcal{M}_1, \mathcal{M}_2$, the weight filtration is described as above with the vector spaces $\Psi_I \mathcal{M}_1, \Psi_I \mathcal{M}_2$. The tensor product VMHS $\mathbb{V} = \mathbb{V}_1 \otimes \mathbb{V}_2$ with its tensor product weight filtration W_\bullet is extended to a mixed Hodge module M with underlying $\mathcal{D}_{\overline{X}}$ -module \mathcal{M} which is exactly $\mathcal{M}_1 \otimes \mathcal{M}_2$ when forgetting the filtrations.

The important fact is that in our previous description of $\mathcal{D}_{\overline{X}}$ -modules near a point of normal crossings, then under tensor product, $\Psi_I \mathcal{M}_1$ and $\Psi_J \mathcal{M}_2$ get paired to $\Psi_{I \cup J} \mathcal{M}$ if $I \cap J = \emptyset$ or 0 else. This is because each Ψ_I corresponds to applying $\text{Gr}_{-1}^{V(i)}$ for $i \in I$ and $\text{Gr}_0^{V(i)}$ for $i \notin I$.

So the compatibility to check between the tensor products of filtrations involve similar computations as in the proof of Proposition 6.6 (the case of curves), that is,

$$(7.5) \quad W_k^1 \otimes W_\ell^2 \subset W_{k+\ell}$$

and

$$(7.6) \quad W_k^1 \otimes Z_i(W_\ell^2) \subset Z_i(W)_{k+\ell}$$

applied several times □

We claim that this is enough for our needs and with a \mathbb{C} -MHC we can construct a \mathbb{C} -MHS on $\widehat{\mathcal{O}}_\rho$ ([Lef19, Proposition 10.6]).

8. APPLICATION TO THE DEFORMATION THEORY OF REPRESENTATIONS OF FUNDAMENTAL GROUPS

In this final section we use the constructions of mixed Hodge diagrams of Lie algebras (i.e. lax symmetric mixed Hodge complexes computing the cohomology with coefficients in a local system of Lie algebras) to describe the formal deformation theory of representation of the fundamental group of a variety X . Thus the result presented here extend and improve directly the results of [Lef19].

8.1. Representations of fundamental groups. Let X be a complex variety, compact or not: in all cases the fundamental group $\pi_1(X, x)$ at some fixed base-point is finitely presentable. Let G be a linear algebraic group defined over the field \mathbf{k} ($\mathbf{k} \subset \mathbb{R}$ or $\mathbf{k} = \mathbb{C}$), with Lie algebra \mathfrak{g} . The representations

$$(8.1) \quad \rho : \pi_1(X, x) \rightarrow G(\mathbf{k})$$

are parametrized by the \mathbf{k} -points of an affine scheme of finite type $\text{Hom}(\pi_1(X, x), G)$ (see [LM85]). Fixing such a representation ρ , let $\widehat{\mathcal{O}}_\rho$ be the completion of the local ring of $\text{Hom}(\pi_1(X, x), G)$ at its \mathbf{k} -point ρ . It has an associated deformation functor Def_ρ , which is a functor from the category of local Artin \mathbf{k} -algebras with residue field \mathbf{k} (simply denoted by $\mathbf{Art}_{\mathbf{k}}$) to the category of sets given by

$$(8.2) \quad \text{Def}_\rho : A \longmapsto \{\tilde{\rho} : \pi_1(X, x) \rightarrow G(A) \mid \tilde{\rho} = \rho \text{ over } G(\mathbf{k})\} = \text{Hom}(\widehat{\mathcal{O}}_\rho, A).$$

Over \mathbb{R} or \mathbb{C} , the representation ρ defines a flat principal G -bundle P over X . The associated bundle

$$(8.3) \quad \text{ad}(\rho) := \text{ad}(P) = P \times_G \mathfrak{g}$$

where G acts on \mathfrak{g} via Ad is then a flat vector bundle with a Lie bracket. So the algebra L of \mathcal{C}^∞ differential forms over X with values in $\text{ad}(\rho)$ has the structure of a differential graded Lie algebra: the bracket combines the wedge product of differential forms and the Lie bracket in \mathfrak{g} , and the differential acts only on differential forms since $\text{ad}(\rho)$ is flat.

To any such DG Lie algebra (over any field \mathbf{k} of characteristic zero) is associated a deformation functor Def_L . For $A \in \mathbf{Art}_{\mathbf{k}}$ with its unique maximal ideal \mathfrak{m}_A , $L^0 \otimes \mathfrak{m}_A$ is a *nilpotent* Lie algebra, thus has a group structure denoted by

$$(8.4) \quad (\exp(L^0 \otimes \mathfrak{m}_A), *)$$

given by the Baker-Campbell-Hausdorff formula, and acts on $L^1 \otimes \mathfrak{m}_A$ by *gauge transformations*. Then Def_L is given by

$$(8.5) \quad \text{Def}_L(A, \mathfrak{m}_A) := \left\{ x \in L^1 \otimes \mathfrak{m}_A \mid 0 = d(x) + \frac{1}{2}[x, x] \in L^2 \otimes \mathfrak{m}_A \right\} / \exp(L^0 \otimes \mathfrak{m}_A).$$

Also, in our case L is equipped with an augmentation $\varepsilon_x : L \rightarrow \mathfrak{g}$ which evaluates differential forms at x . To this is associated an *augmented deformation functor* $\text{Def}_{L, \varepsilon}$ which is a small variation of Def_L , explicitly written by Eyssidieux-Simpson [ES11, § 2.1.1]:

$$(8.6) \quad \text{Def}_{L, \varepsilon_x}(A, \mathfrak{m}_A) := \left\{ (x, e^\alpha) \in (L^1 \otimes \mathfrak{m}_A) \times \exp(\mathfrak{g} \otimes \mathfrak{m}_A) \mid 0 = d(x) + \frac{1}{2}[x, x] \right\} / \exp(L^0 \otimes \mathfrak{m}_A)$$

where, for $e^\lambda \in \exp(L^0 \otimes \mathfrak{m}_A)$,

$$(8.7) \quad e^\lambda \cdot (x, e^\alpha) := (e^\lambda \cdot x, e^\alpha * e^{-\varepsilon_x(\lambda)}).$$

See [Man04] for many more details on formal deformation theory and DG Lie algebras.

Part of the main theorem of Goldman-Millson can be stated as follows:

Theorem 8.1 (Goldman-Millson [GM88]). *For any complex manifold X and representation ρ of $\pi_1(X, x)$ into a linear algebraic group G , and for L the DG Lie algebra of \mathcal{C}^∞ differential forms with values in the flat bundle $\text{ad}(\rho)$, there is an isomorphism of deformation functors*

$$(8.8) \quad \text{Def}_\rho \simeq \text{Def}_{L, \varepsilon_x}.$$

Furthermore, a quasi-isomorphism of DG Lie algebras $L \rightarrow M$ (resp. a quasi-isomorphism augmented over \mathfrak{g}) induces an isomorphism of deformation functors $\text{Def}_L \rightarrow \text{Def}_M$ (resp. $\text{Def}_{L, \varepsilon_x} \rightarrow \text{Def}_{M, \varepsilon_x}$).

In other words $\widehat{\mathcal{O}}_\rho$ can be computed from the data of L above \mathfrak{g} up to quasi-isomorphism. In our previous work, we found canonical formulas compatible with the notion of mixed Hodge complex to construct a functorial MHS on $\widehat{\mathcal{O}}_\rho$ in situations where L has a structure of mixed Hodge diagram of Lie algebras. Implicitly, we consider $\widehat{\mathcal{O}}_\rho$ to be a projective limit

$$(8.9) \quad \widehat{\mathcal{O}}_\rho = \varprojlim \widehat{\mathcal{O}}_\rho / \mathfrak{m}^n$$

of its quotients by powers of its maximal ideal, that are *finite-dimensional*, hence a MHS on $\widehat{\mathcal{O}}_\rho$ means a projective limit of MHS, that are also compatible with its structure of algebra.

Theorem 8.2 ([Lef19]). *When ρ is the monodromy of a representation of $\pi_1(X, x)$ for which one can construct a mixed Hodge diagram of Lie algebras L over \mathbf{k} ($\mathbf{k} \subset \mathbb{R}$ or $\mathbf{k} = \mathbb{C}$), computing the cohomology of X with local coefficients in $\text{ad}(\rho)$, with its augmentation $\varepsilon_x : L \rightarrow \mathfrak{g}$ at x , then one can construct a MHS on $\widehat{\mathcal{O}}_\rho$ over \mathbf{k} , functorial in the data of L, ε_x up to quasi-isomorphism.*

Combining this with the constructions through § 4–7, we deduce:

Theorem 8.3. *Let ρ be the monodromy representation of a graded-polarizable VMHS \mathbb{V} over X , admissible in $X \subset \overline{X}$ and with unipotent monodromy at infinity. Then there is a MHS on $\widehat{\mathcal{O}}_\rho$, functorial in X, x, ρ .*

If \mathbb{V} is defined over the field \mathbf{k} ($\mathbf{k} \subset \mathbb{R}$ or $\mathbf{k} = \mathbb{C}$) and either: X is compact, or X is one-dimensional, or \mathbb{V} is a pure VHS with trivial monodromy at infinity. Then this MHS is defined over \mathbf{k} . Else, \mathbb{V} has to be defined over \mathbb{Q} and the MHS we get is defined only over \mathbb{C} .

Proof. First, if ρ is the monodromy of a VMHS \mathbb{V} over X , then $\text{ad}(\rho) \subset \text{End}(\mathbb{V})$ is again the monodromy of a VMHS by linear algebra. It is graded-polarizable if ρ is, and it is again admissible if ρ is ([SZ85, Appendix]), and of course it is defined over the same base field.

Thus by the constructions of this article, given a compactification $X \subset \overline{X}$ into a compact Kähler manifold, we get in all the listed cases first a MHC of sheaves, then a MHC L by applying $R_{\text{TW}}(\overline{X}, -)$, that computes the cohomology of X with local coefficients in $\text{ad}(\rho)$ and that is lax symmetric monoidal in $\text{ad}(\rho)$. It is defined over \mathbf{k} for the first listed cases when we can do this without mixed Hodge modules, else only the part over \mathbb{C} is proven to be lax symmetric monoidal. Then L has an induced structure of Lie algebra coming from the Lie bracket on $\text{ad}(\rho)$. This is the mixed Hodge diagram required for Theorem 8.2.

To apply our method [Lef19, § 8-9] we also need the augmentation ε_x to be a morphism of mixed Hodge diagrams, when \mathfrak{g} is considered as a mixed Hodge diagram concentrated in degree zero. This is obvious in all these cases.

Furthermore the mixed Hodge diagram L will be functorial in the data of X, \overline{X}, x, ρ for the same reasons as in the classical constructions (i.e. without the Thom-Whitney functors). When working with quasi-projective varieties, it is independent of \overline{X} up to quasi-isomorphism (see for example the proof of [Lef19, Theorem 11.5])

which is enough for the final MHS on $\widehat{\mathcal{O}}_\rho$ to be functorial in X, x, ρ only. For non-quasi-projective varieties, the same argument applies when one fixes an equivalence class of compactifications of X . \square

Remark 8.4. If the monodromy ρ of \mathbb{V} is only quasi-unipotent at infinity, then there is a finite cover $\pi : Y \rightarrow X$ to which the pull-back $\pi^*\rho$ is unipotent. So one can construct the mixed Hodge diagram of Lie algebras in an equivariant way at the level of Y , with the same ideas as in [Lef19, § 11] (the case of finite image): at least if X is quasi-projective, then the cover π extends to a ramified cover $\overline{Y} \rightarrow \overline{X}$ and \overline{Y} is projective, furthermore it is possible to construct such a \overline{Y} smooth and equivariant. Hence if M is the mixed Hodge diagram constructed for $(Y, \overline{Y}, \pi^*\rho)$ then taking the invariants under the finite group of the cover gives the mixed Hodge diagram for (X, \overline{X}, ρ) . It is also possible to lift the augmentation ε_x to M , using the augmentations on Y at all points in $\pi^*(x)$. It is likely that this arguments also holds if \overline{X} is only compact Kähler, but not so obvious and we lack of references.

8.2. Deformation functor of DG Lie and L_∞ algebras. We are going to describe the consequence of L being a mixed Hodge diagram of Lie algebras, with restrictions on the possible weights on cohomology, to its deformation functor.

To have a better understanding of the deformation functor of DG Lie algebras, we will work with the much more powerful L_∞ algebras. Briefly, a L_∞ algebra is given by a graded vector space L with anti-symmetric operations in r variables of degree $r - 2$

$$(8.10) \quad \ell_r : L^{\otimes r} \longrightarrow r, \quad r \geq 1$$

satisfying an infinite list of axioms. Among these, ℓ_1 is a differential d on L and ℓ_2 behaves like a Lie bracket, for which d is a derivation, except that it satisfies the Jacobi identity only *up to homotopy* given by ℓ_3 , and ℓ_3 itself satisfies higher order relations. L_∞ algebras enjoy the following very nice properties:

- (1) L_∞ algebras with $\ell_r = 0$ for all $r \geq 3$ are the same as DG Lie algebras.
- (2) If L is a DG Lie algebra then its cohomology $H(L)$ comes equipped with a structure of L_∞ algebra such that L and $H(L)$ become quasi-isomorphic in the sense of L_∞ algebras. This is called a *homotopy transfer of structure* from L to $H(L)$.
- (3) As a consequence, a DG Lie algebra L is *formal* (i.e. quasi-isomorphic to its cohomology) if and only if there exists a homotopy transfer of structure to $H(L)$ with $\ell_r = 0$ for all $r \neq 2$ (ℓ_1 is always zero on $H(L)$ since it is induced by the differential).

See the lectures of Manetti [Man04] and the book of Loday-Valette [LV12] for much more motivation for L_∞ algebras.

Such a L also has a deformation functor on \mathbf{Art}_k which is given by

$$(8.11) \quad \text{Def}_L : (A, \mathfrak{m}_A) \longmapsto \left\{ x \in L^1 \otimes \mathfrak{m}_A \mid 0 = \sum_{r \geq 1} \frac{\ell_r(x, \dots, x)}{r!} \in L^2 \otimes \mathfrak{m}_A \right\} / \sim$$

where \sim is a certain notion of homotopy equivalence. Again it is invariant under quasi-isomorphism. We see that if $\ell_r = 0$ for $r \geq 3$ we recover the previous deformation functor.

8.3. Homotopy transfer of structure. We need to recall briefly how the homotopy transfer of structure works.

Let L be a L_∞ algebra with operations denoted by μ_r . This includes the case of DG Lie algebras with $\mu_r = 0$ for $r > 2$. What we need to choose is a decomposition in each degree

$$(8.12) \quad L^n := A^n \oplus K^n \oplus B^n$$

where A^n is a complement to $\text{Ker}(d^n) \subset L^n$ and B^n is a complement to $\text{Im}(d^{n-1}) \subset \text{Ker}(d^n)$. Then K^n forms a space of representatives for the cohomology $H^n(L)$. This also determines maps of complexes $i : K^\bullet \rightarrow L^\bullet$ (inclusion), $p : L^\bullet \rightarrow K^\bullet$ (projection), and $h : L^\bullet \rightarrow L^{\bullet-1}$ (homotopy) where h is given by the inverse of the isomorphism $A^{\bullet-1} \rightarrow B^\bullet$ induced by d , extended by 0 to $A \oplus K$. These maps satisfy $pi = \text{id}_K$ and $\text{id}_L - ip = dh - hd$, so L deformation retracts onto K .

Once such a choice is made, it determines operations ℓ_r on $K \simeq H(L)$ with $\ell_1 = 0$ and such that L and $H(L)$ become quasi-isomorphic in the sense of L_∞ algebras via i and p .

One way to describe ℓ_r is via the set RT_r of rooted trees with r leaves, that is, trees that when written vertically, have r leaves thought of as input data and one leave as output data, with internal vertices also presented vertically with at least two input edges and one output edge. Such a tree T corresponds to a composition scheme for a sequence (μ_n) of operations in n variables, by plugging one operation μ_n in an internal vertex with n inputs. One can also label the edges by linear maps. The formula for ℓ_r (see [LV12, § 10.3.4]) is then the sum over all trees $T \in RT_r$ of the following operations in r variables: apply the maps i on the input leaves of T , maps μ_n on the internal vertices with n inputs, maps h on the internal edges, and a map p on the last output edge.

8.4. Higher operations and weights. Let us explain how powerful this point of view is. Because of the homotopy transfer of structure and the invariance by quasi-isomorphism of the deformation functor, then the deformation functor of any DG Lie algebra L can be written directly in $H(L)$. We will work in cases where L has $H^n(L) = 0$ for $n \leq 0$, so the homotopy relation in the formula for $\text{Def}_{H(L)}$ is trivial, and the other terms $H^n(L)$ are finite-dimensional. Hence the formula for $\text{Def}_{H(L)}$ gives us directly a complete local algebra pro-representing Def_L : it is given as the quotient of the power series on $H^1(L)$ by the equations $0 = \sum_{r \geq 2} \ell_r(x, \dots, x)/r!$ seen as power series with values in $H^2(L)$.

If furthermore one can show that only finitely many of the operations $\ell_r : H^1(L)^{\otimes r} \rightarrow H^2(L)$ are non-vanishing, then the formula for $\text{Def}_{H(L)}$ gives a finite presentation of this complete local algebra: it is the completion of the local ring of the germ at 0 inside $H^1(L)$ defined by a finite number of polynomial equations. For brevity we will simply say that the deformation functor is given by a finite number of polynomial equation.

Now we will use this combined with the existence of weights on $H(L)$.

Theorem 8.5. *Let L be a mixed Hodge diagram of Lie algebras. Then on $H(L)$ there is an extra grading $H_\bullet(L)$ over \mathbf{k} that splits the weight filtration, i.e.*

$$(8.13) \quad W_k H^n(L) = \bigoplus_{i \leq k} H_i^n(L),$$

and there are induced higher operations ℓ_r that all respect this grading, i.e.

$$(8.14) \quad \ell_r(H_{k_1}^{n_1}(L), \dots, H_{k_r}^{n_r}(L)) \subset H_{k_1+\dots+k_r}^{n_1+\dots+n_r-2}(L).$$

Proof. The first essential step is to use the theorem of Cirici-Horel [CH17, Theorem 7.8, Theorem 8.2] which shows that (L, W) , over \mathbf{k} , is quasi-isomorphic to $\mathcal{M} := E_1^W(L)$ (the first page of the W -spectral sequence). Since their theorem is formulated as a formality result for mixed Hodge complexes as a symmetric monoidal category, it holds as well for our DG Lie algebras. Since the W -spectral sequence degenerates at E_2 , $H(\mathcal{M}) = H(L)$, and this \mathcal{M} already comes equipped with a weight grading \mathcal{M}_\bullet that on cohomology splits the weight filtration of $H(L)$.

The second essential step is to show that the theorem on homotopy transfer of structure holds with an extra grading, see [BR18, Corollary 5.6] for the case of commutative DG algebras. But it is clear that this also works for L_∞ algebra since the splitting of \mathcal{M} can be taken in a compatible way with its grading and then the maps constructed from the homotopy transfer of structure will respect this grading. So we get higher operations on $H(\mathcal{M})$ that respect the grading. \square

Remark 8.6. Such an argument is intended to be improved if one could show directly a theorem of homotopy transfer of structure for mixed Hodge diagrams (Joana Cirici, personal communication, work in progress). Ideally the operations ℓ_r would be morphisms of mixed Hodge structures, encoding faithfully at the level of $H(L)$ at the same time both structures of MHC and of DG Lie algebra of L .

Now this game of higher operations and weights allows us to show that many operations vanish automatically when they don't respect the weights, and this leads to a quite concrete description of Def_L in many cases.

Theorem 8.7. *Assume that L is a mixed Hodge diagram of Lie algebras with $H^n(L) = 0$ for $n \leq 0$ and such that the weights of $H^1(L)$ are all strictly positive. Then the complete local algebra that pro-represents Def_L has a finite presentation with weights on the generators being the weights of $H^1(L)$ and with weighted-homogeneous relations with weights those of $H^2(L)$.*

Proof. Because of the previous theorem, we compute the deformation functor in $H(L)$. The main point is that if $x_1, \dots, x_r \in H^1(L)$ have weight $k_1, \dots, k_r > 0$ then $\ell_r(x_1, \dots, x_r) \in H^2(L)$ has weight $k_1 + \dots + k_r \geq r$. But $H^2(L)$, being finite-dimensional, has only finitely many weights. Hence for some N big enough all the operations $\ell_r : H^1(L)^{\otimes r} \rightarrow H^2(L)$ vanish for $r > N$. The equations giving the deformation functor

$$(8.15) \quad \frac{\ell_2(x, x)}{2} + \frac{\ell_3(x, x, x)}{3!} + \dots + \frac{\ell_N(x, \dots, x)}{N!} = 0$$

are really a finite number of polynomial equations with weights on the variable x in $H^1(L)$ and the relations take values in $H^2(L)$ respecting those weights. \square

8.5. The augmentation. To apply the above method with the isomorphism of functors of Goldman-Millson, we need to work with the augmented deformation functor $\text{Def}_{L, \varepsilon}$. In [Lef19, § 5] we argue that it is actually the deformation functor of a canonical L_∞ algebra structure on the (desuspended) mapping cone C of ε that is constructed and studied by Fiorenza-Manetti ([FM07]). We will denote by μ_r ($r \geq 1$) these operations.

We write these formula directly in the form we need: C has components

$$(8.16) \quad C^1 = L^1 \oplus \mathfrak{g}, \quad C^n = L^n \quad (n \neq 1).$$

The differential μ_1 is induced by the one of L with the exception of $d^1 : C^0 \rightarrow C^1$ given by

$$(8.17) \quad d^1(x) := (d(x), \varepsilon(x)), \quad x \in L^0.$$

The operation μ_2 is induced in a natural way by the Lie bracket, the only part needed to be described being $C^0 \otimes C^1 \rightarrow C^1$ given by

$$(8.18) \quad \mu_2(x, (y, v)) := \left([x, y], \frac{1}{2} [\varepsilon(x), v] \right), \quad x \in L^0, y \in L^1, v \in \mathfrak{g}.$$

Furthermore, there is one operation μ_r for $r \geq 3$ as follows. Applied to r elements of \mathfrak{g} and $k > 1$ elements of L , μ_{r+k} is 0. Else, applied to r elements u_1, \dots, u_r of \mathfrak{g} and exactly one element x of L it is given by a formula of the type

$$(8.19) \quad \mu_{r+1}(u_1, \dots, u_r, x) = B_r \sum_{\tau} \pm [u_{\tau(1)}, [u_{\tau(2)}, \dots, [u_{\tau(r)}, \varepsilon(x)] \dots]]$$

where the sum is over the symmetric group, with a constant B_r and with signs in the sum. Since this has values in \mathfrak{g} , μ_{r+1} is a non-trivial higher operation $(C^1)^{\otimes r} \otimes C^0 \rightarrow C^1$ for all $r \geq 2$.

Then:

Lemma 8.8 ([Lef19, 5.3]). *For this L_∞ algebra structure on C ,*

$$(8.20) \quad \text{Def}_{L, \varepsilon} := \text{Def}_C$$

We are in situations where ε is surjective and the induced ε on $H^0(L)$ is injective. Hence $H^0(C) = 0$. Let us recall also that there is a long exact sequence for the mapping cone, which reduces here to the short exact sequence

$$(8.21) \quad 0 \longrightarrow \mathfrak{g}/\varepsilon(H^0(L)) \longrightarrow H^1(C) \longrightarrow H^1(L) \longrightarrow 0$$

and of course $H^n(C) = H^n(L)$ for $n \neq 0, 1$. If L is an augmented mixed Hodge diagram of Lie algebras, then the above sequence is a short exact sequence of MHS.

With all of this, we are ready to state our theorem, first without Hodge theory.

Theorem 8.9. *Let $\varepsilon : L \rightarrow \mathfrak{g}$ be an augmented DG Lie algebra, let C be the cone as above with its structure of L_∞ algebra, assuming that ε is surjective on L^0 and that the induced ε on $H^0(L)$ is injective. Then there exist a transferred structure of L_∞ algebra on $H(C)$ given by operations ℓ'_r , and a transferred structure of L_∞ algebra on $H(L)$ given by operations ℓ_r , such that the deformation functor of $H(C)$ can be written as a product*

$$(8.22) \quad \text{Def}_{H(C)} = \text{Def}_{H(L)} \times (\mathfrak{g}/\varepsilon(H^0(L))),$$

in other words for $(A, \mathfrak{m}_A) \in \mathbf{Art}_{\mathbf{k}}$

$$(8.23) \quad \text{Def}_{H(C)}(A) = \left\{ (x, t) \in (H^1(L) \oplus (\mathfrak{g}/\varepsilon(H^0(L)))) \otimes \mathfrak{m}_A \mid 0 = \sum_{r \geq 2} \frac{\ell_r(x, \dots, x)}{r!} \right\}.$$

Proof. For this we prepare a homotopy transfer of structure for L , so we choose a decomposition

$$(8.24) \quad L^n := A^n \oplus K^n \oplus B^n$$

where K forms a space of representatives for the cohomology. This determines again the maps i, p, h and higher operations ℓ_r on $K \simeq H(L)$.

Then we choose such a decomposition for C simply by choosing a subspace $\mathfrak{t} \subset \mathfrak{g}$ complement to $\varepsilon(H^0(L))$. This gives a splitting of C with

$$(8.25) \quad C^1 := A^1 \oplus (K^1 \oplus \mathfrak{t}) \oplus (B^1 \oplus \varepsilon(H^0(L))).$$

and determines again maps i', p', h' (closely related to i, p, h) and higher operations ℓ'_r in $H(C)$ with

$$(8.26) \quad H^1(C) \simeq K^1 \oplus \mathfrak{t} \simeq H^1(L) \oplus (\mathfrak{g}/\varepsilon(H^0(L))).$$

What we then want to show is that the only non-zero operations $\ell'_r : H^1(C)^{\otimes r} \rightarrow H^2(C) = H^2(L)$ are the one induced from ℓ_r , i.e. that

$$(8.27) \quad \text{for } y = (x, t) \in K^1 \oplus \mathfrak{t}, \quad \ell'_r(y, \dots, y) = \ell_r(x, \dots, x).$$

This gives directly the above form of the deformation functor. Such a fact is already clear for $r = 1$ (ℓ'_1 is always zero on cohomology) and for $r = 2$ since ℓ'_2 is induced by μ_2 on cohomology and μ_2 satisfies this.

For higher r we have to understand in more detail what happens in the homotopy transfer of structure for C . We take r elements $y_j = x_j + t_j \in K^1 \oplus \mathfrak{t}$ and want to compute $\ell'(y_1, \dots, y_r)$. Let T be a planar rooted tree as in section 8.3 with r leaves. We first apply a map i' to some n elements y_{j_1}, \dots, y_{j_n} then apply μ_n . If $n = 2$ then what happens is as before, $\mu_2(i(y_{j_1}), i(y_{j_2}))$ equals $\mu_2(i(x_{j_1}), i(x_{j_2}))$ by definition of μ_2 . If $n > 2$ then $\mu_n(i(y_{j_1}), \dots, i(y_{j_n}))$ is zero simply because there is no such higher operation on C (the elements y_{j_1}, \dots, y_{j_n} are all of degree 1). Hence we see that the formula for computing $\ell'(y_1, \dots, y_r)$ is the same as the one for computing $\ell(y_1, \dots, y_r)$. \square

Remark 8.10. If L is formal above \mathfrak{g} , then the above computations show that Def_L reduces to the product of the equation $[x, x] = 0$ in $H^1(L)$ and the vector space $\mathfrak{g}/\varepsilon(H^0(L))$. Thus we recover completely the result of Goldman-Millson [GM88, Theorem 3.5], purely by methods of L_∞ algebras and without invoking their operation \bowtie (kind of homotopy fiber product of groupoids [ES11, § 2.1.1] used to define $\text{Def}_{L,\varepsilon}$ from Def_L).

In geometric situations, this is what we get.

Corollary 8.11. *Let $\varepsilon : L \rightarrow \mathfrak{g}$ be a mixed Hodge diagram of L_∞ algebras, with $H^n(L) = 0$ for $n < 0$, ε surjective on L^0 and injective on $H^0(L)$. Assume that all the weights of $H^1(L)$ are strictly positive. Then again $\text{Def}_{L,\varepsilon}$ has a finite presentation, which is the product of the one of Theorem 8.7 with the vector space $\mathfrak{g}/\varepsilon(H^0(L))$.*

Proof. First we apply, as in the proof of Theorem 8.5, the theorem of Cirici-Horel to the map $\varepsilon : L \rightarrow \mathfrak{g}$ over \mathbf{k} . This map is quasi-isomorphic to an augmented DG Lie algebra $\tau : \mathcal{M} \rightarrow \mathfrak{h}$ with an extra grading (on \mathcal{M} and on \mathfrak{h} , respected by τ) splitting the weight filtration at the level of cohomology. Hence \mathfrak{h} is just a splitting of the weight filtration of \mathfrak{g} . This also defines a weight grading of the cone \mathcal{C} of τ ,

which is quasi-isomorphic to C , compatible with the operations μ_r of the cone, and splitting the weight filtration on $H(C)$.

Then we combine the method of Theorem 8.5 with the previous theorem: we choose a splitting for \mathcal{C} by combining a splitting for \mathcal{M} and a splitting for $\tau(H^0(\mathcal{M})) \subset \mathfrak{h}$, both in a compatible way with the grading. The homotopy transfer of structure with an extra grading then gives us operations ℓ'_r on $H(C)$ respecting the weight grading, and at the level of $H^1(C)^{\otimes r} \rightarrow H^2(L)$ these coincide with the operations ℓ_r of Theorem 8.7. \square

8.6. Consequences. Finally, we come back to the study of the complete local ring $\widehat{\mathcal{O}}_\rho$ of the representation variety of $\pi_1(X, x)$ at a representation ρ for various cases for X and ρ where one can construct the augmented mixed Hodge diagram of Lie algebras L . The previous description of $\text{Def}_{H(C)} = \text{Def}_{L, \varepsilon}$ gives us directly a description of $\widehat{\mathcal{O}}_\rho$ with its weight grading at least in the cases where there are only a finite number of non-vanishing higher operations ℓ_r on $H(L)$.

Recall that when ρ is the monodromy representation of a pure VHS \mathbb{V} then $\text{ad}(\rho) \subset \text{End}(\mathbb{V})$ is a VHS of weight zero, hence in all our cases $H^i(X, \text{ad}(\rho))$ has weights greater or equal to i . This can also be seen at a higher level: the derived category of mixed Hodge modules admits a six-functors formalism and a notion of weights analogous to the one of ℓ -adic sheaves in [Del80] such that for a map f , $R^i f_*$ increases the weights by i .

- (1) If X is compact Kähler and ρ is the monodromy of a polarized VHS over X , then $H^1(L)$ is pure of weight 1 and $H^2(L)$ is pure of weight 2. We already know, or we recover, that L is formal and that there are no operations ℓ_r for $r \geq 3$ and ℓ_2 is the Lie bracket induced on cohomology. We recover well again the result of Goldman-Millson: $\widehat{\mathcal{O}}_\rho$ is given by the product of the equation $[x, x] = 0$ in $H^1(L)$ (quadratic) with the vector space $\mathfrak{g}/\varepsilon(H^0(L))$.
- (2) If X is quasi-projective and ρ has finite image then the weights of L are directly related to the weights of the finite cover $Y \rightarrow X$ over which ρ is trivial. Hence $H^1(L)$ has weights only 1, 2 and $H^2(L)$ has weights only 2, 3, 4. The non-vanishing operations ℓ_r exist only for $r = 2, 3, 4$: only $\ell_2(x, x)$ for x of weight 1, 2 and $\ell_3(x, x, x)$, $\ell_4(x, x, x, x)$ for x of weight 1 produce weights allowed in $H^2(L)$. Furthermore, since ρ has finite image, $\mathfrak{g}/\varepsilon(H^0(L))$ vanishes. Thus we recover completely the result of Kapovich-Millson: $\widehat{\mathcal{O}}_\rho$ has a presentation with generators of weight 1, 2 and relations of weight 2, 3, 4.
- (3) In the above case, assume that $H^1(Y)$ is pure of weight 2. Then $H^1(L)$ is also pure of weight 2. And $H^2(L)$ is again limited to weights 2, 3, 4. Thus the only possible non-zero operation is $\ell_2(x, x)$ for x of weight 2. So we recover the main result of [Lef17]: in this case $\widehat{\mathcal{O}}_\rho$ is quadratic. And we recover some form of *purity implies formality* (see [CH17]): the purity of weights implies some partial formality of L hence it behaves as in the compact case.
- (4) If $X \subset \overline{X}$ (with \overline{X} compact Kähler) and ρ is the monodromy of a polarized VHS over X extendable to \overline{X} , then $H^1(L)$ has again weights 1, 2 and $H^2(L)$ has weights 2, 3, 4. Thus we recover the same result as with finite images except that in this case we also have a non-zero part $\mathfrak{g}/\varepsilon(H^0(L))$.

- (5) If X is a curve and ρ comes from a polarized VHS, then the lowest possible weight of $H^1(L)$ is 1 and the lowest possible weight of $H^2(L)$ is again 2. The higher weights are not so easy to describe, depending on the behavior of the VHS near the singularities, and are not bounded without further hypothesis on the monodromy. This is however enough to conclude that there is a finite weighted-homogeneous presentation for $\widehat{\mathcal{O}}_\rho$.
- (6) In the most general case of $X \subset \overline{X}$ (with \overline{X} compact Kähler) and ρ coming from a pure polarizable VHS over X then again we know that $H^1(L)$ has lowest possible weight 1 and we conclude as for curves, and again we cannot bound the weights in this generality.

If \mathbb{V} is mixed then a priori we cannot ensure that $H^1(L)$ has weights greater or equal to zero without further hypothesis, and we cannot prove this way that there are only finitely many equations, but our methods still allows us to write down weighted-homogeneous equations for $\widehat{\mathcal{O}}_\rho$.

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