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# Causality and network graph in general bilinear state-space representations

Mónika Józsa, Mihály Petreczky and M. Kanat Camlibel

**Abstract**—This paper proposes an extension of the well-known concept of Granger causality, called GB–Granger causality. GB–Granger causality is designed to relate the internal structure of bilinear state-space systems and statistical properties of their output processes. That is, if such a system generates two processes, where one does not GB–Granger cause the other, then it can be interpreted as the interconnection of two subsystems: one that sends information to the other, which does not send information back. This result is an extension of earlier obtained results [1] on the relationship between Granger-causality and the internal structure of linear time-invariant state-space representations.

## I. INTRODUCTION

Detecting interactions among stochastic processes and relating them to the internal structure of the generating systems can be of interest for several applications such as mapping interactions in the brain, predicting economical price movements or understanding social group behaviour. The first step towards detecting such interactions is to propose a formal mathematical definition of the concept of interaction. In this paper, we propose two formalizations of one directional interactions between two stochastic processes. The stochastic processes are assumed to be outputs of a non-linear dynamical system. Both formalizations will try to capture causal interactions, i.e., that one process causes the other one. The first formalization concentrates on the information flow between the dynamical systems that generate the processes. The second one focuses on statistical properties of the processes.

More precisely, let  $\mathbf{y}$  be an output process that is partitioned into two components such as  $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$ . For the first approach, assume that  $\mathbf{y}$  is the output of a dynamical system. Assume that this dynamical system can be represented as an interconnection of two systems: one which generates  $\mathbf{y}_1$  as output, the second which generates  $\mathbf{y}_2$ . Furthermore, assume that the subsystem generating  $\mathbf{y}_1$  sends information to the other subsystem, but there is no informing flowing in the opposite direction. That is, the *network graph*<sup>1</sup> of this dy-

namical system has two nodes and one edge. Then, according to the first approach we say that  $\mathbf{y}_1$  influences  $\mathbf{y}_2$ . This approach offers an intuitive mechanistic explanation of how one component of the output process influences the other. However, the same output process can be generated by systems with different network graphs. As a result, the presence of an interaction between two output components depends on the exact dynamical system representing the output process.

The second approach is based on statistical properties of the joint process  $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$ . A widely used example of this approach is Granger causality [2]. Intuitively,  $\mathbf{y}_1$  Granger causes  $\mathbf{y}_2$  if the best linear predictions of  $\mathbf{y}_2$  based on the past values of  $\mathbf{y}$  are better than those only based on the past values of  $\mathbf{y}_2$ . We then say that  $\mathbf{y}_1$  influences  $\mathbf{y}_2$ , if  $\mathbf{y}_1$  Granger causes  $\mathbf{y}_2$ . Concepts that follow from this second approach lead to definitions that depend only on properties of  $\mathbf{y}$  and do not depend on which dynamical system we use to represent  $\mathbf{y}$ . However, they do not always offer an explanation of the mechanisms according to which the interaction takes place.

In summary, the first approach focuses on the mechanism inside a dynamical system but is too sensitive to the choice of the system itself. The second approach solves the issue with the first; however, it generally does not capture the inner mechanism of the interaction. It is thus of interest to relate these approaches to benefit from the advantages of both.

In [1]–[4], Granger causality was formally related to the network graphs of autoregressive (AR), moving-average (MA) and linear-time-invariant state-space (LTI–SS) models. These results show that Granger causality, despite being defined based on statistical properties of a process, can be related to structural properties of linear models of that process. In most of the fields, however, where Granger causality is applied (e.g. econometrics, neuroscience), linear models are insufficient to represent the observed process.

In this paper, we extend the result on the relation between Granger causality and linear models to an extension of Granger causality and a class of nonlinear models. That is, we define a new concept of causality that can describe interaction between processes that relate to each other in a nonlinear way. Compared to other reformulations of Granger causality, see e.g. [5], our concept is designed to have a structural interpretation in the chosen class of nonlinear systems.

In order to achieve the objective of the paper, we will 1) focus on a specific class of nonlinear dynamical systems; 2) define a new concept of causality as interaction among the components of a process generated by a system chosen in 1) based on statistical properties of the process at hand; and 3) characterize the causality defined in 2) by properties of the inner structure of the system generating the process at hand.

M. Józsa is with IMT Lille Douai, Univ. Lille, Unit de Recherche Informatique Automatique, F-59000 Lille, France and Bernoulli Institute for Mathematics, Computer Science, and Artificial Intelligence, Univ. of Groningen, 9700 AK Groningen, Netherlands (e-mail: monika.jozsa@imt-lille-douai.fr)

M. Petreczky is with the Centre de Recherche en Informatique, Signal et Automatique de Lille (CRISTAL), Avenue Carl Gauss, 59650 Villeneuve-d’Ascq, France (e-mail: mihaly.petreczky@ec-lille.fr)

M. K. Camlibel is with the Johann Bernoulli Institute for Mathematics and Computer Science, and Artificial Intelligence, Univ. of Groningen, 9700 AK Groningen, Netherlands (e-mail: m.k.camlibel@rug.nl)

<sup>1</sup>Informally, by the network graph of a system, we mean a directed graph, whose nodes correspond to subsystems, such that each subsystem generates a component of the output process. Furthermore, there is an edge from one node to the other, if the subsystem corresponding to the source node sends information to the subsystem corresponding to the target node.

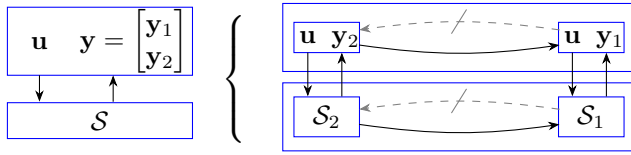


Fig. 1: Illustration of the results: Cascade interconnection structure in a GB-SS representation  $S$  with input  $\mathbf{u}$  and output  $\mathbf{y}$  decomposed into subsystems  $S_1$  and  $S_2$  in the presence of GB-Granger non-causality from  $y_1$  to  $y_2$  with respect to  $\mathbf{u}$ .

As a first step towards nonlinear systems, a natural choice is to study bilinear systems, which include e.g., LTI-SS, switched linear, autoregressive moving-average (ARMA), and jump Markov linear models. Bilinear systems produce richer phenomena than linear systems, yet many analytical tools for linear systems are suitable to analyze them. In this paper, we focus on general bilinear state-space (GB-SS) representations for which stochastic realization theory exists [6]. This theory serves as a basis for the technicalities of the paper.

To formalize causality for the outputs of GB-SS representations, we introduce an extension of Granger causality, called GB-Granger causality, that coincides with Granger causality, when applied to outputs of stochastic LTI-SS models.

In the main results, we consider a GB-SS representation with output process  $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$  and input process  $\mathbf{u}$ . Then, we show that GB-Granger non-causality from  $y_1$  to  $y_2$  with respect to  $\mathbf{u}$  is equivalent to the decomposition of the GB-SS representation into the interconnection of two subsystems, one generating  $y_1$ , and another one generating  $y_2$ , where the former sends no information to the latter (see Figure 1).

The results of this paper are based on realization theory of bilinear systems [6]–[11] and results on Granger causality in linear systems [2], [12]–[16]. We adopt the concept of GB-SS representation from [6] and rely on the realization theory presented there. The advantage of GB-SS representations in [6] is that, contrary to [9], [11], the input process is not necessarily white, which therefore includes, e.g. jump Markov linear systems. However, contrary to [7]–[9], it does not allow additive input terms in the system. **Note that our results depend on realization theory of GB-SS representations. Hence, in order to extend our results to GB-SS representations with additive inputs, realization theory of the latter system class has to be developed. This remains a topic of future research.**

Granger causality between stochastic processes was studied for AR, MA models [2], transfer functions [3], [4], [19], and for stochastic linear state-space representations [1], [12]–[15], [17]. For extending the concept of Granger causality in GB-SS representations, we rely on the ideas from [1], [13], [14], [16]. In contrast to [1], [13], [14], [16], which consider linear state-space representations, in this paper we consider GB-SS representations.

The structure of the paper is as follows: First, we introduce the terminology in Section II, which is followed by a brief summary on realization theory of GB-SS representations in Section III. Then, in Section IV, the main results on GB-Granger causality and GB-SS representations are presented.

Finally, the proofs of the results can be found in Appendix.

## II. PRELIMINARIES AND NOTATIONS

This section presents the terminology adopted from [6] that general bilinear state-space (GB-SS) representations rely on.

We consider discrete-time, square-integrable, multivariate, wide-sense stationary stochastic processes with real entries. Throughout the paper, we fix a probability space  $(\Omega, \mathcal{F}, P)$  and all the random variables and stochastic processes are understood with respect to  $(\Omega, \mathcal{F}, P)$ . The random variable of a process  $\mathbf{z}$  at time  $t$  is denoted by  $\mathbf{z}(t)$ , where  $t$  is from the discrete-time axis of integers  $\mathbb{Z}$ . Using standard notation, the expected value of a random variable  $\mathbf{z}(t)$  is written as  $E[\mathbf{z}(t)]$  and the covariance matrix between two random variables  $\mathbf{z}_1(t)$  and  $\mathbf{z}_2(t)$  is denoted by  $E[(\mathbf{z}_1(t) - E[\mathbf{z}_1(t)])(\mathbf{z}_2(t) - E[\mathbf{z}_2(t)])^T]$ . **Note that if the processes  $\mathbf{z}_1$  and  $\mathbf{z}_2$  are zero-mean, the latter simplifies to  $E[\mathbf{z}_1(t)\mathbf{z}_2^T(t)]$ .** The conditional expectation of a random variable  $\mathbf{z}$  to a  $\sigma$ -algebra  $\mathcal{F}$  is denoted by  $E[\mathbf{z}|\mathcal{F}]$ . When a process  $\mathbf{z}$  or a random variable  $\mathbf{z}(t)$  takes its values from  $\mathbb{R}^n$  then we write  $\mathbf{z} \in \mathbb{R}^n$  and  $\mathbf{z}(t) \in \mathbb{R}^n$ . Consider a process  $\mathbf{z}$  and a present time  $t \in \mathbb{Z}$ . The  $\sigma$ -algebras generated by the random variables in the present, past, and future of  $\mathbf{z}$  are denoted by  $\mathcal{F}_t^{\mathbf{z}} = \sigma(\mathbf{z}(t))$ ,  $\mathcal{F}_{t-}^{\mathbf{z}} = \sigma(\{\mathbf{z}(k)\}_{k=-\infty}^{t-1})$ , and  $\mathcal{F}_{t+}^{\mathbf{z}} = \sigma(\{\mathbf{z}(k)\}_{k=t}^{\infty})$  respectively, where for a set  $Z$  of random variables,  $\sigma(Z)$  denotes the smallest  $\sigma$ -algebra which contains each  $\sigma$ -algebra generated by an element of  $Z$ .

In the rest of this section, we will introduce tools that will help us to define GB-SS representations in Section III.

Throughout the paper, we denote the finite set  $\{1, 2, \dots, d\}$  by  $\Sigma$ , where  $d$  is a positive integer.

Let  $\Sigma^+$  be the set of finite sequences of elements of  $\Sigma$ , i.e., an element of  $\Sigma^+$  is a sequence of the form  $w = \sigma_1 \cdots \sigma_k$ , where  $\sigma_1, \dots, \sigma_k \in \Sigma$ . We define the concatenation operation on  $\Sigma^+$ : if  $w = \sigma_1 \cdots \sigma_k \in \Sigma^+$  and  $v = \hat{\sigma}_1 \cdots \hat{\sigma}_l \in \Sigma^+$ , then the concatenation of  $w$  and  $v$ , denoted by  $wv$ , is defined by  $wv = \sigma_1 \cdots \sigma_k \hat{\sigma}_1 \cdots \hat{\sigma}_l$ . It will be convenient to extend  $\Sigma^+$  by a formal unit element  $\epsilon \notin \Sigma^+$ . We denote this set by  $\Sigma^* = \Sigma^+ \cup \{\epsilon\}$ . The concatenation operation is extended to  $\Sigma^*$  as follows:  $\epsilon\epsilon = \epsilon$ , and for any  $w \in \Sigma^+$ ,  $\epsilon w = w\epsilon = w$ . We define the length of a sequence  $w = \sigma_1 \cdots \sigma_k \in \Sigma^+$  by  $|w| = k$  and the length of  $\epsilon$  by  $|\epsilon| = 0$ . Consider a set of matrices  $\{M_\sigma\}_{\sigma \in \Sigma}$ , where  $M_\sigma \in \mathbb{R}^{n \times n}$ ,  $n \geq 1$  for all  $\sigma \in \Sigma$  and let  $w = \sigma_1 \cdots \sigma_k \in \Sigma^+$ . Then, we denote the matrix  $M_{\sigma_k} \cdots M_{\sigma_1}$  by  $M_w$  and we define  $M_\epsilon = I$ . In addition, for a set of processes  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$  and for  $w = \sigma_1 \cdots \sigma_k \in \Sigma^+$ , we denote the process  $\mathbf{u}_{\sigma_k}(t) \cdots \mathbf{u}_{\sigma_1}(t - |w| + 1)$  by  $\mathbf{u}_w(t)$  and define  $\mathbf{u}_\epsilon(t) \equiv 1$ .

**In order to define GB-SS representations, we introduce the following processes:**

**Definition 1.** Consider a process  $\mathbf{r}$  and a set of processes  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$ . Let  $\sigma \in \Sigma$  and  $w = \sigma_1 \cdots \sigma_k \in \Sigma^+$ . Then, we define the process  $\mathbf{z}_w^{\mathbf{r}}(t) = \mathbf{r}(t - |w|)\mathbf{u}_w(t - 1)$ , which we call the *past* of  $\mathbf{r}$  with respect to  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$  **along  $w$** , and we define the process  $\mathbf{z}_w^{\mathbf{r}+}(t) = \mathbf{r}(t + |w|)\mathbf{u}_w(t + |w| - 1)$ , which we call the *future* of  $\mathbf{r}$  with respect to  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$  **along  $w$** .

Note that for  $w = \epsilon$ , both the past  $\mathbf{z}_\epsilon^{\mathbf{r}}(t)$  and the future  $\mathbf{z}_\epsilon^{\mathbf{r}+}(t)$  of  $\mathbf{r}$  w.r.t.  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$  equal  $\mathbf{r}(t)$ .

Next, we define *admissible sets of processes*, see [6, Definition 1], which will help us to formulate a **Markovian-like property** of the input processes of GB–SS representations.

**Definition 2** (admissible set of processes). A set of processes  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$  is called *admissible* if

- $\{\mathbf{u}_v^T, \mathbf{u}_w^T\}^T$  is wide-sense stationary for all  $v, w \in \Sigma^*$
- there exist  $\{\alpha_\sigma\}_{\sigma \in \Sigma} \in \mathbb{R}$  such that  $\sum_{\sigma \in \Sigma} \alpha_\sigma \mathbf{u}_\sigma(t) \equiv 1$
- there exist (strictly) positive numbers  $\{p_\sigma\}_{\sigma \in \Sigma}$ , such that  $E[\mathbf{u}_{v_1 \sigma_1}(t) \mathbf{u}_{v_2 \sigma_2}(t) | \bigvee_{\sigma \in \Sigma} \mathcal{F}_{t-}^{\mathbf{u}_\sigma}] =$

$$\begin{cases} p_{\sigma_1} \mathbf{u}_{v_1}(t-1) \mathbf{u}_{v_2}(t-1) & \sigma_1 = \sigma_2 \text{ and } v_1 v_2 \in \Sigma^+ \\ 0 & \sigma_1 \neq \sigma_2 \end{cases}$$

for any  $\sigma_1, \sigma_2 \in \Sigma$  and  $v_1, v_2 \in \Sigma^*$ , where  $\bigvee_{\sigma \in \Sigma} \mathcal{F}_{t-}^{\mathbf{u}_\sigma}$  is the smallest  $\sigma$ -algebra, s.t.  $\bigvee_{\sigma \in \Sigma} \mathcal{F}_{t-}^{\mathbf{u}_\sigma} \supseteq \mathcal{F}_{t-}^{\mathbf{u}_\sigma}$  for all  $\sigma \in \Sigma$ .

The next definition introduces a class of processes that the output, state, and noise processes of GB–SS representations belong to. The definition involves the concept of conditionally independent  $\sigma$ -algebras: Two  $\sigma$ -algebras  $\mathcal{F}_1, \mathcal{F}_2$  are conditionally independent w.r.t a third one  $\mathcal{F}_3$ , if for every event  $A_1 \in \mathcal{F}_1$  and  $A_2 \in \mathcal{F}_2$ ,  $P(A_1 \cap A_2 | \mathcal{F}_3) = P(A_1 | \mathcal{F}_3)P(A_2 | \mathcal{F}_3)$  with probability one [20].

**Definition 3** (ZMWSSI process). A stochastic process  $\mathbf{r}$  is called *zero-mean wide-sense stationary* w.r.t. an admissible set of processes  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$  (ZMWSSI) if  $\mathcal{F}_{(t+1)-}^{\mathbf{r}}$  and  $\mathcal{F}_{t+}^{\mathbf{u}}$  are conditionally independent w.r.t.  $\mathcal{F}_{t-}^{\mathbf{u}}$ , and  $[\mathbf{r}^T, (\mathbf{z}_v^{\mathbf{r}})^T, (\mathbf{z}_w^{\mathbf{r}})^T]^T$  is zero-mean wide-sense stationary for all  $v, w \in \Sigma^+$ .

**Definition 4.** SII A process  $\mathbf{r}$  is said to be *square integrable with respect to*  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$  (SII), if for all  $w \in \Sigma^+$ , the process  $\mathbf{z}_w^{\mathbf{r}+}$  is square integrable.

Below, we sum up the notations of this section.

TABLE I: Summary of notation

$\Sigma$	$\{1, 2, \dots, d\}$
$\Sigma^*, \Sigma^+$	the set of all finite sequences of elements of $\Sigma$ with resp. without the empty sequence
$M_w$	the product of matrices $\{M_\sigma\}_{\sigma \in \Sigma}$ along the elements of the sequence $w$
$\mathbf{r}_w(t)$	product of the processes indexed by the elements of the sequence $w$
$\mathbf{z}_w^{\mathbf{r}}(t), \mathbf{z}_w^{\mathbf{r}+}(t)$	the past and future processes of $\mathbf{r}$ w.r.t. $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$ along the sequence $w$
ZMWSII, SSI	class of stochastic processes to which outputs of GB–SS have to belong
$\mathbf{u}_\sigma(t)$	input process indexed by $\sigma \in \Sigma$

### III. GB–SS REPRESENTATIONS

This section introduces general bilinear state-space (GB–SS) representations and some results on their realization theory from [6]. To begin with, we define GB–SS representations.

**Definition 5** (GB–SS representation). A system of the form

$$\begin{aligned} \mathbf{x}(t+1) &= \sum_{\sigma \in \Sigma} (A_\sigma \mathbf{x}(t) + K_\sigma \mathbf{v}(t)) \mathbf{u}_\sigma(t) \\ \mathbf{y}(t) &= C \mathbf{x}(t) + D \mathbf{v}(t), \end{aligned} \quad (1)$$

where  $A_\sigma \in \mathbb{R}^{n \times n}, K_\sigma \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m}, \mathbf{x}(t) \in \mathbb{R}^n, \mathbf{v}(t) \in \mathbb{R}^m, \mathbf{y}(t) \in \mathbb{R}^p$ , and  $\mathbf{u}_\sigma(t) \in \mathbb{R}, \sigma \in \Sigma$  is called *generalized bilinear state-space* (GB–SS) representation of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$  if the following holds:

- $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$  is admissible
- $[\mathbf{x}^T, \mathbf{v}^T]$  is ZMWSSI with respect to  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$
- for  $w \in \Sigma^+, E[\mathbf{z}_w^{\mathbf{v}}(t) \mathbf{v}^T(t)] = 0$  and  $E[\mathbf{z}_w^{\mathbf{x}}(t) \mathbf{v}^T(t)] = 0$
- for  $\hat{\sigma}, \sigma \in \Sigma, E[\mathbf{z}_{\hat{\sigma}}^{\mathbf{x}}(t) (\mathbf{z}_\sigma^{\mathbf{v}}(t))^T] = 0$
- $\sum_{\sigma \in \Sigma} p_\sigma A_\sigma \otimes A_\sigma$  is stable.

We refer to a GB–SS representation (1) as GB–SS representation  $(\{A_\sigma, K_\sigma\}_{\sigma \in \Sigma}, C, D, \mathbf{v}, \{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$  or as GB–SS representation  $(\{A_\sigma, K_\sigma\}_{\sigma \in \Sigma}, C, D, \mathbf{v})$  of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$ , where note that  $\{A_\sigma, K_\sigma, \mathbf{u}_\sigma\}_{\sigma \in \Sigma}$  and  $\mathbf{v}$  determine the state process. Furthermore, notice that  $\mathbf{y}$  is the linear combination of  $\mathbf{x}$  and  $\mathbf{v}$  and thus it is also ZMWSSI w.r.t.  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$ .

Depending on the choice of the input processes, the behaviour of a GB–SS representation can significantly vary. The constraint on the input, formulated in Definition 2, gives scope to choosing  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$ , for example, in the following ways:

- If  $\Sigma = 1$  and  $\mathbf{u}_1(t) \equiv 1$ , then  $\mathbf{u}_1$  is admissible and the GB–SS representation defines an LTI–SS representation.
- If  $\mathbf{u}_\sigma(t)$  is zero-mean, square-integrable, iid process for all  $\sigma \in \Sigma$  and  $\mathbf{u}_{\sigma_1}(t), \mathbf{u}_{\sigma_2}(t)$  are independent for all  $\sigma_1, \sigma_2 \in \Sigma, \sigma_1 \neq \sigma_2$ , then  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$  is admissible.
- If  $\mathbf{u}_\sigma(t) = \chi(\Theta(t) = \sigma)$ , where  $\Theta$  is an iid process taking values in  $\Sigma$ , then  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$  is admissible.

More examples can be found in [6]. Note that Definition 2 gives a stricter definition of admissible set of processes than [6, Definition 1].<sup>2</sup> The results of the paper remain valid with the definition of admissible set of processes in [6]; however, we use Definition 2 in order to avoid technicalities.

#### A. GB–SS representations in forward innovation form

Below, we define innovation processes and innovation GB–SS representations. The latter class of representations plays a key role throughout the rest of the paper.

To this end, we recall from [6] the following notation. The real valued zero-mean square integrable random variables form a Hilbert space  $\mathcal{H}$  with the covariance as the inner product (see [21] for details). Let  $\mathbf{r}$  be a ZMWSSI process w.r.t. a set of admissible processes  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$ . Then, the one-dimensional components of  $\mathbf{r}(t)$  and  $\mathbf{z}_w^{\mathbf{r}}(t)$  (see Definition 1) belong to  $\mathcal{H}$  for all  $t \in \mathbb{Z}$ . We denote the Hilbert spaces generated by the one-dimensional components of  $\mathbf{r}(t)$  and of  $\{\mathbf{z}_w^{\mathbf{r}}(t)\}_{w \in \Sigma^+}$  by  $\mathcal{H}_t^{\mathbf{z}}$  and  $\mathcal{H}_{t,w \in \Sigma^+}^{\mathbf{z}_w^{\mathbf{r}}}$ , respectively. The (orthogonal) linear projection of  $\mathbf{r}(t)$  onto a closed subspace  $\mathcal{M}$  of  $\mathcal{H}$  is meant element-wise and it is denoted by  $E_l[\mathbf{r}(t) | \mathcal{M}]$ . If all the components of  $\mathbf{r}(t)$  are in  $\mathcal{M} \subset \mathcal{H}$ , then we write  $\mathbf{r}(t) \in \mathcal{M}$ .

**Definition 6** (GB–innovation process). The GB–innovation process of a ZMWSSI process  $\mathbf{y}$  w.r.t. the processes  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$  is defined by  $\mathbf{e}(t) = \mathbf{y}(t) - E_l[\mathbf{y}(t) | \mathcal{H}_{t,w \in \Sigma^+}^{\mathbf{z}_w^{\mathbf{y}}}]$ .

**Definition 7** (innovation GB–SS representation). A GB–SS representation (1) is called *innovation GB–SS representation* if the noise process  $\mathbf{v}$  is the GB–innovation process

<sup>2</sup>the set of admissible words used in [6] is here the trivial  $\Sigma^+$  set

$\mathbf{e}(t) = \mathbf{y}(t) - E_l[\mathbf{y}(t)|\mathcal{H}_{t,w \in \Sigma^+}^{\mathbf{z}_w^y}]$  of  $\mathbf{y}$  with respect to the input  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$  and the matrix  $D$  of (1) is the identity matrix.

In the specific case, when  $\Sigma = \{1\}$  and  $\mathbf{u}_1(t) \equiv 1$ , innovation GB–SS representations define innovation LTI–SS representations (called Kalman representation in [1]).

Finally, we make a technical assumption that requires the definition of full rank processes.

**Definition 8.** An output process  $\mathbf{y}$  of a GB–SS representation is called *full rank* if for all  $\sigma \in \Sigma$  and  $t \in \mathbb{Z}$ , the matrix  $E[\mathbf{e}(t)\mathbf{e}^T(t)\mathbf{u}_\sigma^2(t)]$  is strictly positive definite, where  $\mathbf{e}$  is the GB–innovation process of  $\mathbf{y}$  w.r.t the input  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$ .

The next assumption will be in force in the rest of the paper:

**Assumption 1.** *The output process  $\mathbf{y}$  is ZMWSII, SII and it is full rank.*

We define the *dimension* of a GB–SS representation as the dimension of its state process. A GB–SS representation is called *minimal* if it has minimal dimension among all GB–SS representations of the same input-output processes.

*Remark 1 (Realization theory).* According to [6, Theorem 3,5,6], if Assumption 1 holds  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$  has a GB–SS representation, then there exists a minimal GB–SS representation of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$  in forward innovation form. The latter GB–SS representation can be calculated from any GB–SS representation of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$ , using [6, Algorithm 1], or from suitable high-order moments of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$  using [6, Algorithm 2]. Finally, any two are minimal innovation GB–SS representations of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$ , see [6, Section III.B] for the formal definition of isomorphism. That is, without loss of generality, we can restrict attention to minimal GB–SS representations in forward innovation form.

#### IV. GB–GRANGER CAUSALITY IN GB–SS REPRESENTATIONS

In this section, we present the main results of this paper on relating an extended form of Granger causality, called GB–Granger causality, to properties of GB–SS representations. First, we introduce GB–Granger causality, and then, we present its characterization by properties of GB–SS representations. Throughout the rest of the paper,  $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$  is a ZMWSSI process w.r.t. an admissible set of processes  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$ , where  $\mathbf{y}_i \in \mathbb{R}^{p_i}$  for some  $p_i > 0$ ,  $i = 1, 2$ .

##### A. Extending Granger causality

Informally,  $\mathbf{y}_1$  does not Granger cause  $\mathbf{y}_2$ , if the best linear predictions of  $\mathbf{y}_2$  based on the past values of  $\mathbf{y}_2$  are the same as those based only on the past values of  $\mathbf{y}$ . Recall that  $\mathcal{H}_{t-}^{\mathbf{z}}$  denotes the Hilbert space generated by the past  $\{\mathbf{z}(t-k)\}_{k=1}^\infty$  of  $\mathbf{z}$ . Then, Granger causality is defined as follows:

**Definition 9 (Granger causality).** [1, Definition 5] Consider a zero-mean square integrable, wide-sense stationary process  $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$ . We say that  $\mathbf{y}_1$  does not Granger cause  $\mathbf{y}_2$  if for all  $t, k \in \mathbb{Z}$ ,  $k \geq 0$   $E_l[\mathbf{y}_2(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_2}] = E_l[\mathbf{y}_2(t+k)|\mathcal{H}_{t-}^{\mathbf{y}}]$ . Otherwise, we say that  $\mathbf{y}_1$  Granger causes  $\mathbf{y}_2$ .

In contrast to innovation LTI–SS representation, the output process of innovation GB–SS representations cannot be expressed by the linear combination of its own past values. In fact, an innovation GB–SS representation defines a linear relationship between the future of its output w.r.t. the inputs, denoted by  $\mathbf{z}_v^{\mathbf{y}^+}(t)$  and on the past of its output w.r.t. the inputs, denoted by  $\mathbf{z}_w^{\mathbf{y}}(t)$ , see Definition 1. This motivates our extension of Granger causality, where we use the process  $\mathbf{z}_v^{\mathbf{y}^+}(t)$  rather than  $\mathbf{y}(t+|v|)$  and  $\mathbf{z}_w^{\mathbf{y}^-}(t)$  rather than  $\mathbf{y}(t-|v|)$ :

**Definition 10 (GB–Granger causality).** Consider the processes  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T)$ , where  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$  is admissible and  $\mathbf{y}$  is ZMWSSI w.r.t.  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$ . We say that  $\mathbf{y}_1$  does not GB–Granger cause  $\mathbf{y}_2$  w.r.t.  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$  if for all  $v \in \Sigma^*$  and  $t \in \mathbb{Z}$

$$E_l[\mathbf{z}_v^{\mathbf{y}^+}(t)|\mathcal{H}_{t,w \in \Sigma^+}^{\mathbf{z}_w^{\mathbf{y}}}]] = E_l[\mathbf{z}_v^{\mathbf{y}^+}(t)|\mathcal{H}_{t,w \in \Sigma^+}^{\mathbf{z}_w^{\mathbf{y}_2}}]]. \quad (2)$$

Otherwise,  $\mathbf{y}_1$  GB–Granger causes  $\mathbf{y}_2$  w.r.t.  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$ .

Informally,  $\mathbf{y}_1$  does not GB–Granger cause  $\mathbf{y}_2$ , if the best linear predictions of the future of  $\mathbf{y}_2$  w.r.t.  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$  based on the past of  $\mathbf{y}$  w.r.t.  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$  are the same as those based only on the past of  $\mathbf{y}_2$  w.r.t.  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$ .

*Remark 2.* If a process  $\mathbf{y}_1$  does not GB–Granger cause  $\mathbf{y}_2$  then it implies that  $\mathbf{y}_1$  does not Granger causes  $\mathbf{y}_2$ . Moreover, in the specific case, when  $\Sigma = \{1\}$  and  $\mathbf{u}_1(t) \equiv 1$ ,  $\mathbf{z}_v^{\mathbf{y}^+}(t) = \mathbf{y}(t+|v|)$  and  $\mathbf{z}_w^{\mathbf{y}}(t) = \mathbf{y}(t-|w|)$  and thus Definitions 9 and 10 coincide. The relationship between GB–Granger causality and other concepts of causality, such as conditional independence [5], seems to be more involved and remains a topic of future research.

##### B. Main results

Next, we present the main results of this paper on the relation of GB–Granger causality and network graphs of GB–SS representations. The representations in question are minimal innovation GB–SS representations that can be constructed algorithmically (see Algorithm 1 later on in this section).

**Theorem 1.** *With Assumption 1, consider a GB–SS representation of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T)$  and let  $\mathbf{e} = [\mathbf{e}_1^T, \mathbf{e}_2^T]^T$  be the GB–innovation process of  $\mathbf{y}$  w.r.t.  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$ , where  $\mathbf{e}_i \in \mathbb{R}^{p_i}$ ,  $i = 1, 2$ . Then,  $\mathbf{y}_1$  does not GB–Granger cause  $\mathbf{y}_2$  w.r.t.  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$  if and only if there exists a minimal innovation GB–SS representation  $(\{A_\sigma, K_\sigma\}_{\sigma \in \Sigma}, C, I, \mathbf{e})$ ,  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$  such that for all  $\sigma \in \Sigma$*

$$A_\sigma = \begin{bmatrix} A_{\sigma,11} & A_{\sigma,12} \\ 0 & A_{\sigma,22} \end{bmatrix} \quad (3)$$

$$K_\sigma = \begin{bmatrix} K_{\sigma,11} & K_{\sigma,12} \\ 0 & K_{\sigma,22} \end{bmatrix}, C = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix},$$

where  $A_{\sigma,ij} \in \mathbb{R}^{n_i \times n_j}$ ,  $K_{\sigma,ij} \in \mathbb{R}^{n_i \times p_j}$ ,  $C_{ij} \in \mathbb{R}^{p_i, n_j}$  for some  $n_1 \geq 0$ ,  $n_2 > 0$  and  $(\{A_{\sigma,22}, K_{\sigma,22}\}_{\sigma \in \Sigma}, C_{22}, I, \mathbf{e}_2)$  is a minimal innovation GB–SS representation of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y}_2)$ .

The proof can be found in Appendix. If  $\Sigma = \{1\}$  and  $\mathbf{u}_1(t) \equiv 1$ , the GB–SS representation reduces to an LTI–SS representation and Definitions 9 and 10 coincide. As a result, Theorem 3 reduces to earlier results on LTI–SS representations and Granger causality (see [1, Theorem 1]).

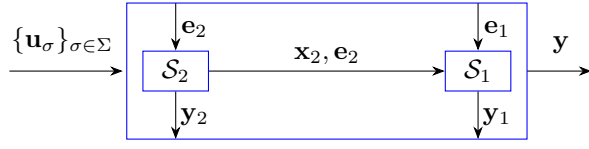


Fig. 2: Cascade interconnection in a GB-SS representation  $(\{A_\sigma, K_\sigma\}_{\sigma \in \Sigma}, C, I, \mathbf{e})$  with system matrices as in (3).

An innovation GB-SS representation  $(\{A_\sigma, K_\sigma\}_{\sigma \in \Sigma}, C, I, \mathbf{e}, \{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$  that satisfies (3) can be viewed as a cascade interconnection of two subsystems. Define the subsystems

$$\begin{aligned} \mathcal{S}_1 \begin{cases} \mathbf{x}_1(t+1) &= \sum_{\sigma \in \Sigma} (A_{\sigma,11} \mathbf{x}_1(t) + K_{\sigma,11} \mathbf{e}_1(t)) \mathbf{u}_\sigma(t) \\ &+ \sum_{\sigma \in \Sigma} (A_{\sigma,12} \mathbf{x}_2(t) + K_{\sigma,12} \mathbf{e}_2(t)) \mathbf{u}_\sigma(t) \\ \mathbf{y}_1(t) &= \sum_{i=1}^2 C_{1i} \mathbf{x}_i(t) + \mathbf{e}_2(t) \end{cases} \\ \mathcal{S}_2 \begin{cases} \mathbf{x}_2(t+1) &= (A_{\sigma,22} \mathbf{x}_2(t) + K_{\sigma,22} \mathbf{e}_2(t)) \mathbf{u}_\sigma(t) \\ \mathbf{y}_2(t) &= C_{22} \mathbf{x}_2(t) + \mathbf{e}_2(t) \end{cases} \end{aligned}$$

Notice that  $\mathcal{S}_2$  sends its state  $\mathbf{x}_2$  and noise  $\mathbf{e}_2$  to  $\mathcal{S}_1$  as an external input while  $\mathcal{S}_1$  does not send information to  $\mathcal{S}_2$ . The corresponding network graph is illustrated in Figure 2.

The necessity part of the proof of Theorem 1 is constructive, and it is based on calculating an innovation GB-SS representation described in Theorem 1. For this calculation, we present Algorithm 1 below, along with the statement of its correctness.

In order to present Algorithm 1, we define a (complete) *lexicographic ordering* ( $\prec$ ) on  $\Sigma^*$ :  $v \prec w$  if either  $|v| < |w|$  or if  $v = \nu_1 \dots \nu_k$ ,  $w = \sigma_1 \dots \sigma_k$  then  $\exists l \in \{1, \dots, k\}$  such that  $\nu_i = \sigma_i$ ,  $i < l$  and  $\nu_l \prec \sigma_l$ . Let the ordered elements of  $\Sigma^*$  be  $v_1 = \epsilon, v_2 = \sigma_1, \dots$  and define  $M(j)$  as the number of words of length at most  $j$ . We then define the observability matrix  $\tilde{\mathcal{O}}_k$  up to  $k$  of a tuple of matrices  $(\{\tilde{A}_\sigma\}_{\sigma \in \Sigma}, \tilde{C})$  as  $\tilde{\mathcal{O}}_k = [(\tilde{C} \tilde{A}_{v_1})^T \dots (\tilde{C} \tilde{A}_{v_k})^T]^T$ , where  $A_\sigma, \sigma \in \Sigma$  is an  $n \times n$  matrix, and  $C$  is a  $p \times n$  matrix.

**Algorithm 1** Block triangular minimal innovation GB-SS representation

**Input**  $\tilde{A}_\sigma \in \mathbb{R}^{n \times n}, \tilde{K}_\sigma \in \mathbb{R}^{n \times m}, \sigma \in \Sigma, \tilde{C} \in \mathbb{R}^{p \times n}$

**Output**  $(\{A_\sigma, K_\sigma\}_{\sigma \in \Sigma}, C)$

**Step 1** Define the sub-matrix consisting of the last  $p_2$  rows of  $\tilde{C}$  by  $\tilde{C}_2 \in \mathbb{R}^{p_2 \times n}$  and take the observability matrix  $\tilde{\mathcal{O}}_{M(n)}$  of  $(\{\tilde{A}_\sigma\}_{\sigma \in \Sigma}, \tilde{C}_2)$  up to  $n$ . If  $\tilde{\mathcal{O}}_{M(n)}$  is not of full column rank then define the non-singular matrix  $T^{-1} = [T_1 \ T_2]$  such that  $T_1 \in \mathbb{R}^{n \times n_1}$  spans the kernel of  $\tilde{\mathcal{O}}_{M(n)}$ . If  $\tilde{\mathcal{O}}_{M(n)}$  is of full column rank, then set  $T = I$ .

**Step 2** Define the matrices  $A_\sigma = T \tilde{A}_\sigma T^{-1}$ ,  $K_\sigma = T \tilde{K}_\sigma$  for  $\sigma \in \Sigma$  and  $C = \tilde{C} T^{-1}$ .

**Lemma 1.** Denote the GB-innovation process of  $\mathbf{y}$  by  $\mathbf{e}$ . Assume that  $(\{\tilde{A}_\sigma, \tilde{K}_\sigma\}_{\sigma \in \Sigma}, \tilde{C}, I, \mathbf{e})$  is a minimal innovation GB-SS representation of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$  of dimension  $n$ . Let  $\{A_\sigma, K_\sigma\}_{\sigma \in \Sigma}$  and  $C$  denote the matrices returned by Algorithm 1

Then  $(\{A_\sigma, K_\sigma\}_{\sigma \in \Sigma}, C, I, \mathbf{e})$  is a minimal innovation GB-SS representation of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$ , and the matrices  $\{A_\sigma\}_{\sigma \in \Sigma}$

and  $C$  are in the form

$$A_\sigma = \begin{bmatrix} A_{\sigma,11} & A_{\sigma,12} \\ 0 & A_{\sigma,22} \end{bmatrix} \quad C = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix} \quad (4)$$

where  $A_{\sigma,ij} \in \mathbb{R}^{n_i \times n_j}$ ,  $C_{ij} \in \mathbb{R}^{p_i \times n_j}$ ,  $i, j = 1, 2$  for some  $n_1 \geq 0$ ,  $n_2 > 0$ . In addition, if  $\mathbf{y}_1$  does not GB-Granger cause  $\mathbf{y}_2$ , then the matrices  $\{K_\sigma\}_{\sigma \in \Sigma}$  are in the form

$$K_\sigma = \begin{bmatrix} K_{\sigma,11} & K_{\sigma,12} \\ 0 & K_{\sigma,22} \end{bmatrix}, \quad (5)$$

where  $K_{\sigma,ij} \in \mathbb{R}^{n_i \times p_j}$ ,  $i, j \in \{1, 2\}$  and  $\mathcal{S}_2 = (\{A_{\sigma,22}, K_{\sigma,22}\}_{\sigma \in \Sigma}, C_{22}, I, \mathbf{e}_2)$  is a minimal innovation GB-SS representation of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y}_2)$ .

The proof of Lemma 1 can be found in Appendix.

*Remark 3.* From Lemma 1, it follows that if  $\mathbf{y}_1$  does not GB-Granger cause  $\mathbf{y}_2$ , then Algorithm 1 calculates the system matrices of the GB-SS representation described in Theorem 1. As it was mentioned in Remark 1, a minimal GB-SS representation in forward innovation form serving as the input of Algorithm 1 can be calculated from any GB-SS representation of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$  using [6, Algorithm 1]. Hence, Algorithm 1 provides a constructive proof of the necessity part of Theorem 1, by calculating a minimal innovation GB-SS representation that characterizes GB-Granger non-causality.

*Remark 4* (Checking GB-Granger causality). Algorithm 1 can be used for checking GB-Granger causality as follows. Assume we know the matrices of a minimal GB-SS representation of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$  in forward innovation form along with the covariance matrices.  $Q_\sigma = E[\mathbf{e}(t) \mathbf{e}^T(t) \mathbf{u}_\sigma^2(t)]$ ,  $\sigma \in \Sigma$ . Check if the matrices  $\{A_\sigma, K_\sigma\}_{\sigma \in \Sigma}$  and  $C$  returned by Algorithm 1 satisfy (4) and (5), and if  $\mathcal{S}_2 = (\{A_{\sigma,22}, K_{\sigma,22}\}_{\sigma \in \Sigma}, C_{22}, I, \mathbf{e}_2)$  is a minimal innovation GB-SS representation of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y}_2)$ . By Lemma 1 and Theorem 1 both tests are positive, if and only if  $\mathbf{y}_1$  does not GB-Granger cause  $\mathbf{y}_2$ . In order to check that  $\mathcal{S}_2 = (\{A_{\sigma,22}, K_{\sigma,22}\}_{\sigma \in \Sigma}, C_{22}, I, \mathbf{e}_2)$  is a minimal GB-SS representation in innovation form, we proceed as follows. It is clear that  $\mathcal{S}_2$  is a realization of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y}_2)$ , hence using [6, Algorithm 1] we can compute a minimal GB-SS representation  $\tilde{\mathcal{S}}_2$  of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$  in forward innovation form and the covariances  $\tilde{Q}_\sigma = E[\mathbf{v}(t) \mathbf{v}^T(t) \mathbf{u}_\sigma^2(t)]$ ,  $\sigma \in \Sigma$  of the innovation process  $\mathbf{v}$  of  $\mathbf{y}_2$ .  $\tilde{\mathcal{S}}_2$  is a minimal GB-SS representation in forward innovation form, if and only if  $\tilde{\mathcal{S}}_2$  and  $\mathcal{S}_2$  have the same dimension and the same noise process, i.e.,  $\mathbf{v} = \mathbf{e}_2$ . For checking the latter, we can use the following lemma.

*Lemma 2.*  $\mathbf{v}(t) = \mathbf{e}_2(t)$  if and only if for all  $i = 1, \dots, p_2$ ,  $\sum_{\sigma \in \Sigma} \alpha_\sigma^2(t) (\tilde{Q}_\sigma)_{i,i} = \sum_{\sigma \in \Sigma} \alpha_\sigma^2(t) (Q_\sigma)_{p_1+i, p_1+i}$  where  $\{\alpha_\sigma\}_{\sigma \in \Sigma}$  are as in Definition 2.

The proof of Lemma 2 is presented in Appendix. Since a minimal GB-SS representation in forward innovation form can be calculated from suitable high-order moments of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$  using [6, Algorithm 2], and the latter moments could be estimated from sampled data, the procedure above could be a starting point of a statistical test for checking GB-Granger causality, similar to the one of [22] for LTI systems. This remains a topic of future research.

*Example 1* (Numerical example). Consider a GBS-SS representation  $(\{\bar{A}_\sigma, \bar{K}_\sigma\}_{\sigma \in \Sigma}, \bar{C}, \bar{D}, \bar{\mathbf{v}})$  with  $d = 2$ ,  $\mathbf{u}_\sigma(t) = \chi(\theta(t) = \sigma)$ ,  $\chi$  is the characteristic function,  $\theta(t) \in \{1, 2\}$  i.i.d process with  $P(\theta(t) = 1) = 0.3$ . Then  $p_1 = 0.3, p_2 = 0.7$ . Assume that the matrices are of the following form.

$$\bar{A}_1 = \begin{bmatrix} 0.8 & 0.9 & -0.8 & 0.3 \\ 1.9 & 0.4 & -1.4 & 1.5 \\ 2.9 & 1.7 & -2.3 & 0.9 \\ 0.9 & 0.4 & -0.6 & 0 \end{bmatrix}, \bar{K}_1 = \begin{bmatrix} 1.1 & 1.5 \\ 1.1 & 0.9 \\ 2.3 & 3 \\ 0.6 & 0.7 \end{bmatrix}$$

$$\bar{A}_2 = \begin{bmatrix} -1.38 & -0.42 & 1.24 & -1.64 \\ -0.66 & -0.58 & 0.68 & -0.52 \\ -2.76 & -1.08 & 2.48 & -2.84 \\ -0.68 & -0.32 & 0.6 & -0.56 \end{bmatrix}, \bar{K}_2 = \begin{bmatrix} 0.2 & 1.24 \\ 0.2 & 0.84 \\ 0.44 & 2.52 \\ 0.12 & 0.56 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} 8.5 & 5.5 & -8 & 11 \\ 3.5 & -1.5 & -2 & 4 \end{bmatrix}$$

$\bar{D} = I_2$ ,  $E[\mathbf{v}(t)\mathbf{v}^T(t)\mathbf{u}_\sigma^2(t)] = p_\sigma I_2$ ,  $\sigma = 2$ , and  $\mathbf{v}(t)$  is a Gaussian white noise process with covariance  $I_2$ . The output of this GB-SS representation is  $\mathbf{y} = [\mathbf{y}_1^T \ \mathbf{y}_2^T]^T$ ,  $\mathbf{y}_i \in \mathbb{R}$ . We transform this GB-SS representation to a minimal GB-SS representation in forward innovation form using [6, Algorithm 1] and then we apply Algorithm 1. The matrices of resulting GB-SS representation  $(\{A_\sigma, K_\sigma\}_{\sigma \in \Sigma}, C, I, \mathbf{e})$  are

$$A_1 = \begin{bmatrix} -1.72 & -2.64 & -2.75 & -0.68 \\ 0.86 & 1.42 & 2.45 & 0.76 \\ 0 & 0 & -0.62 & -0.03 \\ 0 & 0 & 1.11 & -0.18 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.98 & 1.53 & 4.23 & 1.32 \\ -0.56 & -0.9 & -2.47 & -0.89 \\ 0 & 0 & -0.26 & -0.02 \\ 0 & 0 & 1.04 & 0.14 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 2.33 & 3 \\ -1.63 & -1.83 \\ 0 & -0.09 \\ 0 & 0.45 \end{bmatrix}, K_2 = \begin{bmatrix} 0.45 & 2.5 \\ -0.3 & -1.59 \\ 0 & -0.02 \\ 0 & 0.32 \end{bmatrix}$$

$$C = \begin{bmatrix} -2.24 & -5.39 & -14.41 & -6.76 \\ 0 & 0 & -5.86 & 0.43 \end{bmatrix}$$

and  $Q_\sigma = E[\mathbf{e}(t)\mathbf{e}^T(t)\mathbf{u}_\sigma^2(t)] = p_\sigma I_2$ , and hence they satisfy (4) and (5) with  $n_1 = n_2 = 2$ . Following Remark 4, we can check that  $(\{A_{\sigma,22}, K_{\sigma,22}\}_{\sigma \in \Sigma}, C_{22}, I, \mathbf{e}_2)$  is a minimal innovation GB-SS representation of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y}_2)$ :  $\mathbf{e}_2(t)$  is scalar valued and  $E[\mathbf{e}_2^2(t)] = \alpha_1^2(Q_1)_{22} + \alpha_2^2 Q_{22} = 1$ , and the minimal GB-SS representation of  $\mathbf{y}_2$  in forward innovation form has dimension 2 and the variance of its noise process is 1. From this, we can conclude that  $\mathbf{y}_1$  does not GB-Granger cause  $\mathbf{y}_2$ .

## V. CONCLUSIONS

In this paper, we proposed a new concept, called GB-Granger causality for defining causality in a statistical manner between processes that are outputs of GB-SS representations. We showed that GB-Granger causality can be characterized by structural properties of GB-SS representations, namely, absence of GB-Granger causality is equivalent to existence of a GB-SS representation which is a cascade interconnection

of two subsystems. Moreover, we proposed an algorithm for calculating such a GB-S representation. When applied to LTI-SS representations, these result boil down to the known correspondence between Granger causality and structural properties of LTI-SS representations [1], [13], [16].

The results could be used for developing statistical hypothesis testing for the presence of GB-Granger causality, in a similar manner as it was done for linear systems and Granger causality [22]. This extension, which would have potential applications in e.g. neuroscience and econometrics, remains a future work.

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## APPENDIX - PROOFS

*Proof of Lemma 1:* In order to prove Lemma 1, we use the following result:

**Lemma 3.** Consider an innovation GB-SS representation  $(\{A_\sigma, K_\sigma\}_{\sigma \in \Sigma}, C, I, \mathbf{e}, \{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$  with state process  $\mathbf{x}$ . Then,  $E_l[\mathbf{z}_v^{\mathbf{y}^+}(t) | \mathcal{H}_{t,w \in \Sigma^+}^{\mathbf{z}^{\mathbf{y}}}] = CA_v \mathbf{x}(t)$ , for all  $v \in \Sigma^+$ .

*Proof.* Recall that  $\mathcal{H}_{t,w \in \Sigma^+}^{\mathbf{z}^{\mathbf{y}}}$  is the Hilbert space generated by the past  $\{\mathbf{z}_w^{\mathbf{y}}\}_{w \in \Sigma^+}$  of  $\mathbf{y}$  w.r.t.  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$ . From [6, Eq. (38)], we know that  $E[\mathbf{z}_v^{\mathbf{y}^+}(t)(\mathbf{z}_{\sigma w}^{\mathbf{y}}(t))^T] = E[\mathbf{y}(t)(\mathbf{z}_{\sigma w}^{\mathbf{y}}(t))^T] = CA_v A_w G_\sigma$  for all  $\sigma \in \Sigma, v \in \Sigma^+, w \in \Sigma^*$ , where for  $\sigma \in \Sigma$   $G_\sigma = A_\sigma P_\sigma C^T + K_\sigma Q_\sigma$  and  $P_\sigma = E[\mathbf{x}(t)(\mathbf{x}(t))^T \mathbf{u}_\sigma^2(t)]$ . Also, from [6, Lemma 12] we know that  $E[\mathbf{x}(t)(\mathbf{z}_{\sigma w}^{\mathbf{y}}(t))^T] = A_w G_\sigma$  for all  $\sigma \in \Sigma, w \in \Sigma^*$ . Hence,  $E[\mathbf{z}_v^{\mathbf{y}^+}(t)(\mathbf{z}_{\sigma w}^{\mathbf{y}}(t))^T] = CA_v E[\mathbf{x}(t)(\mathbf{z}_{\sigma w}^{\mathbf{y}}(t))^T]$  for all  $v, \sigma, w \in \Sigma^+$ . Considering that  $\mathbf{x}(t) \in \mathcal{H}_{t,w}^{\mathbf{z}^{\mathbf{y}}}$ , and that  $\mathcal{H}_{t,w}^{\mathbf{z}^{\mathbf{y}}} \subseteq \mathcal{H}_{t,w}^{\mathbf{z}^{\mathbf{y}}}$ , we obtain that  $E_l[\mathbf{z}_v^{\mathbf{y}^+}(t) | \mathcal{H}_{t,w \in \Sigma^+}^{\mathbf{z}^{\mathbf{y}}}] = CA_v \mathbf{x}(t)$ .  $\square$

*Cont. proof of Lemma 1:* The following statements should be proven: 1)  $C$  is of the form (4), 2)  $A_\sigma$  is of the form (4), 3) if  $\mathbf{y}_1$  does not GB-Granger cause  $\mathbf{y}_2$ , then first,  $K_\sigma$  is of the form (5), and second, 4)  $(\{A_{\sigma,22}, K_{\sigma,22}\}_{\sigma \in \Sigma}, C_{22}, I, \mathbf{e}_2)$  is a minimal innovation GB-SS representation of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y}_2)$ . Below, we prove statements 1)–4) one by one. **Throughout the proof,  $T = [T_1 \ T_2]^{-1}$  denotes the matrix defined in Step 1. of Algorithm 1.**

1) Since the first  $p_2$  rows of  $\tilde{O}_{M(n)}$  equal  $C_2$  and the columns  $T_1$  span  $\ker \tilde{O}_{M(n)}$ , it follows that  $C_2 T^{-1} = [0 \ C_{22}]$  with some  $C_{22} \in \mathbb{R}^{n_2 \times n_2}$ ,  $0 < n_2 \leq n$ .

2) We first show that  $\ker \tilde{O}_{M(n)} = \ker \tilde{O}_{M(n+1)}$ . Define  $X_k = \ker \tilde{O}_{M(k)}$  for  $k = 0, \dots, n+1$ , where  $\tilde{O}_{M(k)}$  is the observability matrix of  $(\{\tilde{A}_\sigma\}_{\sigma \in \Sigma}, \tilde{C}_2)$  up to  $k$ . Then, either  $\tilde{C}_2 = 0$ , in which case  $\ker \tilde{O}_{M(n)} = \ker \tilde{O}_{M(n+1)}$  trivially holds, or  $\dim(X_0) = \dim(\ker \tilde{C}_2) < n$ . Notice that  $X_{k-1} \supseteq X_k$  for  $k = 1, \dots, n+1$ . **This and  $\dim(X_0) < n$  imply that there exists an  $l \in \{1, \dots, n\}$ , such that  $\dim(X_k) = \dim(X_{k+1})$  and  $X_k = X_{k+1}$  for all  $l \leq k$ .** By using that  $X_n = X_{n+1}$  and that the rows of  $\tilde{O}_{M(n)}$  and  $\tilde{O}_{M(n)} \tilde{A}_\sigma$  are rows of  $\tilde{O}_{M(n+1)}$ , we obtain that  $X_n$  is  $A_\sigma$ -invariant for all  $\sigma \in \Sigma$ . **Since the columns of  $T_1$  span  $X_n$ , we obtain that  $\tilde{A}_\sigma T_1 =$**

$T_1 N \in X_n$  for a suitable matrix  $N \in \mathbb{R}^{n_1 \times n_1}$ . Let  $A_\sigma = T \tilde{A}_\sigma T^{-1} = \begin{bmatrix} A_{\sigma,11} & A_{\sigma,12} \\ A_{\sigma,21} & A_{\sigma,22} \end{bmatrix}$ , where  $A_{\sigma,ij} \in \mathbb{R}^{n_i \times n_j}$  and notice that  $T \tilde{A}_\sigma T^{-1} = [T \tilde{A}_\sigma T_1 \quad \tilde{A}_\sigma T_2] = [T T_1 N \quad \tilde{A}_\sigma T_2]$ . Then,  $(T T_1 N)^T = [N \ 0]$  implies that  $A_{\sigma,21} = 0$ .

3) In order to see that the matrices  $\{K_\sigma\}_{\sigma \in \Sigma}$  are as in (5), we prove a sequence of statements (i)–(ii)–(iii)–(iv)–(v) and (vi), below, where (vi) states that  $\{K_\sigma\}_{\sigma \in \Sigma}$  satisfy (5).

- (i)  $\mathbf{x}_2(t) \in \mathcal{H}_{t,w \in \Sigma^+}^{\mathbf{z}_w^{\mathbf{y}_2}}$
- (ii)  $E[\mathbf{z}_w^{\mathbf{y}}(t)(\mathbf{z}_v^{\mathbf{e}}(t))^T] = 0$  for all  $|v| < |w|$ ,  $w, v \in \Sigma^+$
- (iii)

$$\mathcal{H}_{t,w \in \Sigma^+}^{\mathbf{z}_w^{\mathbf{y}_2}} = \bigoplus_{\sigma \in \Sigma} \left( \mathcal{H}_{t+1,w \in \Sigma^+}^{\mathbf{z}_\sigma^{\mathbf{y}_2}} \oplus \mathcal{H}_{t+1}^{\mathbf{z}_\sigma^{\mathbf{e}_2}} \right),$$

where  $\bigoplus$  denotes the direct sum of orthogonal closed subspaces and  $\mathcal{H}_{t+1,w \in \Sigma^+}^{\mathbf{z}_\sigma^{\mathbf{y}_2}}$  denotes the Hilbert space generated by  $\{\mathbf{z}_\sigma^{\mathbf{y}_2}(t+1)\}_{w \in \Sigma^+}$ , see also (iv) below.

- (iv) There exist  $\{N_\sigma\}_{\sigma \in \Sigma} \in \mathbb{R}^{n_2 \times p_2}$ ,  $\mathbf{r} \in \bigoplus_{\sigma \in \Sigma} \mathcal{H}_{t+1,w \in \Sigma^+}^{\mathbf{z}_\sigma^{\mathbf{y}_2}}$ , such that  $\mathbf{x}_2(t+1) = \mathbf{r} + \sum_{\sigma \in \Sigma} N_\sigma \mathbf{z}_\sigma^{\mathbf{e}_2}(t+1)$ .
- (v) Let  $K_\sigma = [K_{\sigma,21} K_{\sigma,22}]$ , such that  $K_{\sigma,21} \in \mathbb{R}^{n_2 \times p_1}$ ,  $K_{\sigma,22} \in \mathbb{R}^{n_2 \times p_2}$ . Then for  $\sigma \in \Sigma$ ,  $[K_{\sigma,21} K_{\sigma,22}] E[\mathbf{z}_\sigma^{\mathbf{e}}(t+1)(\mathbf{z}_\sigma^{\mathbf{e}}(t+1))^T] = N_\sigma E[\mathbf{z}_\sigma^{\mathbf{e}_2}(t+1)(\mathbf{z}_\sigma^{\mathbf{e}}(t+1))^T]$ .
- (vi)  $K_{\sigma,21} = 0$  for all  $\sigma \in \Sigma$ .

Next, we prove (i)–(vi), one-by-one.

- (i): By using (4), for any  $v \in \Sigma^+$

$$C A_v = \begin{bmatrix} C_{11}(A_v)_{11} & N \\ 0 & C_{22}(A_v)_{22} \end{bmatrix}, \quad (6)$$

where  $(A_v)_{11} \in \mathbb{R}^{n_1 \times n_1}$  and  $(A_v)_{22} \in \mathbb{R}^{n_2 \times n_2}$  are the upper and lower block diagonal sub-matrices of  $A_v$ , and  $N \in \mathbb{R}^{p_1 \times n_2}$  is an appropriate matrix. Notice that for a suitable permutation matrix  $P$ , it holds that  $P \mathcal{O}_{M(n)} = \begin{bmatrix} N_1 & N_2 \\ 0 & O_{M(n)} \end{bmatrix}$ , where  $O_{M(n)}$  is the observability matrix of  $(\{A_{\sigma,22}\}_{\sigma \in \Sigma}, C_{22})$  up to  $n$  and  $N_1, N_2$  are appropriate matrices. Notice now that  $\mathbf{x}(t) = \mathcal{O}_{M(n)}^+ E_l[\mathbf{Z}_n^{\mathbf{y}}(t) | \mathcal{H}_{t,w}^{\mathbf{z}_w^{\mathbf{y}}}]$ , where  $\mathbf{Z}_n^{\mathbf{y}}(t) = [(\mathbf{z}_{v_1}^{\mathbf{y}^+}(t))^T, \dots, (\mathbf{z}_{v_{M(n-1)}}^{\mathbf{y}^+}(t))^T]^T$ , where  $\mathbf{z}_{v_i}^{\mathbf{y}^+}(t)$  is the future of  $\mathbf{y}(t)$  w.r.t. the input along  $v_i$ ,  $i = 1, \dots, M(n-1)$ , (see Definition 1). Since  $P^T P = I$ ,  $(P \mathcal{O}_{M(n)})^+ = \mathcal{O}_{M(n)}^+ P^T$ . Hence,  $\mathbf{x}(t) = (P \mathcal{O}_{M(n)}^+) E_l[P \mathbf{Z}_n^{\mathbf{y}}(t) | \mathcal{H}_{t,w}^{\mathbf{z}_w^{\mathbf{y}}}]$ . Note that  $P \mathbf{Z}_n^{\mathbf{y}}(t) = [(\mathbf{Z}_n^{\mathbf{y}_1}(t))^T \quad (\mathbf{Z}_n^{\mathbf{y}_2}(t))^T]^T$ , where  $\mathbf{Z}_n^{\mathbf{y}_i}(t) = [(\mathbf{z}_{v_1}^{\mathbf{y}_i^+}(t))^T, \dots, (\mathbf{z}_{v_{M(n-1)}}^{\mathbf{y}_i^+}(t))^T]^T$ ,  $i = 1, 2$  and thus  $\mathbf{x}_2(t) = O_{M(n)}^+ E_l[\mathbf{Z}_n^{\mathbf{y}_2}(t) | \mathcal{H}_{t,w \in \Sigma^+}^{\mathbf{z}_w^{\mathbf{y}_2}}]$ . Then, the GB–Granger non-causality condition  $E_l[\mathbf{Z}_n^{\mathbf{y}_2}(t) | \mathcal{H}_{t,w \in \Sigma^+}^{\mathbf{z}_w^{\mathbf{y}_2}}] = E_l[\mathbf{Z}_n^{\mathbf{y}_2}(t) | \mathcal{H}_{t,w \in \Sigma^+}^{\mathbf{z}_w^{\mathbf{y}_2}}]$  implies that  $\mathbf{x}_2(t) \in \mathcal{H}_{t,w \in \Sigma^+}^{\mathbf{z}_w^{\mathbf{y}_2}}$ .

(ii): From [6, Lemma 14] it follows that  $[\mathbf{y}^T, \mathbf{e}^T]^T$  is ZMWSSI. Hence, we can apply [6, Lemma 7] for  $[\mathbf{y}^T, \mathbf{e}^T]^T$ : Let  $w = w_1 \dots w_k \in \Sigma^*$  and  $v = v_1 \dots v_l \in \Sigma^*$ , s.t.  $|v| < |w|$ . If  $w_{k-i} \neq v_{l-i}$  for some  $i = 0, \dots, l-1$ , then [6, Lemma 7] implies that the covariance  $E[\mathbf{z}_w^{\mathbf{y}}(t)(\mathbf{z}_v^{\mathbf{e}}(t))^T] = 0$ . If  $w_{k-i} = v_{l-i}$  for all  $i = 0, \dots, l-1$ , then  $E[\mathbf{z}_w^{\mathbf{y}}(t)(\mathbf{z}_v^{\mathbf{e}}(t))^T] = p_{v_2 \dots v_l} E[\mathbf{z}_{w_1 \dots w_{k-l-1}}^{\mathbf{y}}(t)(\mathbf{z}_{v_1}^{\mathbf{e}}(t))^T] = p_v E[\mathbf{z}_{w_1 \dots w_{k-l-1}}^{\mathbf{y}}(t) \mathbf{e}^T(t)] = 0$ , where for the last equation we used that from Definition 5  $E[\mathbf{z}_{w_1 \dots w_{k-l-1}}^{\mathbf{y}}(t) \mathbf{e}^T(t)] = 0$ .

- (iii): Consider an innovation GB–SS representation of

$(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y}_2)$  and note that by the GB–Granger non-causality condition, the GB–innovation process of  $\mathbf{y}_2$  is  $\mathbf{e}_2$ . Then, by [6, Lemma 16] we can decompose  $\mathcal{H}_{t,w \in \Sigma^+}^{\mathbf{z}_w^{\mathbf{y}_2}}$  as in (iii).

(iv): From (i), we have that  $\mathbf{x}_2(t+1) \in \mathcal{H}_{t+1,w \in \Sigma^+}^{\mathbf{z}_w^{\mathbf{y}_2}}$ . Then, by using (iii),  $\mathbf{x}_2(t+1) = \mathbf{r} + \sum_{\sigma \in \Sigma} N_\sigma \mathbf{z}_\sigma^{\mathbf{e}_2}(t+1)$  for some  $\mathbf{r} \in \bigoplus_{\sigma \in \Sigma} \mathcal{H}_{t+1,w \in \Sigma^+}^{\mathbf{z}_\sigma^{\mathbf{y}_2}}$  and  $\{N_\sigma\}_{\sigma \in \Sigma} \in \mathbb{R}^{n_2 \times p_2}$ .

(v): To shorten the expressions, define  $k = t+1$ . Notice that by using the block triangular form of  $\{A_\sigma\}_{\sigma \in \Sigma}$ , we obtain that  $\mathbf{x}_2(k) = \sum_{\sigma \in \Sigma} A_{\sigma,22} \mathbf{z}_\sigma^{\mathbf{x}_2}(k) + [K_{\sigma,21} K_{\sigma,22}] \mathbf{z}_\sigma^{\mathbf{e}}(k)$ . From [6, Lemma 14], it follows that  $[\mathbf{e}^T, \mathbf{y}^T, \mathbf{x}^T]^T$  is ZMWSSI, and hence  $[\mathbf{e}^T, \mathbf{x}_2^T]^T$  is also ZMWSSI w.r.t.  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$ . By applying [6, Lemma 7] for  $[\mathbf{e}^T, \mathbf{x}_2^T]^T$ , we have that if  $\sigma \neq \sigma^*$ , then  $E[\mathbf{z}_\sigma^{\mathbf{e}}(k)(\mathbf{z}_{\sigma^*}^{\mathbf{x}_2}(k))^T] = E[\mathbf{z}_\sigma^{\mathbf{e}}(k)(\mathbf{z}_{\sigma^*}^{\mathbf{e}}(k))^T] = 0$ . Also, by Definition 5,  $E[\mathbf{z}_{\sigma^*}^{\mathbf{e}}(k)(\mathbf{z}_\sigma^{\mathbf{x}_2}(k))^T] = 0$  for  $\sigma = \sigma^*$ , and since for any  $\sigma \in \Sigma$ ,  $\mathbf{z}_\sigma^{\mathbf{x}_2}$  is formed by a component of  $\mathbf{z}_\sigma^{\mathbf{e}}$ , we have that for  $\sigma = \sigma^*$ ,  $E[\mathbf{z}_{\sigma^*}^{\mathbf{e}}(k)(\mathbf{z}_{\sigma^*}^{\mathbf{x}_2}(k))^T] = 0$ . Hence,  $E[\mathbf{x}_2(k)(\mathbf{z}_\sigma^{\mathbf{e}}(k))^T] = [K_{\sigma,21} K_{\sigma,22}] Q_\sigma$ , where  $Q_\sigma = E[\mathbf{z}_\sigma^{\mathbf{e}}(k)(\mathbf{z}_\sigma^{\mathbf{e}}(k))^T]$ . By using (iv), we also obtain that  $E[\mathbf{x}_2(k)(\mathbf{z}_{\sigma^*}^{\mathbf{e}}(k))^T] = E[\mathbf{r} \mathbf{z}_{\sigma^*}^{\mathbf{e}}(k)^T] + \sum_{\sigma \in \Sigma} N_\sigma E[\mathbf{z}_\sigma^{\mathbf{e}_2}(k)(\mathbf{z}_{\sigma^*}^{\mathbf{e}}(k))^T]$ . Notice that from (ii) and from  $\mathbf{r} \in \bigoplus_{\sigma \in \Sigma} \mathcal{H}_{k,w \in \Sigma^+}^{\mathbf{z}_\sigma^{\mathbf{y}_2}}$ , we know that  $E[\mathbf{r} \mathbf{z}_{\sigma^*}^{\mathbf{e}}(k)^T] = 0$ . Hence,  $E[\mathbf{x}_2(k)(\mathbf{z}_{\sigma^*}^{\mathbf{e}}(k))^T] = N_\sigma E[\mathbf{z}_\sigma^{\mathbf{e}_2}(k)(\mathbf{z}_{\sigma^*}^{\mathbf{e}}(k))^T]$ , which equals  $[K_{\sigma,21} K_{\sigma,22}] Q_\sigma$ . Substituting  $k = t+1$ , we obtain (v).

(vi): Since  $\mathbf{e}_2$  is formed by the last  $p_2$  components of  $\mathbf{e}$ , we have that  $N_\sigma E[\mathbf{z}_\sigma^{\mathbf{e}_2}(t+1)(\mathbf{z}_\sigma^{\mathbf{e}}(t+1))^T] = [0 \ N_\sigma] Q_\sigma$  and hence  $[0 \ N_\sigma] Q_\sigma = [K_{\sigma,21} \ K_{\sigma,22}] Q_\sigma$ . By Assumption 1,  $Q_\sigma$  is positive definite which implies that  $[0 \ N_\sigma] = [K_{\sigma,21} \ K_{\sigma,22}]$ , hence  $K_{\sigma,21} = 0$ .

4) Denote the state process of the minimal innovation GB–SS representation  $\mathcal{G}$  of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$ , that the output matrices of Algorithm 1 define, by  $[\mathbf{x}_1^T, \mathbf{x}_2^T]^T$ , where  $\mathbf{x}_1 \in \mathbb{R}^{n_1}$ ,  $\mathbf{x}_2 \in \mathbb{R}^{n_2}$ . To see that  $\mathcal{G}_2 = (\{A_{\sigma,22}, K_{\sigma,22}\}_{\sigma \in \Sigma}, C_{22}, I, \mathbf{e}_2)$ , with state process  $\mathbf{x}_2$ , defines a minimal innovation GB–SS representation of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y}_2)$ , notice that from the GB–Granger non-causality condition, and from Definition 5, it follows that  $\mathcal{G}_2$  is an innovation GB–SS representation. Assume indirectly that  $\mathcal{G}_2$  is not minimal, i.e. that there exists a minimal innovation GB–SS representation  $\tilde{\mathcal{G}}_2 = (\{\tilde{A}_{\sigma,22}, \tilde{K}_{\sigma,22}\}_{\sigma \in \Sigma}, \tilde{C}_{22}, I, \mathbf{e}_2)$  of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y}_2)$  with state  $\tilde{\mathbf{x}}_2 \in \mathbb{R}^{\tilde{n}_2}$ , where  $\tilde{n}_2 < n_2$ .

From Lemma 3, it follows that  $E_l[\mathbf{Z}_n^{\mathbf{y}_2}(t) | \mathcal{H}_{t,w \in \Sigma^+}^{\mathbf{z}_w^{\mathbf{y}_2}}] = \tilde{O}_{M(n_2)} \tilde{\mathbf{x}}_2(t)$ , where  $\tilde{O}_{M(n_2)}$  is the observability matrix of  $(\{\tilde{A}_{\sigma,22}\}_{\sigma \in \Sigma}, \tilde{C}_{22})$  up to  $n_2$ . Then, by defining  $L = O_{M(n_2)}^+ \tilde{O}_{M(n_2)}$ , where  $O_{M(n_2)}$  is the observability matrix of  $(\{A_{\sigma,22}\}_{\sigma \in \Sigma}, C_{22})$  up to  $n_2$ , we have that  $\mathbf{x}_2 = L \tilde{\mathbf{x}}_2$ . By using  $L$ , we can transform  $\mathcal{G}$  into an innovation GB–SS representation  $\tilde{\mathcal{G}}$  of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y})$  with state process  $[\mathbf{x}_1^T, \tilde{\mathbf{x}}_2^T]^T$ . However,  $\tilde{\mathcal{G}}$  has dimension  $n_1 + \tilde{n}_2 < n_1 + n_2 = n$ , which is a contradiction since  $n$  is the dimension of a minimal innovation GB–SS representation.  $\square$

*Proof of Theorem 1:* The sufficiency part of the proof follows Lemma 1.

To prove the necessity part, let  $(\{A_\sigma, K_\sigma\}_{\sigma \in \Sigma}, C, I, \mathbf{e})$  be a minimal innovation GB–SS representation of



( $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$ ), such that (3) holds and that  $\mathcal{G}_2 = (\{A_{\sigma,22}, K_{\sigma,22}\}_{\sigma \in \Sigma}, C_{22}, I, \mathbf{e}_2)$  is a minimal innovation GB–SS representation of  $(\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}, \mathbf{y}_2)$ . To prove that  $\mathbf{y}_1$  does not GB–Granger causes  $\mathbf{y}_2$ , we need to see that

$$E_l[\mathbf{z}_v^{y_2^+}(t) | \mathcal{H}_{t,w \in \Sigma^+}^{z_w^{y_2}}] = E_l[\mathbf{z}_v^{y_2^+}(t) | \mathcal{H}_{t,w \in \Sigma^+}^{z_w^{y_2}}] \quad (7)$$

for all  $v \in \Sigma^*$ . For  $v = \epsilon$ , (7) directly follows from that  $\mathbf{e}_2$  is the GB–innovation process of  $\mathbf{y}_2$  w.r.t.  $\{\mathbf{u}_\sigma\}_{\sigma \in \Sigma}$ . By (3), the matrices  $\{A_\sigma\}_{\sigma \in \Sigma}$  and  $C$  are block triangular, hence  $CA_v$  is as in (6). It then follows from Lemma 3 that

$$E_l[\mathbf{z}_v^{y_2^+}(t) | \mathcal{H}_{t,w \in \Sigma^+}^{z_w^{y_2}}] = C_{22}(A_v)_{22} \mathbf{x}_2(t). \quad (8)$$

By projecting both side of (8) onto  $\mathcal{H}_{t,w \in \Sigma^+}^{z_w^{y_2}}$ , and by using that  $\mathbf{x}_2(t) \in \mathcal{H}_{t,w \in \Sigma^+}^{z_w^{y_2}}$  (see [6, Theorem 5]), we get that  $E_l[\mathbf{z}_v^{y_2^+}(t) | \mathcal{H}_{t,w \in \Sigma^+}^{z_w^{y_2}}] = C_{22}(A_v)_{22} \mathbf{x}_2(t)$ . By considering (8), the latter implies (7), i.e. that there is no GB–Granger causality from  $\mathbf{y}_1$  to  $\mathbf{y}_2$ .  $\square$

*Proof of Lemma 2.* Note that  $\mathcal{H}_{t,w \in \Sigma^+}^{z_w^{y_2}} \subseteq \mathcal{H}_{t,w \in \Sigma^+}^{z_w^{y_2}}$ , and  $\mathbf{v}(t) = \mathbf{y}_2(t) - E_l[\mathbf{y}_2(t) | \mathcal{H}_{t,w \in \Sigma^+}^{z_w^{y_2}}]$ ,  $\mathbf{e}_2(t) = \mathbf{y}_2(t) - E_l[\mathbf{y}_2(t) | \mathcal{H}_{t,w \in \Sigma^+}^{z_w^{y_2}}]$ , hence by the minimal distance property of orthogonal projections,  $E[(\mathbf{v}(t))_i^2] = E[(\mathbf{e}_2(t))_i^2]$ ,  $i = 1, \dots, p$  if and only if  $\mathbf{e}_2(t) = \mathbf{v}(t)$ , where  $(\mathbf{v}(t))_i, (\mathbf{e}_2(t))_i$  denote the  $i$ th entry of  $\mathbf{v}(t)$  and  $\mathbf{e}_2(t)$  respectively. Note that  $\mathbf{e}_2(t) = \sum_{\sigma \in \Sigma} \alpha_\sigma \mathbf{e}_2(t) \mathbf{u}_\sigma(t)$ ,  $\mathbf{v}(t) = \sum_{\sigma \in \Sigma} \alpha_\sigma \mathbf{v}(t) \mathbf{u}_\sigma(t)$ . As  $\mathbf{e}_2, \mathbf{v}$  are ZMWSII processes,  $E[\mathbf{e}_2(t) \mathbf{e}_2^T(t) \mathbf{u}_{\sigma'}(t) \mathbf{u}_{\sigma'}^T(t)] = 0$ ,  $E[\mathbf{v}(t) \mathbf{v}^T(t) \mathbf{u}_{\sigma'}(t) \mathbf{u}_{\sigma'}^T(t)] = 0$  for all  $\sigma \neq \sigma', \sigma, \sigma' \in \Sigma$ . Hence,  $E[(\mathbf{v}(t))_i^2] = \sum_{\sigma \in \Sigma} \alpha_\sigma^2(t) (Q_\sigma)_{i,i}$ ,  $E[(\mathbf{e}_2(t))_i^2] = \sum_{\sigma \in \Sigma} \alpha_\sigma^2(t) (Q_\sigma)_{p_1+i, p_1+i}$ .  $\square$

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**Mónika Józsa** received the Ph.D. in systems and control at the University of Groningen, Netherlands. Her M.S. degree was in applied mathematics, specialized in stochastic theory from Eötvös Loránd University in Budapest, Hungary in 2013. From 2013 she worked for Research Institute of Agricultural Economics in Budapest. Her research interest focuses on stochastic processes, time series analysis, and realization/identification theory of stochastic dynamical systems.



**Mihály Petreczky** received the Ph.D. degree in mathematics from Vrije Universiteit in Amsterdam, The Netherlands in 2006. In the past, he held appointments at Johns Hopkins University, USA, Eindhoven University of Technology, and Maastricht University, The Netherlands and at Ecole des Mines de Douai, France. He is currently a CNRS researcher at Centre de Recherche en Informatique, Signal et Automatique de Lille (CRISTAL), UMR CNRS 9189. His research interests include control and systems theory of hybrid and LPV systems.



**M. Kanat Camlibel** (M’05) received the Ph.D. degree in mathematical theory of systems and control from Tilburg University in 2001. Currently he is an Associate Professor with Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, where he served as an Assistant Professor between 2007 and 2013. From 2001 to 2007, he held Post-Doctoral/Assistant Professor positions with the University of Groningen, Tilburg University, and Eindhoven Technical University. His research interests include differential variational inequalities, complementarity problems, optimization, piecewise affine dynamical systems, switched linear systems, constrained linear systems, multiagent systems, model reduction, and geometric theory of linear systems.