

Some linguistic issues on the teaching of mathematical proof

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There are different meanings of proof-related words and their connotations in different languages. This study aims to reveal issues of the relationship between natural and mathematical language in the teaching of mathematical proof. For this purpose, we examine the grammatical characteristics of language from Japanese and international perspectives, as well as linguistics issues associated with statements with quantifications. A pilot study shows that natural language may influence how statements are formulated by students in mathematical discourse.

Keywords: Mathematical proof, statement, quantification, ordinary language, theory.

Introduction

Proof-related words such as “proof” and “argumentation” are used by different people in different ways. According to recent publications within mathematics education research, the meaning of “proof” is still a subject of debate among researchers (e.g. Balacheff, 2008; Cabassut et al., 2012; Mariotti, Durand-Guerrier, & Stylianides, 2018; Reid, 2015; Reid & Knipping, 2010; Stylianides, Bieda, & Morselli, 2016). In an international context, it is especially important for researchers to consider cultural dimensions that may affect the use of such words in different countries (e.g. Reid, 2015; Sekiguchi & Miyazaki, 2000). As discussed by Shinno et al. (2018), there are two words for “proof” used in Japan: *shōmei* and *ronshō*, as follows.

Some Japanese works distinguish the word *ronshō* from *shōmei* by referring to their relationship with the system of mathematics. [...], such that *shōmei* is related to deriving consequences from premises for establishing the truth of a proposition, while *ronshō* is related to the (axiomatic) system in which logical relations between propositions take place. (Shinno et al., 2018, p. 26)

Sometimes, *ronshō* is used as a special type of *shōmei* (Shinno et al., 2018). This distinction is similar but not identical to the distinction between the French words: *preuve* and *démonstration*, as Balacheff (1987) discussed. According to this distinction by Balacheff, *preuve* and *démonstration* are translated as the English words “proof” and “mathematical proof”, respectively. As a commentary on Shinno et al. (2018), Koichu (2018) also discusses two proof-related words in Hebrew: *hohaha* and *hanmaka*.

Two Hebrew words are frequently used in the official Israeli mathematics curricula: *hohaha* (translates as proof) and *hanmaka* (translates as both argumentation and reasoning). Similarly to Japan, Israeli curricula treat *hohaha* as a particular case of *hanmaka* where the former notion

presumes mathematical rigor and the latter one is somewhat vague and alludes to providing an argument without strictly prescribing what type of argument may be used. (Koichu, 2018, p. 26)

Identifying such proof-related terminologies and their differences and similarities across languages might reveal underlying cultural issues in the educational system or curricula of different countries. For Japan, the word *ronshō* indicates an emphasis on a systematic approach to mathematics, and the word *shōmei* is reserved for validating a general statement (Shinno et al., 2018). This is likely due to the quasi-axiomatic nature of the content and sequencing of the Japanese geometry curriculum and to the fact that all statements required to be proven in the textbook relate to general objects (Miyakawa, 2017). In the case of Hebrew, “the *hahaha* and *hanmaka* notions are complementary in Israeli middle school textbooks, as well as in the intermediate-level high school curriculum” (Koichu, 2018, p. 26). Koichu also mentions that in Miyakawa’s (2017) finding, it “sounds as if the idea of *shōmei* was put forward in Israeli textbooks, but *ronshō* was in the minds of the textbook writers as part of their horizon knowledge” (p. 26). Thus, a linguistic or terminological comparison of the teaching of proof in different countries might be possible. In doing so, it behooves researchers to investigate how such linguistic differences may affect curricula and textbooks (or classroom teaching) in each country.

The meaning of proof-related words differs from culture to culture, depending on the language used in a specific country. Nevertheless, it is possible for researchers to identify similar connotations across different languages. In order to further discuss cultural and linguistic dimensions of proof, the present study attempts to reveal issues concerning the relationship between natural and mathematical language on the teaching of mathematical proof.

Theoretical background and methodology

Mariotti (2006) states, “it is not possible to grasp the sense of a mathematical proof without linking it to the other two elements: statement and theory” (p. 184). Within this triad (statement, proof, and theory), the first two elements, statement and proof, can be subjected to investigation in terms of their different linguistic formulations; the third element, theory, is of a more epistemological nature. A model (a reference epistemological model) proposed by Shinno et al. (2018) consists of three layers —“real-world logic”, “local theory”, and “axiomatic theory”—which are characterized in terms of the epistemological nature of theory, wherein the statement is formulated, and the proof is carried out. For the purposes of international discourse, it is important for researchers to make a distinction and/or connection between the linguistic formulation (representation) of the statement/proof and its epistemological nature (Balacheff, 2008). For example, the linguistic distinction between *shōmei* and *ronshō* can be understood through the epistemological distinction between local and axiomatic theory (Shinno et al., 2018).

This paper pays special attention to linguistic formulations of the statement to be proven in Japanese textbooks. Concerning the linguistic aspects of proof, it is important to take into account the relationship between natural and mathematical language, because word usage in natural language may influence how statements and proofs are formulated and understood by students in mathematical discourse. This topic has been discussed in various ways (e.g. Barton & Neville-Barton, 2004; Cousin & Miyakawa, 2017; Duval, 2017; Mejia-Ramos & Inglis, 2011; Pimm, 1987; Planas, Morgan, & Schütte, 2018). In this paper, we analyse some typical statements from Japanese textbooks to

clarify how ordinary Japanese language is used in mathematical discourse. We also present a pilot study which allowed us to investigate Japanese undergraduate students' difficulties with linguistics issues related to statement with quantifications.

The relation between natural and mathematical language in Japanese textbooks

The grammatical characteristics of the Japanese language

Mathematics sentences such as $5 + 3 = 8$ are introduced at the beginning of the first grade (6- to 7-year-old students) in Japan. As seen in Figure 1, the textbook (both Japanese and English versions) shows how to read the sentence ("5 plus 3 equals 8"), alongside other representations such as blocks.

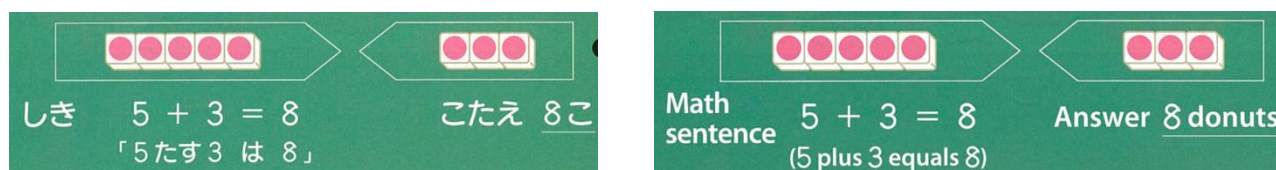


Figure 1: Math sentence in a Japanese textbook (Shimizu & Funakoshi, 2011, p. 39)

The sentence “5 plus 3 equals 8” is seen as mathematical discourse in English, which translates as “5 *tasu* (たす) 3 *wa* (は) 8” in Japanese. Here “*tasu*” refers to “plus”, and “*wa*” refers to “equals”. From a grammatical point of view, however, this Japanese sentence is not proper. In ordinary Japanese, it should be “5 *ni* 3 *wo tasu to* 8 *ni naru*” (please focus on only the bold words in this sentence, since others are particles which have no counterparts in the English sentence). Here, “*tasu*” means “plus (or add)”, but “*naru*” means “makes (or becomes)” which can be the same as “equals”. This is because, in Japanese grammar, the verb generally appears at the end of the sentence. If the sentence “5 *ni* 3 *wo tasu to* 8 *ni naru*” above is translated literally into mathematical symbols, it would be like “5 3 + 8 =” which contradicts the syntax of mathematics (Hirabayashi, 1994). This fact can be seen as a linguistic peculiarity of Japanese grammatical characteristics. This may cause an obstacle in the understanding of mathematical sentences on the basis of Japanese language, because grammatical order of some statements, which appears at a more advanced level such as AE and EA statements, is crucial for interpreting what a given statement mean in mathematical discourse.

Difficulties related to the linguistic and logical aspects from an international perspective

One characteristic of statements in Japanese textbooks for lower secondary schools is that the statement to be proven is considered a universal proposition (Miyakawa, 2017). However, universal quantification using ordinary language and words such as “any” and “all” is rarely encountered in lower secondary textbooks across domains; that is to say, universality is rarely formulated in written form. Therefore, Japanese students and teachers are required to interpret a universal proposition without any quantification. This is probably because the Japanese language does not use articles (Shinno et al., 2018). As Pimm (1987) comments, the situation is similar in Russian, in which

there is no article marker distinguishing *a* from *the*, the definite from the indefinite. Yet this language contains a sophisticated mathematics register fully capable of distinguishing the meaning ‘there exists’ from ‘there exists a unique’. (Pimm, 1987, p. 81)

Japanese does not have such “a sophisticated mathematics register”, at least in secondary school textbooks. Additionally, according to Durand-Guerrier et al. (2016), it is difficult even for people who

speak English (and many European languages) to distinguish an implicit universal quantifier from other meanings of an indefinite article:

A well-known difficulty is related to the meaning of ‘a’ that can either refer to an individual, a generic element, or an implicit universal quantifier. (Durand-Guerrier et al., 2016, p. 89)

Beyond the grammatical aspects of a given language, this issue is related to students’ interpretations of universal quantifications. We will develop this issue when introducing the pilot study.

Most Japanese students have few opportunities to consider quantification in their own language. Most do not encounter quantifiers formulated in mathematical terms or symbols such as “ \forall ” or “ \exists ”, based on predicate logic, until they reach university. To this point, and as a reply to Shinno et al. (2018), Czocher, Dawkins, and Weber (2018) argue that “universal statements in American classrooms are often not explicitly quantified, either” (p. 25). Further, Czocher et al. (2018) introduce an excerpt from a textbook used in the United States:

In a popular undergraduate textbook, Hammack (2013) directly addressed this point:

Now we come to the very important point. In mathematics, whenever $P(x)$ and $Q(x)$ are open sentences concerning elements x in some set S (depending on context), an expression of form $P(x) \Rightarrow Q(x)$ is understood to be the statement $\forall x \in S (P(x) \Rightarrow Q(x))$. In other words, if a conditional statement is not explicitly quantified then there is an implied universal quantifier in front of it. This is done because statements of the form $\forall x \in S, P(x) \Rightarrow Q(x)$ are so common in mathematics that we would get tired of putting the $\forall x \in S$ in front of them. (Hammack, 2013, p. 46).

We interpret this excerpt to mean that a focus on universal statements is an invariant part of mathematical practice. (Czocher et al., 2018, p. 25)

While we agree with both Hammack’s assertion, “if a conditional statement is not explicitly quantified then there is an implied universal quantifier in front of it”, and Czocher et al. (2018)’s commentary, it seems to us that this book is intended to introduce undergraduate students to predicate logic. At the undergraduate level, Japanese students also need to pay attention to the similar point made in the excerpt. We wonder if this phrase still holds true for general statements taught at lower grades in the United States. It might be difficult to examine, however, because “proving opportunities are limited or non-existent in the curriculum as enacted in many American classrooms” (Czocher et al., 2018, p. 24), even though, as they also mention, it is important to make empirical comparisons of the usage of universal quantifications across different countries.

As described above, most students do not encounter quantifiers formulated in mathematical terms or symbols until they reach the undergraduate level. It is worth investigating whether language influences students’ understanding in the transition from secondary school to university. Here, we focus on the linguistic aspects of universal and existential quantifications to be formulated in a given statement, so-called AE and EA statements, as below (e.g. Dubinsky & Yiparaki, 2000)).

- AE statement: *For every x , there exists a y such that $f(x, y)$.*
- EA statement: *There exists an x such that for every y , $f(x, y)$*

Previous studies, conducted in English-speaking countries, reported that students faced difficulties in distinguishing the two kinds of statements (e.g. Dubinsky & Yiparaki, 2000). Difficulties related to

AE/EA statements are also prevalent in the Japanese mathematics (education) community, and there is an assumption that these difficulties are affected by ordinary language (e.g. Hosoi, 1981). These difficulties are related to what Selden and Selden (1995) call “unpacking (logical structure of) informal statements”. Let us briefly explain the linguistic issues of the universal and existential quantifiers of AE/EA statements. In terms of English, they can be explained as follows.

In everyday language, statements where the words ‘there is’ precede the words ‘for every’ are almost always interpreted as if the words ‘for every’ precede the words ‘there is’.
(Durand-Guerrier et al., 2012, p. 376)

Unlike in ordinary English sentences, it is difficult to interpret this exchange between the words “there is” and “for every” in Japanese. In English, both phrases precede the *variables* such as “for every X ” and “there is Y ”. In Japanese, the words “for every” precede the variable, but the phrase “there is” does not precede. Thus, the grammatical order of the English phrase, “there is Y ”, is not preserved in Japanese (cf. Hosoi, 1981), because “is” or “exist” cannot precede the object “ Y ”. Rather, the phrase “ x exists” is more natural than “there exists x ”. Due to this linguistic issue, it is unusual for Japanese speakers to read and write the words according to the order of EA statements, as in English. In Japanese, when we intend to write two kinds of statements in a distinct way, different formulations or representations by ordinary Japanese are possible, but all of them contain vague words or additional notations (like brackets) to preserve the order.

A pilot study: Japanese students’ interpretations of AE and EA statements

To investigate Japanese undergraduate students’ difficulties or gaps associated with their interpretations of statements involving multiple quantifiers, we conducted a pilot study targeting 47 undergraduate students (as native Japanese speakers) who had already learnt mathematical statements with multiple quantifiers. The questionnaire included both AE and EA statements formulated in two ways: mathematical and English. The participants were asked to translate each statement into Japanese and to judge whether a given statement was true or false. Figure 2 indicates the two types of questions. Different formulations of the same statement are found in 1A and 2A and 1B and 2B; 1A and 2A are AE statements (true), while 1B and 2B are EA statements (false).

The participants were divided into two groups. Twenty-four participants were asked to translate 1A and 1B into the Japanese form, while 23 participants were asked to translate 2A and 2B into the Japanese form. Both groups were then asked to judge whether a given statement was true or false and to write down their justifications. The results are shown in Table 1.

- Type 1: Mathematical form \rightarrow Japanese form (N=24)
 - 1A: $\forall n \in \mathbb{N}, \exists K \in \mathbb{N}, 2n < K$
 - 1B: $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = 0$
- Type 2: English form \rightarrow Japanese form (N=23)
 - 2A: For every natural number n , there exists a natural number K such that $2n < K$
 - 2B: Let x and y be real numbers. There exists an x such that for every y , $x + y = 0$

Figure 2: Two types of questions

	1A (true)	1B (false)	2A (true)	2B (false)
True	23	2	21	20
False	1	22	2	3

Table 1: Result of participants' true/false judgments

These results show that most participants could adequately interpret AE statements (both 1A and 2A). However, there are considerable variations in their interpretations of EA statements. Although many participants could judge that statement 1B (in mathematical form) was false, most of them (20/23) failed to interpret statement 2B (in English form) in their own language. It is worth noting how ordinary language influenced their discrepant interpretations. Figure 3 shows a typical example of this confusion, wherein the participant interpreted the false EA statement (2B) as a true AE statement in Japanese. The box on the right in Figure 3 provides our back-translation. It is important to note that this translation is not “correct” in terms of the English language, because the order of Japanese phrases is preserved, and the term “*tonaru* (と なる)” which corresponds to “such that” is not translated. The point here is that the condition which comes after “such that” in English precedes “*tonaru*” in ordinary Japanese language, and Japanese people may not clearly know from this sentence whether the first phrase “For every y ” is included in the condition of the existential proposition (EA statement) or not (AE statement). The meaning of the sentence would be clearer if we put the two first phrases into brackets or quotations, but this is not common.

<p>和訳:</p> <p>x, y を実数とする。</p> <p>すべての y に対して、$x + y = 0$ となる x が存在する。</p> <p>真偽: <u>真</u></p> <p>理由:</p> <p>$x = -y$ とおけば、命題は成り立つ。</p>	<p>(Japanese translation):</p> <p><i>Let x and y be real numbers.</i></p> <p><i>For every y, $x + y = 0$, x exists.</i></p> <p>(True/False): <i>True</i></p> <p>(Justification): <i>Set $x = -y$, the statement holds.</i></p>
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Figure 3: A typical example of incorrect interpretation

Conclusion

The logical difference between AE and EA statements may not always be preserved when the two statements are written in everyday Japanese sentences. The pilot study shows that many students are not very conscious of this linguistic characteristic; therefore, most of them failed to interpret EA statements (English) in the Japanese language. To create an opportunity for further international comparative study, it is important to elaborate a meta theoretical language (e.g., a reference epistemological model) for research from international perspectives, since a researcher may need to use different languages for different purposes (Chellougui, Thu, & Winsløw, 2016). Without such a common ground, it would be more challenging for international readers to share “invariant aspects of mathematical reasoning” (Czocher et al., 2018) which we initially associated with a Japanese context.

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