Proof comprehension of undergraduate students and the relation to individual characteristics

Silke Neuhaus¹ and Stefanie Rach¹

¹Otto von Guericke University Magdeburg, Germany; silke.neuhaus@ovgu.de
²Otto von Guericke University Magdeburg, Germany; stefanie.rach@ovgu.de

In tertiary education, proof comprehension, that means reading and understanding written proofs, is an important activity in learning processes. However, to our knowledge there are no (empirical) studies analyzing the influence of reading strategies and the students' individual characteristics on proof comprehension, yet. We developed a proof comprehension test in analysis based on an assessment model of Mejia-Ramos et al. (2012) to test the relation of students’ individual characteristics to proof comprehension. To get a first in-depth look which factors influence proof comprehension we analyzed the data of 64 students in their second semester in university. Additionally, we asked the students about their use of reading strategies. The results show that proof comprehension correlates with prior knowledge and with the use of single reading strategies. Possible consequences for mathematical higher education are discussed.

Keywords: Proof comprehension, reading strategies, individual characteristics.

Introduction

Learning how to prove is one of the main components students have to encounter in tertiary mathematics programs. In order to understand, what a proof is and how to construct proofs, students read many proofs for example in their lectures or in their text books. Those proofs are given to the students by their lecturers, not only to show them, that an assumption is true, but also to teach proof methods and general ideas of mathematics that are used in the proof (Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012; Weber 2012). In many papers, understanding proof covers and is measured by different aspects like comprehending the written text, validating if a given argument can be categorized as a proof, evaluating of given arguments or even constructing a new proof. Stylianides (2015) questioned the validity and comparability of those different research findings:

“Take, for example, a study that draws conclusions about students’ understanding of proof based on students’ argument constructions in response to a number of proving tasks. This study is likely to report a poorer picture of students’ understanding of proof than another study that considered also students’ evaluations of their own constructions (…)” (Stylianides, 2015, p. 213)

Selden and Selden (2015) mention four concepts related to proof or proving in mathematics education research literature: proof construction, proof validation, proof evaluation, and proof comprehension. Proof construction is about doing proofs of given statements. Distinguishing if a written argument is a (correct) proof is defined as proof validation, and assessing if a given proof is nicely written is meant by proof evaluation. Lastly proof comprehension means to understand a written and correct proof. In this contribution, we focus on proof comprehension in the field of “analysis”, concrete which individual characteristics and reading strategies may influence proof comprehension.
Theoretical Framework and related literature

The definition of proof comprehension has to cover many different aspects and is connected to the question of how to measure proof comprehension. A first approach in this direction was done by Mejia-Ramos et al. (2012) by developing an assessment model for proof comprehension. Their model consists of seven dimensions, divided into two categories without a hierarchical order. Understanding a proof on a local basis includes to understand the meaning of the proof’s terms and statements, the logical status of statements and proof framework and the justifications of claims. To understand the proof on a holistic basis, one has to summarize the proof via the high-level ideas, identify the proof’s modular structure, transfer the general ideas or methods to another context and illustrate the proof with examples. Based on this assessment model, Mejia-Ramos and colleagues constructed proof comprehension tests and also gave an instruction manual on how to develop and validate such proof comprehension tests (Mejia-Ramos, Lew, La Torre, & Weber, 2017; Mejia-Ramos et al., 2012). Likewise, Hodds, Alcock, & Inglis (2014) developed proof comprehension tests and measured a positive influence of their self-explanation training on the proof comprehension of undergraduate students. To solve items of a proof comprehension test correctly, learners have to read the proof and sometimes to combine the content of the proof with prior knowledge. Items should be avoided that can be solved by referring to the prior knowledge only and not to the proof. These items would only assess prior knowledge and not proof comprehension.

In order to analyze students’ behavior while trying to comprehend written proofs, Weber (2015) videotaped and analyzed four excellent students in their senior year while reading proofs. He identified five strategies which the students used to foster proof comprehension: “(i) trying to prove a theorem before reading its proof, (ii) identifying the proof framework being used in the proof, (iii) breaking the proof into parts or sub-proofs, (iv) illustrating difficult assertions in the proof with an example, and (v) comparing the method used in the proof with one’s own approach” (Weber, 2015, p. 289). In a follow-up study reported in the same paper, Weber asked 83 mathematicians, whether they desired their students to implement those strategies, and most mathematicians did. In a separated study, Weber & Mejia-Ramos (2013) found similar strategies while watching major students read proofs and also present results indicating that a majority of students claimed not to use these strategies. In contrast to this Weber & Mejia-Ramos (2011, 2014) interviewed mathematicians why and in which way they read proofs. One reason to read proofs was to discover the proofs’ methods which may be useful for the mathematicians own research. To get the main ideas, the mathematicians zoom in on a proof’s problematic parts and zoom out and look at the high-level structure.

As far as we are aware, there is no (empirical) study analyzing whether the use of those strategies has indeed an influence on proof comprehension and whereas students benefit from the strategies mathematicians use. Psycho-linguistic studies showed that general reading comprehension correlates with prior knowledge and the use of reading strategies (e.g. Carell, 1983; Crowley & Azevedo, 2007; Pearson, Hansen, & Gordon, 1979.). We find similar results in mathematics education research referring to other concepts of proving. For example, Sommerhoff et al. (2016) analyzed the influence of domain specific and domain general cognitive student prerequisites on proof validation and showed, that conceptual mathematical knowledge influences the students’
proof validation skills. Those results suggest a possible relation between prior knowledge and the use of reading strategies on proof comprehension. Additionally, psycho linguistic researchers looked at the relation between individual characteristics and reading or text comprehension. For example, individual interest has a positive influence on text comprehension (see e.g. the overview of Schiefele, 1992). Despite this, we have not yet found an analysis concerning the relation between individual characteristics like interest or self-concept and proof comprehension.

All in all, we assume that the analysis of the relationship between reading strategies, individual characteristics and proof comprehension provide a deeper insight into the role of proof comprehension in undergraduate learning processes. Besides, such an analysis probably contributes to develop interventions supporting undergraduate students improving their proof comprehension.

**Research questions**

In this study we analyze the relation between proof comprehension, the use of reading strategies, and individual characteristics like interest in proof. Therefore, we focus on the following research questions:

1. Are proof reading strategies correlated with proof comprehension?
   
   According to the results of Weber (2015) and Weber & Mejia-Ramos (2013) we expected the above mentioned five reading strategies to influence proof comprehension. In addition, we expected some influence of the reading strategies based on the self-explanation training of Hodds et al. (2014).

2. Do individual characteristics (like self-concept, interest or prior knowledge) correlate with proof comprehension?
   
   General reading comprehension research showed relations between reading comprehension, prior knowledge and some individual characteristics, especially interest. We expected prior knowledge to influence significantly proof comprehension and we assumed at least a little relation between proof comprehension and interest in proof or interest in university mathematics.

**Method**

**Sample**

Our sample consist of 64 students (21 students in a bachelors’ program, 34 students in a teacher education program, to 9 students we have no information concerning their study program) in their second semester at a German university. Both types of students heard the same lectures till now. The mean value theorem, which was used in the proof comprehension test, was part of their Analysis I lecture, but with a slightly different proof.

**Measuring proof comprehension**

Based on the assessment model of Mejia-Ramos et al. (2012), we developed a proof comprehension test to the mean value theorem, which states that if f is a continuous function on a closed interval $[a, b] \subset \mathbb{R}$ and differentiable on the open interval $(a, b)$, then there exists a point $c \in (a, b)$, such that the tangent in the point $c$ equals the secant through the endpoints a and b of the interval. The
proof of the theorem uses a helper function $h$, and distinguishes two cases, $h$ constant and $h$ not constant, to show the assertion.

As the proof is not written in detail, for example not all justifications are explained, we could develop items based on the assessment model of Mejia-Ramos et al. (2012). First, we constructed open-ended items for every dimension of the assessment model. Then we wanted to change most of the items to multiple-choice items because the objectivity of the test is raised in this way. Still, most of the items remained as open items, because it was not possible to create good distractors. The first version of the test was given to experts of the khdm (Kompetenzzentrum für Hochschuldidaktik in Mathematik, in english: Centre for Higher Mathematics Education) for a review. We improved the test based on their answers and comments. Afterwards the new version was given to 52 students at the end of their first semester in an analysis lecture. Their answers were analyzed, and the test again revised. The final test had 10 items, 5 items referring to a more local comprehension and 5 to holistic comprehension. Four of them were multiple-choice Items. This is an example of a multiple-choice item referring to holistic comprehension:

<table>
<thead>
<tr>
<th>We prove $h(a) = h(b)$. Why is this useful in the proof?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h(a) = h(b)$ shows that $h$ is differentiable in $(a, b)$. (13.0%)</td>
</tr>
<tr>
<td>If $h(a) \neq h(b)$, then the maximum or minimum of $h$ could be on the endpoints of $[a, b]$. (right answer, 53.7%)</td>
</tr>
<tr>
<td>If $h(a) \neq h(b)$, then we could exclude the case „$h$ constant“. (24.1%)</td>
</tr>
<tr>
<td>$h(a) = h(b)$ shows $f(a) = f(b)$ and therefore $f$ could be constant. (9.3%)</td>
</tr>
</tbody>
</table>

Figure 1: Item in the proof comprehension test (percentage of chosen answers, $N = 62$)

We took care, that the items could not be answered by prior knowledge only. Each item was scored dichotomously, i.e. one point for a solution which was accepted as correct and zero points for other solutions. Concerning the multiple-choice items only one answer was correct (1 point). The open items were coded separately by two people given 1 point for a correct answer with only small mistakes (for example a missing declaration of variables used in the answer) and zero points for wrong answers. A missing answer was coded zero points if the student still answered items afterwards so that we assumed that the student had have enough time for answering the question.

**Measuring the use of proof-reading strategies**

In addition to the proof comprehension test, we developed different scales to assess students’ use of proof-reading strategies on a four-point Likert scale from 1 (disagree) to 4 (agree). In total we developed 22 items covering different kinds of reading strategies, using ideas from a German reading comprehension test (Schlagmüller & Schneider, 2007), the self-explanation training from Hodds et al. (2014), and the expected effective reading strategies (i) – (iii) from the study of Weber (2015) mentioned above. We assigned the proof-reading strategies to 5 different scales, which are listed in table 1.
<table>
<thead>
<tr>
<th>Before reading (4 items)</th>
<th>“I think about the assertion before I read the proof.”</th>
</tr>
</thead>
<tbody>
<tr>
<td>While reading (4 items)</td>
<td>“While reading the text, I highlight important propositions.”</td>
</tr>
<tr>
<td>Elaborated reading (5 items)</td>
<td>“I build connections between the different ideas of the proof.”</td>
</tr>
<tr>
<td>Reproductive reading (4 items)</td>
<td>“I sum up each line of the proof in my own words.”</td>
</tr>
<tr>
<td>After reading (5 items)</td>
<td>“I discuss the proof with my learning group.”</td>
</tr>
</tbody>
</table>

**Table 1: Learning strategies sorted in scales**

**Measuring individual characteristics**

Afterwards the students got a test to examine their prior knowledge in analysis (Rach & Heinze, 2017). The eight test items were coded dichotomously and students could gain up to 8 points in this test. The sum of their answers was computed to a test score for further analysis. Moreover, we asked the students to state their German and mathematics school grades and their grade of their analysis I course. At last we asked the students to estimate statements about other individual characteristics. They rated statements covering their interest in school and university mathematics (Ufer, Rach, & Kosiol, 2017), their interest in proof (Ufer, Rach, & Kosiol, 2017), their self-concept referring to proof (Ufer, Rach, & Kosiol, 2017) and to school respectively university mathematics in general (Ufer, Rach, & Kosiol, 2017) and their satisfaction referring to their university career (Schäfele & Jacob-Ebbinghaus, 2006). Because of practical constraints, some characteristics were only covered by one item. The scales of the individual characteristics (interest and self-concept) had good till very good reliabilities, so the individual mean value of a single student on a scale was computed if the student answered half of the items of the scale at least. In contrast, the reliabilities of the predicted scales for the proof-reading strategies were not good, so we only inserted single items in the analysis.

**Results**

The reliability of the proof comprehension test ($M = 3.59$, $SD = 1.83$, $Max = 10$ points, $\alpha = .57$) was medium. The single items’ mean value was $M = .15 - .76$ ($SD = 0.36 - 0.50$, $Max = 1$). An explorative factor analysis did not show any empirical evidence for different factors of proof comprehension, a local or a holistic understanding, so we measured proof comprehension by the sum of the whole test items as a test score. The following correlations were analyzed with Pearson correlation coefficient.

**RQ 1: Are proof reading strategies correlated with proof comprehension?**

Analyses show that there were only two correlations between proof comprehension and the use of single proof-reading strategies (negative correlation: “I skim read the proof and concentrate on important aspects” ($r = -.28$, $p < .05$), positive: “I divide the proof into coherent parts” ($r = .27$, $p < .05$)). The positive correlation refers to Weber’s (2015) third suggested useful proof-reading strategy. The correlation between Weber’s (ii) proof reading strategy and proof comprehension is nearly significant ($r = .21$, $p < .1$). As there are not many correlations, we looked at the mean values of the proof-reading strategies for further analysis. The absolute interpretation of mean values is
dangerous but carefully interpreting, it can give us a first insight into students reading strategies. Some of the proof-reading strategy items have high mean values, meaning that nearly all students said they do this while reading the proof, for example “I think about the assertion, before I read the proof” or “I read the proof several times”. Whereas the use of other strategies was only mentioned by a few people, for example “I prove the assertion on my own before reading the proof” which is the first of Weber’s (2015) assumed, useful reading strategies.

RQ 2: Do individual characteristics (like interest, self-concept or prior knowledge) correlate with proof comprehension?

As expected, proof comprehension correlates positively with the grade in the last analysis lecture ($r = .39, p < .01$) and with the prior knowledge in analysis ($r = .46, p < .01$). There is no correlation of the students’ final school grade or their last mathematics school grade. The proof comprehension only correlates negatively with the students’ interest in school mathematics ($r = -.27, p < .05$), there were no other significant correlations.

Discussion

Proof comprehension is an important activity in university mathematics, however students struggle a lot by performing this activity. To get a better insight into proof comprehension, we developed a proof comprehension test in the field of analysis to analyze the relation between proof comprehension and students’ individual characteristics, based on the assessment model of Mejia-Ramos et al. (2012). Our results show no empirical evidence for a separated local or global comprehension of proof. This supports the results of our first study (Neuhaus & Rach, 2018) we made with students at the beginning of their university career. We cannot see any differences between local or holistic items concerning the students’ answers. Mejia-Ramos et al. (2012) also don’t view the different types of their assessment model as part of a hierarchy. So while an empirical distinction between local and holistic proof comprehension might be difficult, we think that a theoretical separation is still helpful to get the whole picture of proof comprehension and to develop items for proof comprehension tests.

The correlation of the proof comprehension with prior knowledge was expected and shows that undoubtedly, for understanding proofs, a solid basis of concept understanding in the field is needed. However, there were no positive correlations between proof comprehension and the students’ interest in university mathematics or their mathematical self-confidence, which is somewhat surprising. We assumed that at least interest in proofs would predict proof comprehension with a small influence. Instead of this, a negative correlation between interest in school mathematics and proof comprehension suggests that interest in school mathematics and interest in university mathematics have to be analyzed separately.

Only one of Weber’s (2015) expected useful proof-reading strategies and none of the reading strategies based on Hodds et al.’s self-explanation training (2014) correlate with proof comprehension, but as only single items were included in the analysis, further studies need to be done. Another difference is, that Weber (2015) looked at good students reading proofs and predicted the used reading strategies to be useful to generate proof comprehension, but maybe only better students can benefit from those strategies. Thus, for the practical context, one should be
aware that not for every student every strategy is helpful. In addition, only few students declared to use one of Weber’s suggested helpful strategies and some other strategies, fitting the results of Weber & Mejia-Ramos (2013). Maybe the strategies need to be given to and practiced by the students in advance so they could benefit more from proofs given to them in their lectures.

Our proof comprehension test only refers to one proof in one mathematical topic (in this case analysis). For a broader view, proof comprehension has to be measured by different proofs in different topics. Additionally, we only assessed a partial part of the students’ proof comprehension as they were allowed to look at the proof while answering the proof comprehension test and maybe the questions indicated them to look to the proof in more detail. Furthermore, the sample is rather small and the students only report a self-estimation for their individual characteristics and their use of the proof reading strategies. There is no real evidence in which way and how often they really use the strategies. That’s why we want to reproduce the presented results with a bigger sample. Additionally, research is needed on the effectiveness of proof-reading strategies, the correlations between proof comprehension and other concepts of proving, like proof validation, and proof comprehension in different topics. With our study, we did a first step to get a deeper look into the concept of proof comprehension that may help us in the future to support students in this activity.

References


Stylianides, A. J. (2015). The role of mode of representation in students’ argument constructions. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 79–85). Prague, Czech Republic.


